Numerical Integration

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About Our Project

Purpose & Aim

The purpose of our project is to introduce the topic of Numerical Integration and expand not only our own knowledge on the background of this topic, but also that of others who wish to delve deeper into calculus & Computational Mathematics.

Through this project, we also aim to help others get a better grasp of the topic better through the variety of rules and visualization in our project.

Audience

This program is dedicated for anyone of any particular age. It is especially made to those wanting to get a better and clearer understanding and need help solving or visualizing a certain problem/ function at hand.

Scope

This application includes a diverse amount of features including graphing, computing, background, as well as GUI.

Our project will revolve around the big topic Numerical Integration and explore on subtopics including Trapezium Rule, Simpson's ⅓ Rule, as well as Simpson's ⅜ Rule.

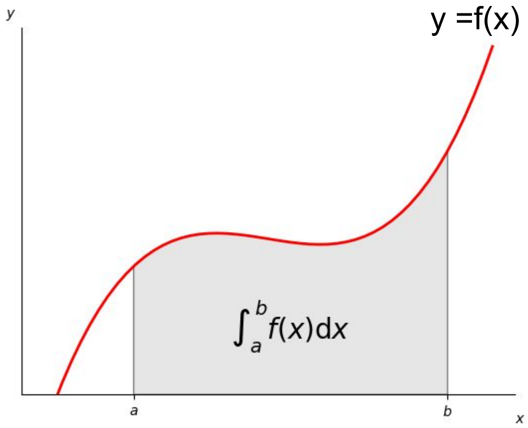
Mathematical Background

Integration

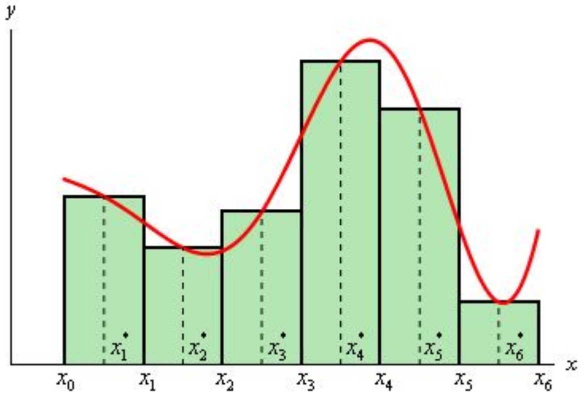
Backrground Info

$$\int ax^n\backslash\, dx = \frac{ax^{n+1}}{n+1} + c$$

$$\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

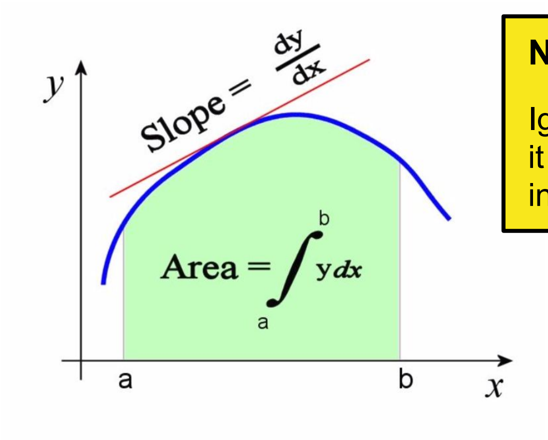


- Integration is the opposite of differentiation.
- Typically we divide the curve using rectangles to estimate the numerical integration of the function.
- The "S" symbol at the beginning is used to indicate "the integral of"
- "dx" at the end is used to indicate "with respect to x" similar to the symbol "dy/dx" in differentiation
- Indefinite Integral:
  - Integrals without limits of integration
  - Don't Forget to include "+ c" at the end of the solution
- Definite Integral:
  - Integrals with limits of integration
  - Don't need to include " + c" at the end of the solution



Steps on how to calculate the area under the graph:

1. Integrate the curve function f(x)
2. Substitute boundaries of x (limits)
3. Subtract one from another



Note:

Ignore c because it is a definite integral

When integrating by parts,

Tips on what to make as u:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}dx$$

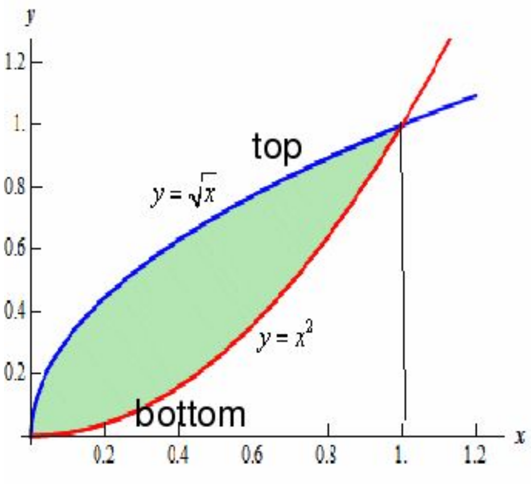
Remember: **LATE**

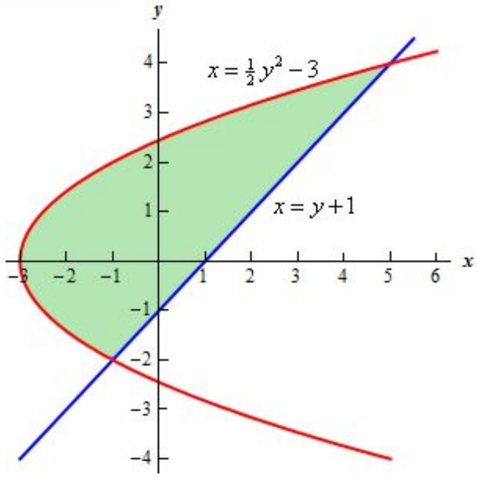
Logarithm **A**lgebra **T**rigonometry **e**

Area between two curves

In the case of finding the area enclosed between two curves, we must first subtract the function of the curve on the top or left from the one on the bottom or the right side if the graph.

The result of the subtraction will then be your new f(x) that you should integrate and substitute into to obtain the area.





USEFUL FORMULAS

$f(x)$	$\int f(x)\,dx$	
x^n	$\dfrac{x^{n+1}}{n+1}$	$(n \neq -1)$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$	
e^x	e^x	
$\sin x$	$-\cos x$	
$\cos x$	$\sin x$	
$\sec^2 x$	$\tan x$	
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$	$(x > a)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert$	$(\lvert x\rvert < a)$

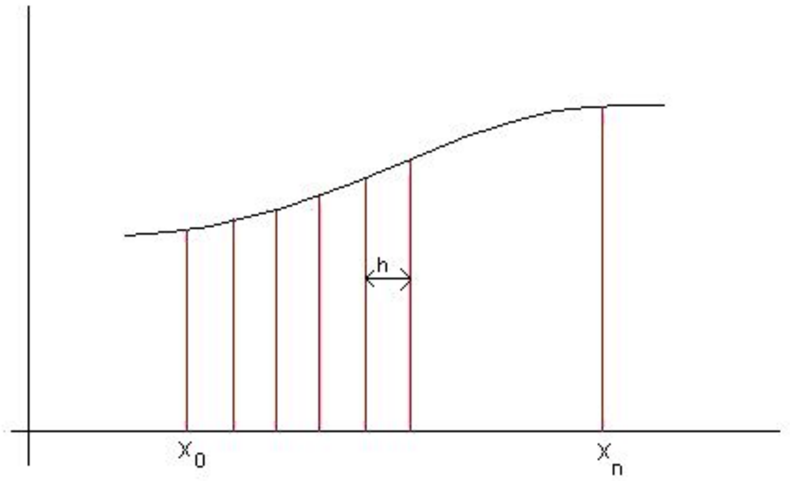
$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert$$

$f(x)$	$\int f(x)\,dx$	
$\sec x$	$\ln\lvert\sec x + \tan x\rvert = \ln\lvert\tan(\frac{1}{2}x + \frac{1}{4}\pi)\rvert$	$(\lvert x\rvert < \frac{1}{2}\pi)$
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x + \cot x\rvert = \ln\lvert\tan(\frac{1}{2}x)\rvert$	$(0 < x < \pi)$
$\sinh x$	$\cosh x$	
$\cosh x$	$\sinh x$	
$\operatorname{sech}^2 x$	$\tanh x$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$(\lvert x\rvert < a)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$	$(x > a)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$	

Trapezium Rule

Background info



The trapezium rule is a way of estimating the area under a curve. We know that the area under a curve is given by integration, so the trapezium rule gives a method of estimating integrals. This is useful when we come across integrals that we don't know how to evaluate.

The trapezium rule works by splitting the area under a curve into a number of trapeziums. The sum of area between each intervals of point xo through xn with its length (h) indicates the integration estimation.

Formula

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right],$$

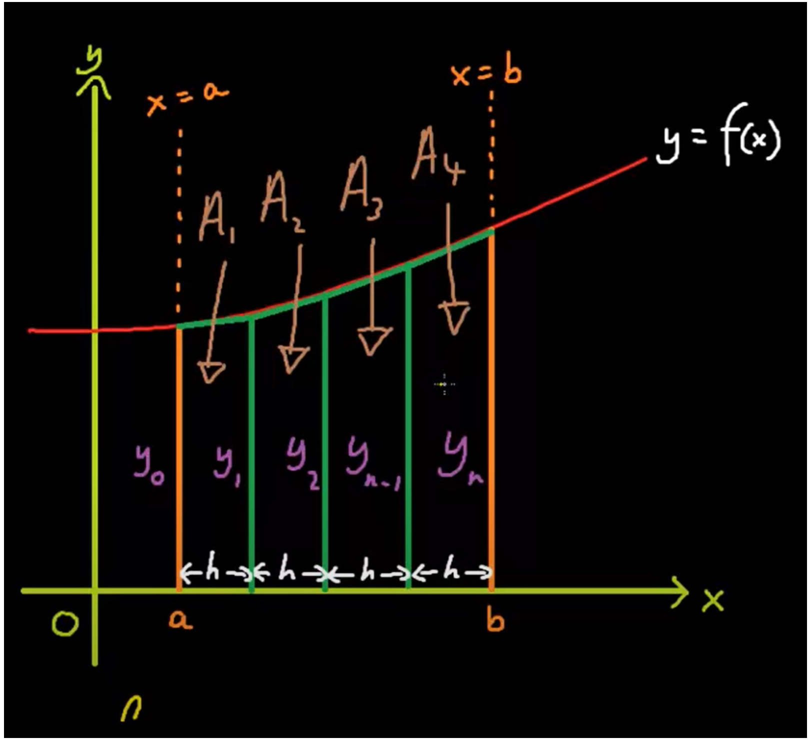
$$\Delta x = \frac{b-a}{n}$$

n = even

Snippet of our code:

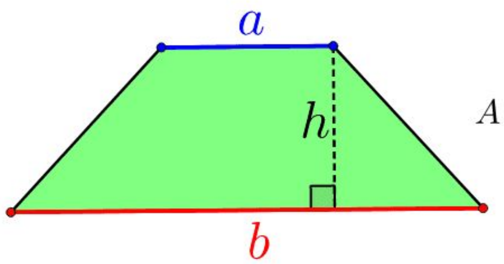
```
# Method to implement the Trapezium Rule
def trapezoidal(f, a, b, n):
    h = float ((b-a)/n)
    s = 0.0
    s += f(a)/2.0
    for i in range (1, n):
        s += f(a + i*h)
    s += f(b)/2.0
    return s *(h/2)
```

Formula Derivation



$$\int_a^b Y\,dx = A1 + A2 + A3 + A4$$

Area of a trapezium



$$A = \frac{1}{2}h(a+b)$$

So looking at the diagram,

A1 = h* (yo + y1) /2
A2 = h* (y1 + y2) /2
A3 = h*(y2 + yn-1) /2
A4 = h*(yn-1 + yn) /2

$$\int_a^b y\,dx = A1 + A2 + A3 + A4$$

$$= h^*(yo + y1)/2 + h^*(y1 + y2)/2 + h^*(y2 + yn-1)/2 + h^*(yn-1 + yn)/2$$

Factorize the formula

$$= h/2\,[\,(yo + yq) + (y1 + y2) + (y2 + yn-1) + (yn-1 + yn)\,]$$
$$= h/2\,[yo + y1 + y1 + y2 + y2 + yn-1 + yn-1 + yn\,]$$
$$= h/2\,[\,yo + 2^*y1 + 2^*y2 + 2^*yn-1 + yn\,]$$
$$= h/2\,[\,yo + 2\,(y1 + y2 + yn-1) + yn\,]$$

Finally we have this formula:

$$\boxed{\int_a^b y\,dx = h/2\,[yo + 2(y1 + y2 + \dots + yn-1) + yn]}$$

Simpson’s Rule

Background Info

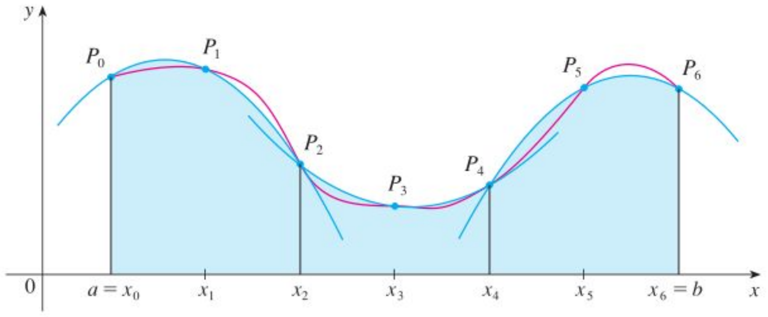


FIGURE 7

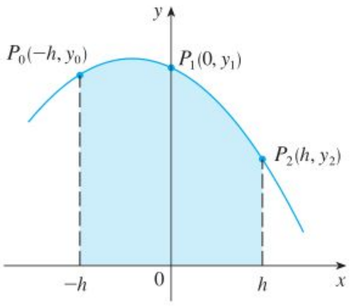


FIGURE 8

The Simpson's Rule is an extension of the Trapezium Rule and just like it's brother, the Simpson's rule is also used to calculate the numerical approximation of definite integrals.

However, unlike the trapezium rule, Simpson's rule doesn't rely on trapezoids but on parabolas as shown in figure 7. When using this method, we split the function into different parabolas and calculate the integration of each parabola before adding up the results.

Figure 8 shows an example of one of the parabolas of the function in figure 7 shifted to be symmetric about the origin. Shifting the parabola doesn't alter the result as the area of the curve will always remain the same.

We start with a typical Parabola Equation

$$\int_a^b Ax^2 + Bx + C1dx$$

We can write it this way

$$= \int_{-h}^{h} \left(Ax^2 + C\right) dx + \int_{-h}^{h} Bx\, dx$$

Even Function

- Odd Function
- Symmetrical integral = 0

$$= 2\left[\frac{Ax^3}{3} + Cx\right]\Bigg|_{-h}^{h}$$

$$= 2\left[\left(\frac{Ah^3}{3} + Ch\right) - 0\right]$$

Area under parabola:

$$= \frac{2Ah^3}{3} + 2Ch$$

After Factorization:

$$\frac{h}{3}\left(2Ah^2 + 6C\right)$$

$$y_o = A(-h)^2 + B(-h) + C$$
$$= Ah^2 - Bh + C$$

$$y_1 = A(0)^2 + B(0) + C$$
$$= C$$

$$y_2 = A(h)^2 + B(h) + C$$
$$= Ah^2 + Bh + C$$

$$y_o + 4y_1 + y_2 = Ah^2 - Bh + C + 4C + Ah^2 + Bh + C$$
$$= 2Ah^2 + 6C$$

It looks similar now.

We can now express area in terms of y_o , y_1 , y_2

Area under parabola: $\frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$

Area of 2nd parabola: $\frac{h}{3}\left(y_2 + 4y_3 + y_4\right)$

Area of last parabola: $\frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right)$

Don't Forget to refer to figure 7 and figure 8 for reference of the graph

After combining everything, we get:

$$= \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

If answer of area is a negative number, we ignore the (-) sign but still make use of the value

Finally, we get this formula:

$$\boxed{\frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 4f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right)}$$

Simpson's ⅓

- f(x) is approximated by a second order polynomial (quadratic);
- the quadratic interpolant being P(x).
- If a function is highly oscillatory or lacks derivatives at certain points, then the above rule may fail to produce accurate results.

Fun Fact:
In the formula, you will find a factor of 1 / 3. That's why, it is called **Simpson's 1/3 Rule**.

Simpson's ⅜

- Does the same exact thing as Simpson's ⅓
- f(x) is approximated by a third order polynomial (cubic);
- Uses one additional value compared to Simpson's ⅓
- Results in more accurate calculations than the ⅓ Rule.

Simpson’s 1/3

Formula

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left\{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right\}$$

$$\Delta x = \frac{b-a}{n}$$

n = multiple of 3

Snippet of our code:

```
def simpson1_3(f,a,b,n):
    h = float ((b-a)/n)
    x = a + h
    k = 0.0
    for i in range (1,n/2 + 1):
        k += 4*f(x)
        x += 2*h
    for i in range (1,n/2):
        k += 2*f(x)
        x += 2*h
    return (h/3)*(f(a)+f(b)+k)
```

Simpson’s 3/8

Formula

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left\{f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right\}$$

Snippet of our code:

```
def simpson3_8(f,a,b,n):
    h = float ((b-a)/n)
    s = f(a)+f(b)
    for i in range (1,n):
        if (i%3 == 0):
            s = s + 2 * f(a + i * n)
        else :
            s = s + 3 * f(a + i * n)
    return ((float(3*n)/8)*s )
```

Gaussian Quadrature

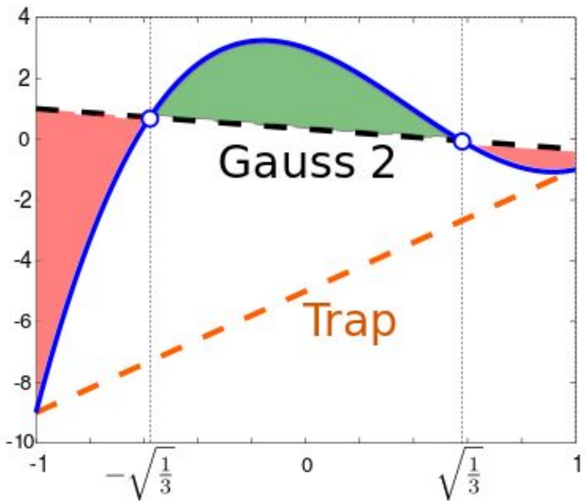
Background Info

When calculating an approximate integral in the form of:

$$\int_a^b w(x)f(x)dx$$

It is best to use gaussian integration formulas. This is because the weighting function, *w(x)*, can carry singular points. The distinction between Newton-Cotes rules and Gaussian integration formulas is that the nodes in Newton-Caton rules were evenly spaced. Meanwhile, the nodes in gaussian quadrature are weighted.

There are different formulas to quadratic integration, however we will focus on using Gauss-Legendre Quadrature formula.



Two-Point Gaussian Quadrature Rule

General formula: $I = w_o f_o + w_1 f_1$

We can assume this gives an exact result of integrating third order polynomial, giving us four unknowns which are: *w0, x0, w1, x1*

$$f(x) = Ax^3 + Bx^2 + Cx + D$$

After substituting f(x) into the general formula, we would find that

$$\int_{-1}^{+1} f(x)dx = w_0 f(x_0) + w_1 f(x_1)$$

$$\int_{-1}^{+1} [Ax^3 + Bx^2 + Cx + D]dx = w_o[Ax_o^3 + Bx_o^2 + Cx_o + D] + w_1[Ax_1^3 + Bx_1^2 + Cx_1 + D]$$

$$\Rightarrow$$

$$\left[\frac{Ax^4}{4} + \frac{Bx^3}{3} + \frac{Cx^2}{2} + Dx\right]_{-1}^{+1} = w_o(Ax_o^3 + Bx_o^2 + Cx_o + D) + w_1(Ax_1^3 + Bx_1^2 + Cx_1 + D)$$

$$\Rightarrow$$

$$A[w_o x_o^3 + w_1 x_1^3] + B\left[w_o x_o^2 + w_1 x_1^2 - \frac{2}{3}\right] + C[w_o x_o + w_1 x_1] + D[w_o + w_1 - 2] = 0$$

Since the coefficients of A, B, C and D are arbitrary

$$w_0 x_0^3 + \ w_1 w_1^3 = 0$$

$$w_0 x_0^2 + \ w_1 x_1^2 - \frac{2}{3} = \ 0$$

$$w_0 x_0 + \ w_1 x_1 = 0$$

$$w_0 + w_1 - 2 = 0$$

The weighting factors and function arguments for two point gaussian quadrature integration as we have summarised.

$$w_0 = 1 \ \ and \ \ w_1 = 1$$

$$x_0 = \ -\sqrt{\frac{1}{3}} \ \ and \ x_1 = \ \sqrt{\frac{1}{3}}$$

Two-Point Formula

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

$$= \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right)$$

Three-Point Formula

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots\ldots + c_n f(x_n)$$

Example

Question

Given this value, find the solution using all of the methods

x	f(x)
0	1.0
0.25	0.9896
0.5	0.9589
0.75	0.9089
1.0	0.8415

SOLUTION

Trapezium Rule

$\int Ydx = h2(y0 + 2(y1 + y2 + y3) + y4)$

$\int Ydx = 0.252(1 + 2*(0.9896 + 0.9589 + 0.9089) + 0.8415)$

$\int Ydx = 0.252(1 + 2*(2.8574) + 0.8415)$

$\int Ydx = 0.94454$

Solution By Trapezoidal Rule is 0.94454

Simpson's ⅓ Rule

$\int Ydx = h3(y0 + 4(y1 + y3) + 2(y2) + y4)$

$\int Ydx=0.253(1+4*(0.9896+0.9089)+2*(0.9589)+0.8415)$

$\int Ydx=0.253(1+4*(1.8985)+2*(0.9589)+0.8415)$

$\int Ydx=0.94611$

Solution By Simpson's 13 Rule is 0.94611

Simpson's ⅜ Rule

$\int Ydx = 3h*8(y0 + 2(y3) + 3(y1+y2)+y4)$

$\int Ydx=3*0.258(1+2*(0.9089)+3*(0.9896+0.9589)+0.8415)$

$\int Ydx=3*0.258(1+2*(0.9089)+3*(1.9485)+0.8415)$

$\int Ydx=0.89108$

Solution By Simpson's 38 Rule is 0.89108

Project Demonstration

Kindly watch the video attached:

Link : https://codecollab.io/@callistarl/Numerical%20Integration

Resources & References:

- https://www.instructables.com/id/How-to-Make-a-Numerical-Integration-Program-in-Pyt/
- https://revisionmaths.com/advanced-level-maths-revision/pure-maths/calculus/trapezium-rule
- https://www.freecodecamp.org/news/simpsons-rule/
- https://revisionmaths.com/advanced-level-maths-revision/pure-maths/calculus/integration-techniques
- https://www.cambridgeinternational.org/Images/417318-list-of-formulae-and-statistical-tables.pdf
- https://www.znotes.org/caie/igcse/add-maths-0606/theory?chapterId=zuD9uGsh3vJieBop2&cardId=9q3XEMnFT3Edmqqor
- http://atozmath.com/example/CONM/NumeInte.aspx?he=e
- https://youtu.be/YwrO4kvp5XI
- https://youtu.be/7MoRzPObRf0
- https://www.youtube.com/watch?v=7A_csP9drJw
- https://www.youtube.com/watch?v=8exB6Ly3nx0
- https://www.math.arizona.edu/~alexa/m129/simpson.pdf
- https://runestone.academy/runestone/books/published/thinkcspy/GUIandEventDrivenProgramming/02_standard_dialog_boxes.html
- https://docs.python.org/3/library/tkinter.ttk.html#scrollable-widget-options

Thank You

That’s it from us, hope it was informative...

Enjoy our project !!!