CHAPTER II

# The Čech–de Rham Complex

## §8 The Generalized Mayer–Vietoris Principle

### Reformulation of the Mayer–Vietoris Sequence

Let $U$ and $V$ be open sets on a manifold. In Section 2, we saw that the sequence of inclusions

$$U \cup V \leftarrow U \coprod V \leftleftarrows U \cap V$$

gives rise to an exact sequence of differential complexes

$$0 \to \Omega^*(U \cup V) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0$$

called the *Mayer–Vietoris sequence*. The associated long exact sequence

$$\cdots \to H^q(U \cup V) \xrightarrow{\alpha} H^q(U) \oplus H^q(V) \xrightarrow{\delta} H^q(U \cap V) \xrightarrow{d^*} H^{q+1}(U \cup V) \to \cdots$$

allows one to compute in many cases the cohomology of the union $U \cup V$ from the cohomology of the open subsets $U$ and $V$. In this section, the Mayer-Vietoris sequence will be generalized from two open sets to countably many open sets. The main ideas here are due to Weil [1].

To make this generalization more transparent, we first reformulate the Mayer-Vietoris sequence for two open sets as follows. Let $\mathfrak{U}$ be the open cover $\{U, V\}$. Consider the double complex $C^*(\mathfrak{U}, \Omega^*) = \bigoplus K^{p,q} = \bigoplus C^p(\mathfrak{U}, \Omega^q)$ where

$$K^{0,q} = C^0(\mathfrak{U}, \Omega^q) = \Omega^q(U) \oplus \Omega^q(V),$$

$$K^{1,q} = C^1(\mathfrak{U}, \Omega^q) = \Omega^q(U \cap V),$$

$$K^{p,q} = 0, \quad p \geq 2.$$

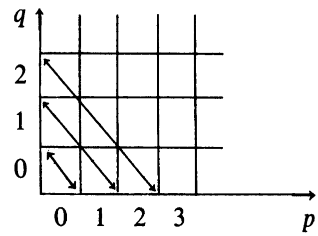
$$
\begin{array}{c|c|c|c}
q\uparrow & & & \\
 & \vdots & \vdots & \vdots \\
3 & & & \\
2 & \Omega^2(U)\oplus\Omega^2(V) & \Omega^2(U\cap V) & 0 \\
d\uparrow\ 1 & \Omega^1(U)\oplus\Omega^1(V) & \Omega^1(U\cap V) & 0 \\
0 & \Omega^0(U)\oplus\Omega^0(V) & \Omega^0(U\cap V) & 0 \\
\hline
 & 0 & 1 & 2\quad p
\end{array}
$$

$$\xrightarrow{\delta}$$

This double complex is equipped with two differential operators, the exterior derivative $d$ in the vertical direction and the difference operator $\delta$ in the horizontal direction. Of course, $\delta$ is 0 after the first column. Because $d$ and $\delta$ are independent operators, they commute.

In general given a doubly graded complex $K^{*,*}$ with commuting differentials $d$ and $\delta$, one can form a singly graded complex $K^*$ by summing along the antidiagonal lines

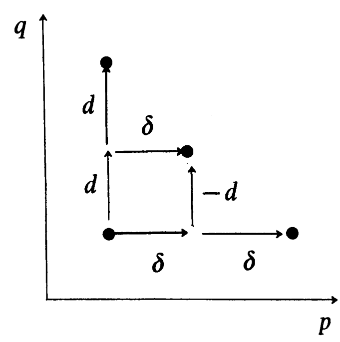
$$K^n = \bigoplus_{p+q=n} K^{p,q}$$




and defining the differential operator to be

$$D = D' + D'' \text{ with } D' = \delta,\ D'' = (-1)^p d \text{ on } K^{p,q}.$$

Remark on the Definition of D.




If $D$ were naively defined as $\tilde{D} = d + \delta$, it would not be a differential operator since $\tilde{D}^2 = 2d\delta \neq 0$. However, if we alternate the sign of $d$ from one column to the next, then as is apparent from the diagram above,

$$D^2 = d^2 + \delta d - d\delta + \delta^2 = 0.$$

In the sequel we will use the same symbol $C^*(\mathfrak{U}, \Omega^*)$ to denote the double complex and its associated single complex. In this setup, the Mayer-Vietoris principle assumes the following form.

**Theorem 8.1.** *The double complex $C^*(\mathfrak{U}, \Omega^*)$ computes the de Rham cohomology of $M$:*

$$H_D\{C^*(\mathfrak{U}, \Omega^*)\} \simeq H^*_{DR}(M).$$

PROOF. In one direction there is the natural map

$$r\colon \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \subset C^*(\mathfrak{U}, \Omega^*)$$

given by the restriction of forms. Our first observation is that $r$ is a chain map, i.e., that the following diagram is commutative:

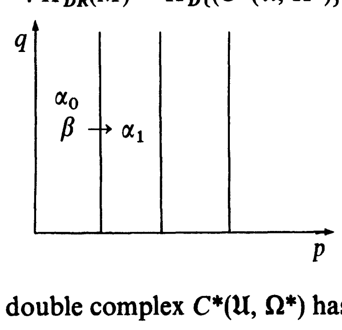
$$\begin{tikzcd} \Omega^*(M) \arrow[r, "r"] & C^*(\mathfrak{U}, \Omega^*) \\ \Omega^*(M) \arrow[r, "r"'] \arrow[u, "d"] & C^*(\mathfrak{U}, \Omega^*) \arrow[u, "D"'] \end{tikzcd}.$$

This is because

$$\begin{aligned} Dr &= (\delta + (-1)^p\, d)r \quad [\text{here } p = 0] \\ &= dr \\ &= rd\,. \end{aligned}$$

Consequently $r$ induces a map in cohomology

$$r^* : H^*_{DR}(M) \to H_D\{(C^*(\mathfrak{U}, \Omega^*)\}.$$




A $q$-cochain $\alpha$ in the double complex $C^*(\mathfrak{U}, \Omega^*)$ has two components

$$\alpha = \alpha_0 + \alpha_1, \qquad \alpha_0 \in K^{0,q}, \qquad \alpha_1 \in K^{1,q-1}.$$

By the exactness of the Mayer–Vietoris sequence there exists a $\beta$ such that $\delta\beta = \alpha_1$. With this choice of $\beta$, $\alpha - D\beta$ has only the $(0, q)$-component. Thus, *every cochain in $C^*(\mathfrak{U}, \Omega^*)$ is $D$-cohomologous to a cochain with only the top component.*

We now show $r^*$ to be an isomorphism.

*Step 1. $r^*$ is surjective.*

By the remark above we may assume that a given cohomology class in $H_D\{C^*(\mathfrak{U}, \Omega^*)\}$ is represented by a cocycle $\phi$ with only the top component. In this case

$$D\phi = 0 \qquad \text{if and only if} \qquad d\phi = \delta\phi = 0.$$

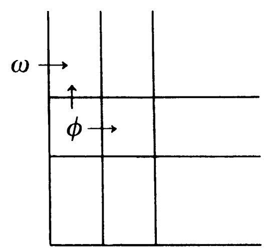
So $\phi$ is a global closed form.

*Step 2. $r^*$ is injective.*

Suppose $r(\omega) = D\phi$ for some cochain $\phi$ in $C^*(\mathfrak{U}, \Omega^*)$. Again by the remark above we may write $\phi = \phi' + D\phi''$, where $\phi'$ has only the top component. Then

$$r(\omega) = D\phi' = d\phi', \quad \delta\phi' = 0.$$

So $\omega$ is the exterior derivative of a global form on $M$.




□

## Generalization to Countably Many Open Sets and Applications

Instead of a cover with two open sets as in the usual Mayer-Vietoris sequence, consider the open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in J}$ of $M$, where the index set $J$ is a countable *ordered* set. Of course $J$ may be finite. Denote the pairwise intersections $U_\alpha \cap U_\beta$ by $U_{\alpha\beta}$, triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ by $U_{\alpha\beta\gamma}$, etc. There is a sequence of inclusions of open sets

$$M \leftarrow \coprod U_{\alpha_0} \underset{\overset{\partial_1}{\longleftarrow}}{\overset{\partial_0}{\longleftarrow}} \coprod_{\alpha_0 < \alpha_1} U_{\alpha_0\alpha_1} \underset{\overset{\partial_2}{\longleftarrow}}{\overset{\overset{\partial_0}{\longleftarrow}}{\overset{\partial_1}{\longleftarrow}}} \coprod_{\alpha_0 < \alpha_1 < \alpha_2} U_{\alpha_0\alpha_1\alpha_2} \begin{smallmatrix}\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow\end{smallmatrix} \cdots$$

where $\partial_i$ is the inclusion which "ignores" the $i$th open set; for example,

$$\partial_0 : U_{\alpha_0\alpha_1\alpha_2} \hookrightarrow U_{\alpha_1\alpha_2}$$

This sequence of inclusions of open sets induces a sequence of restrictions of forms

$$\Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \underset{\longrightarrow}{\overset{\delta_0}{\underset{\delta_1}{\longrightarrow}}} \prod_{\alpha_0 < \alpha_1} \Omega^*(U_{\alpha_0\alpha_1}) \underset{\overset{\delta_2}{\longrightarrow}}{\overset{\overset{\delta_0}{\longrightarrow}}{\overset{\delta_1}{\longrightarrow}}} \prod_{\alpha_0 < \alpha_1 < \alpha_2} \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \begin{smallmatrix}\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow\end{smallmatrix} \cdots$$

where $\delta_0$, for instance, is induced from the inclusion

$$\partial_0 : \coprod_\alpha U_{\alpha\beta\gamma} \to U_{\beta\gamma}$$

and therefore is the restriction

$$\delta_0 : \Omega^*(U_{\beta\gamma}) \to \prod_\alpha \Omega^*(U_{\alpha\beta\gamma}).$$

We define the difference operator $\delta : \prod\Omega^*(U_{\alpha_0\alpha_1}) \to \prod\Omega^*(U_{\alpha_0\alpha_1\alpha_2})$ to be the alternating difference $\delta_0 - \delta_1 + \delta_2$. Thus

$$(\delta\xi)_{\alpha_0\alpha_1\alpha_2} = \xi_{\alpha_1\alpha_2} - \xi_{\alpha_0\alpha_2} + \xi_{\alpha_0\alpha_1}.$$

More generally the difference operator is defined as follows.

**Definition 8.2.** If $\omega \in \prod\Omega^q(U_{\alpha_0\ldots\alpha_p})$, then $\omega$ has "components" $\omega_{\alpha_0\ldots\alpha_p} \in \Omega^q(U_{\alpha_0\ldots\alpha_p})$ and

$$(\delta\omega)_{\alpha_0\ldots\alpha_{p+1}} = \sum_{i=0}^{p+1}(-1)^i\omega_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_{p+1}},$$

where on the right-hand side the restriction operation to $U_{\alpha_0\ldots\alpha_{p+1}}$ has been suppressed and the caret denotes omission.

**Proposition 8.3.** $\delta^2 = 0$.

Proof. Basically this is true because in $(\delta^2\omega)_{\alpha_0\ldots\alpha_{p+2}}$ we omit two indices $\alpha_i$, $\alpha_j$ twice with opposite signs. To be precise,

$$\begin{aligned}(\delta^2\omega)_{\alpha_0\ldots\alpha_{p+2}} &= \sum(-1)^i(\delta\omega)_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_{p+2}}\\ &= \sum_{j<i}(-1)^i(-1)^j\omega_{\alpha_0\ldots\hat{\alpha}_j\ldots\hat{\alpha}_i\ldots\alpha_{p+2}}\\ &\quad+\sum_{j>i}(-1)^i(-1)^{j-1}\omega_{\alpha_0\ldots\hat{\alpha}_i\ldots\hat{\alpha}_j\ldots\alpha_{p+2}}\\ &= 0.\end{aligned}$$

□

*Convention.* Up until now the indices in $\omega_{\alpha_0\ldots\alpha_p}$ are all in increasing order $\alpha_0 < \ldots < \alpha_p$. More generally we will allow indices in any order, even with repetitions, subject to the convention that when two indices are interchanged, the form becomes its negative:

$$\omega_{\ldots\alpha\ldots\beta\ldots} = -\omega_{\ldots\beta\ldots\alpha\ldots}.$$

In particular a form with repeated indices is 0. In the following exercise the reader is asked to check that this convention is consistent with the definition of the difference operator $\delta$ above.

*Exercise 8.4.* Suppose $\alpha < \beta$. Then $(\delta\omega)_{\ldots\beta\ldots\alpha\ldots}$ may be defined either as $-(\delta\omega)_{\ldots\alpha\ldots\beta\ldots}$ or by the difference operator formula (8.2). Show that these two definitions agree.

**Proposition 8.5.** (The Generalized Mayer–Vietoris Sequence). *The sequence*

$$0 \longrightarrow \Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \xrightarrow{\delta} \prod \Omega^*(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \prod \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \xrightarrow{\delta} \cdots$$

*is exact; in other words, the $\delta$-cohomology of this complex vanishes identically.*

PROOF. Clearly $\Omega^*(M)$ is the kernel of the first $\delta$ since an element of $\prod \Omega^*(U_{\alpha_0})$ is a global form on $M$ if and only if its components agree on the overlaps.

Now let $\{\rho_\alpha\}$ be a partition of unity subordinate to the open cover $\mathfrak{U} = \{U_\alpha\}$. Suppose $\omega \in \prod \Omega^*(U_{\alpha_0 \ldots \alpha_p})$ is a $p$-cocycle. Define a $(p-1)$-cochain $\tau$ by

$$\tau_{\alpha_0 \ldots \alpha_{p-1}} = \sum_\alpha \rho_\alpha \omega_{\alpha\alpha_0 \ldots \alpha_{p-1}}.$$

Then

$$\begin{aligned}(\delta\tau)_{\alpha_0 \ldots \alpha_p} &= \sum_i (-1)^i \tau_{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_p} \\ &= \sum_{i,\alpha} (-1)^i \rho_\alpha \omega_{\alpha\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_p}.\end{aligned}$$

Because $\omega$ is a cocycle,

$$(\delta\omega)_{\alpha\alpha_0 \ldots \alpha_p} = \omega_{\alpha_0 \ldots \alpha_p} + \sum_i (-1)^{i+1} \omega_{\alpha\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_p} = 0.$$

So

$$\begin{aligned}(\delta\tau)_{\alpha_0 \ldots \alpha_p} &= \sum_\alpha \rho_\alpha \sum_i (-1)^i \omega_{\alpha\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_p} \\ &= \sum_\alpha \rho_\alpha \omega_{\alpha_0 \ldots \alpha_p} \\ &= \omega_{\alpha_0 \ldots \alpha_p}.\end{aligned}$$

This shows that every cocycle is a coboundary. The exactness now follows from Proposition 8.3. □

In fact, the definition of $\tau$ in this proof gives a homotopy operator on the complex. Write $K\omega$ for $\tau$:

(8.6) $$(K\omega)_{\alpha_0 \ldots \alpha_{p-1}} = \sum_\alpha \rho_\alpha \omega_{\alpha\alpha_0 \ldots \alpha_{p-1}}.$$

Then

$$\begin{aligned}(\delta K\omega)_{\alpha_0 \ldots \alpha_p} &= \sum (-1)^i (K\omega)_{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_p} \\ &= \sum (-1)^i \rho_\alpha \omega_{\alpha\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_p} \\ (K\delta\omega)_{\alpha_0 \ldots \alpha_p} &= \sum \rho_\alpha (\delta\omega)_{\alpha\alpha_0 \ldots \alpha_p} \\ &= \left(\sum \rho_\alpha\right) \omega_{\alpha_0 \ldots \alpha_p} + \sum (-1)^{i+1} \rho_\alpha \omega_{\alpha\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_p} \\ &= \omega_{\alpha_0 \ldots \alpha_p} - (\delta K\omega)_{\alpha_0 \ldots \alpha_p}.\end{aligned}$$

Therefore, $K$ is an operator from $\prod \Omega^*(U_{\alpha_0 \ldots \alpha_p})$ to $\prod \Omega^*(U_{\alpha_0 \ldots \alpha_{p-1}})$ such that

$$\delta K + K \delta = 1. \tag{8.7}$$

As in the proof of the Poincaré lemma, the existence of a homotopy operator on a differential complex implies that the cohomology of the complex vanishes.

For future reference we note here that if $\phi$ is a cocycle, then by (8.7), $\delta K \phi = \phi$. So on cocycles $K$ is a right inverse to $\delta$. Given such $\phi$, the set of all solutions $\xi$ of $\delta \xi = \phi$ consists of $K\phi + \delta$-coboundaries.

The Mayer-Vietoris sequence may be arranged as an augmented double complex

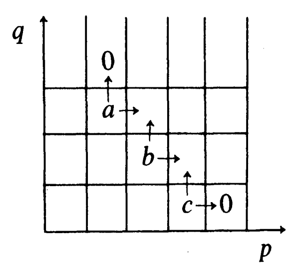
$$\begin{array}{r|c|c|c|c|}
q \uparrow & & & & \\
0 \longrightarrow \Omega^2(M) \xrightarrow{r} & K^{0,2} & K^{1,2} & & \\ \hline
0 \longrightarrow \Omega^1(M) \xrightarrow{r} & K^{0,1} & K^{1,1} & & \\ \hline
0 \longrightarrow \Omega^0(M) \xrightarrow{r} & K^{0,0} & K^{1,0} & & \\ \hline
\multicolumn{5}{r}{\longrightarrow p} \\
\end{array},$$

where $K^{p,q} = C^p(\mathfrak{U}, \Omega^q) = \prod \Omega^q(U_{\alpha_0 \ldots \alpha_p})$ consists of the "$p$-cochains of the cover $\mathfrak{U}$ with values in the $q$-forms." The horizontal maps of the double complex are the difference operators $\delta$ and the vertical ones the exterior derivatives $d$. As before, the double complex may be made into a single complex with the differential operator given by

$$D = D' + D'' = \delta + (-1)^p d.$$

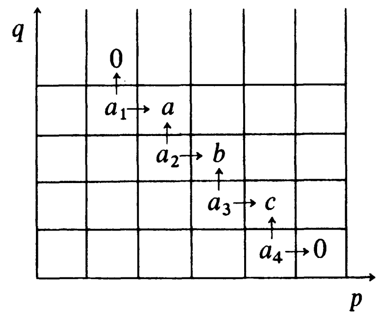
A $D$-cocycle is a string such as $\phi = a + b + c$ with

$$\begin{aligned} da &= 0, \\ \delta a &= \pm db \\ \delta b &= \pm dc \\ \delta c &= 0, \end{aligned}$$




(To be precise we should write $\delta a = -D''b$, $\delta b = -D''c$.) So a $D$-cocycle may be pictured as a "zig-zag."

A $D$-coboundary is a string such as $\phi = a + b + c$ in the figure below, where $a = \delta a_1 + D'' a_2$, etc.




The double complex

$$C^*(\mathfrak{U}, \Omega^*) = \bigoplus_{p,\, q \geq 0} C^p(\mathfrak{U}, \Omega^q)$$

is called the *Čech–de Rham complex*, and an element of the Čech–de Rham complex is called a *Čech–de Rham cochain*. We sometimes refer to a Čech–de Rham cochain more simply as a $D$-cochain.

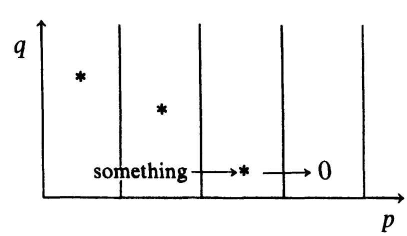
The fact that all the rows of the augmented complex are exact is the key ingredient in the proof of the following.

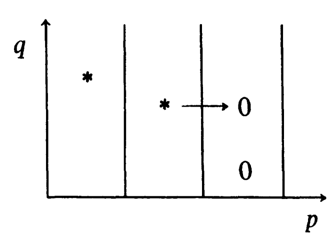
**Proposition 8.8** (Generalized Mayer–Vietoris Principle). *The double complex $C^*(\mathfrak{U}, \Omega^*)$ computes the de Rham cohomology of $M$; more precisely, the restriction map $r : \Omega^*(M) \to C^*(\mathfrak{U}, \Omega^*)$ induces an isomorphism in cohomology:*

$$r^* : H^*_{DR}(M) \to H_D\{C^*(\mathfrak{U}, \Omega^*)\}.$$

Proof. Since $Dr = (\delta + d)\, r = dr = rd$, $r$ is a chain map, and so it induces a map $r^*$ in cohomology.

*Step 1. $r^*$ is surjective.*

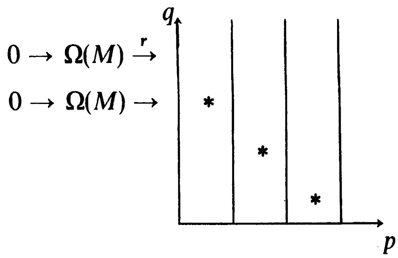


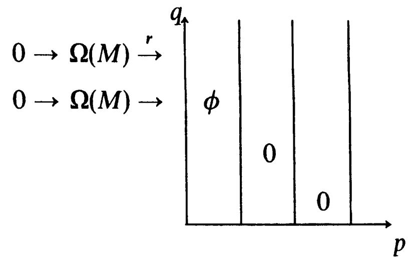




Let $\phi$ be a cocycle relative to $D$. By $\delta$-exactness the lowest component of $\phi$ is $\delta$ of something. By subtracting $D$(something) from $\phi$, we can remove the lowest component of $\phi$ and still stay in the same cohomology class as $\phi$.

After iterating this procedure enough times we can move $\phi$ in its cohomology class to a cocycle $\phi'$ with only the top component. $\phi'$ is a closed global form because $d\phi' = 0$ and $\delta\phi' = 0$.

*Step 2. $r^*$ is injective.*




If $r(\omega) = D\phi$, we can shorten $\phi$ as before by subtracting boundaries until it consists of only the top component. Then because $\delta\phi$ is 0, it is actually a global form on $M$. So $\omega$ is exact. □

The proof of this proposition is a very general argument from which we may conclude: *if all the rows of an augmented double complex are exact, then the D-cohomology of the complex is isomorphic to the cohomology of the initial column.*

It is natural to augment each column by the kernel of the bottom $d$, denoted $C^*(\mathfrak{U}, \mathbb{R})$. The vector space $C^p(\mathfrak{U}, \mathbb{R})$ consists of the locally constant functions on the $(p+1)$-fold intersections $U_{\alpha_0 \dots \alpha_p}$.

$$
\begin{array}{l|l|l|l}
0 \to \Omega^2(M) \xrightarrow{r} & \prod \Omega^2(U_{\alpha_0}) & & \\
0 \to \Omega^1(M) \to & \prod \Omega^1(U_{\alpha_0}) & & \\
0 \to \Omega^0(M) \to & \prod \Omega^0(U_{\alpha_0}) & \prod \Omega^0(U_{\alpha_0 \alpha_1}) & \prod \Omega^0(U_{\alpha_0 \alpha_1 \alpha_2}) \\
\hline
 & \quad i\uparrow & \quad i\uparrow & \quad i\uparrow \qquad p \\
 & C^0(\mathfrak{U}, \mathbb{R}) \to & C^1(\mathfrak{U}, \mathbb{R}) \to & C^2(\mathfrak{U}, \mathbb{R}) \to \\
 & \quad \uparrow & \quad \uparrow & \quad \uparrow \\
 & \quad 0 & \quad 0 & \quad 0
\end{array}
$$

The bottom row

$$C^0(\mathfrak{U}, \mathbb{R}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathbb{R}) \xrightarrow{\delta} C^2(\mathfrak{U}, \mathbb{R}) \xrightarrow{\delta}$$

is a differential complex, and the homology of this complex, $H^*(\mathfrak{U}, \mathbb{R})$, is called the *Čech cohomology of the cover* $\mathfrak{U}$. This is a purely combinatorial object. Note that the argument for the exactness of the generalized Mayer-Vietoris sequence breaks down for the complex $C^*(\mathfrak{U}, \mathbb{R})$, because here the cochains are locally constant functions so that partitions of unity are not applicable.

If the augmented columns of the complex $C^*(\mathfrak{U}, \Omega^*)$ are exact, then the same argument as in (8.8) will yield an isomorphism between the Čech cohomology and the cohomology of the double complex

$$H^*(\mathfrak{U}, \mathbb{R}) \xrightarrow{\sim} H_D\{C^*(\mathfrak{U}, \Omega^*)\},$$

and consequently an isomorphism between de Rham cohomology and Čech cohomology

$$H^*_{DR}(M) \simeq H^*(\mathfrak{U}, \mathbb{R}).$$

Now the failure of the $p^{\text{th}}$ column to be exact is measured by the cohomology groups

$$\prod_{\substack{q \geqslant 1 \\ \alpha_0 < \cdots < \alpha_p}} H^q(U_{\alpha_0 \ldots \alpha_p}).$$

So if the cover is such that all finite nonempty intersections are contractible, e.g., a good cover, then all augmented columns will be exact. We have proven

**Theorem 8.9.** *If $\mathfrak{U}$ is a good cover of the manifold $M$, then the de Rham cohomology of $M$ is isomorphic to the Čech cohomology of the good cover*

$$H^*_{DR}(M) \simeq H^*(\mathfrak{U}, \mathbb{R}).$$

Let us recapitulate here what has transpired so far. First, the basic sequence of inclusions

$$M \leftarrow U_\alpha \leftleftarrows U_{\alpha\beta} \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow}} U_{\alpha\beta\gamma} \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}} \cdots$$

gives rise to the diagram

$$\begin{tikzcd}
\begin{array}{l}\text{differential}\\ \text{geometry of}\\ \text{forms}\end{array} & 0 \arrow[r] & \Omega^*(M) \arrow[r, "r"] & C^*(\mathfrak{U}, \Omega^*) \\
& & & C^*(\mathfrak{U}, \mathbb{R}) \arrow[u] \\
& & & 0 \arrow[u] \\
& & & \begin{array}{c}\text{combinatorics}\\ \text{of the cover}\end{array}
\end{tikzcd}$$

Along the left-hand side is the differential geometry of forms on $M$, along the bottom is the combinatorics of the cover $\mathfrak{U} = \{U_\alpha\}$, and in the double complex itself the two are mixed. As the complex is the generalized Mayer–Vietoris sequence, the augmented rows are exact, for *any* cover. It follows that the de Rham cohomology of $M$ is always isomorphic to the cohomology of the double complex:

$$H^*_{DR}(M) \simeq H_D\{C^*(\mathfrak{U}, \Omega^*)\}.$$

If in addition $\mathfrak{U}$ is a *good* cover, then by the Poincaré lemma the augmented columns are exact. In that case the Čech cohomology of the cover is also isomorphic to the cohomology of the double complex:

$$H^*(\mathfrak{U}, \mathbb{R}) \simeq H_D\{C^*(\mathfrak{U}, \Omega^*)\}.$$

Hence there is an isomorphism between de Rham and Čech. This result provides us with a way of computing the de Rham cohomology by means of combinatorics, since from Section 5 we know that every manifold has a good cover. All three complexes here can be given product structures, in which case the isomorphisms between them are actually isomorphisms of algebras, as will be shown in (14.28).

A priori there is no reason why different covers of $M$ should have the same Čech cohomology. However, it follows from Theorem 8.9 that

**Corollary 8.9.1.** *The Čech cohomology $H^*(\mathfrak{U}, \mathbb{R})$ is the same for all good covers $\mathfrak{U}$ of $M$.*

If a manifold is compact, then it has a *finite* good cover. For such a cover the Čech cohomology $H^*(\mathfrak{U}, \mathbb{R})$ is clearly finite-dimensional. Thus,

**Corollary 8.9.2.** *The de Rham cohomology $H^*_{DR}(M)$ of a compact manifold is finite-dimensional.*

In fact,

**Corollary 8.9.3.** *Whenever $M$ has a finite good cover, its de Rham cohomology $H^*_{DR}(M)$ is finite-dimensional.*

Both the proof here and the induction argument in Section 5 of the finite dimensionality of the de Rham cohomology rest on the Mayer-Vietoris sequence, but they are otherwise independent of each other.

# §9 More Examples and Applications of the Mayer–Vietoris Principle

In the previous section we used the Mayer-Vietoris principle to show the isomorphism of the de Rham cohomology of a manifold and the Čech cohomology of a good cover; from this, various corollaries follow. In this section, after some examples in which the combinatorics of a good cover is used to compute the de Rham cohomology, we give an explicit isomorphism from Čech to de Rham: given a Čech cocycle, we construct the corresponding global closed differential form by means of a collating formula (9.5) based on the homotopy operator $K$ of (8.6). To conclude the section, we give as another application of the Mayer–Vietoris principle a proof of the Künneth formula valid under the hypothesis that one of the factors has finite-dimensional cohomology.

## Examples: Computing the de Rham Cohomology from the Combinatorics of a Good Cover

Let $\mathfrak{U} = \{U_\alpha\}$ be an open cover of a manifold $M$. The *nerve* of $\mathfrak{U}$ is a simplicial complex constructed as follows. To every open set $U_\alpha$, we associate a vertex $\alpha$. If $U_\alpha \cap U_\beta$ is nonempty, we connect the vertices $\alpha$ and $\beta$ with an edge. If $U_\alpha \cap U_\beta \cap U_\gamma$ is nonempty, we fill in the face of the triangle $\alpha\beta\gamma$. Repeating this procedure for all finite intersections gives the nerve of $\mathfrak{U}$, denoted $N(\mathfrak{U})$. For the basics of simplicial complexes, see Croom [1].

EXAMPLE 9.1 (The circle). Let $\mathfrak{U} = \{U_0, U_1, U_2\}$ be the good cover of the circle as shown in Figure 9.1. The Čech complex has two terms:

$$C^0(\mathfrak{U}, \mathbb{R}) = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} = \{(\omega_0, \omega_1, \omega_2) \mid \omega_\alpha \text{ is a constant on } U_\alpha\},$$

$$C^1(\mathfrak{U}, \mathbb{R}) = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} = \{(\eta_{01}, \eta_{02}, \eta_{12}) \mid \eta_{\alpha\beta} \text{ is a constant on } U_{\alpha\beta}\}.$$

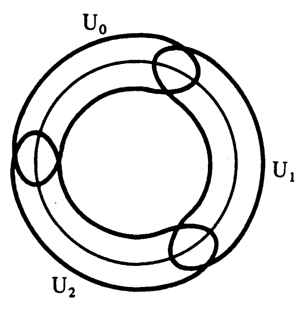


Figure 9.1

The coboundary $\delta : C^0 \to C^1$ is given by $(\delta\omega)_{\alpha\beta} = \omega_\beta - \omega_\alpha$. Therefore,

$$\ker \delta = \{(\omega_0, \omega_1, \omega_2) \mid \omega_0 = \omega_1 = \omega_2\} = \mathbb{R}$$

and

$$H^0(S^1) = \mathbb{R}.$$

Since $\operatorname{im} \delta = \mathbb{R}^2$, $H^1(S^1) = \mathbb{R}^3/\operatorname{im} \delta = \mathbb{R}$.

EXAMPLE 9.2 (A nontrivial 1-cocycle on the circle). If a 1-cocycle $\eta = (\eta_{01}, \eta_{02}, \eta_{12})$ is a coboundary, then $\eta_{01} - \eta_{02} + \eta_{12} = 0$. So $\eta = (1, 0, 0)$ is a nontrivial 1-cocycle on the circle.

EXAMPLE 9.3 (The 2-sphere). Cover the lower hemisphere of Figure 9.2 with three open sets as in Figure 9.3. Together with the upper hemisphere $U_0$, this gives a good cover of the entire sphere. The nerve of the cover is the surface of a tetrahedron as depicted in Figure 9.4. The Čech complex has

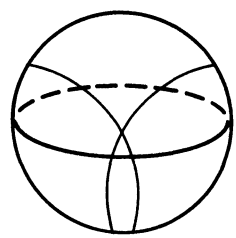


Figure 9.2

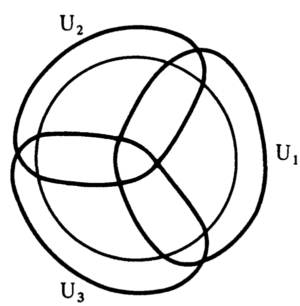



Figure 9.3

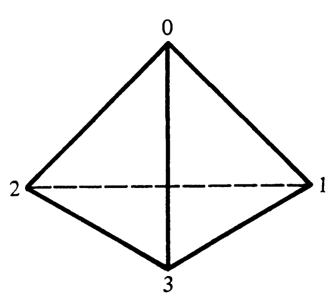



Figure 9.4

three terms:

$$\begin{array}{ccccc} C^0(\mathfrak{U}, \mathbb{R}) & \xrightarrow{\delta_0} & C^1(\mathfrak{U}, \mathbb{R}) & \xrightarrow{\delta_1} & C^2(\mathfrak{U}, \mathbb{R}) \\ \| & & \| & & \| \\ \underset{0}{\mathbb{R}}\oplus\underset{1}{\mathbb{R}}\oplus\underset{2}{\mathbb{R}}\oplus\underset{3}{\mathbb{R}} & & \underset{01}{\mathbb{R}}\oplus\underset{02}{\mathbb{R}}\oplus\underset{03}{\mathbb{R}}\oplus\underset{12}{\mathbb{R}}\oplus\underset{13}{\mathbb{R}}\oplus\underset{23}{\mathbb{R}} & & \underset{012}{\mathbb{R}}\oplus\underset{013}{\mathbb{R}}\oplus\underset{023}{\mathbb{R}}\oplus\underset{123}{\mathbb{R}} \end{array}$$

$$\ker \delta_0 = \{(\omega_0, \omega_1, \omega_2, \omega_3) \mid \omega_0 = \omega_1 = \omega_2 = \omega_3\} = \mathbb{R}$$

So im $\delta_0 = \mathbb{R}^3$ and $H^0(S^2) = \mathbb{R}$. If $\eta$ is in ker $\delta_1$, then $\eta$ is completely determined by $\eta_{01}, \eta_{02}$, and $\eta_{03}$. Therefore ker $\delta_1 = \mathbb{R}^3$ and

$$H^1(S^2) = \ker \delta_1/\text{im } \delta_0 = 0.$$

Since im $\delta_1 = C^1/\ker \delta_1 = \mathbb{R}^3$,

$$H^2(S^2) = \mathbb{R}^4/\text{im } \delta_1 = \mathbb{R}.$$

## Explicit Isomorphisms between the Double Complex and de Rham and Čech

We saw in Proposition 8.8 that the Čech–de Rham complex $C^*(\mathfrak{U}, \Omega^*)$ and the de Rham complex $\Omega^*(M)$ have the same cohomology. Actually, what is true is that these two complexes are chain homotopic. To be more precise, there is a chain map

(9.4) $$f: C^*(\mathfrak{U}, \Omega^*) \to \Omega^*(M)$$

such that

(a) $f \circ r = 1$, and
(b) $r \circ f$ is chain homotopic to the identity.

We may think of $f$ as a recipe for collating together the components of a Čech–de Rham cochain into a global form. The not very intuitive formulas below were obtained, after repeated tries, by a careful bookkeeping of the inductive steps in the proof of Proposition 8.8.

**Proposition 9.5** (The Collating Formula). *Let $K$ be the homotopy operator defined in* (8.6). *If $\alpha = \sum_{i=0}^n \alpha_i$ is an n-cochain and $D\alpha = \beta = \sum_{i=0}^{n+1} \beta_i$, then*

$$f(\alpha) = \sum_{i=0}^{n} (-D''K)^i\alpha_i - \sum_{i=1}^{n+1} K(-D''K)^{i-1}\beta_i \in C^0(\mathfrak{U}, \Omega^n)$$

*is a global form satisfying the properties above. The homotopy operator*

$$L: C^*(\mathfrak{U}, \Omega^*) \to C^*(\mathfrak{U}, \Omega^*)$$

*such that $1 - r \circ f = DL + LD$ is given by*

$$L\alpha = \sum_{p=0}^{n-1} (L\alpha)_p,$$

*where*

$$(L\alpha)_p = \sum_{i=p+1}^{n} K(-D''K)^{i-(p+1)}\alpha_i \in C^p(\mathfrak{U}, \Omega^{n-1-p}).$$

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |
| $\beta_0$ | | | | | | |
| $\alpha_0$ | $\beta_1$ | | | | | |
| | $\alpha_1$ | $\beta_2$ | | | | |
| | | $\alpha_2$ | $\beta_3$ | | | |
| | | | | | | |
| | | | | | $\alpha_n$ | $\beta_{n+1}$ |

REMARK. To strip away some of the mysteries in the expression for $f(\alpha)$, it may be helpful to observe that the operator $D''K$ sends an element of $C^p(\mathfrak{U}, \Omega^q)$ into $C^{p-1}(\mathfrak{U}, \Omega^{q+1})$, so that $(D''K)^i\alpha_i$ and $K(D''K)^{i-1}\beta_i$ are collections of $n$-forms on the open sets $U_\alpha$. The collating formula says that a suitable linear combination of these local $n$-forms, with $\pm 1$ as coefficients, is the restriction of a global form.

The proof of Proposition 9.5 requires the following technical lemma.

**Lemma 9.6.** *For* $i \geq 1$,

$$\delta(D''K)^i = (D''K)^i\,\delta - (D''K)^{i-1}D''.$$

PROOF OF LEMMA 9.6. Since $\delta$ anticommutes with $D''$ and since $\delta K + K\delta = 1$,

$$\begin{aligned}\delta(D''K)(D''K)^{i-1} &= -D''\delta K(D''K)^{i-1}\\ &= -D''(1 - K\delta)(D''K)^{i-1}\\ &= (D''K)\delta(D''K)^{i-1}.\end{aligned}$$

So we can commute $D''K$ and $\delta$ until we reach $(D''K)^{i-1}\delta(D''K)$. Then,

$$\begin{aligned}\delta(D''K)^i &= (D''K)^{i-1}\delta(D''K)\\ &= -(D''K)^{i-1}D''(1 - K\delta)\\ &= -(D''K)^{i-1}D'' + (D''K)^i\delta. \qquad \square\end{aligned}$$

PROOF OF PROPOSITION 9.5. To show that $f(\alpha)$ is a global form, we compute $\delta f(\alpha)$. Using the lemma above and the fact that $\delta\alpha_i + D''\alpha_{i+1} = \beta_{i+1}$, this is a straightforward exercise which we leave to the reader.

*Exercise 9.7.* Show that $\delta f(\alpha) = 0$.

Next we check that $f$ is a chain map.

$$f(D\alpha) = f(\beta) = \sum_{i=0}^{n+1} (-1)^i (D''K)^i \beta_i.$$

$$df(\alpha) = D''f(\alpha) = \beta_0 + \sum_{i=1}^{n+1} (-1)^i (D''K)^i \beta_i.$$

So

$$f(D\alpha) = df(\alpha).$$

The verification of Property (a) is easy, since if $\alpha$ is a global form, then $\alpha = \alpha_0$ and

$$f \circ r(\alpha) = f(\alpha) = \alpha_0 = \alpha.$$

Property (b) follows from the fact that

$$1 - r \circ f = DL + LD.$$

As its verification is straightforward and not very illuminating, we shall omit it. The skeptical reader may wish to carry it out for himself. Apart from the definitions, the only facts needed are Lemma 9.6 and the chain-homotopy formula (8.7). □

REMARK. Actually the existence of the chain-homotopy inverse $f$ and the homotopy operator $L$ is guaranteed by a general principle in the theory of chain complexes (See Spanier [1, Ch. 4, Sec. 2; in particular, Cor. 11, p. 167]).

We can now give an explicit description of the various isomorphisms that follow from the generalized Mayer-Vietoris principle. For example, by applying the collating formula (9.5), we get

**Proposition 9.8** (Explicit Isomorphism between de Rham and Čech). *If $\eta \in C^n(\mathfrak{U}, \mathbb{R})$ is a Čech cocycle, then the global closed form corresponding to it is given by $f(\eta) = (-1)^n (D''K)^n \eta$.*

EXAMPLE 9.9. Let $\mathfrak{U}$ be a good cover of the circle $S^1$. We shall construct from a generator of the Čech cohomology $H^1(\mathfrak{U}, \mathbb{R})$ a differential form representing a generator of the de Rham cohomology $H^1_{DR}(S^1)$.

As we saw in Example 9.2, a nontrivial 1-cocycle on $S^1$ is

$$\eta = (\eta_{01}, \eta_{02}, \eta_{12}) = (1, 0, 0).$$

If $\{\rho_\alpha\}$ is a partition of unity, then

$$K\eta = (-\rho_1, \rho_0, 0).$$

So the generator $-D''K\eta$ of $H^1_{DR}(S^1)$ is represented by $-d(-\rho_1)$, a bump form on $U_0 \cap U_1$ with total integral 1.

*Exercise 9.10.* The real projective plane $\mathbb{R}P^2$ is obtained by identifying the boundary of a disc as shown in Figure 9.5. Find a good cover for $\mathbb{R}P^2$ and

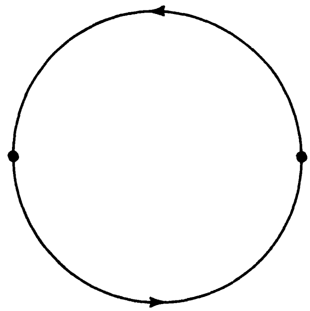


Figure 9.5

compute its de Rham cohomology from the combinatorics of the cover. One possible good cover has the nerve depicted in Figure 9.6.

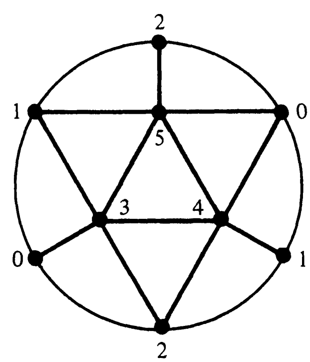



Figure 9.6

*Exercise 9.11.* Let Figure 9.7 be the nerve of a good cover $\mathfrak{U}$ on the torus, where the arrows indicate how the vertices are ordered. Write down a nontrivial 1-cocycle in $C^1(\mathfrak{U}, \mathbb{R})$.

## The Tic-Tac-Toe Proof of the Künneth Formula

We now apply the main theorems of the preceding section to give another proof of the Künneth formula. This proof, admittedly more involved in its

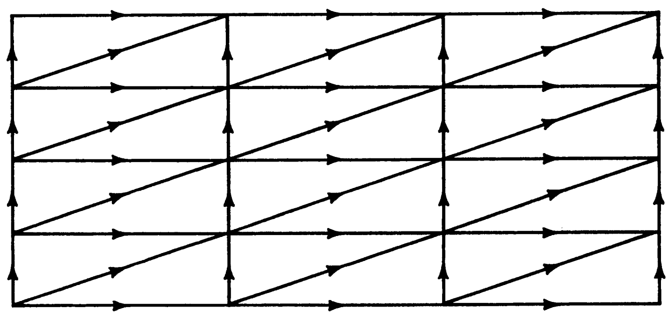


Figure 9.7

construction than the Mayer–Vietoris argument of Section 5, is a prototype for the spectral sequence argument of Chapter III. It will also allow us to replace the requirement that $M$ has a finite good cover by the slightly weaker hypothesis that $F$ has finite-dimensional cohomology.

Before commencing the proof we make some general remarks about a technique for studying maps. Let $\pi : E \to M$ be a map of manifolds. A cover $\mathfrak{U}$ on $M$ induces a cover $\pi^{-1}\mathfrak{U}$ on $E$, and we have the inclusions

$$\begin{array}{ccccccc} E & \leftarrow & \coprod \pi^{-1}U_\alpha & \leftleftarrows & \coprod \pi^{-1}U_{\alpha\beta} & \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} & \cdots \\ {\scriptstyle\pi}\big\downarrow & & & & & & \\ M & \leftarrow & \coprod U_\alpha & \leftleftarrows & \coprod U_{\alpha\beta} & \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} & \cdots. \end{array}$$

In general $U_\alpha \cap U_\beta \neq \phi$ is not equivalent to $\pi^{-1}U_\alpha \cap \pi^{-1}U_\beta \neq \phi$. However, if $\pi$ is surjective, then the two statements are equivalent, so that in this case the combinatorics of the covers $\mathfrak{U}$ and $\pi^{-1}\mathfrak{U}$ are the same. The double complex of the inverse cover computes the cohomology of $E$, which can then be related to the cohomology of $M$, because the inverse cover comes from a cover on $M$. This idea will be systematically exploited throughout this chapter and the next.

A quick example of how the inverse cover $\pi^{-1}\mathfrak{U}$ may be used to study maps is the following. Note that although the inverse image of a good cover is usually not a good cover, for a vector bundle $\pi : E \to M$ the "goodness" of the cover is preserved. Since the de Rham cohomology is determined by the combinatorics of a good cover, this implies that

$$H^*_{DR}(E) \simeq H^*_{DR}(M).$$

Of course, this also follows from the homotopy axiom for the de Rham cohomology (Corollary 4.1.2.2).

**Proposition 9.12** (Künneth Formula). *If $M$ and $F$ are two manifolds and $F$ has finite-dimensional cohomology, then the de Rham cohomology of the product $M \times F$ is*

$$H^*(M \times F) = H^*(M) \otimes H^*(F).$$

Proof. Let $\mathfrak{U} = \{U_\alpha\}$ be a good cover for $M$ and $\pi: M \times F \to M$ the projection onto the first factor. Then $\pi^{-1}\mathfrak{U} = \{\pi^{-1}U_\alpha\}$ is some sort of a cover for $E = M \times F$, though in general not a good cover. There is a natural map

$$\begin{array}{c} C^*(\pi^{-1}\mathfrak{U}, \Omega^*) \\ \uparrow \pi^* \\ C^*(\mathfrak{U}, \Omega^*) \end{array}$$

which pulls back differential forms on open sets. Choose a basis for $H^*(F)$, say $\{[\omega_\alpha]\}$, and choose differential forms $\omega_\alpha$ representing them. These may be used to define a map of double complexes

$$\begin{array}{c} C^*(\pi^{-1}\mathfrak{U}, \Omega^*) \\ \uparrow \pi^*_{\mathfrak{U}} \\ H^*(F) \otimes C^*(\mathfrak{U}, \Omega^*) \end{array}$$

by

$$\pi^*_{\mathfrak{U}}([\omega_\alpha] \otimes \phi) = \rho^*\omega_\alpha \wedge \pi^*\phi$$

where $\rho$ is the projection on the fiber

$$\begin{array}{ccc} E & \xrightarrow{\rho} & F \\ \downarrow \pi & & \\ M. & & \end{array}$$

Since $H^*(F)$ is a vector space, $H^*(F) \otimes C^*(\mathfrak{U}, \Omega^*)$ is a number of copies of $C^*(\mathfrak{U}, \Omega^*)$ and the differential operator $D$ on the double complex $C^*(\mathfrak{U}, \Omega^*)$ induces an operator on $H^*(F) \otimes C^*(\mathfrak{U}, \Omega^*)$ whose cohomology is

$$H^*(F) \otimes H_D\{C^*(\mathfrak{U}, \Omega^*)\} = H^*(F) \otimes H^*(M).$$

Since the $D$-cohomology of $C^*(\pi^{-1}\mathfrak{U}, \Omega^*)$ is $H^*(E)$, if we can show that

$$\begin{array}{c} C^*(\pi^{-1}\mathfrak{U}, \Omega^*) \\ \uparrow \pi^*_{\mathfrak{U}} \\ H^*(F) \otimes C^*(\mathfrak{U}, \Omega^*) \end{array}$$

induces an isomorphism in $D$-cohomology, the Künneth formula will follow.

The proof now divides into two steps:

*Step 1.*

*For a good cover $\mathfrak{U}$, the map $\pi^*_{\mathfrak{U}}$ induces an isomorphism in $H_d$ of these complexes.*

*Step 2.*

*Whenever a homomorphism $f: K \to K'$ of double complexes induces $H_d$-isomorphism, it also induces $H_D$-isomorphism.* (By a homomorphism of double complexes, we mean a vector-space homomorphism which preserves bidegrees and commutes with $d$ and $\delta$.)

PROOF OF STEP 1. The $p^{\text{th}}$ column $C^p(\pi^{-1}\mathfrak{U}, \Omega^*)$ consists of forms on the $(p+1)$-fold intersections $\amalg \pi^{-1} U_{\alpha_0 \dots \alpha_p}$ and $C^p(\mathfrak{U}, \Omega^*)$ consists of forms on $\amalg U_{\alpha_0 \dots \alpha_p}$. The $d$-cohomology of $C^p(\pi^{-1}\mathfrak{U}, \Omega^*)$ is

$$(9.12.1) \qquad \prod H^*(\pi^{-1} U_{\alpha_0 \dots \alpha_p}) \simeq H^*(F) \otimes \prod H^*(U_{\alpha_0 \dots \alpha_p}),$$

the isomorphism being given by the wedge product of pullbacks. So $\pi_{\mathfrak{U}}^*$ induces an isomorphism of the $d$-cohomology of $C^*(\pi^{-1}\mathfrak{U}, \Omega^*)$ and $H^*(F) \otimes C^*(\mathfrak{U}, \Omega^*)$. □

*Exercise 9.13.* Give a proof of Step 2.

REMARK. This argument for the Künneth formula also proves the Leray-Hirsch theorem (5.11), but again instead of assuming that $M$ has a finite good cover, we require the cohomology of $F$ to be finite-dimensional. If both $M$ and $F$ have infinite-dimensional cohomology, the isomorphism in (9.12.1) may not be valid.

The following example shows that some sort of finiteness hypothesis is necessary for the Künneth formula or the Leray-Hirsch theorem to hold.

EXAMPLE 9.14 (Counterexample to the Künneth formula when both $M$ and $F$ have infinite-dimensional cohomology). Let $M$ and $F$ each be the set $\mathbb{Z}^+$ of all positive integers. Then

$$H^0(M \times F) = \{\text{square matrices of real numbers } (a_{ij}),\ i, j \in \mathbb{Z}^+\}.$$

But $H^0(M) \otimes H^0(F)$ consists of *finite* sums of matrices $(a_{ij})$ of rank 1. These two vector spaces are not equal, since a finite sum of matrices of rank 1 has finite rank, but $H^0(M \times F)$ contains matrices of infinite rank.

# §10 Presheaves and Čech Cohomology

## Presheaves

The functor $\Omega^*(\ )$ which assigns to every open set $U$ on a manifold the differential forms on $U$ is an example of a *presheaf*. By definition a *presheaf* $\mathscr{F}$ on a topological space $X$ is a function that assigns to every open set $U$ in

$X$ an abelian group $\mathscr{F}(U)$ and to every inclusion of open sets

$$i_U^V : V \to U$$

a group homomorphism, called the *restriction*,

$$\mathscr{F}(i_U^V) : \mathscr{F}(U) \to \mathscr{F}(V)$$

satisfying the following properties:

(a) $\mathscr{F}(i_V^V) =$ identity map
(b) transitivity: $\mathscr{F}(i_V^W)\ \mathscr{F}(i_U^V) = \mathscr{F}(i_U^W)$.

The restriction $\mathscr{F}(i_U^V) : \mathscr{F}(U) \to \mathscr{F}(V)$ is often denoted $\rho_V^U$. A *homomorphism* of two presheaves, $f : \mathscr{F} \to \mathscr{G}$, is a collection of maps $f_U : \mathscr{F}(U) \to \mathscr{G}(U)$ which commute with the restrictions:

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{f_U} & \mathscr{G}(U) \\ \rho_V^U \downarrow & & \downarrow \rho_V^U \\ \mathscr{F}(V) & \xrightarrow[f_V]{} & \mathscr{G}(V) \end{array}$$

Let Open($X$) be the category whose objects are the open sets in $X$ and whose morphisms are inclusions of open sets. In functorial language, a presheaf is simply a contravariant functor from the category Open($X$) to the category of Abelian groups, and a homomorphism of two presheaves, $f : \mathscr{F} \to \mathscr{G}$, is a *natural transformation* from the functor $\mathscr{F}$ to the functor $\mathscr{G}$.

We define the *constant presheaf with group G* to be the presheaf $\mathscr{F}$ which associates to every open set $U$ the locally constant functions: $U \to G$, and to every inclusion of open sets $V \subset U$ the restriction of functions: $\mathscr{F}(U) \to \mathscr{F}(V)$.

EXAMPLE. By abuse of notation, the constant presheaf with group $\mathbb{R}$ will also be denoted by $\mathbb{R}$.

EXAMPLE 10.1. Let $\pi : E \to M$ be a fiber bundle with fiber $F$. Define a presheaf $\mathscr{H}^q$ on $M$ by $\mathscr{H}^q(U) = H^q(\pi^{-1}U)$, and if $V \subset U$ is an inclusion, let

$$\rho_V^U : H^q(\pi^{-1}U) \to H^q(\pi^{-1}V)$$

be the natural restriction map. For $U$ contractible, $\pi^{-1}U \simeq U \times F$, so by the Künneth formula

$$\mathscr{H}^q(U) \simeq H^q(U \times F) \simeq H^q(F).$$

Moreover, if $V \subset U$ is an inclusion of contractible open sets, then $\rho_V^U : H^q(\pi^{-1}U) \to H^q(\pi^{-1}V)$ is an isomorphism. The presheaf $\mathscr{H}^q$ is an example of a *locally constant presheaf on a good cover*, to be defined in Section 13.

## Čech Cohomology

Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in J}$ be an open cover of the topological space $X$. The 0-*cochains on U with values in the presheaf* $\mathscr{F}$ are functions which assign to each open set $U_\alpha$ an element of $\mathscr{F}(U_\alpha)$, i.e., $C^0(\mathfrak{U}, \mathscr{F}) = \Pi_{\alpha \in J} \mathscr{F}(U_\alpha)$. Similarly the 1-cochains are elements of

$$C^1(\mathfrak{U}, \mathscr{F}) = \prod_{\alpha < \beta} \mathscr{F}(U_\alpha \cap U_\beta)$$

and so on.

The sequence of inclusions

$$U_\alpha \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} U_{\alpha\beta} \begin{smallmatrix}\leftarrow\\ \leftarrow\\ \leftarrow\end{smallmatrix} \cdots$$

gives rise to a sequence of group homomorphisms

$$\prod \mathscr{F}(U_\alpha) \rightrightarrows \prod \mathscr{F}(U_{\alpha\beta}) \begin{smallmatrix}\rightarrow\\ \rightarrow\\ \rightarrow\end{smallmatrix} \cdots .$$

We define $\delta : C^p(\mathfrak{U}, \mathscr{F}) \to C^{p+1}(\mathfrak{U}, \mathscr{F})$ to be the alternating difference of the $\mathscr{F}(\partial_i)$'s; for example,

$$\delta : C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F})$$

is given by

$$\delta = \mathscr{F}(\partial_0) - \mathscr{F}(\partial_1).$$

In general

$$\delta : C^p(\mathfrak{U}, \mathscr{F}) \to C^{p+1}(\mathfrak{U}, \mathscr{F})$$

is given by

$$\delta = \mathscr{F}(\partial_0) - \mathscr{F}(\partial_1) + \cdots + (-1)^{p+1} \mathscr{F}(\partial_{p+1}).$$

Explicitly, if $\omega \in C^p(\mathfrak{U}, \mathscr{F})$, then

$$(\delta\omega)_{\alpha_0 \ldots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_{p+1}}, \tag{10.2}$$

where on the right-hand side the restriction of $\omega_{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_{p+1}}$ to $U_{\alpha_0 \ldots \alpha_{p+1}}$ is suppressed. It follows from the transitivity of the restriction homomorphism that $\delta^2 = 0$ (proof as in Proposition 8.3). Thus $C^*(\mathfrak{U}, \mathscr{F})$ is a differential complex with differential operator $\delta$. The cohomology of this complex, denoted by $H_\delta C^*(\mathfrak{U}, \mathscr{F})$ or $H^*(\mathfrak{U}, \mathscr{F})$, is called the *Čech cohomology of the cover* $\mathfrak{U}$ *with values in* $\mathscr{F}$.

REMARK 10.3. If $\mathscr{F}$ is a *covariant* functor from the category Open($X$) to the category of Abelian groups, and $\mathfrak{U}$ is an open cover of $X$, one can define analogously a chain complex $C_*(\mathfrak{U}, \mathscr{F})$ and its homology $H_*(\mathfrak{U}, \mathscr{F})$. Apart from the direction of the arrows, the only difference from the case of a presheaf is in the definition of the coboundary operator $\delta : C_p(\mathfrak{U}, \mathscr{F}) \to C_{p-1}(\mathfrak{U}, \mathscr{F})$, which is now given by

$$(\delta\omega)_{\alpha_0 \ldots \alpha_{p-1}} = \sum_\alpha \omega_{\alpha\alpha_0 \ldots \alpha_{p-1}} \in \mathscr{F}\left(U_{\alpha_0 \ldots \alpha_{p-1}}\right).$$

One verifies easily that this $\delta$ also satisfies $\delta^2 = 0$. The functor $\mathscr{H}_c^q$ which associates to every open set $U$ on a manifold the compact cohomology $H_c^q(U)$ is covariant.

Because of the antisymmetry convention on the subscripts, in this formula there is no sign in the sum. Of course, if we had written each term $\omega_{\alpha_0 \ldots \alpha_{p-1}}$ with the subscript $\alpha$ inserted in the $i$-th place, then there would be a sign: $\sum_i (-1)^i \omega_{\alpha_0 \ldots \alpha \ldots \alpha_{p-1}}$.

Returning to the discussion of the Čech cohomology of a presheaf $\mathscr{F}$, recall that the cover $\mathfrak{V} = \{V_\beta\}_{\beta \in J}$ is a *refinement* of the cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$, written $\mathfrak{U} < \mathfrak{V}$, if there is a map $\phi : J \to I$ such that $V_\beta \subset U_{\phi(\beta)}$. The refinement $\phi$ induces a map

$$\phi^\# : C^q(\mathfrak{U}, \mathscr{F}) \to C^q(\mathfrak{V}, \mathscr{F})$$

in the obvious manner:

$$(\phi^\#\omega)\left(V_{\beta_0 \ldots \beta_q}\right) = \omega\left(U_{\phi(\beta_0) \ldots \phi(\beta_q)}\right).$$

**Lemma 10.4.1.** *$\phi^\#$ is a chain map, i.e., it commutes with $\delta$.*

PROOF.
$$\begin{aligned}(\delta(\phi^\#\omega))(V_{\beta_0 \ldots \beta_{q+1}}) &= \sum (-1)^i (\phi^\#\omega)(V_{\beta_0 \ldots \hat{\beta}_i \ldots \beta_{q+1}}) \\ &= \sum (-1)^i \omega(U_{\phi(\beta_0) \ldots \widehat{\phi(\beta_i)} \ldots \phi(\beta_{q+1})}) \\ (\phi^\#\delta\omega)(V_{\beta_0 \ldots \beta_{q+1}}) &= (\delta\omega)(U_{\phi(\beta_0) \ldots \phi(\beta_{q+1})}) \\ &= \sum (-1)^i \omega(U_{\phi(\beta_0) \ldots \widehat{\phi(\beta_i)} \ldots \phi(\beta_{q+1})}).\end{aligned}$$

□

**Lemma 10.4.2.** *Given $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ an open cover and $\mathfrak{V} = \{V_\beta\}_{\beta \in J}$ a refinement, if $\phi$ and $\psi$ are two refinement maps: $J \to I$, then there is a homotopy operator between $\phi^\#$ and $\psi^\#$.*

PROOF. Define $K : C^q(\mathfrak{U}, \mathscr{F}) \to C^{q-1}(\mathfrak{V}, \mathscr{F})$ by

$$(K\omega)(V_{\beta_0 \ldots \beta_{q-1}}) = \sum (-1)^i \omega(U_{\phi(\beta_0) \ldots \phi(\beta_i)\psi(\beta_i) \ldots \psi(\beta_{q-1})}).$$

*Exercise 10.5.* Show that

$$\psi^\# - \phi^\# = \delta K + K\delta.$$

□

A *direct system of groups* is a collection of groups $\{G_i\}_{i \in I}$ indexed by a directed set $I$ such that for any pair $a < b$ there is a group homomorphism $f_b^a : G_a \to G_b$ satisfying

(1) $f_a^a = \text{identity}$,
(2) $f_c^a = f_c^b \circ f_b^a$, if $a < b < c$.

On the disjoint union $\amalg G_i$ we introduce an equivalence relation $\sim$ by decreeing two elements $g_a$ in $G_a$ and $g_b$ in $G_b$ to be equivalent if for some upper bound $c$ of $a$ and $b$, we have $f_c^a(g_a) = f_c^b(g_b)$ in $G_c$. The *direct limit* of the direct system, denoted by $\lim_{i \in I} G_i$, is the quotient of $\amalg G_i$ by the equivalence relation $\sim$; in other words, two elements of $\amalg G_i$ represent the same element in the direct limit if they are "eventually equal". We make the direct limit into a group by defining $[g_a] + [g_b] = [f_c^a(g_a) + f_c^b(g_b)]$, where $[g_a]$ is the equivalence class of $g_a$.

It follows from the two lemmas above that if $\mathfrak{U} < \mathfrak{V}$, then there is a well-defined map in cohomology

$$H^*(\mathfrak{U}, \mathscr{F}) \to H^*(\mathfrak{V}, \mathscr{F}),$$

making $\{H^*(\mathfrak{U}, \mathscr{F})\}_{\mathfrak{U}}$ into a direct system of groups. The direct limit of this direct system

$$H^*(X, \mathscr{F}) = \lim_{\mathfrak{U}} H^*(\mathfrak{U}, \mathscr{F})$$

is the *Čech cohomology of $X$ with values in the presheaf $\mathscr{F}$.*

**Proposition 10.6.** *Let $\mathbb{R}$ be the constant presheaf on a manifold $M$. Then the Čech cohomology of $M$ with values in $\mathbb{R}$ is isomorphic to the de Rham cohomology.*

PROOF. Since the good covers are cofinal in the set of all covers of $M$ (Corollary 5.2), we can use only good covers in the direct limit

$$H^*(M, \mathbb{R}) = \lim_{\mathfrak{U}} H^*(\mathfrak{U}, \mathbb{R}).$$

By Theorem 8.9,

$$H^*(\mathfrak{U}, \mathbb{R}) \simeq H^*_{DR}(M)$$

for any good cover of $M$. Moreover, it is easily seen that this isomorphism is compatible with refinement of good covers. Therefore, there is an isomorphism

$$H^*(M, \mathbb{R}) \simeq H^*_{DR}(M).$$

□

*Exercise 10.7* (*Cohomology with Twisted Coefficients*). Let $\mathscr{F}$ be the presheaf on $S^1$ which associates to every open set the group $\mathbb{Z}$. We define the restriction homomorphism on the good cover $\mathfrak{U} = \{U_0, U_1, U_2\}$ (Figure 10.1) by

$$\rho_{01}^0 = \rho_{01}^1 = 1,$$
$$\rho_{12}^1 = \rho_{12}^2 = 1,$$
$$\rho_{02}^2 = -1, \ \rho_{02}^0 = 1,$$

where $\rho_{ij}^i$ is the restriction from $U_i$ to $U_i \cap U_j$. Compute $H^*(\mathfrak{U}, \mathscr{F})$. (Cf. presheaf on an open cover, p. 142.)

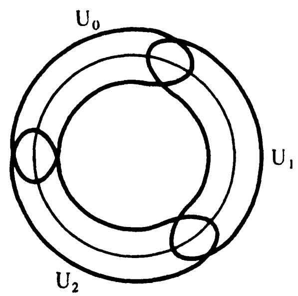



Figure 10.1

# §11 Sphere Bundles

Let $\pi: E \to M$ be a fiber bundle with fiber the sphere $S^n$, $n \geq 1$. As the structure group we normally take the largest group possible, namely the diffeomorphism group $\mathrm{Diff}(S^n)$, but sometimes we also consider sphere bundles with structure group $O(n+1)$. These two notions are not equivalent; there are examples of sphere bundles whose structure groups cannot be reduced to the orthogonal group. Thus, every vector bundle defines a sphere bundle, but not conversely. By the Leray-Hirsch theorem if there is a closed global $n$-form on $E$ whose restriction to each fiber generates the cohomology of the fiber, then the cohomology of $E$ is

$$H^*(E) = H^*(M) \otimes H^*(S^n).$$

It is therefore of interest to know when such a global form exists.

In Section 6 we constructed the global angular form $\psi$ on a rank 2 vector bundle with structure group $SO(2)$. This form $\psi$ was seen to have the following two properties:

(a) $\psi$ restricts to the volume form on each fiber, i.e., a generator of $H_c^2(\text{fiber})$
(b) $d\psi = -\pi^* e$

where $e$ is the Euler class. Exactly the same procedure defines the angular form and the Euler class of a circle bundle with structure group $SO(2)$.

Consequently, for such a bundle also, if the Euler class vanishes, then $\psi$ is closed and satisfies the condition of the Leray–Hirsch theorem.

We now consider more generally a sphere bundle with structure group $\mathrm{Diff}(S^n)$ or $O(n+1)$. We will see that the existence of a global form as above entails overcoming two obstructions: orientability and the Euler class.

## Orientability

In this section the base space of the bundle is assumed to be connected. A sphere bundle with fiber $S^n$, $n \geq 1$, is said to be *orientable* if for each fiber $F_x$ it is possible to choose a generator $[\sigma_x]$ of $H^n(F_x)$ satisfying the *local compatibility condition*: around any point there is a neighborhood $U$ and a generator $[\sigma_U]$ of $H^n(E|_U)$ such that for any $x$ in $U$, $[\sigma_U]$ restricted to the fiber $F_x$ is the chosen generator $[\sigma_x]$; equivalently, there is an open cover $\{U_\alpha\}$ of $M$ and generators $[\sigma_\alpha]$ of $H^n(E|_{U_\alpha})$ so that $[\sigma_\alpha] = [\sigma_\beta]$ in $H^n(E|_{U_\alpha \cap U_\beta})$.

Since a generator of the top cohomology of a fiber is an $n$-form with total integral 1, there are two possible generators, depending on the orientation of the fiber. A priori all that one could say is that $[\sigma_\alpha] = \pm[\sigma_\beta]$ on $U_\alpha \cap U_\beta$. For an orientable sphere bundle either choice of a consistent system of generators is called an *orientation* of the sphere bundle. A bundle with a given orientation is said to be *oriented*. An $S^0$-bundle over a manifold $M$ is a double cover of $M$; such a bundle over a connected base space is said to be *orientable* if and only if the total space has two connected components.

CAVEAT. The fact that the cohomology classes $\{[\sigma_\alpha]\}$ agree on overlaps does not mean that they piece together to form a global cohomology class. A global cohomology class must be represented by a global form; the equality of cohomology classes $[\sigma_\alpha] = [\sigma_\beta]$ implies only that the forms $\sigma_\alpha$ and $\sigma_\beta$ differ by an exact form.

Recall that in Section 7 we called a vector bundle of rank $n+1$ orientable if and only if it can be given by transition functions with values in $SO(n+1)$. We now study the relation between the orientability of a sphere bundle and the orientability of a vector bundle.

Let $E$ be a vector bundle of rank $n+1$ endowed with a Riemannian metric so that its structure group is reduced to $O(n+1)$. Its unit sphere bundle $S(E)$ is the fiber bundle whose fiber at $x$ consists of all the unit vectors in $E_x$ and whose transition functions are the same as those of $E$. $S(E)$ is an $S^n$-bundle with structure group $O(n+1)$.

REMARK 11.1. Fix an orientation on the sphere $S^n$. If the linear transformation $g$ is in the special orthogonal group $SO(n+1)$ and $[\sigma]$ is a generator of $H^n(S^n)$ with $\int_{S^n} \sigma = 1$, then the image $g(S^n)$ is the sphere $S^n$ with the same orientation, so that

$$\int_{S^n} g^*\sigma = \int_{g(S^n)} \sigma = \int_{S^n} \sigma = 1.$$

Thus *for an orthogonal transformation $g$, $g^*\sigma$ and $\sigma$ represent the same cohomology class if and only if $g$ has positive determinant.*

**Proposition 11.2.** *A vector bundle $E$ is orientable if and only if its sphere bundle $S(E)$ is orientable.*

Proof. ($\Rightarrow$) Fix a generator $\sigma$ on $S^n$ and fix a trivialization $\{(U_\alpha, \phi_\alpha)\}$ for $E$ so that the transition functions $g_{\alpha\beta}$ assume values in $SO(n+1)$. Let

$$\rho_\alpha : U_\alpha \times S^n \to S^n$$

be the projection and let $\pi^{-1}(x)$ be the fiber of the sphere bundle $\pi : S(E) \to M$ at $x$. Define $[\sigma_\alpha]$ in $H^n(S(E)|_{U_\alpha})$ by

$$[\sigma_\alpha] = \phi_\alpha^* \rho_\alpha^*[\sigma].$$

To avoid cumbersome notations we will write $[\sigma_\alpha]|_x$ and $\phi_\alpha|_x$ for the restrictions $[\sigma_\alpha]|_{\pi^{-1}(x)}$ and $\phi_\alpha|_{\pi^{-1}(x)}$ respectively. Then for every $x$ in $U_\alpha$,

$$[\sigma_\alpha]|_x = (\phi_\alpha|_x)^*[\sigma].$$

For $x \in U_\alpha \cap U_\beta$,

$$\begin{aligned} & [\sigma_\beta]|_x = [\sigma_\alpha]|_x \\ \text{iff}\quad & (\phi_\beta|_x)^*[\sigma] = (\phi_\alpha|_x)^*[\sigma] \\ \text{iff}\quad & [\sigma] = ((\phi_\beta|_x)^*)^{-1}(\phi_\alpha|_x)^*[\sigma] \\ \text{iff}\quad & [\sigma] = g_{\alpha\beta}(x)^*[\sigma]. \end{aligned}$$

Since $g_{\alpha\beta}(x)$ has positive determinant, $[\sigma] = g_{\alpha\beta}(x)^*[\sigma]$ by (11.1). Therefore, $[\sigma_\beta] = [\sigma_\alpha]$ on $U_\alpha \cap U_\beta$ and the sphere bundle $S(E)$ is orientable.
($\Leftarrow$) Conversely, let $\{U_\alpha, [\sigma_\alpha]\}$ be an orientation on the sphere bundle $S(E)$ and let $(S^n, \sigma)$ be an oriented sphere in $\mathbb{R}^{n+1}$, where $\sigma$ is a nontrivial top form on $S^n$. Choose the trivializations for $S(E)$

$$\phi_\alpha : S(E)|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times S^n$$

in such a way that $\phi_\alpha$ preserves the metric and $\phi_\alpha^* \rho_\alpha^*[\sigma] = [\sigma_\alpha]$. Then at any point $x$ in $U_\alpha \cap U_\beta$, the transition function $g_{\alpha\beta}(x)$ pulls $[\sigma]$ to itself and so $g_{\alpha\beta}(x)$ must be in $SO(n+1)$. □

Remark 11.3. Since $SO(1) = \{1\}$, a line bundle $L$ over a connected base space is orientable if and only if it is trivial. In this case the sphere bundle $S(L)$ consists of two connected components.

**Proposition 11.4.** *A vector bundle $E$ is orientable if and only if its determinant bundle* det $E$ *is orientable.*

Proof. Let $\{g_{\alpha\beta}\}$ be the transition functions of $E$. Then the transition functions of det $E$ are $\{\det g_{\alpha\beta}\}$. An orthogonal matrix $g_{\alpha\beta}$ assumes values in $SO(n+1)$ if and only if $\det g_{\alpha\beta}$ is positive, so the proposition follows. □

Whether $E$ is orientable or not, the 0-sphere bundle $S(\det E)$ is always a 2-sheeted covering of $M$. Combining Corollary 11.3 and Proposition 11.4, we see that over a connected base space a vector bundle $E$ is orientable if and only if $S(\det E)$ is disconnected. Since a simply connected base space cannot have any connected covering space of more than one sheet, we have proven the following.

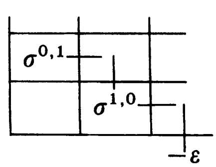
**Proposition 11.5.** *Every vector bundle over a simply connected base space is orientable.*

In particular, the tangent bundle of a simply connected manifold is orientable. Since a manifold is orientable if and only if its tangent bundle is (Example 6.3), this gives

**Corollary 11.6.** *Every simply connected manifold is orientable.*

## The Euler Class of an Oriented Sphere Bundle

We first consider the case of a circle bundle $\pi : E \to M$ with structure group Diff$(S^1)$. As stated in the introduction to this section, our problem is to find a closed global 1-form on $E$ which restricts to a generator of the cohomology on each fiber. As a first approximation, in each $U_\alpha$ of a good cover $\{U_\alpha\}$ of $M$ we choose a generator $[\sigma_\alpha]$ of $H^1(E|_{U_\alpha})$. The collection $\{\sigma_\alpha\}$ is an element $\sigma^{0,1}$ in the double complex $C^*(\pi^{-1}\mathfrak{U}, \Omega^*)$:



.

From the isomorphism between the cohomology of $E$ and the cohomology of this double complex,

$$H^*_{DR}(E) \simeq H_D\{C^*(\pi^{-1}\mathfrak{U}, \Omega^*)\},$$

we see that to find a global form which restricts to the $d$-cohomology class of $\sigma^{0,1}$ it suffices to extend $\sigma^{0,1}$ to a $D$-cocycle. The first step of the extension requires that $(\delta\sigma^{0,1})_{\alpha\beta} = \sigma_\beta - \sigma_\alpha$ be exact, i.e., $[\sigma_\alpha] = [\sigma_\beta]$ for all $\alpha$, $\beta$.

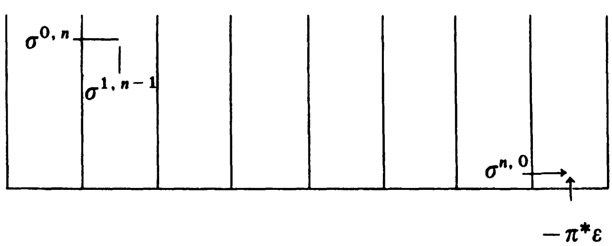
This is precisely the orientability condition. Assume the bundle $E$ to be oriented with orientation $\sigma^{0,1}$, so that $\delta\sigma^{0,1} = d\sigma^{1,0}$ for some $\sigma^{1,0}$ in $C^1(\pi^{-1}\mathfrak{U}, \Omega^0)$. Then $\sigma^{0,1} + \sigma^{1,0}$ is a $D$-cocycle if and only if $\delta\sigma^{1,0} = 0$. Since

$$d(\delta\sigma^{1,0}) = \delta(d\sigma^{1,0}) = \delta(\delta\sigma^{0,1}) = 0,$$

$\delta\sigma^{1,0}$ actually comes from an element $-\varepsilon$ of the cochain group $C^2(\pi^{-1}\mathfrak{U}, \mathbb{R})$. Now since the open covers $\mathfrak{U}$ and $\pi^{-1}\mathfrak{U}$ have the same combinatorics, i.e., $\pi^{-1}U_{\alpha_0\ldots\alpha_p}$ is nonempty if and only if $U_{\alpha_0\ldots\alpha_p}$ is, $C^*(\pi^{-1}\mathfrak{U}, \mathbb{R}) = C^*(\mathfrak{U}, \mathbb{R})$ and we may regard $\varepsilon$ as an element of $C^2(\mathfrak{U}, \mathbb{R})$. In fact, because $\delta\varepsilon = 0$, $\varepsilon$ defines a Čech cohomology class in $H^2(\mathfrak{U}, \mathbb{R})$. By the isomorphism between the Čech cohomology of a good cover and de Rham cohomology, $\varepsilon$ corresponds to a cohomology class $e(E)$ in $H^2(M)$. For a circle bundle with structure group $SO(2)$, this class turns out to be the Euler class of Section 6, as will be shown later. So for an oriented circle bundle $E$ with structure group $\mathrm{Diff}(S^1)$ we also call $e(E)$ the *Euler class*.

The discussion above generalizes immediately to any sphere bundle with fiber $S^n$, $n \geq 1$. Such a sphere bundle is orientable if and only if it is possible to find an element $\sigma^{0,n}$ in $C^0(\pi^{-1}\mathfrak{U}, \Omega^n)$ which extends one step down toward being a $D$-cocycle:

$$\delta\sigma^{0,n} = d\sigma^{1,n-1} = -D''\sigma^{1,n-1}.$$




There is no obstruction to extending $\sigma^{1,n-1}$ one step further, since every closed $(n-1)$-form on $E|_{U_{\alpha_0,\alpha_1,\alpha_2}}$ is exact. In general, extension is possible until we hit a nontrivial cohomology of the fiber. Thus for an oriented sphere bundle $E$ we can extend all the way down to $\sigma^{n,0}$ in such a manner that if

$$\sigma = \sigma^{0,n} + \sigma^{1,n-1} + \cdots + \sigma^{n,0},$$

then

$$D\sigma = \delta\sigma^{n,0}.$$

Since $d(\delta\sigma^{n,0}) = \delta(d\sigma^{n,0}) = \pm\delta(\delta\sigma^{n-1,1}) = 0$,

$$D\sigma = \delta\sigma^{n,0} = i(-\varepsilon)$$

for some $\varepsilon$ in $C^{n+1}(\pi^{-1}\mathfrak{U}, \mathbb{R}) \simeq C^{n+1}(\mathfrak{U}, \mathbb{R})$, where $i$ is the inclusion $C^{n+1}(\pi^{-1}\mathfrak{U}, \mathbb{R}) \to C^{n+1}(\pi^{-1}\mathfrak{U}, \Omega^0)$. Clearly $\delta\varepsilon = 0$, so $\varepsilon$ defines a cohomology class $e(E)$ in $H^{n+1}(\mathfrak{U}, \mathbb{R}) \simeq H^{n+1}(M)$, the *Euler class* of the oriented $S^n$-bundle $E$

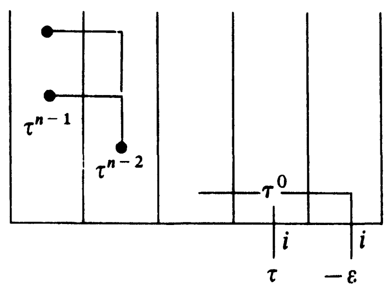
with orientation $\sigma^{0,n}$. The Euler class of an oriented $S^0$-bundle is defined to be 0. Note that the Euler class depends on the orientation $\{[\sigma_\alpha]\}$ of $E$; the opposite orientation would give $-e(E)$ instead.

If $E$ is an oriented vector bundle, the complement $E^0$ of its zero section has the homotopy type of an oriented sphere bundle. The Euler class of $E$ is defined to be that of $E^0$. Equivalently, if $E$ is endowed with a Riemannian metric, then the unit sphere bundle $S(E)$ of $E$ makes sense and we may define the Euler class of $E$ to be that of its unit sphere bundle. This latter definition is independent of the metric and in fact agrees with the definition in terms of $E^0$, since for any metric on $E$, the unit sphere bundle $S(E)$ has the homotopy type of $E^0$.

In the next two propositions we show that the Euler class is well defined.

**Proposition 11.7.** *For a given orientation $\{[\sigma_\alpha]\}$ the Euler class is independent of the choice of $\sigma^{j,n-j}$, $j = 0, \ldots, n$.*

PROOF.




Let $\bar{\sigma}^{0,n}$ be another cochain in $C^0(\pi^{-1}\mathfrak{U}, \Omega^n)$ which represents the orientation $\{[\sigma_\alpha]\}$. Then $\bar{\sigma}^{0,n} - \sigma^{0,n} = d\tau^{n-1}$ for some $\tau^{n-1}$ in $C^0(\pi^{-1}\mathfrak{U}, \Omega^{n-1})$. Since $d(\delta\tau^{n-1})$ and $d(\bar{\sigma}^{1,n-1} - \sigma^{1,n-1})$ are equal, $\delta\tau^{n-1}$ and $\bar{\sigma}^{1,n-1} - \sigma^{1,n-1}$ differ by $d\tau^{n-2}$ for some $\tau^{n-2}$ in $C^1(\pi^{-1}\mathfrak{U}, \Omega^{n-2})$. Again,

$$d(\delta\tau^{n-2}) = -d(\bar{\sigma}^{2,n-2} - \sigma^{2,n-2}),$$

so

$$(\delta\tau^{n-2}) - (\bar{\sigma}^{2,n-2} - \sigma^{2,n-2}) = d\tau^{n-3}$$

for some $\tau^{n-3}$ in $C^2(\pi^{-1}\mathfrak{U}, \Omega^{n-3})$. Eventually we get

$$\delta\tau^0 - (\bar{\sigma}^{n,0} - \sigma^{n,0}) = i\tau,\ \tau \in C^n(\pi^{-1}\mathfrak{U}, \mathbb{R}).$$

Taking $\delta$ of both sides, we have

$$\bar{\varepsilon} - \varepsilon = \delta\tau.$$

So $\bar{\varepsilon}$ and $\varepsilon$ define the same Čech cohomology class. □

**Proposition 11.8.** *The Euler class $e(E)$ is independent of the choice of the good cover.*

PROOF. Write $\varepsilon_{\mathfrak{U}}$ for the cocycle in $H^{n+1}(\mathfrak{U}, \mathbb{R})$ which defines the Euler class in terms of the good cover $\mathfrak{U}$. If a good cover $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, then there is a commutative diagram

$$\begin{tikzcd}
H^{n+1}(\mathfrak{U}, \mathbb{R}) \arrow[rr] \arrow[dr, "\simeq"'] & & H^{n+1}(\mathfrak{V}, \mathbb{R}) \arrow[dl, "\simeq"] \\
& H^{n+1}_{DR}(M) &
\end{tikzcd} .$$

$\varepsilon_{\mathfrak{U}}$ and $\varepsilon_{\mathfrak{V}}$ give the same element in $H^{n+1}_{DR}(M)$, because if we choose the $\sigma^{0,n}$ on $\pi^{-1}\mathfrak{V}$ to be the restriction of the $\sigma^{0,n}$ on $\pi^{-1}\mathfrak{U}$, the cocycle $\varepsilon_{\mathfrak{V}}$ in $C^{n+1}(\mathfrak{V}, \mathbb{R})$ will be the restriction of the cocycle $\varepsilon_{\mathfrak{U}}$ in $C^{n+1}(\mathfrak{U}, \mathbb{R})$, so that as elements of the Čech cohomology $H^{n+1}(M, \mathbb{R})$ they are equal. Given two arbitrary good covers $\mathfrak{U}$ and $\mathfrak{V}$, we can take a common refinement $\mathfrak{W}$; then $\varepsilon_{\mathfrak{U}} = \varepsilon_{\mathfrak{W}} = \varepsilon_{\mathfrak{V}}$ in $H^{n+1}(M, \mathbb{R})$. So the Euler class is independent of the cover.

□

If the Euler class $e(E) \in H^{n+1}(M)$ vanishes, its representative $\varepsilon \in C^{n+1}(\mathfrak{U}, \mathbb{R})$ is a $\delta$-coboundary; this permits one to alter $\sigma^{n,0}$ so that $D\sigma = 0$. The $D$-cocycle $\sigma$ then corresponds to a global form which restricts to the $d$-cohomology class of $\sigma^{0,n}$. In sum, then, there is a global form that restricts to a generator on each fiber if and only if

(a) $E$ is orientable, and
(b) the Euler class $e(E)$ vanishes.

For $E$ a product bundle, the extension stops at the $\sigma^{0,n}$ stage so that $\varepsilon = 0$. In this sense the Euler class is a measure of the twisting of an oriented sphere bundle. However, as we will see in the proposition below, $E$ need not be a product bundle for its Euler class to vanish.

**Proposition 11.9.** *If the oriented sphere bundle E has a section, then its Euler class vanishes.*

PROOF. Let $s$ be a section of $E$. It follows from $\pi \circ s = 1$ that $s^*\pi^* = 1$. We saw in the construction of the Euler class that

$$-\pi^*\varepsilon = D\sigma$$

for some $D$-cochain $\sigma$. Applying $s^*$ to both sides gives

$$-\varepsilon = Ds^*\sigma,$$

so $e$ is a coboundary in $H^*(M)$.

□

The converse of this proposition is not true. In general a cohomology class is too "coarse" an invariant to yield information on the existence of geometrical constructs. In (23.16) we will show the existence of a sphere bundle whose Euler class vanishes, but which does not admit any section.

We now show that for a circle bundle $\pi : E \to M$ with structure group $SO(2)$ the definitions of the Euler class in Section 6 and in this section agree. We briefly recall here the earlier construction. If $\theta_\alpha$ is the angular coordinate over $U_\alpha$, then $[d\theta_\alpha/2\pi]$ is a generator of $H^1(E|_{U_\alpha})$. Furthermore,

$$\frac{d\theta_\beta}{2\pi} - \frac{d\theta_\alpha}{2\pi} = \pi^* \frac{d\phi_{\alpha\beta}}{2\pi} = \pi^*\xi_\beta - \pi^*\xi_\alpha \text{ for some 1-form } \xi_\alpha \text{ over } U_\alpha.$$

The Euler class of the circle bundle $E$ was defined to be the cohomology class of the global form $\{d\xi_\alpha\}$.

In the present context these cochains fit into the double complex $C^*(\pi^{-1}\mathfrak{U}, \Omega^*)$ of $E$ as shown in the diagram below.

$$\begin{array}{r|c|c|c|l}
 & & & & \\
\hline
\Omega^*(E) & \dfrac{d\theta_\alpha}{2\pi} & \dfrac{\pi^* d\phi_{\alpha\beta}}{2\pi} & & C^*(\pi^{-1}\mathfrak{U}, \Omega^*) \\
\hline
 & & \dfrac{\pi^*\phi_{\alpha\beta}}{2\pi} & -\pi^*\varepsilon & \\
\hline
\multicolumn{1}{r}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\uparrow} & \\
\multicolumn{1}{r}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{-\pi^*\varepsilon} & \\
\multicolumn{1}{r}{} & \multicolumn{3}{c}{C^*(\pi^{-1}\mathfrak{U}, \mathbb{R})} &
\end{array}$$

By the explicit isomorphism between de Rham and Čech (Proposition 9.8), the differential form on $M$ corresponding to the Čech cocycle $\varepsilon$ is $(-D''K)^2\varepsilon$. Since $\xi_\beta - \xi_\alpha = (1/2\pi)\, d\phi_{\alpha\beta}$, $\delta\xi = (1/2\pi)\, d\phi$, so by (8.7), we may take $\xi$ to be $(1/2\pi)\, Kd\phi$. Also note that since $\delta(\phi/2\pi) = -\varepsilon$,

$$-K\varepsilon = \phi/2\pi \text{ (modulo a } \delta\text{-coboundary).}$$

Hence

$$\begin{aligned}
(-D''K)^2\varepsilon &= -dKdK\varepsilon \\
&= dKd((\phi/2\pi) + \delta\tau) \quad \text{for some } \tau \\
&= dKd(\phi/2\pi) + dKd\delta\tau \\
&= d\xi + dKd\delta\tau.
\end{aligned}$$

Here

$$\begin{aligned}
dKd\delta\tau &= dK\delta d\tau && \text{because } d \text{ commutes with } \delta \\
&= d(1 - \delta K)d\tau && \text{by (8.7)} \\
&= -\delta dKd\tau.
\end{aligned}$$

Since $Kd\tau \in \Omega^1(M)$, $dKd\tau$ is a global exact form, so $\delta dKd\tau = 0$. Hence $(-D''K)^2\varepsilon = d\xi$, showing that the two definitions of the Euler class could be made to agree on the level of forms.

## The Global Angular Form

In Section 6 we exhibited on an oriented circle bundle the global angular form $\psi$ which has the following properties:

(a) its restriction to each fiber is a generator of the cohomology of the fiber;
(b) $d\psi = -\pi^*e$, where $e$ represents the Euler class of the circle bundle.

Using the collating formula (9.5) we will now construct such a form on any oriented $S^n$-bundle.

Let $\mathfrak{U} = \{U_\alpha\}$ be an open cover of $M$. Recall that the Euler class of $E$ is defined by the following diagram:

$$\begin{array}{|c|c|c|c|c|c}
\alpha_0 & & & & & \\
 & \alpha_1 & & & & \\
 & & \alpha_2 & & & \\
 & & & & \alpha_n & -\pi^*\varepsilon \\
\hline
\end{array}$$

where $\alpha_0 \in C^0(\pi^{-1}\mathfrak{U}, \Omega^n)$ is the orientation of $E$,

$$\delta\alpha_i = -D''\alpha_{i+1}, \qquad i = 0, \ldots, n-1,$$

and

$$\delta\alpha_n = -\pi^*\varepsilon.$$

Hence

$$D(\alpha_0 + \cdots + \alpha_n) = -\pi^*\varepsilon.$$

Here $\alpha_i$ is what we formerly wrote as $\sigma^{i,\,n-i}$.

If $\{\rho_\alpha\}$ is a partition of unity subordinate to the open cover $\mathfrak{U} = \{U_\alpha\}$, then $\{\pi^*\rho_\alpha\}$ is a partition of unity subordinate to the inverse cover $\pi^{-1}\mathfrak{U} = \{\pi^{-1}U_\alpha\}$. Using these data we can define a homotopy operator $K$ on the double complex $C^*(\mathfrak{U}, \Omega^*)$ and also one on $C^*(\pi^{-1}\mathfrak{U}, \Omega^*)$ as in (8.6). We denote both operators by $K$. Both $K$ satisfy

$$\delta K + K\delta = 1.$$

Since

$$\begin{aligned}
(K\pi^*\omega)_{\alpha_0\ldots\alpha_{p-1}} &= \sum(\pi^*\rho_\alpha)(\pi^*\omega)_{\alpha\alpha_0\ldots\alpha_{p-1}} \\
&= \pi^* \sum\rho_\alpha\omega_{\alpha\alpha_0\ldots\alpha_{p-1}} \\
&= (\pi^*K\omega)_{\alpha_0\ldots\alpha_{p-1}},
\end{aligned}$$

$K$ commutes with $\pi^*$.

*Exercise 11.10.* If $s : M \to E$ is a section, show that $Ks^* = s^*K$.

By the collating formula (9.5),

$$(11.11) \qquad \psi = \sum_{i=0}^{n} (-1)^i (D''K)^i \alpha_i \quad + (-1)^{n+1} K(D''K)^n(-\pi^*\varepsilon)$$

is a global form on $E$. Furthermore,

$$\begin{aligned} d\psi &= (-1)^{n+1} dK(D''K)^n(-\pi^*\varepsilon) \\ &= -\pi^*(-1)^{n+1}(D''K)^{n+1}\varepsilon \quad \text{since } \pi^* \text{ commutes with } D''K \\ (11.12) \qquad &= -\pi^* e \quad \text{by Proposition 9.8.} \end{aligned}$$

In formula (11.11) since the restriction of $\pi^*((-1)^{n+1}K(D''K)^n\varepsilon)$ to a fiber is 0, the restriction of the global form $\psi$ to each fiber is $d$-cohomologous to $\alpha_0|_{\text{fiber}}$, hence is a generator of the cohomology of the fiber. The global $n$-form $\psi$ on the sphere bundle $E$ satisfies the properties $(a)$ and $(b)$ stated earlier. We call it the *global angular form* on the sphere bundle.

REMARK 11.12.1. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $M$ which trivializes the $n$-sphere bundle $E$ and let $\psi$ and $e$ be defined by (11.11) and (11.12). Then $\operatorname{Supp} d\psi \subset \cup \pi^{-1}(U_{\alpha_0 \ldots \alpha_n})$ and $\operatorname{Supp} e$ is contained in the union $\cup U_{\alpha_0 \ldots \alpha_n}$ of the $(n+1)$-fold intersections.

PROOF. By (8.6), $\operatorname{Supp}(K\omega)_{\alpha_0 \ldots \alpha_{p-1}} \subset \cup_\alpha \operatorname{Supp} \omega_{\alpha\alpha_0 \ldots \alpha_{p-1}} \subset \cup_\alpha U_{\alpha\alpha_0 \ldots \alpha_{p-1}}$. Since $\operatorname{Supp} \varepsilon \subset \cup U_{\alpha_0 \ldots \alpha_n}$, the remark follows from (11.11) and (11.12). □

*Exercise 11.13.* Use the existence of the global angular form $\psi$ to prove Proposition 11.9.

## Euler Number and the Isolated Singularities of a Section

Let $\pi : E \to M$ be an oriented $(k-1)$-sphere bundle over a compact oriented manifold of dimension $k$. Since $H^k(M) \simeq \mathbb{R}$, the Euler class of $E$ may be identified with the number $\int_M e(E)$, which is by definition the *Euler number* of $E$. The Euler number of the manifold $M$ is defined to be that of its unit tangent bundle $S(T_M)$ relative to some Riemannian structure on $M$. While the Euler number of an orientable sphere bundle is defined only up to sign, depending on the orientations of both $E$ and $M$, the Euler number of the orientable manifold $M$ is unambiguous, since reversing the orientation of $M$ also reverses that of the tangent bundle.

In general the sphere bundle $E$ will not have a global section; however, there may be a section $s$ over the complement of a finite number of points $x_1, \ldots, x_q$ in $M$. In fact, as we will show in Proposition 11.14, if the sphere bundle has structure group $O(k)$, then such a "partial" section $s$ always exists. In this section we will explain how one may compute the Euler class of $E$ in terms of the behavior of the section $s$ near the singularities $x_1, \ldots, x_q$.

**Proposition 11.14.** *Let $\pi : E \to M$ be a $(k-1)$-sphere bundle over a compact manifold of dimension $k$. Suppose the structure group of $E$ can be reduced to $O(k)$. Then $E$ has a section over $M - \{x_1, \ldots, x_q\}$ for some finite number of points in $M$.*

PROOF. Since the structure group of $E$ is $O(k)$, we can form a Riemannian vector bundle $E'$ of rank $k$ whose unit sphere bundle is $E$. A section $s'$ of $E'$ over $M$ gives rise to a partial section $s$ of $E : s(x) = s'(x)/\|s'(x)\|$, where $\| \ \|$ denotes the length of a vector in $E'$. Let $Z$ be the zero locus of $s'$; $s$ is only a partial section in the sense that it is not defined over $Z$. Thus to prove the proposition, we only have to show that the vector bundle $E'$ has a section that vanishes over a finite number of points.

This is an easy consequence of the transversality theorem which states that given a submanifold $Z$ in a manifold $Y$, every map $f: X \to Y$ becomes transversal to $Z$ under a slight perturbation (Guillemin and Pollack [1, p. 68]). Furthermore, we may assume that a small perturbation of a section $t$ of $E'$ is again a section, as follows. Suppose $f$ is a perturbation of $t$ and $f$ is transversal to the zero section. Then $g = \pi \circ f$ is a perturbation of $\pi \circ t$, which is the identity. Thus, for a sufficiently small perturbation, $g$ will be close to the identity and so must be a diffeomorphism. For such an $f$, define $s'(x) = f(g^{-1}(x))$. Then $\pi \circ s' = 1$ and $s'$ is transversal to $s_0(M)$, i.e., $S = s'(M)$ intersects $S_0 = s_0(M)$ transversally. Applying this procedure to the zero section of $E'$, i.e., choosing $t = s_0$, will yield the desired transversal section $s'$ for $E'$. Since

$$\dim S + \dim S_0 = \dim E',$$

$S \cap S_0$ consists of a discrete set of points. By the compactness of $S$, it must be a finite set of points. □

REMARK 11.15. It follows from the rudiments of obstruction theory that this proposition is true even if the structure group of the sphere bundle cannot be reduced to an orthogonal group. For a beautiful account of obstruction theory, see Steenrod [1, Part III].

Suppose $s$ is a section over a punctured neighborhood of a point $x$ in $M$. Choose this neighborhood sufficiently small so that it is diffeomorphic to a punctured disc in $\mathbb{R}^k$ and $E$ is trivial over it. Let $D_r$ be the open neighborhood of $x$ corresponding to the ball of radius $r$ in $\mathbb{R}^k$ under the diffeomorphism above. As an open subset of the oriented manifold $M$, $D_r$ is also oriented. Choose the orientation on the sphere $S^{k-1}$ in such a way that the isomorphism $E|_{D_r} \simeq D_r \times S^{k-1}$ is orientation-preserving, where $D_r \times S^{k-1}$ is given the product orientation. (If $A$ and $B$ are two oriented manifolds with orientation forms $\omega_A$ and $\omega_B$, then the *product orientation* on $A \times B$ is given by $(p_1^*\omega_A) \wedge (p_2^*\omega_B)$, where $p_1$ and $p_2$ are the projections of $A \times B$ onto $A$ and $B$ respectively.) The *local degree* of the section $s$ at $x$ is defined to be the degree of the composite map

$$\partial \bar{D}_r \xrightarrow{s} E|_{\bar{D}_r} = \bar{D}_r \times S^{k-1} \xrightarrow{\rho} S^{k-1}$$

where $\rho$ is the projection and $\bar{D}_r$ is the closure of $D_r$.

**Theorem 11.16.** *Let $\pi : E \to M$ be an oriented $(k-1)$-sphere bundle over a compact oriented manifold of dimension k. If E has a section over $M - \{x_1, \ldots, x_q\}$, then the Euler number of E is the sum of the local degrees of s at $x_1, \ldots, x_q$.*

PROOF. We first show that it is possible to move the support of the Euler class away from finitely many points.

**Lemma.** *Let M be a manifold and $\{U_\alpha\}_{\alpha \in I}$ an open cover of M. Given finitely many points $x_1, \ldots, x_q$ on M, there is a refinement $\{V_\alpha\}_{\alpha \in I}$ of $\{U_\alpha\}_{\alpha \in I}$ such that $V_\alpha \subset U_\alpha$ and each $x_i$ has a neighborhood $W_i$ which is disjoint from all but one of the $V_\alpha$'s.*

PROOF OF LEMMA. Suppose $x_1 \in U_1$. Let $W_1$ be an open neighborhood of $x_1$ such that $x_1 \in W_1 \subset \overline{W}_1 \subset U_1$. We define a new open cover $\{U'_\alpha\}_{\alpha \in I}$ by setting $U'_1 = U_1$ and $U'_\alpha = U_\alpha - \overline{W}_1$ for $\alpha \neq 1$. (Check that this is indeed an open cover of $M$.) The neighborhood $W_1$ of $x_1$ is contained in $U'_1$ but disjoint from all $U'_\alpha$, $\alpha \neq 1$.

Next suppose $x_2 \in U'_2$. Let $W_2$ be an open neighborhood of $x_2$ such that $x_2 \in W_2 \subset \overline{W}_2 \subset U'_2$. As before define a new open cover $\{U''_\alpha\}_{\alpha \in I}$ by setting $U''_2 = U'_2$ and $U''_\alpha = U'_\alpha - \overline{W}_2$ for $\alpha \neq 2$. Since $U''_\alpha \subset U'_\alpha$, the open neighborhood $W_1$ of $x_1$ is disjoint from all $U''_\alpha$, $\alpha \neq 1$. By definition, the open neighborhood $W_2$ of $x_2$ is disjoint from all $U''_\alpha$, $\alpha \neq 2$. Repeating this process to $x_3, \ldots, x_q$ in succession yields the open cover $\{V_\alpha\}$ of the lemma. □

Now let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $M$ which trivializes $E$. By the lemma we may assume that each $x_i$ has a neighborhood $W_i$ which is contained in exactly one $U_\alpha$. Construct the global angular form $\psi$ and the form $e$ relative to $\{U_\alpha\}_{\alpha \in I}$. By Remark 11.12.1, since $\operatorname{Supp} e \subset \cup U_{\alpha_0 \ldots \alpha_{k-1}}$, the form $e$ must vanish on $W_i$ for all $i = 1, \ldots, q$. So $e$ is supported away from the points $x_1, \ldots, x_q$.

For each $i$ choose an open ball $D_i$ around the point $x_i$ so that $\overline{D}_i \subset W_i$. Then

(11.16.1)
$$\begin{aligned} \int_M e = \int_{M - \cup D_i} e &= \int_{M - \cup D_i} s^*\pi^* e && \text{since } s \text{ is a global section over } M - \cup D_i \\ &= -\int_{M - \cup D_i} s^* d\psi && \text{because } \pi^* e = -d\psi \\ &= \sum_i \int_{\partial \overline{D}_i} s^*\psi && \text{by Stokes' theorem and the fact that } \partial \overline{D}_i \text{ has the opposite orientation as } \partial(M - \cup D_i). \end{aligned}$$

Although the global angular form is not closed, by our construction $d\psi = 0$ on $E|_{W_i}$, so $\psi$ defines a cohomology class in $H^{k-1}(E|_{W_i})$, which is in fact the generator. Let $\sigma$ be the generator of $S^{k-1}$. Then $\rho^*\sigma$ restricts to the generator on each fiber of $E|_{W_i}$. So $\rho^*\sigma$ and the angular form $\psi$ define the same cohomology class in $H^{k-1}(E|_{W_i})$, i.e.,

$$\psi - \rho^*\sigma = d\tau$$

for some $(k-2)$-form $\tau$ on $E|_{W_i}$. Thus on $\bar{D}_i$,

$$s^*\psi - s^*\rho^*\sigma = s^*\, d\tau$$

and

$$\int_{\partial \bar{D}_r} s^*\,\psi - \int_{\partial \bar{D}_r} s^*\rho^*\sigma = \int_{\partial \bar{D}_r} ds^*\tau = 0 \qquad \text{by Stokes' theorem.}$$

Therefore,

$$\int_{\partial \bar{D}_r} s^*\psi = \text{local degree of the section } s \text{ at } x_i.$$

Together with (11.16.1), this gives

$$\int_M e = \sum_i (\text{local degree of } s \text{ at } x_i). \qquad \square$$

This theorem can also be phrased in terms of vector bundles. Let $\pi : E \to M$ be an oriented rank $k$ vector bundle over a manifold of dimension $k$ and $s$ a section of $E$ with a finite number of zeros. The *multiplicity* of a zero $x$ of $s$ is defined to be the local degree of $x$ as a singularity of the section $s/\| s \|$ of the unit sphere bundle of $E$ relative to some Riemannian structure on $E$. (This definition of the index is independent of the Riemannian structure because the local degree is a homotopy invariant.) Since the Euler class $e(E)$ of $E$ is a $k$-form on $M$, it is Poincaré dual to $nP$, where $n = \int_M e(E)$ and $P$ is a point on $M$. Thus we have the following.

**Theorem 11.17.** *Let $\pi : E \to M$ be an oriented rank $k$ vector bundle over a compact oriented manifold of dimension $k$. Let $s$ be a section of $E$ with a finite number of zeros. The Euler class of $E$ is Poincaré dual to the zeros of $s$, counted with the appropriate multiplicities.*

EXAMPLE 11.18 (The Euler class of the unit tangent bundle to $S^2$). Let $S(T_{S^2})$ be the unit tangent bundle to $S^2$. It is a circle bundle over $S^2$:

$$\begin{array}{ccc} S^1 & \to & S(T_{S^2}) \\ & & \downarrow \\ & & S^2 \end{array}$$

Fix a unit tangent vector $v$ at the north pole. We can define a smooth vector field on $S^2$-{south pole} by parallel translating $v$ along the great circles from the north pole to the south pole (see Figure 11.1). (Parallel translation along a great circle on $S^2$ is prescribed by the following two conditions:

(a) the tangent field to the great circle is parallel;
(b) the angles are preserved under parallel translation.)

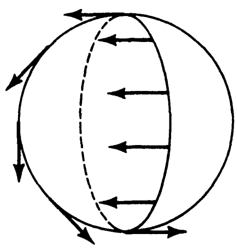


Figure 11.1

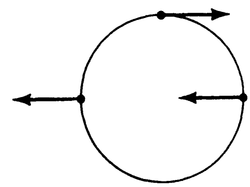


Figure 11.2

This gives a section $s$ of $S(T_{S^2})$ over $S^2$-{south pole}. On a small circle around the south pole, the vector field looks like Figure 11.2, i.e., as we go around the circle 90°, the vectors rotate through 180°; therefore, the local degree of $s$ at the south pole is 2. By Theorem 11.16, the Euler number of the unit tangent bundle to $S^2$ is 2.

*Exercise 11.19.* Show that the Euler class of an oriented sphere bundle with even-dimensional fibers is zero, at least when the sphere bundle comes from a vector bundle.

Since the Euler class is the obstruction to finding a closed global angular form on an oriented sphere bundle, by the Leray-Hirsch theorem we have the following corollary of Exercise 11.19.

**Proposition 11.20.** *If $\pi : E \to M$ is an orientable $S^{2n}$-bundle, then*

$$H^*(E) = H^*(M) \otimes H^*(S^{2n}).$$

*Exercise 11.21.* Compute the Euler class of the unit tangent bundle of the sphere $S^k$ by finding a vector field on $S^k$ and computing its local degrees.

## Euler Characteristic and the Hopf Index Theorem

In this section we show that the Euler number $\int_M e(T_M)$ is the same as the Euler characteristic $\chi(M) = \sum(-1)^q \dim H^q(M)$ and deduce as a corollary the Hopf index theorem. The manifold $M$ is assumed to be compact and oriented.

Let $\{\omega_i\}$ be a basis of the vector space $H^*(M)$, $\{\tau_j\}$ be the dual basis under Poincaré duality, i.e., $\int_M \omega_i \wedge \tau_j = \delta_{ij}$, and let $\pi$ and $\rho$ be the two projections of $M \times M$ to $M$:

$$\begin{tikzcd} & M \times M \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ M & & M. \end{tikzcd}$$

By the Künneth formula, $H^*(M \times M) = H^*(M) \otimes H^*(M)$ with $\{\pi^*\omega_i \wedge \rho^*\tau_j\}$ as an additive basis. So the Poincaré dual $\eta_\Delta$ of the diagonal $\Delta$ in $M \times M$ is some linear combination $\eta_\Delta = \sum c_{ij}\pi^*\omega_i \wedge \rho^*\tau_j$.

**Lemma 11.22.** $\eta_\Delta = \sum(-1)^{\deg \omega_i}\pi^*\omega_i \wedge \rho^*\tau_i$.

Proof. We compute $\int_\Delta \pi^*\tau_k \wedge \rho^*\omega_l$ in two ways. On the one hand, we can pull this integral back to $M$ via the diagonal map $\iota : M \to \Delta \subset M \times M$:

$$\int_\Delta \pi^*\tau_k \wedge \rho^*\omega_l = \int_M \iota^*\pi^*\tau_k \wedge \iota^*\rho^*\omega_l = \int_M \tau_k \wedge \omega_l = (-1)^{(\deg \tau_k)(\deg \omega_l)}\delta_{kl}.$$

On the other hand, by the definition of the Poincaré dual of a closed oriented submanifold (5.13),

$$\begin{aligned}\int_\Delta \pi^*\tau_k \wedge \rho^*\omega_l &= \int_{M\times M} \pi^*\tau_k \wedge \rho^*\omega_l \wedge \eta_\Delta\\ &= \sum_{i,j} c_{ij}\int_{M\times M} \pi^*\tau_k \wedge \rho^*\omega_l \wedge \pi^*\omega_i \wedge \rho^*\tau_j\\ &= \sum_{i,j} c_{ij}(-1)^{(\deg \tau_k + \deg \omega_l)(\deg \omega_i)}\int_{M\times M} \pi^*(\omega_i \wedge \tau_k)\rho^*(\omega_l \wedge \tau_j)\\ &= (-1)^{(\deg \tau_k + \deg \omega_l)\deg \omega_k}\, c_{kl}.\end{aligned}$$

Therefore

$$c_{kl} = \begin{cases} 0 & \text{if } k \neq l\\ (-1)^{\deg \omega_k} & \text{if } k = l.\end{cases}$$

□

**Lemma 11.23.** *The normal bundle $N_\Delta$ of the diagonal $\Delta$ in $M \times M$ is isomorphic to the tangent bundle $T_\Delta$.*

Proof. Since the diagonal map $\iota: M \to M \times M$ sends $M$ diffeomorphically onto $\Delta$, $\iota^*T_\Delta = T_M$. It follows from the commutative diagram

$$\begin{array}{ccccccccc} & & (v,v) & \mapsto & (v,v) & & & & \\ 0 & \to & T_\Delta & \to & T_{M\times M}|_\Delta & \to & N_\Delta & \to & 0\\ & & \wr\!\wr & & \wr\!\wr & & & & \\ 0 & \to & T_M & \to & T_M \oplus T_M & \to & T_M & \to & 0\\ & & v & \mapsto & (v,v) & & & & \end{array}$$

that $N_\Delta \simeq T_M \simeq T_\Delta$.

□

Recall that the Poincaré dual of a closed oriented submanifold $S$ is represented by the same form as the Thom class of a tubular neighborhood of $S$ (see (6.23)). Thus

$$\int_\Delta \eta_\Delta = \int_\Delta \Phi(N_\Delta) \quad \text{where } \Phi(N_\Delta) \text{ is the Thom class of the normal bundle } N_\Delta \text{ regarded as a tubular neighborhood of } \Delta \text{ in } M \times M$$

$$= \int_\Delta e(N_\Delta) \quad \text{since the Thom class restricted to the zero section of the bundle is the Euler class (proved for rank 2 bundles in Prop. 6.41 on p. 74; the general case will be shown later, in Prop. 12.4 on p. 128.)}$$

$$= \int_\Delta e(T_\Delta)$$

$$= \int_M e(T_M)\,.$$

So *the self-intersection number of the diagonal* $\Delta$ *in* $M \times M$ *is the Euler number of* $M$. (By Poincaré duality, $\int_\Delta \eta_\Delta = \int_{M\times M} \eta_\Delta \wedge \eta_\Delta$ is the self-intersection number of $\Delta$ in $M \times M$.)

Now the right-hand side of Lemma 11.22 evaluated on the diagonal $\Delta$ is

$$\begin{aligned}
\int_\Delta \eta_\Delta &= \sum_i (-1)^{\deg \omega_i} \int_\Delta \pi^*\omega_i \wedge \rho^*\tau_i \\
&= \sum_i (-1)^{\deg \omega_i} \int_M \iota^*\pi^*\omega_i \wedge \iota^*\rho^*\tau_i \\
&= \sum_i (-1)^{\deg \omega_i} \int_M \omega_i \wedge \tau_i \\
&= \sum_i (-1)^{\deg \omega_i} \\
&= \sum_q (-1)^q \dim H^q(M) \\
&= \chi(M).
\end{aligned}$$

Therefore,

**Proposition 11.24.** *The Euler number of a compact oriented manifold* $\int_M e(T_M)$ *is equal to its Euler characteristic* $\chi(M) = \sum(-1)^q \dim H^q$.

It is now a simple matter to derive the Hopf index theorem. Let $V$ be a vector field with isolated zeros on $M$. The *index* of $V$ at a zero $u$ is defined to be the local degree at $u$ of $V/\|V\|$ as a section of the unit tangent bundle of $M$ relative to some Riemannian metric on $M$. By Theorem 11.16 the sum of the indices of $V$ is the Euler number of $M$. The equality of the Euler number and the Euler characteristic then yields the following.

**Theorem 11.25** (Hopf Index Theorem). *The sum of the indices of a vector field on a compact oriented manifold $M$ is the Euler characteristic of $M$.*

*Exercise 11.26* (*Lefschetz fixed-point formula*). Let $f: M \to M$ be a smooth map of a compact oriented manifold into itself. Denote by $H^q(f)$ the induced map on the cohomology $H^q(M)$. The *Lefschetz number* of $f$ is defined to be

$$L(f) = \sum_q (-1)^q \operatorname{trace} H^q(f).$$

Let $\Gamma$ be the graph of $f$ in $M \times M$.

(a) Show that

$$\int_\Delta \eta_\Gamma = L(f).$$

(b) Show that if $f$ has no fixed points, then $L(f)$ is zero.

(c) At a fixed point $P$ of $f$ the derivative $(Df)_P$ is an endomorphism of the tangent space $T_P M$. We define the *multiplicity* of the fixed point $P$ to be

$$\sigma_P = \operatorname{sgn} \det((Df)_P - I).$$

Show that if the graph $\Gamma$ is transversal to the diagonal $\Delta$ in $M \times M$, then

$$L(f) = \sum_P \sigma_P,$$

where $P$ ranges over the fixed points of $f$. (For an explanation of the meaning of the multiplicity $\sigma_P$, see Guillemin and Pollack [1, p. 121].)

# §12 Thom Isomorphism and Poincaré Duality Revisited

In this section we study the Thom isomorphism and Poincaré duality from the tic-tac-toe point of view. The results obtained here are more general than those of Sections 5 and 6 in two ways:

(a) $M$ need not have a finite good cover,

and

(b) the orientability assumption on the vector bundle $E$ has been dropped.

## The Thom Isomorphism

Let $\pi : E \to M$ be a rank $n$ vector bundle. $E$ is not assumed to be orientable. We are interested in the cohomology of $E$ with compact support in the vertical direction, $H^*_{cv}(E) = H^*\{\Omega^*_{cv}(E)\}$. Recall that

(a) $H^*_c(\mathbb{R}^n) = \begin{cases} \mathbb{R} \text{ in dimension } n \\ 0 \text{ otherwise,} \end{cases}$

(b) (Poincaré lemma) $H^*_{cv}(M \times \mathbb{R}^n) = H^{*-n}(M)$.

Let $\mathfrak{U}$ be a good cover of the base manifold $M$. We augment the double complex $C^*(\pi^{-1}\mathfrak{U}, \Omega^*_{cv})$ by adding a column consisting of the kernels of the first $\delta$:

$$\begin{array}{l} 0 \to \Omega^2_{cv}(E) \to \\ 0 \to \Omega^1_{cv}(E) \to \\ 0 \to \Omega^0_{cv}(E) \to \end{array}$$

Using a partition of unity from the base, it can be shown that all the rows of this agumented double complex are exact. The proof is identical to that of the generalized Mayer-Vietoris sequence in (8.5) and will not be repeated here. From the exactness of the rows of the augmented complex, it follows as in (8.8) that the cohomology of the initial column is the total cohomology of the double complex, i.e.,

$$H^*_{cv}(E) \simeq H_D\{C^*(\pi^{-1}\mathfrak{U}, \Omega^*_{cv})\}.$$

On the other hand,

$$\begin{aligned} H_d^{p,q}\{C^*(\pi^{-1}\mathfrak{U}, \Omega^*_{cv})\} &= H^q_{cv}(\textstyle\coprod \pi^{-1}U_{\alpha_0\ldots\alpha_p}) \\ &= \textstyle\prod H^q_{cv}(\pi^{-1}U_{\alpha_0\ldots\alpha_p}) \\ &= C^p(\mathfrak{U}, \mathscr{H}^q_{cv}), \end{aligned}$$

where $\mathscr{H}^q_{cv}$ is the presheaf given by

$$\mathscr{H}^q_{cv}(U) = H^q_{cv}(\pi^{-1}U).$$

By the Poincaré lemma for compactly supported cohomology, if $U$ is contractible, then

$$\mathscr{H}^q_{cv}(U) = \begin{cases} \mathbb{R} & \text{if } q = n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $H_d$ and also $H_\delta^{p,q}H_d = H_\delta^p\{C^*(\mathfrak{U}, \mathscr{H}^q_{cv})\} = H^p(\mathfrak{U}, \mathscr{H}^q_{cv})$ have entries only in the $n$th row.

**Proposition 12.1.** *Given any double complex $K$, if $H_\delta H_d(K)$ has entries only in one row, then $H_\delta H_d$ is isomorphic to $H_D$.*

This proposition will be substantially generalized in Section 14, for it is simply an example of a degenerate spectral sequence. Its proof is a technical exercise which we defer to the end of this section. Combined with the preceding discussion, it gives

$$H_{cv}^*(E) = H_D^* = \bigoplus_{p+q=*} H^p(\mathfrak{U}, \mathscr{H}_{cv}^q) = H^{*-n}(\mathfrak{U}, \mathscr{H}_{cv}^n).$$

This is the Thom isomorphism for a not necessarily orientable vector bundle.

**Theorem 12.2** (Thom Isomorphism). *For $\pi : E \to M$ any vector bundle of rank $n$ over $M$ and $\mathfrak{U}$ a good cover of $M$,*

$$H_{cv}^*(E) \simeq H^{*-n}(\mathfrak{U}, \mathscr{H}_{cv}^n),$$

*where $\mathscr{H}_{cv}^n$ is the presheaf $\mathscr{H}_{cv}^n(U) = H_{cv}^n(\pi^{-1}U)$.*

We now deduce the orientable version of the Thom isomorphism from this. So suppose $\pi : E \to M$ is an *orientable* vector bundle of rank $n$ over $M$. This means there exist forms $\sigma_\alpha$ on the sphere bundles $S(E)|_{U_\alpha}$ which restrict to a generator on each fiber and such that on overlaps $U_\alpha \cap U_\beta$ their cohomology classes agree: $[\sigma_\alpha] = [\sigma_\beta]$. Now choose a Riemannian metric on $E$ so that the "radius" $r$ is well-defined on each fiber and any function of the radius $r$ is a global function on $E$. Let $\rho(r)$ be the function shown in Figure 12.1. Then $(d\rho)\sigma_\alpha$ is a form on $E|_{U_\alpha}$, where we regard $\sigma_\alpha$ as a form on the complement of the zero section. Furthermore, $[(d\rho)\sigma_\alpha] \in H_{cv}^n(E|_{U_\alpha})$ restricts to a generator of the compactly supported cohomology of the fiber and $[(d\rho)\sigma_\alpha] = [(d\rho)\sigma_\beta]$ on $U_\alpha \cap U_\beta$. Since the fiber has no cohomology in dimensions less than $n$, $\sigma^{0,n} = \{(d\rho)\,\sigma_\alpha\}$ can be extended to a $D$-cocycle. This $D$-cocycle corresponds to a global closed form $\Phi$ on $E$, the *Thom class* of $E$, which restricts to a generator on each fiber. Now $\mathscr{H}_{cv}^n(U)$ is generated by $\Phi|_U$ and for $V \subset U$ the restriction map from $\mathscr{H}_{cv}^n(U)$ to $\mathscr{H}_{cv}^n(V)$ sends

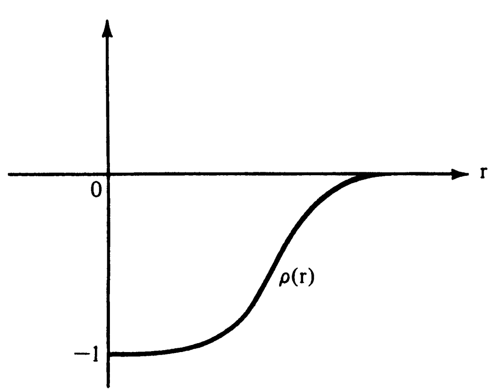



Figure 12.1

$\Phi|_U$ to $\Phi|_V$. Hence, via the map which sends $\Phi|_U$, for every open set $U$, to the generator 1 of the constant presheaf $\mathbb{R}$, the presheaf $\mathscr{H}_{cv}^n$ is isomorphic to $\mathbb{R}$. The Thom isomorphism theorem then assumes the form

$$H_{cv}^*(E) \simeq H^{*-n}(\mathfrak{U}, \mathscr{H}_{cv}^n) = H^{*-n}(\mathfrak{U}, \mathbb{R}) = H^{*-n}(M), \tag{12.2.1}$$

for an orientable rank $n$ vector bundle $E$. This agrees with Proposition 6.17. It holds in particular when $M$ is simply connected, since by (11.5), every vector bundle over a simply connected manifold is orientable.

From the explicit formula (11.11) for the global angular form on an oriented sphere bundle, we can derive a formula for the Thom class of an oriented vector bundle. Let $f : E^0 \to S(E)$ be a deformation retraction of the complement of the zero section in $E$ onto the unit sphere bundle. If $\psi_S$ is the global angular form on $S(E)$, then $\psi = f^*\psi_S \in H^{n-1}(E^0)$ is the global angular form on $E^0$. It has the property that

$$d\psi = -\pi^* e,$$

where $e$ represents the Euler class of the bundle $E$.

**Proposition 12.3.** *The cohomology class of*

$$\Phi = d(\rho(r) \cdot \psi) \in \Omega_{cv}^n(E)$$

*is the Thom class of the oriented vector bundle $E$.*

PROOF. Note that

$$\Phi = d\rho(r) \cdot \psi - \rho(r)\pi^* e. \tag{12.3.1}$$

For the same reasons as in the discussion following (6.40), $\Phi$ is a closed global form on $E$ with compact support in the vertical direction. Its restriction to the fiber at $p$ is $d\rho(r) \cdot \iota_p^*\psi$, where $\iota_p : E_p \to E$ is the inclusion and $\iota_p^*\psi$ gives a generator of $H^{n-1}(\mathbb{R}^n - \{0\}) = H^{n-1}(S^{n-1})$. Since

$$\int_{\mathbb{R}^n} d\rho(r) \cdot \iota_p^*\psi = \int_{\mathbb{R}^1} d\rho(r) \int_{S^{n-1}} \iota_p^*\psi = 1,$$

by (6.18), $\Phi$ is the Thom class of $E$. □

If $s$ is the zero section of $E$, then $s^*d\rho = 0$ and $s^*\rho = -1$. By (12.3.1),

$$s^*\Phi = -(s^*\rho)s^*\pi^*e = e.$$

Thus,

**Proposition 12.4.** *The pullback of the Thom class of an oriented rank n vector bundle via the zero section to the base manifold is the Euler class.*

REMARK 12.4.1. From the formula for the Thom class (12.3), it is clear that by making the support of $\rho(r)$ sufficiently close to 0, the Thom class $\Phi$ can be made to have support arbitrarily close to the zero section of the vector bundle.

REMARK 12.4.2. In fact, in Proposition 12.4 *any* section will pull the Thom class back to the Euler class. Let $s$ be a section of the oriented vector bundle $E$ and $s^* : H^*_{cv}(E) \to H^*(M)$ the induced map in cohomology. Note that $s^*$ can be written as the composition of the natural maps $i : H^*_{cv}(E) \to H^*(E)$ and $\bar{s}^* : H^*(E) \to H^*(M)$. As a map from $M$ into $E$, the section $s$ is homotopic to the zero section $s_0$. By the homotopy axiom for de Rham cohomology (Cor. 4.1.2), $\bar{s}^* = \bar{s}_0^*$. Hence, $s^* = s_0^*$.

Using the description of the Euler class as the pullback of the Thom class, it is easy to prove the Whitney product formula.

**Theorem 12.5** (Whitney Product Formula for the Euler Class). *If $E$ and $F$ are two oriented vector bundles, then $e(E \oplus F) = e(E)e(F)$.*

PROOF. By Proposition 6.19, the Thom class of $E \oplus F$ is

$$\Phi(E \oplus F) = \pi_1{}^*\Phi(E) \wedge \pi_2^*\Phi(F)$$

where $\pi_1$ and $\pi_2$ are the projections of $E \oplus F$ onto $E$ and $F$ respectively. Let $s$ be the zero section of $E \oplus F$. Then $\pi_1 \circ s$ and $\pi_2 \circ s$ are the zero sections of $E$ and $F$. By Proposition 12.4,

$$e(E \oplus F) = s^*\Phi(E \oplus F) = s^*\pi_1^* (E) \wedge s^*\pi_2^*\Phi(F) = e(E)e(F).$$

□

*Exercise 12.6.* Let $\pi : E \to M$ be an oriented vector bundle.

(a) Show that $\pi^*e = \Phi$ as cohomology classes in $H^*(E)$, but not in $H^*_{cv}(E)$.

(b) Prove that $\Phi \wedge \Phi = \Phi \wedge \pi^*e$ in $H^*_{cv}(E)$.

## Euler Class and the Zero Locus of a Section

Let $\pi : E \to M$ be a vector bundle and $S_0$ the image of the zero section in $E$. A section $s$ of $E$ is transversal if its image $S = s(M)$ intersects $S_0$ transversally. The purpose of this section is to derive an interpretation of the Euler class of an oriented vector bundle as the Poincaré dual of the zero locus of a transversal section. This is an analogue of Theorem 11.17, but it differs from Theorem 11.17 in two ways: (1) there is no hypothesis on the rank of $E$; (2) the section is now assumed to be transversal.

**Proposition 12.7.** *Let $\pi : E \to M$ be any vector bundle and $Z$ the zero locus of a transversal section. Then $Z$ is a submanifold of $M$ and its normal bundle in $M$ is $N_{Z/M} \simeq E|_Z$.*

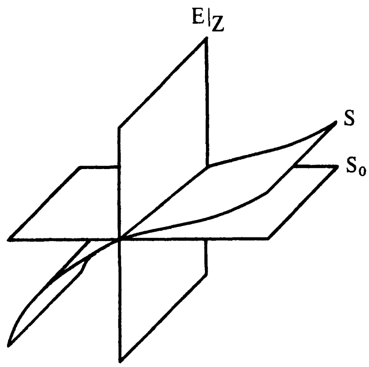



Figure 12.2

PROOF. Write $S = s(M)$ for the image of the section $s$ (see Figure 12.2). Because $S$ intersects $S_0$ transversally, $S \cap S_0$ is a submanifold of $S$ by the transversality theorem (Guillemin and Pollack [1, p. 28]). Under the diffeomorphism $s : M \to S$, $Z$ is mapped homeomorphically to $S \cap S_0$. So $Z$ can be made into a submanifold of $M$.

To compute the normal bundle of $Z$, we first note that because $E$ is locally trivial, its tangent bundle on $S_0$ has the following canonical decomposition

$$T_E|_{S_0} = E|_{S_0} \oplus T_{S_0}.$$

By the transversality of $S \cap S_0$,

$$T_S + T_{S_0} = T_E = E \oplus T_{S_0} \text{ on } S \cap S_0.$$

Hence the projection $T_S \to E$ over $S \cap S_0$ is surjective with kernel $T_S \cap T_{S_0}$. Again by the transversality of $S \cap S_0$, $T_S \cap T_{S_0} = T_{S \cap S_0}$. So we have an exact sequence over $Z \simeq S \cap S_0$:

$$0 \to T_Z \to T_S|_Z \to E|_Z \to 0.$$

Hence $N_{Z/M} \simeq E|_Z$. □

In the proposition above, if $E$ and $M$ are both oriented, then the zero locus $Z$ of a transversal section is naturally an *oriented* manifold, oriented in such a way that

$$E|_Z \oplus T_Z = T_M|_Z$$

has the direct sum orientation.

**Proposition 12.8.** *Let $\pi : E \to M$ be an oriented vector bundle over an oriented manifold $M$. Then the Euler class $e(E)$ is Poincaré dual to the zero locus of a transversal section.*

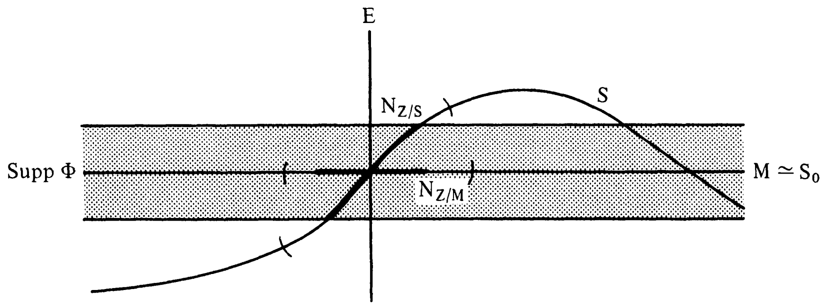



Figure 12.3

PROOF. We will identify $M$ with the image $S_0$ of the zero section. If $S$ is the image in $E$ of the transversal section $s : M \to E$, then the zero locus of $s$ is $Z = S \cap S_0$. $Z$ is a closed oriented submanifold of $M$ and by Proposition 12.7, its normal bundle in $M$ is $N_{Z/M} = E|_Z$. Since $S$ is diffeomorphic to $M$, the normal bundle $N_{Z/S}$ of $Z$ in $S$ is also $E|_Z$. The normal bundles $N_{Z/M}$ and $N_{Z/S}$ will be identified with the tubular neighborhoods of $Z$ in $M$ and in $S$ respectively, as in Figure 12.3.

Choose the Thom class $\Phi$ of $E$ to have support so close to the zero section (Remark 12.4.1) that $\Phi$ restricted to the tubular neighborhood $N_{Z/S}$ in $S$ has compact support in the vertical direction. In Figure 12.3 the support of $\Phi$ is in the shaded region. We will now show that $s^*\Phi$ is the Thom class of the tubular neighborhood $N_{Z/M}$ in $M$.

Let $E_z$, $S_z$, and $M_z$ be the fibers of $E|_Z \simeq N_{Z/S} \simeq N_{Z/M}$ respectively above the point $z$ in $Z$. Because $\Phi$ has compact support in $S_z$, $s^*\Phi$ has compact support in $M_z$. Furthermore,

$$\begin{aligned}\int_{M_z} s^*\Phi &= \int_{S_z} \Phi && \text{by the invariance of the integral under the orientation-preserving diffeomorphism } s : M_z \to S_z\\ &= \int_{E_z} \Phi && \text{because } E_z \text{ is homotopic to } S_z \text{ modulo the region in } E \text{ where } \Phi \text{ is zero}\\ &= 1 && \text{by the definition of the Thom class.}\end{aligned}$$

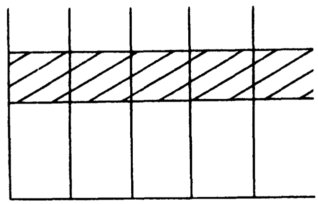
So $s^*\Phi$ is the Thom class of $N_{Z/M}$. By Proposition 12.4, $s^*\Phi = e(E)$. Since by (6.24) the Thom class of $N_{Z/M}$ is Poincaré dual to $Z$ in $M$, the Euler class $e(E)$ is Poincaré dual to $Z$ in $M$.

□

## A Tic-Tac-Toe Lemma

In this section we will prove the technical lemma (Proposition 12.1) that if $H_\delta H_d$ of a double complex $K$ has entries in only one row, then $H_\delta H_d$ is isomorphic to the total cohomology $H_D(K)$. With this tic-tac-toe lemma we will re-examine the Mayer–Vietoris principle of Section 8.

PROOF OF PROPOSITION 12.1.



We first define a map $h\colon H_\delta H_d \to H_D$. Recall that $D = D' + D'' = \delta + (-1)^p d$. An element $[\phi]$ in $H_\delta^{p,q} H_d$ may be represented by a $D$-cochain $\phi$ of degree $(p, q)$ such that

$$D''\phi = 0$$

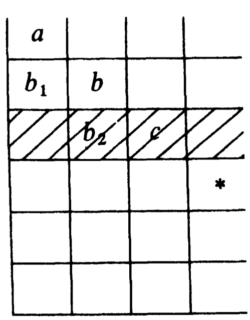
$$\delta\phi = -D''\phi_1 \text{ for some } \phi_1.$$

This is summarized by the diagram

$$\begin{array}{ccc} 0 & & \\ D''\uparrow & & \\ \phi & \xrightarrow{\delta} & \delta\phi + D''\phi_1 = 0 \\ & & \uparrow D'' \\ & & \phi_1 \end{array}$$

Since $H_\delta^{p+2,q-1} H_d = 0$, $\delta\phi_1 = -D''\phi_2$ for some $\phi_2$. Continuing in this manner, we see that $\phi$ can be extended downward to a $D$-cocycle $\phi + \phi_1 + \cdots + \phi_n$. The map $h$ is defined by sending $[\phi]$ to $[\phi + \phi_1 + \cdots + \phi_n]$.

Next we define the inverse map $g : H_D \to H_\delta H_d$. Let $\omega$ be a cocycle in $H_D$. As the image of $\omega$ we cannot simply take the component of $\omega$ in the nonzero row because $d$ of it may not be zero. Suppose $\omega = a + b + c + \cdots$ as shown.




We will move $\omega$ in its $D$-cohomology class so that it has nothing above the nonzero row. Since $da = 0$ and $\delta a = -D''b$, $a$ represents a cocycle in $H_\delta H_d$. But $H_\delta H_d = 0$ at the position of $a$, so $a$ is 0 in $H_\delta H_d$; this implies that

$a = D''a_1$ for some $a_1$. Then $\omega - Da_1$ has no components in the first column. Thus we may assume $\omega = b + c + \cdots$. Again $b$ is 0 in $H_\delta H_d$, so that $b = \delta b_1 + D''b_2$, where $D''b_1 = 0$. Then $\omega - D(b_1 + b_2) = (c - \delta b_2) + \cdots$ starts at the nonzero row.

$$\begin{tikzcd} 0 & & \\ b_1 \arrow[u] \arrow[r] & b & \\ & b_2 \arrow[u] \arrow[r] & c \end{tikzcd}$$

Thus given $[\omega] \in H_D$, we may pick $\omega$ to have no components above the nonzero row of $H_\delta H_d$, say $\omega = c + \cdots$. Then $dc = 0$ and the map $g : H_D \to H_\delta H_d$ is defined by sending $[\omega]$ to $[c]$.

Provided they are well-defined, $h$ and $g$ are clearly inverse to each other.

*Exercise 12.9.* Show that $h$ and $g$ are well-defined.

□

Using Proposition 12.1 we can give more succinct proofs of the main results of Section 8. Let $\mathfrak{U} = \{U_\alpha\}$ be an open cover of the manifold $M$ and $C^p(\mathfrak{U}, \Omega^q) = \prod \Omega^q(U_{\alpha_0 \cdots \alpha_p})$. By the exactness of the Mayer-Vietoris sequence, $H_\delta$ of the Čech–de Rham complex $C^*(\mathfrak{U}, \Omega^*)$ is

$$\begin{array}{c|c|c|c|l} q & & & & \\ & \vdots & & & \\ & \Omega^2(M) & & & \\ & \Omega^1(M) & & & \\ & \Omega^0(M) & & & \\ \hline & 0 & 1 & 2 & p \end{array}$$

so that $H_d H_\delta$ is

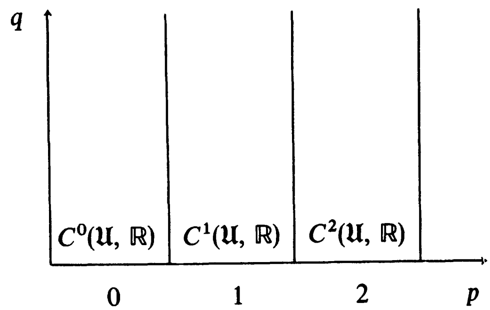
$$\begin{array}{c|c|c|c|l} q & & & & \\ & \vdots & & & \\ & H^2(M) & & & \\ & H^1(M) & & & \\ & H^0(M) & & & \\ \hline & 0 & 1 & 2 & p \end{array}.$$

Since $H_d H_\delta$ has only one nonzero column, we conclude from Proposition 12.1 that

$$H_D^*\{C^*(\mathfrak{U}, \Omega^*)\} \simeq H_{DR}^*(M)$$

for any cover $\mathfrak{U}$. This is the generalized Mayer–Vietoris principle (Proposition 8.8).

Now if $\mathfrak{U}$ is a good cover, $H_d$ of the Čech–de Rham complex is

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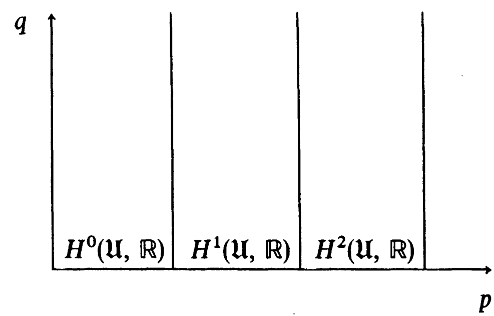
and $H_\delta H_d$ is

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Again because $H_\delta H_d$ has only one nonzero row,

$$H_D^*\{C^*(\mathfrak{U}, \Omega^*)\} \simeq H^*(\mathfrak{U}, \mathbb{R}).$$

This gives the isomorphism between de Rham cohomology and the Čech cohomology of a good cover with coefficients in the constant presheaf $\mathbb{R}$.

*Exercise 12.10.* Let $\mathbb{C}P^n$ have homogeneous coordinates $z_0, \ldots, z_n$. Define $U_i = \{z_i \neq 0\}$. Then $\mathfrak{U} = \{U_0, \ldots, U_n\}$ is an open cover of $\mathbb{C}P^n$, although not a good cover. Compute $H^*(\mathbb{C}P^n)$ from the double complex $C^*(\mathfrak{U}, \Omega^*)$. Find elements in $C^*(\mathfrak{U}, \Omega^*)$ which represent the generators of $H^*(\mathbb{C}P^n)$.

*Exercise 12.11.* Apply the Thom isomorphism (12.2) to compute the cohomology with compact support of the open Möbius strip (cf. Exercise 4.8).

## Poincaré Duality

In the same spirit as above, we now give a version of Poincaré duality, in terms of the Čech–de Rham complex, for a not necessarily orientable manifold. Let $M$ be a manifold of dimension $n$ and $\mathfrak{U} = \{U_\alpha\}$ any open cover of $M$. Define the coboundary operator

$$\delta : \bigoplus \Omega_c^*(U_{\alpha_0 \dots \alpha_p}) \to \bigoplus \Omega_c^*(U_{\alpha_0 \dots \alpha_{p-1}})$$

by the formula

$$(\delta\omega)_{\alpha_0 \dots \alpha_{p-1}} = \sum_\alpha \omega_{\alpha\alpha_0 \dots \alpha_{p-1}}$$

where on the right-hand side we mean the extension by zero of $\omega_{\alpha\alpha_0 \dots \alpha_{p-1}}$ to a form on $U_{\alpha_0 \dots \alpha_{p-1}}$. To ensure that each component of $\delta\omega$ has compact support, the groups here are direct sums rather than direct products, so that $\omega \in \bigoplus \Omega\,(U_{\alpha_0 \dots \alpha_p})$ by definition has only a finite number of nonzero components.

**Proposition 12.12** (Generalized Mayer–Vietoris Sequence for Compact Supports). *Suppose the open cover $\mathfrak{U} = \{U_\alpha\}$ of the manifold $M$ satisfies the local finite condition*:

(*) *each open set $U_\alpha$ intersects only finitely many $U_\beta$'s.*

*Then the sequence*

$$0 \leftarrow \Omega_c^*(M) \xleftarrow{\text{sum}} \bigoplus \Omega_c^*(U_{\alpha_0}) \leftarrow \bigoplus \Omega_c^*(U_{\alpha_0\alpha_1}) \leftarrow \cdots \leftarrow \bigoplus \Omega_c^*(U_{\alpha_0 \dots \alpha_p}) \leftarrow \cdots$$

*is exact.*

PROOF. We first show $\delta^2 = 0$. Let $\omega$ be in $\bigoplus \Omega_c^*(U_{\alpha_0 \dots \alpha_p})$. Then

$$(\delta^2\omega)_{\alpha_0 \dots \alpha_{p-2}} = \sum_\alpha (\delta\omega)_{\alpha\alpha_0 \dots \alpha_{p-2}} = \sum_\alpha \sum_\beta \omega_{\beta\alpha\alpha_0 \dots \alpha_{p-2}}$$
$$= 0, \text{ since } \omega_{\alpha\beta \dots} = -\omega_{\beta\alpha \dots}.$$

Now suppose $\delta\omega = 0$. We will show that $\omega$ is a $\delta$-coboundary. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the cover $\mathfrak{U}$. Define

$$\tau_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \rho_{\alpha_i} \omega_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}.$$

Note that $\tau_{\alpha_0 \dots \alpha_{p+1}}$ has compact support. Moreover, there are only finitely many $(\beta, \alpha_0, \dots, \alpha_p)$ for which $\rho_\beta \omega_{\alpha_0 \dots \alpha_p} \neq 0$, since $\omega_{\alpha_0 \dots \alpha_p} \neq 0$ for finitely many $(\alpha_0, \dots, \alpha_p)$ and by (*) each $U_{\alpha_0 \dots \alpha_p} \subset U_{\alpha_0}$ intersects only finitely many $U_\beta$. Therefore, $\tau$ has finitely many nonzero components, and $\tau \in \bigoplus \Omega_c^*(U_{\alpha_0 \dots \alpha_{p+1}})$. Then

$$(\delta\tau)_{\alpha_0 \dots \alpha_p} = \sum_\alpha \tau_{\alpha\alpha_0 \dots \alpha_p}$$
$$= \sum_\alpha \left( \rho_\alpha \omega_{\alpha_0 \dots \alpha_p} + \sum_i (-1)^{i+1} \rho_{\alpha_i} \omega_{\alpha\alpha_0 \dots \hat{\alpha}_i \dots \alpha_p} \right)$$
$$= \omega_{\alpha_0 \dots \alpha_p} + \sum_i (-1)^{i+1} \rho_{\alpha_i} (\delta\omega)_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_p}$$
$$= \omega_{\alpha_0 \dots \alpha_p}.$$

□

*Exercise 12.12.1.* Show that the definition of $\tau$ in the proof above provides a homotopy operator for the compact Mayer–Vietoris sequence (12.12). More precisely, if $\omega$ is in $\oplus\Omega_c^*(U_{\alpha_0\ldots\alpha_p})$ and

$$(K\omega)_{\alpha_0\ldots\alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \rho_{\alpha_i}\omega_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_{p+1}},$$

then

$$\delta K + K\delta = 1.$$

Consider the double complex $C^p(\mathfrak{U}, \Omega_c^q)$, where $\mathfrak{U}$ satisfies the local finite condition (*):

$$\begin{array}{c|c|c|c|c}
q & & & & \\
2 & & & & \\
1 & & & & \\
0 & \oplus\ \Omega_c^0(U_{\alpha_0}) \leftarrow & \oplus\ \Omega_c^0(U_{\alpha_0\alpha_1}) \leftarrow & & \\
\hline
 & 0 & 1 & 2 & p
\end{array}$$

In this double complex the $\delta$-operator goes in the wrong direction, so we define a new complex

$$K^{-p,q} = C^p(\mathfrak{U}, \Omega_c^q).$$

$$\begin{array}{c|c|c|c}
 & & & q \\
\to & \oplus\ \Omega_c^q(U_{\alpha_0\alpha_1\alpha_2}) \to & \oplus\ \Omega_c^q(U_{\alpha_0\alpha_1}) \to & \oplus\ \Omega_c^q(U_{\alpha_0}) \\
\hline
 & -2 & -1 & 0 \quad p
\end{array}$$

By the exactness of the rows, $H_\delta(K)$ is

$$\begin{array}{c|c|c|c|c}
 & & q & \vdots & \\
 & & & \Omega_c^2(M) & \\
 & & & \Omega_c^1(M) & \\
 & & & \Omega_c^0(M) & \\
\hline
-2 & -1 & 0 & & p
\end{array}$$

Since $H_d H_\delta$ has only one nonzero column, it follows from Proposition 12.1 that

$$H_D(K) = H_d H_\delta(K) = H_c(M). \tag{12.13}$$

On the other hand, if $\mathfrak{U}$ is a good cover, then $H_d(K)$ is

$$\begin{array}{c|c|c|c|c|}
 & & & q\uparrow & \\
\to \bigoplus_{\alpha_0<\dots<\alpha_p} \mathbb{R} & \to \quad \cdots & \to \bigoplus_{\alpha_0<\alpha_1} \mathbb{R} & \to \bigoplus_{\alpha_0} \mathbb{R} & \\
\hline
 & & -1 & 0 & p \to
\end{array}$$

$$H_d^{-p,q}(K) = C^p(\mathfrak{U}, \mathscr{H}_c^q)$$

where $\mathscr{H}_c^q$ is the covariant functor which associates to every open set $U$ the compact cohomology $H_c^q(U)$ and to every inclusion $i$, the extension by zero, $i_*$; moreover,

$$H_d^{-p,q}(K) = 0 \quad \text{for } q \neq n.$$

Again by Proposition 12.1,

$$H_D^*(K) = H_\delta^{*-n,n} H_d = H_{n-*}(\mathfrak{U}, \mathscr{H}_c^n). \tag{12.14}$$

Here $H_{n-*}(\mathfrak{U}, \mathscr{H}_c^n)$ is the $(n - *)$-th Čech homology of the cover $\mathfrak{U}$ with coefficients in the covariant functor $\mathscr{H}_c^n$ (cf. Remark 10.3). Comparing (12.13) and (12.14) gives

**Theorem 12.15** (Poincaré Duality). *Let $M$ be a manifold of dimension $n$ and $\mathfrak{U}$ any good cover of $M$ satisfying the local finite condition (*) of Proposition 12.12. Here $M$ is not assumed to be orientable. Then*

$$H_c^*(M) \simeq H_{n-*}(\mathfrak{U}, \mathscr{H}_c^n),$$

*where $\mathscr{H}_c^n$ is the covariant functor $\mathscr{H}_c^n(U) = H_c^n(U)$.*

*Exercise 12.16.* By applying Poincaré duality (12.15), compute the compact cohomology of the open Möbius strip (cf. Exercise 4.8).

# §13 Monodromy

## When Is a Locally Constant Presheaf Constant?

In the preceding section we saw that the compact vertical cohomology $H_{cv}^*(E)$ of a vector bundle $E$ may be computed as the cohomology of the base with coefficients in the presheaf $\mathscr{H}_{cv}^n$. When the presheaf $\mathscr{H}_{cv}^n$ is the constant presheaf $\mathbb{R}^n$, $H^*_{cv}(E)$ is expressible in terms of the de Rham cohomology of the base manifold (Proposition 10.6). In this case the problem of computing $H^*_{cv}(E)$ is greatly simplified. It is therefore important to determine the conditions under which a presheaf such as $\mathscr{H}^n_{cv}$ is constant.

First we need to review some basic definitions from the theory of simplicial complexes (see, for instance, Munkres [2]). Recall that if an $n$-simplex in an Euclidean space has vertices $v_0, \ldots, v_n$, then its *barycenter* is the point $(v_0 + \cdots + v_n)/(n + 1)$. For example, the barycenter of an edge is its midpoint and the barycenter of a triangle (a 2-simplex) is its center. The *first barycentric subdivision* of a simplex $\sigma$ is the simplicial complex having all the barycenters of $\sigma$ as vertices. By applying the barycentric subdivision to each simplex of a simplical complex $K$, we obtain a new simplicial complex $K'$, called the *first barycentric subdivision* of $K$. The *support* of $K$, denoted $|K|$, is the underlying topological space of $K$, and the *k-skeleton* of $K$ is the subcomplex consisting of all the simplices of dimension less than or equal to $k$. The complex $K$ and its barycentric subdivision $K'$ have the same support. The *star* of a vertex $v$ in $K$, denoted st$(v)$, is the union of all the closed simplices in $K$ having $v$ as a vertex.

Next we introduce the notion of a presheaf on a good cover. Let $X$ be a topological space and $\mathfrak{U} = \{U_\alpha\}$ a good cover of $X$. The *presheaf $\mathscr{F}$ on $\mathfrak{U}$* is defined to be a functor $\mathscr{F}$ on the subcategory of Open$(X)$ consisting of all finite intersections $U_{\alpha_0 \ldots \alpha_p}$ of open sets in $\mathfrak{U}$. Equivalently, if $N(\mathfrak{U})$ is the nerve of $\mathfrak{U}$, the presheaf $\mathscr{F}$ on $\mathfrak{U}$ is the assignment of an appropriate group to the barycenter of each simplex in $N(\mathfrak{U})$; for example, the group attached to the barycenter of the 2-simplex representing $U \cap V \cap W$ is $\mathscr{F}(U \cap V \cap W)$. Each inclusion, say $U \cap V \to U$, becomes an arrow in the picture, $\mathscr{F}(U) \to \mathscr{F}(U \cap V)$, and the transitivity of the arrows says that Figure 13.1 is a commutative diagram.

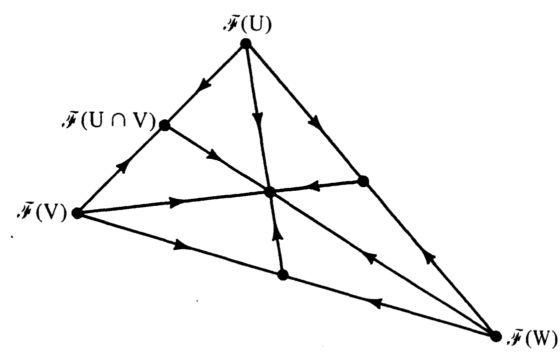



Figure 13.1

Two presheaves $\mathscr{F}$ and $\mathscr{G}$ are *isomorphic relative to a good cover* $\mathfrak{U} = \{U_\alpha\}$ if for each $W = U_{\alpha_0 \ldots \alpha_p}$ there is an isomorphism

$$h_W : \mathscr{F}(W) \to \mathscr{G}(W)$$

compatible with all arrows. In other words, there is a natural equivalence of functors $\mathscr{F} \to \mathscr{G}$ where $\mathscr{F}$ and $\mathscr{G}$ are regarded as functors on the subcategory of Open($X$) consisting of all finite intersections $U_{\alpha_0 \ldots \alpha_n}$ of open sets in $\mathfrak{U}$. The *constant presheaf with group G on a good cover* $\mathfrak{U}$ is defined as in Section 10; it associates to every open set $U_{\alpha_0 \ldots \alpha_p}$ the group of locally constant and hence constant functions: $U_{\alpha_0 \ldots \alpha_p} \to G$. Thus, for a constant presheaf on a good cover, all the groups are $G$ and all the arrows are the identity map. We say that a presheaf $\mathscr{F}$ is *locally constant on a good cover* $\mathfrak{U}$ if all the groups are isomorphic and all the arrows are isomorphisms.

Of course, if two presheaves $\mathscr{F}$ and $\mathscr{G}$ are isomorphic on a good cover $\mathfrak{U}$, then the cohomology groups $H^*(\mathfrak{U}, \mathscr{F})$ and $H^*(\mathfrak{U}, \mathscr{G})$ are isomorphic.

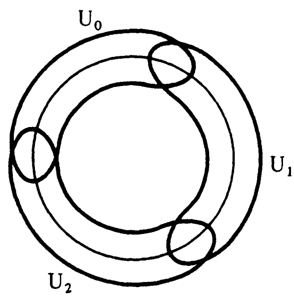


Figure 13.2

EXAMPLE 13.1 (A locally constant presheaf on $\mathfrak{U}$ which is not constant). Let $\mathfrak{U} = \{U_0, U_1, U_2\}$ be a good cover of the circle $S^1$ (see Figure 13.2). Define a presheaf $\mathscr{F}$ by

$$\mathscr{F}(U) = \mathbb{Z} \text{ for all open sets } U,$$

$$\rho_{01}^0 = \rho_{01}^1 = \rho_{12}^1 = \rho_{12}^2 = 1,$$

$$\rho_{02}^2 = -1,\ \rho_{02}^0 = 1.$$

$\mathscr{F}$ is locally constant but not constant on $\mathfrak{U}$ because $\rho_{02}^2$ is not the identity.

Let $\mathscr{F}$ be a locally constant presheaf with group $G$ on a good cover $\mathfrak{U} = \{U_\alpha\}$. Fix isomorphisms

$$\phi_\alpha : \mathscr{F}(U_\alpha) \xrightarrow{\sim} G.$$

If $U_\alpha$ and $U_\beta$ intersect, then from the diagram

$$\begin{array}{ccc}
\mathscr{F}(U_\alpha) & \xrightarrow[\sim]{\phi_\alpha} & G \\
\rho^\alpha_{\alpha\beta}\downarrow & & | \\
\mathscr{F}(U_\alpha \cap U_\beta) & & | \\
\rho^\beta_{\alpha\beta}\uparrow & & \downarrow \\
\mathscr{F}(U_\beta) & \xrightarrow[\sim]{\phi_\beta} & G
\end{array}$$

we obtain an automorphism of $G$, namely $\phi_\beta(\rho^\beta_{\alpha\beta})^{-1}\rho^\alpha_{\alpha\beta}\phi_\alpha^{-1}$. Write $\rho^\alpha_\beta : \mathscr{F}(U_\alpha) \to \mathscr{F}(U_\beta)$ for the isomorphism $(\rho^\beta_{\alpha\beta})^{-1} \circ \rho^\alpha_{\alpha\beta}$. Choose some vertex $U_0$ as the base point of the nerve $N(\mathfrak{U})$. For $U_0 U_1 \ldots U_r U_0$ a loop based at $U_0$ we get an automorphism of $G$ by following along the edges

$$\begin{array}{ccc}
\mathscr{F}(U_0) & \xrightarrow[\sim]{\phi_0} & G \\
\downarrow & & \downarrow \\
\mathscr{F}(U_1) & \xrightarrow[\sim]{\phi_1} & G \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathscr{F}(U_0) & \xrightarrow[\sim]{\phi_0} & G.
\end{array}$$

This gives a map from {loops at $U_0$} to Aut $G$. We claim that if a loop bounds a 2-chain, then the associated automorphism of $G$ is the identity. Consider the example of the 2-simplex as shown in Figure 13.3.

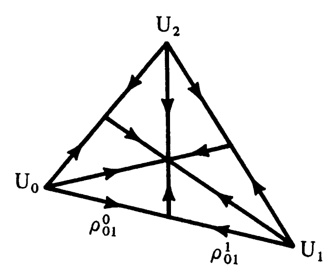
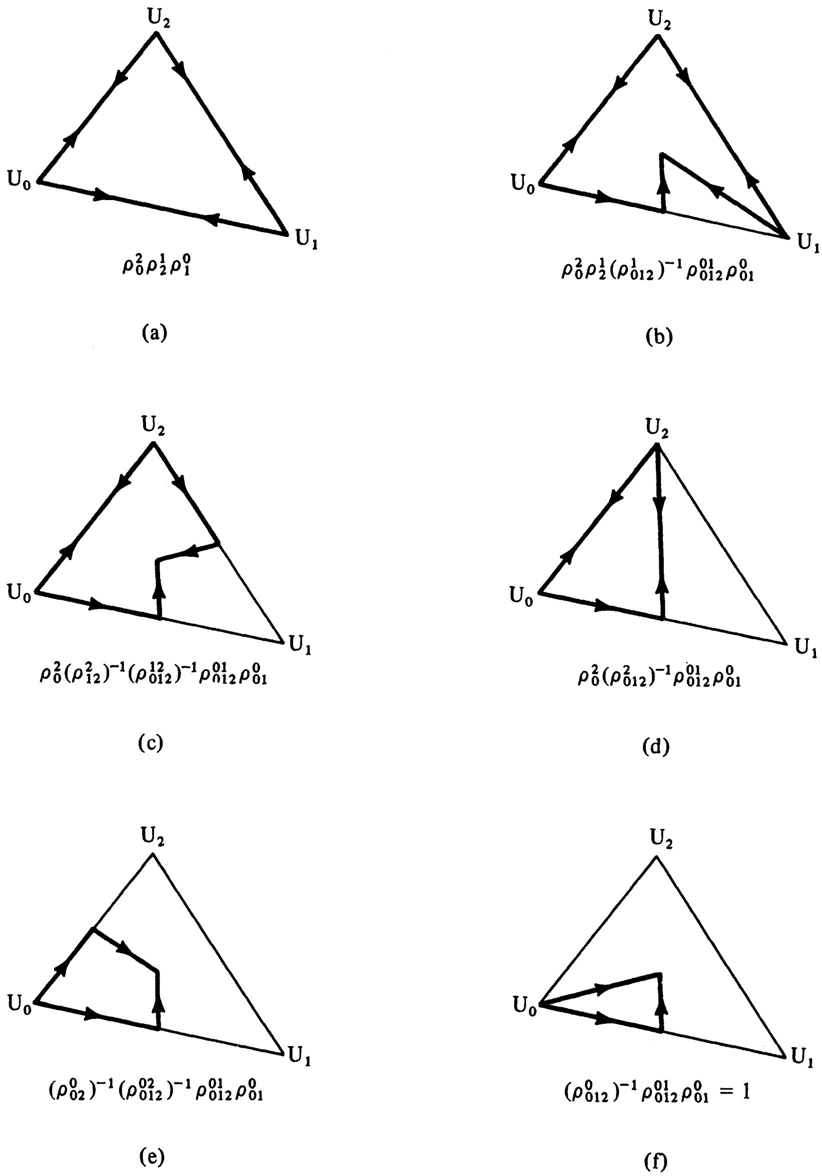



Figure 13.3




$\rho_0^2\rho_2^1\rho_1^0$

(a)

$\rho_0^2\rho_2^1(\rho_{012}^1)^{-1}\rho_{012}^{01}\rho_{01}^0$

(b)

$\rho_0^2(\rho_{12}^2)^{-1}(\rho_{012}^{12})^{-1}\rho_{012}^{01}\rho_{01}^0$

(c)

$\rho_0^2(\rho_{012}^2)^{-1}\rho_{012}^{01}\rho_{01}^0$

(d)

$(\rho_{02}^0)^{-1}(\rho_{012}^{02})^{-1}\rho_{012}^{01}\rho_{01}^0$

(e)

$(\rho_{012}^0)^{-1}\rho_{012}^{01}\rho_{01}^0 = 1$

(f)

Figure 13.4

The associated automorphism of the loop $U_0U_1U_2$ is $\phi_0(\rho_0^2\rho_2^1\rho_1^0)\phi_0^{-1}$ so it is a matter of showing that $\rho_0^2\rho_2^1\rho_1^0$ is the identity. This is clear from the sequence of pictures in Figure 13.4, where we use heavy solid lines to indicate maps which, by the commutativity of the arrows, are all equal to $\rho_0^2\rho_2^1\rho_1^0$.

More generally, the same procedure shows that the map $\rho_0^\alpha \ldots \rho_\beta^0$ around any bounding loop is the identity. Hence there is a homomorphism

$$\rho : \pi_1(N(\mathfrak{U})) = \frac{\{\text{loops}\}}{\{\text{bounding loops}\}} \to \operatorname{Aut} G,$$

called the *monodromy representation* of the presheaf $\mathscr{F}$. Here $\pi_1(N(\mathfrak{U}))$ denotes the edge path group of the nerve $N(\mathfrak{U})$ as a simplicial complex.

**Theorem 13.2.** *Let $\mathfrak{U}$ be a good cover on a connected topological space $X$ and $N(\mathfrak{U})$ its nerve. If $\pi_1(N(\mathfrak{U})) = 0$, then every locally constant presheaf on $\mathfrak{U}$ is constant.*

PROOF. Suppose $\pi_1(N(\mathfrak{U})) = 0$, i.e., every loop bounds some 2-chain. For each open set $U_\alpha$, choose a path from $U_0$ to $U_\alpha$, say $U_0 U_{\alpha_1} \ldots U_{\alpha_r} U_\alpha$, and define $\psi_\alpha = \phi_0 (\rho_\alpha^{\alpha_r} \ldots \rho_{\alpha_2}^{\alpha_1} \rho_{\alpha_1}^0)^{-1} : \mathscr{F}(U_\alpha) \to G$.

$$\begin{array}{c} \phi_0 \\ \mathscr{F}(U_0) \xrightarrow{\sim} G \\ \downarrow \\ \mathscr{F}(U_\alpha) \end{array}$$

$\psi_\alpha$ is well-defined independent of the chosen path, because as we have seen, around a bounding loop the map $\rho_0^\alpha \ldots \rho_\beta^0$ is the identity.

Now carry out the barycentric subdivision of the nerve $N(\mathfrak{U})$ to get a new simplicial complex $K$ so that every open set $U_{\alpha_0 \ldots \alpha_p}$ corresponds to a vertex of $K$. Clearly $\pi_1(N(\mathfrak{U})) = \pi_1(K)$. By the same procedure as in the preceding paragraph we can define isomorphisms

$$\psi_{\alpha_0 \ldots \alpha_p} : \mathscr{F}(U_{\alpha_0 \ldots \alpha_p}) \to G$$

for all nonempty $U_{\alpha_0 \ldots \alpha_p}$. The maps $\psi_{\alpha_0 \ldots \alpha_p}$ give an isomorphism of the presheaf $\mathscr{F}$ to the constant presheaf $G$ on the cover $\mathfrak{U}$. □

REMARK 13.2.1. If the group $G$ of a locally constant presheaf has no automorphisms except the identity, then there is no monodromy. In particular, every locally constant presheaf with group $\mathbb{Z}_2$ is constant.

REMARK 13.3. Recall that a *simplicial map* between two simplicial complexes $K$ and $L$ is a map $f$ from the vertices of $K$ to the vertices of $L$ such that if $v_0, \ldots, v_n$ span a simplex in $K$, then $f(v_0), \ldots, f(v_n)$ span a simplex in $L$. A simplicial map $f$ from $K$ to $L$ induces a map $f : |K| \to |L|$ by linearity:

$$f(\sum \lambda_i v_i) = \sum \lambda_i f_i(v_i).$$

By abuse of language we refer to either of these maps as a simplicial map.

For the proof of the next theorem we assemble here some standard facts from the theory of simplicial complexes.

(a) The edge path group of a simplicial complex is the same as that of its 2-skeleton (Seifert and Threlfall [1, §44, p. 167]).

(b) The edge path group of a simplicial complex is the same as the topological fundamental group of its support (Seifert and Threlfall [1, §44, p. 165]).

(c) (The Simplicial Approximation Theorem). Let $K$ and $L$ be two simplicial complexes. Then every map $f: |K| \to |L|$ is homotopic to a simplicial map $g: |K^{(k)}| \to |L|$ for some integer $k$, where $K^{(k)}$ is the $k$-th barycentric subdivision of $K$(Croom [1, p. 49]).

Because of (b) we also refer to the edge path group of a simplicial complex as its fundamental group.

None of these facts are difficult to prove. They all depend on the following very intuitive principle from obstruction theory.

**The Extension Principle.** *A map from the union of all the faces of a cube into a* contractible *space can be extended to the entire cube.*

Aside. With a little homotopy theory the extension principle can be refined as follows. Let $X$ be a topological space and $I^k$ the unit $k$-dimensional cube. If $\pi_q(X) = 0$ for all $q \leq k - 1$, then any maps from the boundary of $I^k$ into $X$ can be extended to the entire cube $I^k$.

In section 5 we defined a good cover on a manifold to be an open cover $\{U_\alpha\}$ for which all finite intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ are diffeomorphic to a Euclidean space. By a *good cover on a topological space* we shall mean an open cover for which all finite intersections are contractible.

Remark. Thus, on a manifold there are two notions of a good cover. These two notions are not equivalent. Let us call a noncompact boundaryless manifold an *open* manifold. Then there are contractible open 3-manifolds not homeomorphic to $\mathbb{R}^3$. In 1935 J. H. C. Whitehead found the first example of such a manifold [J. H. C. Whitehead, A certain $n$-manifold whose group is unity, *Quart. J. Math. Oxford* **6** (1935), 268–279]. D. R. McMillan, Jr. constructed infinitely many more in [D. R. McMillan, Jr., Some contractible open 3-manifolds, *Transactions of the A. M. S.* **102** (1962), 372–382]. For an open cover on a manifold to be a good cover we will always require the more restrictive hypothesis that the finite nonempty intersections be diffeomorphic to $\mathbb{R}^n$. This is because in order to prove Poincaré duality, whether by the Mayer–Vietoris argument of Section 5 or by the tic-tac-toe game of Section 12, we need the compact Poincaré lemma (Corollary 4.7), which is not always true for an open set with merely the homotopy type of $\mathbb{R}^n$.

**Theorem 13.4.** *Suppose the topological space $X$ has a good cover $\mathfrak{U}$. Then the fundamental group of $X$ is isomorphic to the fundamental group $\pi_1(N(\mathfrak{U}))$ of the nerve of the good cover.*

PROOF. Write $N_2(\mathfrak{U})$ for the 2-skeleton of the nerve $N(\mathfrak{U})$. Let $U_i$, $U_{ij}$, and $U_{ijk}$ be the barycenters of the vertices, edges, and faces of $N_2(\mathfrak{U})$ and let $N_2'(\mathfrak{U})$ be its barycentric subdivision. As the first step in the proof of the theorem we will define a map $f$ from $|N_2'(\mathfrak{U})|$ to $X$. We will then show that this map induces an isomorphism of fundamental groups.

To this end choose a point $p_i$ in each open set $U_i$ in $\mathfrak{U}$, a point $p_{ij}$ in each nonempty pairwise intersection $U_{ij}$, and a point $p_{ijk}$ in each nonempty triple intersection $U_{ijk}$. Also, fix a contraction $c_i$ of $U_i$ to $p_i$ and a contraction $c_{ij}$ of $U_{ij}$ to $p_{ij}$. These contractions exist because $\mathfrak{U}$ is a good cover. By decree the map $f$ sends $U_i$, $U_{ij}$, and $U_{ijk}$ to $p_i$, $p_{ij}$, and $p_{ijk}$ respectively.

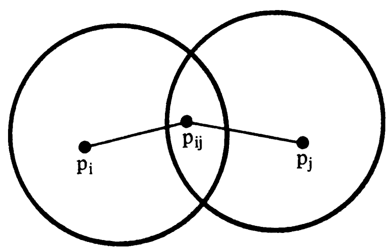



Figure 13.5

Next we define $f$ on the edges of $|N_2'(\mathfrak{U})|$. The contraction $c_i$ takes $p_{ij}$ to $p_i$ and gives a well-defined path between $p_i$ and $p_{ij}$. Similarly, the contraction $c_j$ gives a well-defined path between $p_j$ and $p_{ij}$ (see Figure 13.5). Furthermore, for each point $p_{ijk}$ the six contractions $c_i$, $c_j$, $c_k$, $c_{ij}$, $c_{ik}$, and $c_{jk}$ produce six paths in $X$ joining $p_{ijk}$ to $p_i$, $p_j$, $p_k$, $p_{ij}$, $p_{ik}$, and $p_{jk}$ respectively (see Figure 13.6).

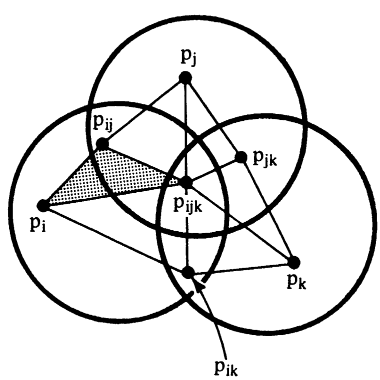



Figure 13.6

The map $f$ shall send the edges of $|N'_2(\mathfrak{U})|$ to the paths just defined; for example, the edge $U_i U_{ijk}$ is sent to the path joining $p_i$ and $p_{ijk}$.

Finally we define $f$ on the faces of $|N'_2(\mathfrak{U})|$. Since each "triangle" $p_i p_{ij} p_{ijk}$ lies entirely inside the open set $U_i$ (such a triangle may be *degenerate*; i.e., it may only be a point or a segment), the triangle may be "filled in" in a well-defined manner: to fill in the triangle $p_i p_{ij} p_{ijk}$, use the contraction $c_i$ to contract the edge $p_{ij} p_{ijk}$ to $p_i$ (see Figure 13.6). This "filled-in" triangle will be the image of the triangle $U_i U_j U_{ijk}$ under $f$. In summary, with the choice of the points $p_i$, $p_{ij}$, $p_{ijk}$ and the contractions $c_i$, $c_{ij}$ fixed, we have defined a map $f: |N'_2(\mathfrak{U})| \to X$. We will now show that the induced map of fundamental groups, $f_*: \pi_1(|N'_2(\mathfrak{U})|) \to \pi_1(X)$ is an isomorphism.

Step 1 (Surjectivity of $f_*$). Take $p_0$ in $U_0$ to be the base point of $X$. Let $\gamma: S^1 \to X$ be a loop in $X$ based at $p_0$. We would like to deform $\gamma$ to a map of the form $f_*(\bar{\gamma})$, where $\bar{\gamma}: S^1 \to |N_2(\mathfrak{U})|$ is a loop in $|N_2(\mathfrak{U})|$ based at $U_0$.

Regard $S^1$ as the unit interval $I$ with its endpoints identified. To define $\bar{\gamma}$, we first subdivide the unit interval into equal pieces, so that it becomes a simplicial complex $K$ with vertices $q_0, \ldots, q_n$ (Figure 13.7).

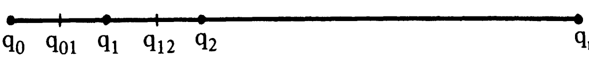



Figure 13.7

By making the pieces sufficiently small, we can ensure that the star of $q_i$ in the barycentric subdivision $K'$ of $K$ is mapped entirely into an open set $U_{\alpha(i)}$:

$$\gamma(\mathrm{st}(q_i)) \subset U_{\alpha(i)}.$$

To simplify the notation, write $j$ instead of $i+1$, so that $q_i q_j$ is a 1-simplex in $K$. Let $q_{ij}$ be the midpoint of $q_i q_j$. Define $\bar{\gamma}: S^1 \to |N_2(\mathfrak{U})|$ by sending the segment $q_i q_j$ to the segment $U_{\alpha(i)} U_{\alpha(j)}$; it follows that $\bar{\gamma}(q_i) = U_{\alpha(i)}$ and $f_*(\bar{\gamma})(q_i) = p_{\alpha(i)}$.

Next define a map $F$ on the sides of the square $I^2$ by (see Figure 13.8)

$$F\big|_{\text{bottom side}} = F(x, 0) = \gamma(x),$$

$$F\big|_{\text{top side}} = F(x, 1) = f_*\bar{\gamma}(x),$$

and

$$F\big|_{\text{vertical sides}} = F(0, t) = F(1, t) = p_0.$$

The problem now is to extend $F: \partial I^2 \to X$ to the entire square. Subdivide the square by joining with vertical segments the vertices $(q_i, 0)$, $(q_{ij}, 0)$ on the bottom edge to the corresponding vertices on the top edge. Since $F(q_i, 0) = \gamma(q_i)$ and $F(q_i, 1) = f_*\bar{\gamma}(q_i) = p_{\alpha(i)}$, they both lie in $U_{\alpha(i)}$. Since $U_{\alpha(i)}$ is contractible, by the extension principle $F$ can be extended to the

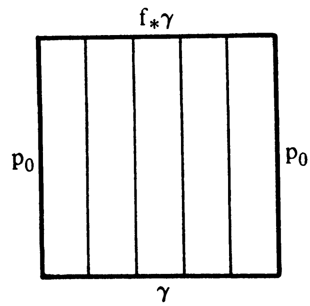



Figure 13.8

vertical segment $\{q_i\} \times I$. Similarly, $F$ can be extended to the vertical segment $\{q_{ij}\} \times I$. Thus in Figure 13.8, $F$ is defined on the boundary of each rectangle and maps that boundary entirely into a contractible open set $U_\alpha$. By the extension principle again, $F$ can be extended over each rectangle. In this way $F$ is extended to the entire square $I^2$.

STEP 2 (Injectivity of $f_*$). Suppose $\gamma: I \to |N_2(\mathfrak{U})|$ is a loop such that $f_*(\gamma)$ is null-homotopic in $X$. This means there is a map $H$ from the square $I^2$ to $X$ as in Figure 13.9.

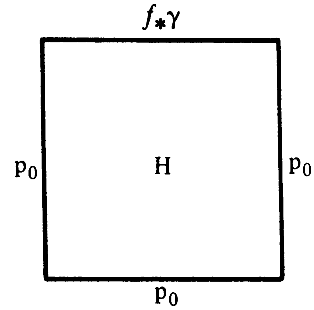



Figure 13.9

By the simplicial approximation theorem we may assume that $\gamma$ is a simplicial map from some subdivision $L$ of the top edge of the square to $|N_2(\mathfrak{U})|$. Now subdivide the square $I^2$ repeatedly to get a triangulation $K$ with the property that if $q_i$ is a vertex of $K$ and $\mathrm{st}(q_i)$ is the star of $q_i$ in the barycentric subdivision $K'$, then

$$H(\mathrm{st}(q_i)) \subset U_{\alpha(i)}$$

for some open set $U_{\alpha(i)}$ in $\mathfrak{U}$. In the process of the subdivision new vertices are introduced on the top edge only by repeated bisection of the edge; furthermore, the function $\alpha$ on the vertices of the top edge may be chosen as follows. Consider for example the 1-simplex $q_1 q_2$. If $q_k$ is a new vertex to the left of the midpoint $q_{12}$, choose $\alpha(k) = \alpha(1)$; otherwise, choose $\alpha(k) = \alpha(2)$.

Define

$$\bar{H}: I^2 = |K| \to |N_2'(\mathfrak{U})|$$

to be the simplicial map with

$$\bar{H}(q_i) = U_{\alpha(i)}.$$

The restriction $\beta$ of $\bar{H}$ to the top edge of the square agrees with $\gamma$ on the vertices of $L$. Furthermore, by construction $\beta$ is homotopic to $\gamma$ in $|N_2(\mathfrak{U})|$, and $\bar{H}$ is a null-homotopy for $\beta$. Therefore, $f_*: \pi_1(|N_2(\mathfrak{U})|) \to \pi_1(X)$ is injective. Since the nerve $N(\mathfrak{U})$ and its 2-skeleton $N_2(\mathfrak{U})$ have the same fundamental group (Remark 13.3 (a)), the theorem is proved. □

## Examples of Monodromy

EXAMPLE 13.5. Let $S^1$ be the unit circle in the complex plane with good cover $\mathfrak{U} = \{U_0, U_1, U_2\}$ as in Figure 13.10. The map $\pi : z \to z^2$ defines a fiber bundle $\pi : S^1 \to S^1$ each of whose fibers consists of two distinct points. Let $F = \{A, B\}$ be the fiber above the point 1. The cohomology $H^*(F)$ consists of all functions on $\{A, B\}$, i.e., $H^*(F) = \{(a, b) \in \mathbb{R}^2\}$.

Fix an isomorphism $H^*(\pi^{-1}U_0) \simeq H^*(F)$. We have the diagram

$$\begin{tikzcd}
H^*(\pi^{-1}U_0) \arrow[r, "\simeq"] \arrow[d] & H^*(F) \arrow[dddddd, dashed] \\
H^*(\pi^{-1}U_{01}) & \\
H^*(\pi^{-1}U_1) \arrow[u] \arrow[d] & \\
H^*(\pi^{-1}U_{12}) & \\
H^*(\pi^{-1}U_2) \arrow[u] \arrow[d] & \\
H^*(\pi^{-1}U_{02}) & \\
H^*(\pi^{-1}U_0) \arrow[u] \arrow[r, "\simeq"] & H^*(F).
\end{tikzcd}$$

If we start with a generator, say (1, 0), of $H^*(F)$ and follow it around the diagram, we do not end up with the same generator; in fact, we get (0, 1). In general $(a, b)$ goes to $(b, a)$. Therefore the presheaf $\mathscr{H}^*(U) = H^*(\pi^{-1}U)$ is not a constant presheaf.

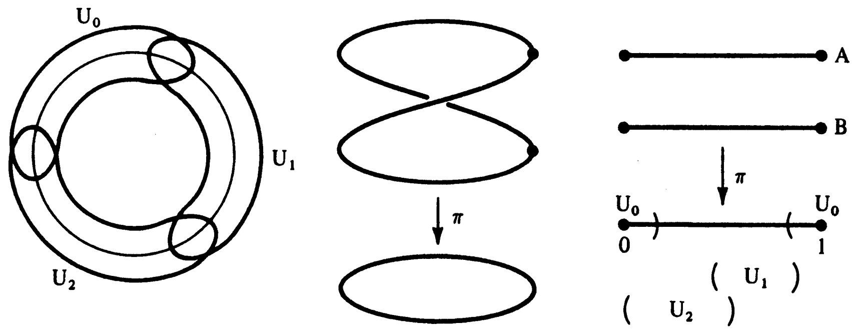



Figure 13.10

*Exercise 13.6.* Since $H_d$ of the double complex $C^*(\pi^{-1}\mathfrak{U}, \Omega^*)$ in Example 13.5 has only one nonzero row, we see by the generalized Mayer-Vietoris principle and Proposition 12.1 that

$$H^*(S^1) = H_D^*\{C^*(\pi^{-1}\mathfrak{U}, \Omega^*)\} = H_\delta H_d = H^*(\mathfrak{U}, \mathscr{H}^0).$$

Compute the Čech cohomology $H^*(\mathfrak{U}, \mathscr{H}^0)$ directly.

EXAMPLE 13.7. The universal covering $\pi : \mathbb{R}^1 \to S^1$ given by $\pi(x) = e^{2\pi i x}$ is a fiber bundle with fiber a countable set of points. The action of the loop downstairs on the homology $H_0$(fiber) is translation by $1 : x \mapsto x + 1$. In cohomology a loop downstairs sends the function on the fiber with support at $x$ to the function with support at $x + 1$. (See Figure 13.11.)

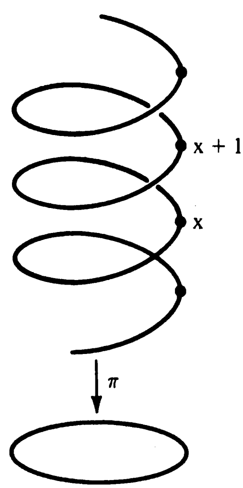



Figure 13.11

*Exercise 13.8.* As in Example 13.5, with $\mathfrak{U}$ being the usual good cover of $S^1$,

$$H^*(\mathbb{R}^1) = H_D^*\{C^*(\pi^{-1}\mathfrak{U}, \Omega^*)\} = H_\delta H_d = H^*(\mathfrak{U}, \mathscr{H}^0).$$

Compute $H^*(\mathfrak{U}, \mathscr{H}^0)$ directly.

EXAMPLE 13.9. In the previous two examples, the fundamental group of the base acts on $H_0$ of the fiber. We now give an example in which it acts on $H_2$.

The *wedge* $S^m \vee S^n$ of two spheres $S^m$ and $S^n$ is the union of $S^m$ and $S^n$ with one point identified. Let $X$ be $S^1 \vee S^2$ as shown in Figure 13.12 and let $\tilde{X}$ be the universal covering of $X$. Note that although $H^*(X)$ is finite, $H^*(\tilde{X})$ is infinite. We define a fiber bundle over the circle $S^1$ with fiber $\tilde{X}$ by setting.

$$E = \tilde{X} \times I/(x, 0) \sim (s(x), 1)$$

where $s$ is the deck transformation of the universal cover $\tilde{X}$ which shifts everything one unit up. The projection $\pi : E \to S^1$ is given by $\pi(\tilde{x}, t) = t$. The fundamental group of the base $\pi_1(S^1)$ acts on $H_2$(fiber) by shifting each sphere one up.

*Exercise 13.10.* Find the homotopy type of the space $E$.

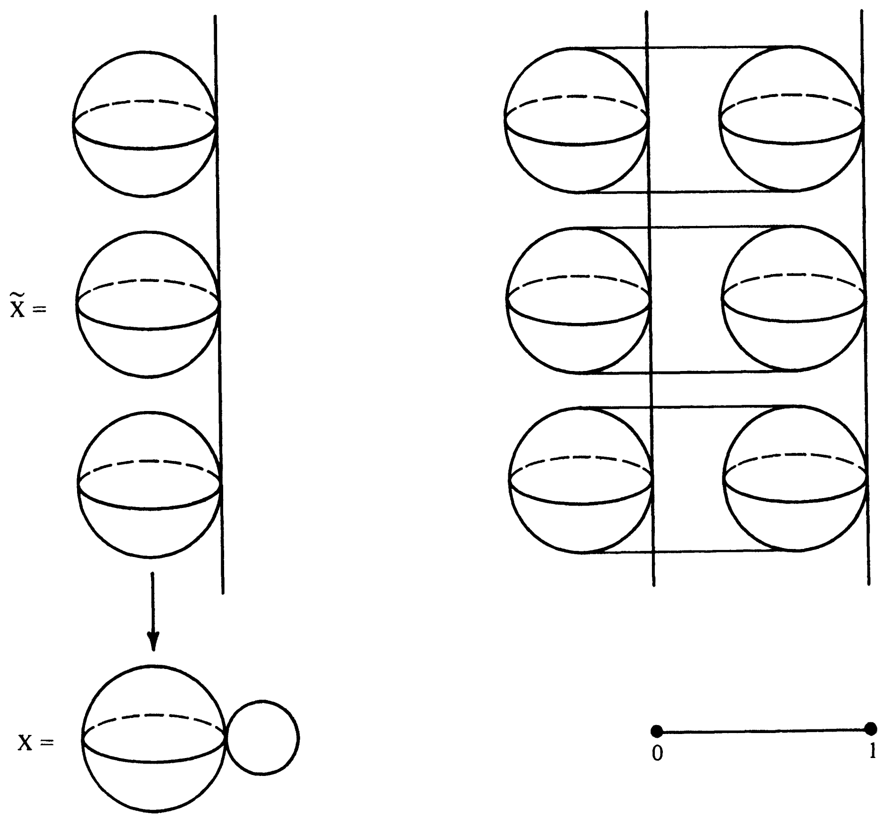



Figure 13.12