CHAPTER I

de Rham Theory

§1 The de Rham Complex on $\mathbb{R}^n$

To start things off we define in this section the de Rham cohomology and compute a few examples. This will turn out to be the most important diffeomorphism invariant of a manifold. So let $x_1, \dots, x_n$ be the linear coordinates on $\mathbb{R}^n$. We define Ω^* to be the algebra over $\mathbb{R}$ generated by $dx_1, \dots, dx_n$ with the relations

$$\begin{cases} (dx_i)^2 = 0 \\ dx_i\, dx_j = -dx_j\, dx_i,\ i \neq j. \end{cases}$$

As a vector space over $\mathbb{R}$, Ω^* has basis

$$1,\ dx_i,\ \underset{i<j}{dx_i\, dx_j},\ \underset{i<j<k}{dx_i\, dx_j\, dx_k},\ \dots,\ dx_1 \dots dx_n.$$

The C^∞ *differential forms* on $\mathbb{R}^n$ are elements of

$$\Omega^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n\} \underset{\mathbb{R}}{\otimes} \Omega^*.$$

Thus, if ω is such a form, then ω can be uniquely written as $\sum f_{i_1 \dots i_q}\, dx_{i_1} \dots dx_{i_q}$ where the coefficients $f_{i_1 \dots i_q}$ are C^∞ functions. We also write $\omega = \sum f_I\, dx_I$. The algebra $\Omega^*(\mathbb{R}^n) = \bigoplus_{q=0}^{n} \Omega^q(\mathbb{R}^n)$ is naturally graded, where $\Omega^q(\mathbb{R}^n)$ consists of the C^∞ q-forms on $\mathbb{R}^n$. There is a differential operator

$$d : \Omega^q(\mathbb{R}^n) \longrightarrow \Omega^{q+1}(\mathbb{R}^n),$$

defined as follows:

i) if $f \in \Omega^0(\mathbb{R}^n)$, then $df = \sum \partial f/\partial x_i\, dx_i$
ii) if $\omega = \sum f_I\, dx_I$, then $d\omega = \sum df_I\, dx_I$.

EXAMPLE 1.1. If $\omega = x\,dy$, then $d\omega = dx\,dy$.

This d, called the *exterior differentiation*, is the ultimate abstract extension of the usual gradient, curl, and divergence of vector calculus on $\mathbb{R}^3$, as the example below partially illustrates.

EXAMPLE 1.2. On $\mathbb{R}^3$, $\Omega^0(\mathbb{R}^3)$ and $\Omega^3(\mathbb{R}^3)$ are each 1-dimensional and $\Omega^1(\mathbb{R}^3)$ and $\Omega^2(\mathbb{R}^3)$ are each 3-dimensional over the C^∞ functions, so the following identifications are possible:

$$\begin{array}{ccccc} \{\text{functions}\} & \simeq & \{\text{0-forms}\} & \simeq & \{\text{3-forms}\} \\ f & \leftrightarrow & f & \leftrightarrow & f\,dx\,dy\,dz \end{array}$$

and

$$\begin{array}{ccccc} \{\text{vector fields}\} & \simeq & \{\text{1-forms}\} & \simeq & \{\text{2-forms}\} \end{array}$$
$$X = (f_1, f_2, f_3) \leftrightarrow f_1\,dx + f_2\,dy + f_3\,dz \leftrightarrow f_1\,dy\,dz - f_2\,dx\,dz + f_3\,dx\,dy.$$

On functions,

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz.$$

On 1-forms,

$$d(f_1\,dx + f_2\,dy + f_3\,dz)$$
$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dy\,dz - \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) dx\,dz + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy.$$

On 2-forms,

$$d(f_1\,dy\,dz - f_2\,dx\,dz + f_3\,dx\,dy) = \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz.$$

In summary,

$$d(\text{0-forms}) = \text{gradient},$$
$$d(\text{1-forms}) = \text{curl},$$
$$d(\text{2-forms}) = \text{divergence}.$$

The *wedge product* of two differential forms, written $\tau \wedge \omega$ or $\tau \cdot \omega$, is defined as follows: if $\tau = \sum f_I\,dx_I$ and $\omega = \sum g_J\,dx_J$, then

$$\tau \wedge \omega = \sum f_I g_J\,dx_I\,dx_J.$$

Note that $\tau \wedge \omega = (-1)^{\deg\tau\,\deg\omega}\omega \wedge \tau$.

Proposition 1.3. *d is an antiderivation, i.e.,*

$$d(\tau \cdot \omega) = (d\tau) \cdot \omega + (-1)^{\deg\tau}\,\tau \cdot d\omega.$$

PROOF. By linearity it suffices to check on monomials

$$\tau = f_I\, dx_I,\ \omega = g_J\, dx_J.$$

$$\begin{aligned} d(\tau \cdot \omega) &= d(f_I g_J)\, dx_I\, dx_J = (df_I)g_J\, dx_I\, dx_J + f_I\, dg_J\, dx_I\, dx_J \\ &= (d\tau) \cdot \omega + (-1)^{\deg \tau}\, \tau \cdot d\omega. \end{aligned}$$

On the level of functions $d(fg) = (df)g + f(dg)$ is simply the ordinary product rule. □

Proposition 1.4. $d^2 = 0$.

PROOF. This is basically a consequence of the fact that the mixed partials are equal. On functions,

$$d^2 f = d\left(\sum_i \frac{\partial f}{\partial x_i}\, dx_i\right) = \sum_{i,j} \frac{\partial^2 f}{\partial x_j\, \partial x_i}\, dx_j\, dx_i.$$

Here the factors $\partial^2 f/\partial x_j \partial x_i$ are symmetric in i, j while $dx_j\, dx_i$ are skew-symmetric in i, j; hence $d^2 f = 0$. On forms $\omega = f_I\, dx_I$,

$$d^2\omega = d^2(f_I\, dx_I) = d(d f_I\, dx_I) = 0$$

by the previous computation and the antiderivation property of d. □

The complex $\Omega^*(\mathbb{R}^n)$ together with the differential operator d is called the *de Rham complex* on $\mathbb{R}^n$. The kernel of d are the *closed* forms and the image of d, the *exact* forms. The de Rham complex may be viewed as a God-given set of differential equations, whose solutions are the closed forms. For instance, finding a closed 1-form $f\, dx + g\, dy$ on $\mathbb{R}^2$ is tantamount to solving the differential equation $\partial g/\partial x - \partial f/\partial y = 0$. By Proposition 1.4 the exact forms are automatically closed; these are the trivial or "uninteresting" solutions. A measure of the size of the space of "interesting" solutions is the definition of the de Rham cohomology.

Definition. The q-th *de Rham cohomology* of $\mathbb{R}^n$ is the vector space

$$H^q_{DR}(\mathbb{R}^n) = \{\text{closed } q\text{-forms}\}/\{\text{exact } q\text{-forms}\}.$$

We sometimes suppress the subscript DR and write $H^q(\mathbb{R}^n)$. If there is a need to distinguish between a form ω and its cohomology class, we denote the latter by $[\omega]$.

Note that all the definitions so far work equally well for any open subset U of $\mathbb{R}^n$; for instance,

$$\Omega^*(U) = \{C^\infty \text{ functions on } U\} \underset{\mathbb{R}}{\otimes} \Omega^*.$$

So we may also speak of the de Rham cohomology $H^*_{DR}(U)$ of U.

EXAMPLES 1.5.

(a) $n = 0$

$$H^q = \begin{cases} \mathbb{R} & q = 0 \\ 0 & q > 0. \end{cases}$$

(b) $n = 1$

Since $(\ker d) \cap \Omega^0(\mathbb{R}^1)$ are the constant functions,

$$H^0(\mathbb{R}^1) = \mathbb{R}.$$

On $\Omega^1(\mathbb{R}^1)$, $\ker d$ are all the 1-forms.

If $\omega = g(x)dx$ is a 1-form, then by taking

$$f = \int_0^x g(u)\, du,$$

we find that

$$df = g(x)\, dx.$$

Therefore every 1-form on $\mathbb{R}^1$ is exact and

$$H^1(\mathbb{R}^1) = 0.$$

(c) Let U be a disjoint union of m open intervals on $\mathbb{R}^1$. Then

$$H^0(U) = \mathbb{R}^m$$

and

$$H^1(U) = 0.$$

(d) In general

$$H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } 0, \\ 0 & \text{otherwise.} \end{cases}$$

This result is called the *Poincaré lemma* and will be proved in Section 4.

The de Rham complex is an example of a *differential complex*. For the convenience of the reader we recall here some basic definitions and results on differential complexes. A direct sum of vector spaces $C = \bigoplus_{q \in \mathbb{Z}} C^q$ indexed by the integers is called a *differential complex* if there are homomorphisms

$$\cdots \longrightarrow C^{q-1} \xrightarrow{d} C^q \xrightarrow{d} C^{q+1} \longrightarrow \cdots$$

such that $d^2 = 0$. d is the *differential operator* of the complex C. The *cohomology* of C is the direct sum of vector spaces $H(C) = \bigoplus_{q \in \mathbb{Z}} H^q(C)$, where

$$H^q(C) = (\ker d \cap C^q)/(\operatorname{im} d \cap C^q).$$

A map $f: A \to B$ between two differential complexes is a *chain map* if it commutes with the differential operators of A and B : $f\, d_A = d_B\, f$.

A sequence of vector spaces

$$\cdots \longrightarrow V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \longrightarrow \cdots$$

is said to be *exact* if for all i the kernel of f_i is equal to the image of its predecessor f_{i-1}. An exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is called a *short exact sequence*. Given a short exact sequence of differential complexes

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in which the maps f and g are chain maps, there is a long exact sequence of cohomology groups

$$\begin{tikzcd} H^q(A) \arrow[r, "f^*"] & H^q(B) \arrow[r, "g^*"] & H^q(C) \arrow[d, "d^*"] \\ & & H^{q+1}(A) \arrow[r] & \cdots \end{tikzcd}$$

In this sequence f^* and g^* are the naturally induced maps and $d^*[c]$, $c \in C^q$, is obtained as follows:

$$\begin{tikzcd} & \uparrow & & \uparrow & & \uparrow \\ 0 \arrow[r] & A^{q+1} \arrow[u] \arrow[r, "f"] & B^{q+1} \arrow[u] \arrow[r, "g"] & C^{q+1} \arrow[u] \arrow[r] & 0 \\ 0 \arrow[r] & A^q \arrow[u] \arrow[r, "f"] & B^q \arrow[u, "d"] \arrow[r, "g"] & C^q \arrow[u, "d"] \arrow[r] & 0 \\ & \arrow[u] & \arrow[u] & \arrow[u] \end{tikzcd}$$

By the surjectivity of g there is an element b in B^q such that $g(b) = c$. Because $g(db) = d(gb) = dc = 0$, $db = f(a)$ for some a in A^{q+1}. This a is easily checked to be closed. $d^*[c]$ is defined to be the cohomology class $[a]$ in $H^{q+1}(A)$. A simple diagram-chasing shows that this definition of d^* is independent of the choices made.

Exercise. Show that the long exact sequence of cohomology groups exists and is exact. (See, for instance, Munkres [2, §24].)

Compact Supports

A slight modification of the construction of the preceding section will give us another diffeomorphism invariant of a manifold. For now we again restrict our attention to $\mathbb{R}^n$. Recall that the *support* of a continuous function f on a topological space X is the closure of the set on which f is not zero, i.e., $\operatorname{Supp} f = \overline{\{p \in X \mid f(p) \neq 0\}}$. If in the definition of the de Rham complex we use only the C^∞ functions with compact support, the resulting complex is called the *de Rham complex* $\Omega_c^*(\mathbb{R}^n)$ *with compact supports*:

$$\Omega_c^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n \text{ with compact support}\} \underset{\mathbb{R}}{\otimes} \Omega^*.$$

The cohomology of this complex is denoted by $H_c^*(\mathbb{R}^n)$.

EXAMPLE 1.6.

(a) $H_c^*(\text{point}) = \begin{cases} \mathbb{R} & \text{in dimension } 0, \\ 0 & \text{elsewhere.} \end{cases}$

(b) *The compact cohomology of* $\mathbb{R}^1$. Again the closed 0-forms are the constant functions. Since there are no constant functions on $\mathbb{R}^1$ with compact support,

$$H_c^0(\mathbb{R}^1) = 0.$$

To compute $H_c^1(\mathbb{R}^1)$, consider the integration map

$$\int_{\mathbb{R}^1} : \Omega_c^1(\mathbb{R}^1) \longrightarrow \mathbb{R}^1.$$

This map is clearly surjective. It vanishes on the exact 1-forms df where f has compact support, for if the support of f lies in the interior of $[a,b]$, then

$$\int_{\mathbb{R}^1} \frac{df}{dx}\, dx = \int_a^b \frac{df}{dx}\, dx = f(b) - f(a) = 0.$$

If $g(x)\, dx \in \Omega_c^1(\mathbb{R}^1)$ is in the kernel of the integration map, then the function

$$f(x) = \int_{-\infty}^{x} g(u)\, du$$

will have compact support and $df = g(x)\, dx$. Hence the kernel of $\int_{\mathbb{R}^1}$ are precisely the exact forms and

$$H_c^1(\mathbb{R}^1) = \frac{\Omega_c^1(\mathbb{R}^1)}{\ker \int_{\mathbb{R}^1}} = \mathbb{R}^1.$$

REMARK. If $g(x)\, dx \in \Omega_c^1(\mathbb{R}^1)$ does not have total integral 0, then

$$f(x) = \int_{-\infty}^{x} g(u)\, du$$

will not have compact support and $g(x)\, dx$ will not be exact.

(c) More generally,

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } n \\ 0 & \text{otherwise.} \end{cases}$$

This result is the *Poincaré lemma for cohomology with compact support* and will be proved in Section 4.

Exercise 1.7. Compute $H_{DR}^*(\mathbb{R}^2 - P - Q)$ where P and Q are two points in $\mathbb{R}^2$. Find the closed forms that represent the cohomology classes.

§2 The Mayer-Vietoris Sequence

In this section we extend the definition of the de Rham cohomology from $\mathbb{R}^n$ to any differentiable manifold and introduce a basic technique for computing the de Rham cohomology, the Mayer-Vietoris sequence. But first we have to discuss the functorial nature of the de Rham complex.

The Functor Ω^*

Let $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ be the standard coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A smooth map $f: \mathbb{R}^m \to \mathbb{R}^n$ induces a pullback map on C^∞ functions $f^*: \Omega^0(\mathbb{R}^n) \to \Omega^0(\mathbb{R}^m)$ via

$$f^*(g) = g \circ f.$$

We would like to extend this pullback map to all forms $f^*: \Omega^*(\mathbb{R}^n) \to \Omega^*(\mathbb{R}^m)$ in such a way that it commutes with d. The commutativity with d defines f^* uniquely:

$$f^*(\textstyle\sum g_I \, dy_{i_1} \ldots dy_{i_q}) = \sum (g_I \circ f) \, df_{i_1} \ldots df_{i_q},$$

where $f_i = y_i \circ f$ is the i-th component of the function f.

Proposition 2.1. *With the above definition of the pullback map f^* on forms, f^* commutes with d.*

PROOF. The proof is essentially an application of the chain rule.

$$df^*(g_I \, dy_{i_1} \ldots dy_{i_q}) = d((g_I \circ f) \, df_{i_1} \ldots df_{i_q}) = d(g_I \circ f) \, df_{i_1} \ldots df_{i_q}.$$

$$\begin{aligned} f^* d(g_I \, dy_{i_1} \ldots dy_{i_q}) &= f^*\left(\sum_{i=1}^n \frac{\partial g_I}{\partial y_i} dy_i \, dy_{i_1} \ldots dy_{i_q} \right) \\ &= \sum_{i=1}^n \left(\left(\frac{\partial g_I}{\partial y_i} \circ f \right) df_i \right) df_{i_1} \ldots df_{i_q} \\ &= d(g_I \circ f) \, df_{i_1} \ldots df_{i_q}. \end{aligned}$$

□

Let $x_1, \ldots, x_n$ be the standard coordinate system and $u_1, \ldots u_n$ a new coordinate system on $\mathbb{R}^n$, i.e., there is a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $u_i = x_i \circ f = f^*(x_i)$. By the chain rule, if g is a smooth function on $\mathbb{R}^n$, then

$$\sum_i \frac{\partial g}{\partial u_i} du_i = \sum_{i,j} \frac{\partial g}{\partial u_i} \frac{\partial u_i}{\partial x_j} dx_j = \sum_j \frac{\partial g}{\partial x_j} dx_j.$$

So dg is independent of the coordinate system.

Exercise 2.1.1. More generally show that if $\omega = \sum g_I \, du_I$, then $d\omega = \sum dg_I \, du_I$.

Thus the exterior derivative d is independent of the coordinate system on $\mathbb{R}^n$.

Recall that a *category* consists of a class of *objects* and for any two objects A and B, a set $\mathrm{Hom}(A, B)$ of *morphisms* from A to B, satisfying the following properties. If f is a morphism from A to B and g a morphism from B to C, then the *composite morphism* $g \circ f$ from A to C is defined; furthermore, the composition operation is required to be associative and to have an identity 1_A in $\mathrm{Hom}(A, A)$ for every object A. The class of all groups together with the group homomorphisms is an example of a category.

A *covariant functor* F from a category $\mathscr{K}$ to a category $\mathscr{L}$ associates to every object A in $\mathscr{K}$ an object $F(A)$ in $\mathscr{L}$, and every morphism $f: A \to B$ in $\mathscr{K}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathscr{L}$ such that F preserves composition and the identity:

$$F(g \circ f) = F(g) \circ F(f)$$

$$F(1_A) = 1_{F(A)}.$$

If F reverses the arrows, i.e., $F(f) : F(B) \to F(A)$, it is said to be a *contravariant functor.*

In this fancier language the discussion above may be summarized as follows: Ω^* *is a contravariant functor from the category of Euclidean spaces* $\{\mathbb{R}^n\}_{n \in \mathbb{Z}}$ *and smooth maps*: $\mathbb{R}^m \to \mathbb{R}^n$ *to the category of commutative differential graded algebras and their homomorphisms. It is the unique such functor that is the pullback of functions on* $\Omega^0(\mathbb{R}^n)$. Here the commutativity of the graded algebra refers to the fact that

$$\tau\omega = (-1)^{\deg \tau \deg \omega} \omega\tau.$$

The functor Ω^* may be extended to the category of differentiable manifolds. For the fundamentals of manifold theory we recommend de Rham [1, Chap. I]. Recall that a differentiable structure on a manifold is given by an *atlas*, i.e., an open cover $\{U_\alpha\}_{\alpha \in A}$ of M in which each open set U_α is homeomorphic to $\mathbb{R}^n$ via a homeomorphism $\phi_\alpha : U_\alpha \xrightarrow{\sim} \mathbb{R}^n$, and on the overlaps $U_\alpha \cap U_\beta$ the transition functions

$$g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

are diffeomorphisms of open subsets of $\mathbb{R}^n$; furthermore, the atlas is required to be maximal with respect to inclusions. All manifolds will be assumed to be Hausdorff and to have a countable basis. The collection $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ is called a *coordinate open cover* of M and ϕ_α is the *trivialization* of U_α. Let $u_1, \ldots, u_n$ be the standard coordinates on $\mathbb{R}^n$. We can write $\phi_\alpha = (x_1, \ldots, x_n)$, where $x_i = u_i \circ \phi_\alpha$ are a coordinate system on U_α. A function f on U_α is *differentiable* if $f \circ \phi_\alpha^{-1}$ is a differentiable function on $\mathbb{R}^n$. If f is a differentiable function on U_α, the partial derivative $\partial f/\partial x_i$ is defined to be the i-th partial of the pullback function $f \circ \phi_\alpha^{-1}$ on $\mathbb{R}^n$:

$$\frac{\partial f}{\partial x_i}(p) = \frac{\partial(f \circ \phi_\alpha^{-1})}{\partial u_i}(\phi_\alpha(p)).$$

The *tangent space* to M at p, written T_pM, is the vector space over $\mathbb{R}$ spanned by the operators $\partial/\partial x_1(p), \ldots, \partial/\partial x_n(p)$, and a smooth *vector field* on U_α is a linear combination $X_\alpha = \sum f_i\, \partial/\partial x_i$ where the f_i's are smooth functions on U_α. Relative to another coordinate system $(y_1, \ldots, y_n)$, $X_\alpha = \sum g_j\, \partial/\partial y_i$ where $\partial/\partial x_i$ and $\partial/\partial y_j$ satisfy the *chain rule*:

$$\frac{\partial}{\partial x_i} = \sum \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}.$$

A C^∞ vector field on M may be viewed as a collection of vector fields X_α on U_α which agree on the overlaps $U_\alpha \cap U_\beta$.

A *differential form* ω on M is a collection of forms ω_U for U in the atlas defining M, which are compatible in the following sense: if i and j are the inclusions

$$\begin{tikzcd} U \cap V \arrow[r, "i"] \arrow[dr, "j"'] & U \\ & V \end{tikzcd}$$

then $i^*\omega_U = j^*\omega_V$ in $\Omega^*(U \cap V)$. By the functoriality of Ω^*, the exterior derivative and the wedge product extend to differential forms on a manifold. Just as for $\mathbb{R}^n$ a smooth map of differentiable manifolds $f : M \to N$ induces in a natural way a pullback map on forms $f^* : \Omega^*(N) \to \Omega^*(M)$. In this way Ω^* becomes a contravariant functor on the category of differentiable manifolds.

A *partition of unity* on a manifold M is a collection of non-negative C^∞ functions $\{\rho_\alpha\}_{\alpha \in I}$ such that

(*a*) Every point has a neighborhood in which $\Sigma\rho_\alpha$ is a finite sum.
(*b*) $\Sigma\rho_\alpha = 1$.

The basic technical tool in the theory of differentiable manifolds is the existence of a partition of unity. This result assumes two forms:

(1) Given an open cover $\{U_\alpha\}_{\alpha \in I}$ of M, there is a partition of unity $\{\rho_\alpha\}_{\alpha \in I}$ such that the support of ρ_α is contained in U_α. We say in this case that $\{\rho_\alpha\}$ is a partition of unity *subordinate* to the open cover $\{U_\alpha\}$.

(2) Given an open cover $\{U_\alpha\}_{\alpha \in I}$ of M, there is a partition of unity $\{\rho_\beta\}_{\beta \in J}$ with *compact support*, but possibly with an index set J different from I, such that the support of ρ_β is contained in some U_α.

For a proof see Warner [1, p. 10] or de Rham [1, p. 3].

Note that in (1) the support of ρ_α is not assumed to be compact and the index set of $\{\rho_\alpha\}$ is the same as that of $\{U_\alpha\}$, while in (2) the reverse is true. We usually cannot demand simultaneously compact support and the same index set on a noncompact manifold M. For example, consider the open cover of $\mathbb{R}^1$ consisting of precisely one open set, namely $\mathbb{R}^1$ itself. This open cover clearly does not have a partition of unity with compact support subordinate to it.

The Mayer-Vietoris Sequence

The Mayer-Vietoris sequence allows one to compute the cohomology of the union of two open sets. Suppose $M = U \cup V$ with U, V open. Then there is a sequence of inclusions

$$M \leftarrow U \coprod V \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} U \cap V$$

where $U \coprod V$ is the disjoint union of U and V and ∂_0 and ∂_1 are the inclusions of $U \cap V$ in V and in U respectively. Applying the contravariant functor Ω^*, we get a sequence of restrictions of forms

$$\Omega^*(M) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \underset{\partial_1^*}{\overset{\partial_0^*}{\rightrightarrows}} \Omega^*(U \cap V),$$

where by the restriction of a form to a submanifold we mean its image under the pullback map induced by the inclusion. By taking the difference of the last two maps, we obtain the *Mayer-Vietoris sequence*

$$\begin{array}{lccccc} (2.2) \qquad 0 \longrightarrow \Omega^*(M) \longrightarrow & \Omega^*(U) \oplus \Omega^*(V) & \longrightarrow & \Omega^*(U \cap V) \longrightarrow 0 \\ & (\omega, \tau) & \mapsto & \tau - \omega \end{array}$$

Proposition 2.3. *The Mayer-Vietoris sequence is exact.*

PROOF. The exactness is clear except at the last step. We first consider the case of functions on $M = \mathbb{R}^1$. Let f be a C^∞ function on $U \cap V$ as shown in Figure 2.1. We must write f as the difference of a function on U and a function on V. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to the open cover $\{U, V\}$. Note that $\rho_V f$ is a function on U—to get a function on an open set we must multiply by the partition function of the other open set. Since

$$(\rho_U f) - (-\rho_V f) = f,$$

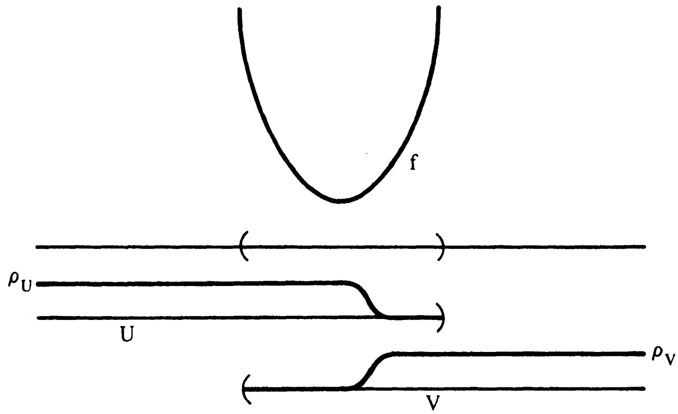


Figure 2.1

we see that $\Omega^0(U) \oplus \Omega^0(V) \to \Omega^0(\mathbb{R}^1)$ is surjective. For a general manifold M, if $\omega \in \Omega^q(U \cap V)$, then $(-\rho_V \omega, \rho_U \omega)$ in $\Omega^q(U) \oplus \Omega^q(V)$ maps onto ω. □

The Mayer-Vietoris sequence

$$0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0$$

induces a long exact sequence in cohomology, also called a Mayer-Vietoris sequence:

$$(2.4)\qquad \begin{tikzcd} {} \arrow[r] & H^{q+1}(M) \arrow[r] & H^{q+1}(U) \oplus H^{q+1}(V) \arrow[r] & H^{q+1}(U \cap V) \arrow[r] & {} \\ {} \arrow[r] & H^q(M) \arrow[r] & H^q(U) \oplus H^q(V) \arrow[r] & H^q(U \cap V) \arrow[ull, "d^*"] & \end{tikzcd}$$

We recall again the definition of the coboundary operator d^* in this explicit instance. The short exact sequence gives rise to a diagram with exact rows

$$\begin{array}{ccccccccc} & & \uparrow & & \uparrow & & \uparrow & & \\ 0 \to & & \Omega^{q+1}(M) & \to & \Omega^{q+1}(U) \oplus \Omega^{q+1}(V) & \to & \Omega^{q+1}(U \cap V) & \to 0 & \\ & & d\uparrow & & d\uparrow & & d\uparrow & & \\ 0 \to & & \Omega^q(M) & \to & \Omega^q(U) \oplus \Omega^q(V) & \to & \Omega^q(U \cap V) & \to 0 & \\ & & & & \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid & & \\ & & & & \xi & & \omega & & d\omega = 0 \end{array}$$

Let $\omega \in \Omega^q(U \cap V)$ be a closed form. By the exactness of the rows, there is a $\xi \in \Omega^q(U) \oplus \Omega^q(V)$ which maps to ω, namely, $\xi = (-\rho_V \omega, \rho_U \omega)$. By the commutativity of the diagram and the fact that $d\omega = 0$, $d\xi$ goes to 0 in $\Omega^{q+1}(U \cap V)$, i.e., $-d(\rho_V \omega)$ and $d(\rho_U \omega)$ agree on the overlap $U \cap V$. Hence $d\xi$ is the image of an element in $\Omega^{q+1}(M)$. This element is easily seen to be closed and represents $d^*[\omega]$. As remarked earlier, it can be shown that $d^*[\omega]$ is independent of the choices in this construction. Explicitly we see that the coboundary operator is given by

$$d^*[\omega] = \begin{cases} [-d(\rho_V \omega)] & \text{on } U \\ [d(\rho_U \omega)] & \text{on } V. \end{cases} \tag{2.5}$$

We define the *support* of a form ω on a manifold M to be $\operatorname{Supp} \omega = \overline{\{ p \in M \mid \omega(p) \neq 0\}}$. Note that in the Mayer-Vietoris sequence $d^*\omega \in H^*(M)$ has support in $U \cap V$.

EXAMPLE 2.6 (The cohomology of the circle). Cover the circle with two open sets U and V as shown in Figure 2.2. The Mayer-Vietoris sequence gives

$$\begin{array}{cccccc}
 & S^1 & & U\coprod V & & U\cap V\\
H^2 & 0 & & 0 & & 0\\
H^1 & & \longrightarrow & 0 & \longrightarrow & 0\\
 & & & d^* & & \\
H^0 & & \longrightarrow & \mathbb{R}\oplus\mathbb{R} & \xrightarrow{\ \delta\ } & \mathbb{R}\oplus\mathbb{R}
\end{array}$$

The difference map δ sends (ω, τ) to $(\tau - \omega, \tau - \omega)$, so $\operatorname{im} \delta$ is 1-dimensional. It follows that $\ker \delta$ is also 1-dimensional. Therefore,

$$H^0(S^1) = \ker \delta = \mathbb{R}$$
$$H^1(S^1) = \operatorname{coker} \delta = \mathbb{R}.$$

We now find an explicit representative for the generator of $H^1(S^1)$. If $\alpha \in \Omega^0(U \cap V)$ is a closed 0-form which is not the image under δ of a closed form in $\Omega^0(U) \oplus \Omega^0(V)$, then $d^*\alpha$ will represent a generator of $H^1(S^1)$. As α we may take the function which is 1 on the upper piece of $U \cap V$ and 0 on

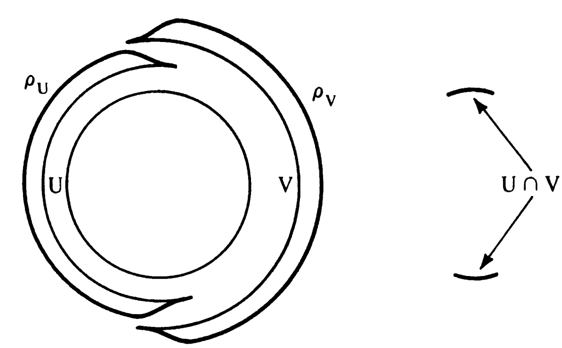


Figure 2.2

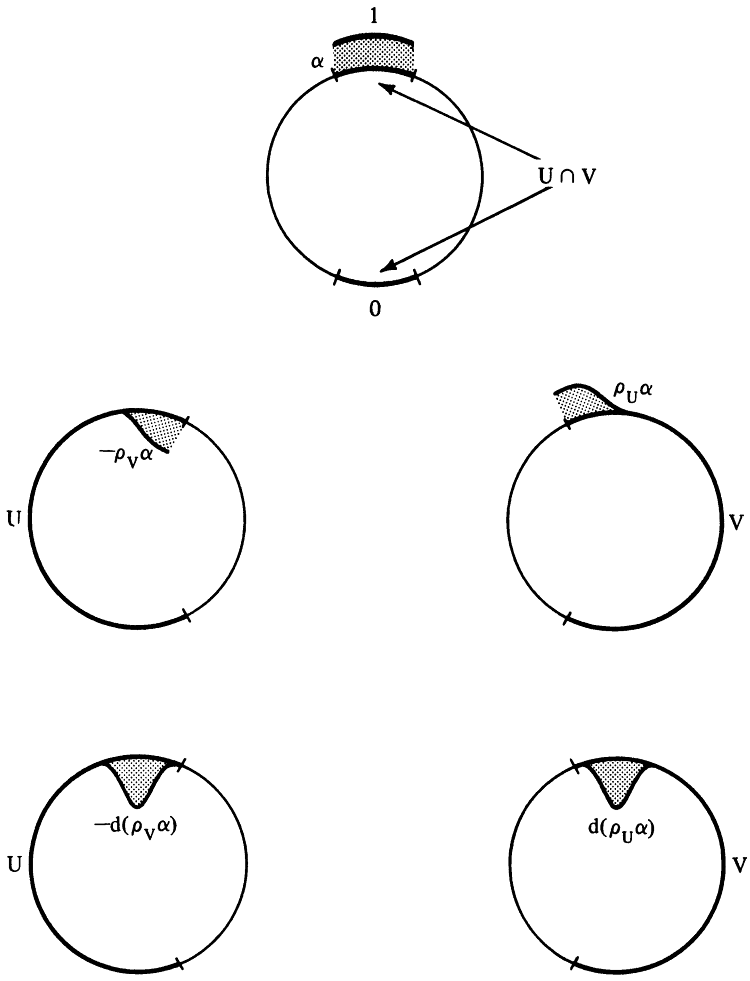


Figure 2.3

the lower piece (see Figure 2.3). Now α is the image of $(-\rho_V\alpha, \rho_U\alpha)$. Since $-d(\rho_V\alpha)$ and $d\rho_U\alpha$ agree on $U \cap V$, they represent a global form on S^1; this form is $d^*\alpha$. It is a bump 1-form with support in $U \cap V$.

The Functor Ω_c^* and the Mayer-Vietoris Sequence for Compact Supports

Again, before taking up the Mayer-Vietoris sequence for compactly supported cohomology, we need to discuss the functorial properties of $\Omega_c^*(M)$, the algebra of forms with compact support on the manifold M. In general the pullback by a smooth map of a form with compact support need not have compact support; for example, consider the pullback of functions under the projection $M \times \mathbb{R} \to M$. So Ω_c^* is not a functor on the category of manifolds and smooth maps. However if we consider not all smooth maps, but only an appropriate subset of smooth maps, then Ω_c^* can be made into a functor. There are two ways in which this can be done.

(a) Ω_c^* is a *contravariant* functor under *proper maps.* (A map is *proper* if the inverse image of every compact set is compact.)
(b) Ω_c^* is a *covariant* functor under *inclusions of open sets.*

If $j : U \to M$ is the inclusion of the open subset U in the manifold M, then $j_* : \Omega_c^*(U) \to \Omega_c^*(M)$ is the map which extends a form on U by zero to a form on M.

It is the covariant nature of Ω_c^* which we shall exploit to prove Poincaré duality for noncompact manifolds. So from now on we assume that Ω_c^* refers to the covariant functor in (b). There is also a Mayer-Vietoris sequence for this functor. As before, let M be covered by two open sets U and V. The sequence of inclusions

$$M \leftarrow U \textstyle\coprod V \leftleftarrows U \cap V$$

gives rise to a sequence of forms with compact support

$$\begin{array}{ccccc} \Omega_c^*(M) & \xleftarrow[\text{sum}]{} & \Omega_c^*(U) \oplus \Omega_c^*(V) & \xleftarrow[\substack{\text{signed}\\ \text{inclusion}}]{\delta} & \Omega_c^*(U \cap V) \\ & & & & \\ & & (-j_*\omega, j_*\omega) & \longleftarrow\!\shortmid & \omega \end{array}$$

Proposition 2.7. *The Mayer-Vietoris sequence of forms with compact support*

$$0 \leftarrow \Omega_c^*(M) \leftarrow \Omega_c^*(U) \oplus \Omega_c^*(V) \leftarrow \Omega_c^*(U \cap V) \leftarrow 0$$

is exact.

Proof. This time exactness is easy to check at every step. We do it for the last step. Let ω be a form in $\Omega_c^*(M)$. Then ω is the image of $(\rho_U \omega, \rho_V \omega)$ in $\Omega_c^*(U) \oplus \Omega_c^*(V)$. The form $\rho_U \omega$ has compact support because Supp $\rho_U \omega \subset$ Supp $\rho_U \cap$ Supp ω and by a lemma from general topology, a closed subset of a compact set in a Hausdorff space is compact. This shows the surjectivity of the map $\Omega_c^*(U) \oplus \Omega_c^*(V) \to \Omega_c^*(M)$. Note that whereas in the previous Mayer-Vietoris sequence we multiply by ρ_V to get a form on U, here $\rho_U \omega$ is a form on U. $\square$

Again the Mayer-Vietoris sequence gives rise to a long exact sequence in cohomology:

(2.8)
$$\begin{tikzcd} \arrow[r] & H_c^{q+1}(M) & H_c^{q+1}(U) \oplus H_c^{q+1}(V) \arrow[l] & H_c^{q+1}(U \cap V) \arrow[l] & \arrow[l] \\ & H_c^{q}(M) \arrow[urr, "d_*"] & H_c^{q}(U) \oplus H_c^{q}(V) \arrow[l] & H_c^{q}(U \cap V) \arrow[l] & \arrow[l] \end{tikzcd}$$

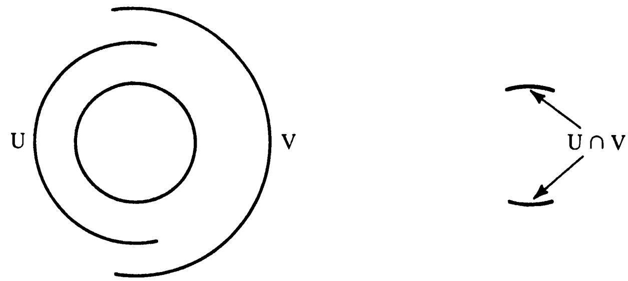


Figure 2.4

Example 2.9 (The cohomology with compact support of the circle). Of course since S^1 is compact, the cohomology with compact support $H_c^*(S^1)$ should be the same as the ordinary de Rham cohomology $H^*(S^1)$. Nonetheless, as an illustration we will compute $H_c^*(S^1)$ from the Mayer-Vietoris sequence for compact supports:

$$\begin{tikzcd}
 & S^1 & U\coprod V & U\cap V \\
H_c^2 & & 0 & 0 \\
H_c^1 & & \mathbb{R}\oplus\mathbb{R} \arrow[l] & \mathbb{R}\oplus\mathbb{R} \arrow[l, "\delta"'] \\
H_c^0 & & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Here the map δ sends $\omega = (\omega_1, \omega_2) \in H_c^1(U \cap V)$ to $(-(j_U)_*\omega, (j_V)_*\omega) \in H_c^1(U) \oplus H_c^1(V)$, where j_U and j_V are the inclusions of $U \cap V$ in U and in V respectively. Since im δ is 1-dimensional,

$$H_c^0(S^1) = \ker \delta = \mathbb{R}$$

$$H_c^1(S^1) = \operatorname{coker} \delta = \mathbb{R}.$$

§3 Orientation and Integration

Orientation and the Integral of a Differential Form

Let $x_1, \ldots, x_n$ be the standard coordinates on $\mathbb{R}^n$. Recall that the Riemann integral of a differentiable function f with compact support is

$$\int_{\mathbb{R}^n} f\,|dx_1 \ldots dx_n| = \lim_{\Delta x_i \to 0} \sum f \Delta x_1 \ldots \Delta x_n.$$

We define the integral of an n-form with compact support $\omega = f\,dx_1 \ldots dx_n$ to be the Riemann integral $\int_{\mathbb{R}^n} f\,|dx_1 \ldots dx_n|$. Note that contrary to the usual calculus notation we put an absolute value sign in the Riemann integral; this is to emphasize the distinction between the Riemann integral of a function and the integral of a differential form. While the order of $x_1, \ldots, x_n$ matters in a differential form, it does not in a Riemann integral; if π is a permutation of $\{1, \ldots, n\}$, then

$$\int f\, dx_{\pi(1)} \ldots dx_{\pi(n)} = (\operatorname{sgn} \pi) \int f |dx_1 \ldots dx_n|,$$

but

$$\int f |dx_{\pi(1)} \ldots dx_{\pi(n)}| = \int f |dx_1 \ldots dx_n|.$$

In a situation where there is no possibility of confusion, we may revert to the usual calculus notation.

So defined, the integral of an n-form on $\mathbb{R}^n$ depends on the coordinates $x_1, \ldots, x_n$. From our point of view a change of coordinates is given by a diffeomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ with coordinates $y_1, \ldots, y_n$ and $x_1, \ldots, x_n$ respectively:

$$x_i = x_i \circ T(y_1, \ldots, y_n) = T_i(y_1, \ldots, y_n).$$

We now study how the integral $\int \omega$ transforms under such diffeomorphisms.

Exercise 3.1. Show that $dT_1 \ldots dT_n = J(T) dy_1 \ldots dy_n$, where $J(T) = \det(\partial x_i / \partial y_j)$ is the Jacobian determinant of T.

Hence,

$$\int_{\mathbb{R}^n} T^*\omega = \int_{\mathbb{R}^n} (f \circ T)\, dT_1 \ldots dT_n = \int_{\mathbb{R}^n} (f \circ T) J(T) |dy_1 \ldots dy_n|$$

relative to the coordinate system $y_1, \ldots, y_n$. On the other hand, by the change of variables formula,

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) |dx_1 \ldots dx_n| = \int_{\mathbb{R}^n} (f \circ T) |J(T)| |dy_1 \ldots dy_n|$$

Thus

$$\int_{\mathbb{R}^n} T^*\omega = \pm \int_{\mathbb{R}^n} \omega$$

depending on whether the Jacobian determinant is positive or negative. In general if T is a diffeomorphism of open subsets of $\mathbb{R}^n$ and if the Jacobian determinant $J(T)$ is everywhere positive, then T is said to be *orientation-preserving*. The integral on $\mathbb{R}^n$ is not invariant under the whole group of diffeomorphisms of $\mathbb{R}^n$, but only under the subgroup of orientation-preserving diffeomorphisms.

Let M be a differentiable manifold with atlas $\{(U_\alpha, \phi_\alpha)\}$. We say that the atlas is *oriented* if all the transition functions $g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ are orientation-preserving, and that the manifold is *orientable* if it has an oriented atlas.

Proposition 3.2. *A manifold M of dimension n is orientable if and only if it has a global nowhere vanishing n-form.*

Proof. Observe that $T : \mathbb{R}^n \to \mathbb{R}^n$ is orientation-preserving if and only if $T^* dx_1 \dots dx_n$ is a positive multiple of $dx_1 \dots dx_n$ at every point.

($\Leftarrow$) Suppose M has a global nowhere-vanishing n-form ω. Let $\phi_\alpha : U_\alpha \xrightarrow{\sim} \mathbb{R}^n$ be a coordinate map. Then $\phi_\alpha^* dx_1 \dots dx_n = f_\alpha \omega$ where f_α is a nowhere-vanishing real-valued function on U_α. Thus f_α is either everywhere positive or everywhere negative. In the latter case replace ϕ_α by $\psi_\alpha = T \circ \phi_\alpha$, where $T: \mathbb{R}^n \to \mathbb{R}^n$ is the orientation-reversing diffeomorphism $T(x_1, x_2, \dots, x_n) = (-x_1, x_2, \dots, x_n)$. Since $\psi_\alpha^* dx_1 \dots dx_n = \phi_\alpha^* T^* dx_1 \dots dx_n = -\phi_\alpha^* dx_1 \dots dx_n = (-f_\alpha)\omega$, we may assume f_α to be positive for all α. Hence, any transition function $\phi_\beta \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ will pull $dx_1 \dots dx_n$ to a positive multiple of itself. So $\{(U_\alpha, \phi_\alpha)\}$ is an oriented atlas.

($\Rightarrow$) Conversely, suppose M has an oriented atlas $\{(U_\alpha, \phi_\alpha)\}$. Then

$$(\phi_\beta \phi_\alpha^{-1})^* (dx_1 \dots dx_n) = \lambda \, dx_1 \dots dx_n$$

for some positive function λ. Thus

$$\phi_\beta^* \, dx_1 \dots dx_n = (\phi_\alpha^* \lambda)(\phi_\alpha^* \, dx_1 \dots dx_n).$$

Denoting $\phi_\alpha^* dx_1 \dots dx_n$ by ω_α, we see that $\omega_\beta = f\omega_\alpha$ where $f = \phi_\alpha^* \lambda = \lambda \circ \phi_\alpha$ is a positive function on $U_\alpha \cap U_\beta$.

Let $\omega = \sum \rho_\alpha \omega_\alpha$ where ρ_α is a partition of unity subordinate to the open cover $\{U_\alpha\}$. At each point p in M, all the forms ω_α, if defined, are positive multiples of one another. Since $\rho_\alpha \geq 0$ and not all ρ_α can vanish at a point, ω is nowhere vanishing. □

Any two global nowhere vanishing n-forms ω and ω' on an orientable manifold M of dimension n differ by a nowhere vanishing function: $\omega = f\omega'$. If M is connected, then f is either everywhere positive or everywhere negative. We say that ω and ω' are *equivalent* if f is positive. Thus on a connected orientable manifold M the nowhere vanishing n-forms fall into two equivalence classes. Either class is called an *orientation* on M, written $[M]$. For example, the standard orientation on $\mathbb{R}^n$ is given by $dx_1 \dots dx_n$.

Now choose an orientation $[M]$ on M. Given a top form τ in $\Omega_c^n(M)$, we define its integral by

$$\int_{[M]} \tau = \sum_\alpha \int_{U_\alpha} \rho_\alpha \tau$$

where $\int_{U_\alpha} \rho_\alpha \tau$ means $\int_{\mathbb{R}^n} (\phi_\alpha^{-1})^*(\rho_\alpha \tau)$ for some orientation-preserving trivialization $\phi_\alpha : U_\alpha \xrightarrow{\sim} \mathbb{R}^n$; as in Proposition 2.7, $\rho_\alpha \tau$ has compact support. By the orientability assumption, the integral over a coordinate patch $\int_{U_\alpha} \omega$ is well defined. With a fixed orientation on M understood, we will often write $\int_M \tau$ instead of $\int_{[M]} \tau$. Reversing the orientation results in the negative of the integral.

Proposition 3.3. *The definition of the integral $\int_M \tau$ is independent of the oriented atlas $\{(U_\alpha, \phi_\alpha)\}$ and the partition of unity $\{\rho_\alpha\}$.*

PROOF. Let $\{V_\beta\}$ be another oriented atlas of M, and $\{\chi_\beta\}$ a partition of unity subordinate to $\{V_\beta\}$. Since $\sum_\beta \chi_\beta = 1$,

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \tau = \sum_{\alpha, \beta} \int_{U_\alpha} \rho_\alpha \chi_\beta \tau.$$

Now $\rho_\alpha \chi_\beta \tau$ has support in $U_\alpha \cap V_\beta$, so

$$\int_{U_\alpha} \rho_\alpha \chi_\beta \tau = \int_{V_\beta} \rho_\alpha \chi_\beta \tau.$$

Therefore

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \tau = \sum_{\alpha, \beta} \int_{V_\beta} \rho_\alpha \chi_\beta \tau = \sum_\beta \int_{V_\beta} \chi_\beta \tau. \qquad \square$$

A *manifold M of dimension n with boundary* is given by an atlas $\{(U_\alpha, \phi_\alpha)\}$ where U_α is homeomorphic to either $\mathbb{R}^n$ or the upper half space $\mathbb{H}^n = \{(x_1, \ldots, x_n) \mid x_n \geq 0\}$. The boundary ∂M of M is an $(n-1)$-dimensional manifold. An oriented atlas for M induces in a natural way an oriented atlas for ∂M. This is a consequence of the following lemma.

Lemma 3.4. *Let $T : \mathbb{H}^n \to \mathbb{H}^n$ be a diffeomorphism of the upper half space with everywhere positive Jacobian determinant. T induces a map $\bar{T}$ of the boundary of $\mathbb{H}^n$ to itself. The induced map $\bar{T}$, as a diffeomorphism of $\mathbb{R}^{n-1}$, also has positive Jacobian determinant everywhere.*

PROOF. By the inverse function theorem an interior point of $\mathbb{H}^n$ must be the image of an interior point. Hence T maps the boundary to the boundary. We will check that $\bar{T}$ has positive Jacobian determinant for $n = 2$; the general case is similar.

Let T be given by

$$x_1 = T_1(y_1, y_2)$$
$$x_2 = T_2(y_1, y_2).$$

Then $\bar{T}$ is given by

$$x_1 = T_1(y_1, 0).$$

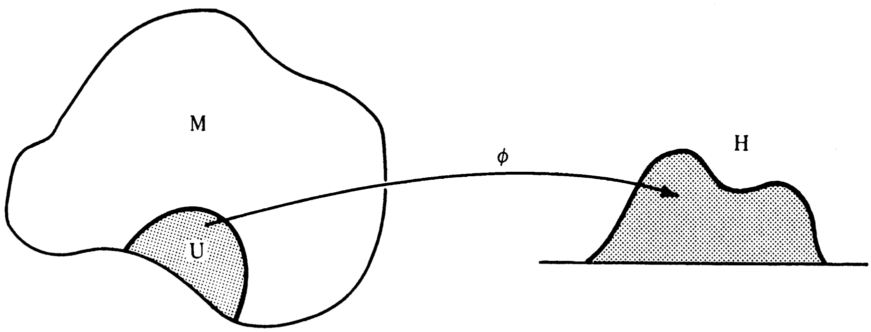


Figure 3.1

By assumption

$$\begin{vmatrix} \dfrac{\partial T_1}{\partial y_1}(y_1, 0) & \dfrac{\partial T_1}{\partial y_2}(y_1, 0) \\ \dfrac{\partial T_2}{\partial y_1}(y_1, 0) & \dfrac{\partial T_2}{\partial y_2}(y_1, 0) \end{vmatrix} > 0.$$

Since $0 = T_2(y_1, 0)$ for all y_1, $\partial T_2/\partial y_1\ (y_1, 0) = 0$; since T maps the upper half plane to itself,

$$\frac{\partial T_2}{\partial y_2}(y_1, 0) > 0.$$

Therefore

$$\frac{\partial T_1}{\partial y_1}(y_1, 0) > 0.$$ □

Let the upper half space $\mathbb{H}^n = \{x_n \geq 0\}$ in $\mathbb{R}^n$ be given the standard orientation $dx_1 \dots dx_n$. Then the *induced orientation* on its boundary $\partial \mathbb{H}^n = \{x_n = 0\}$ is by definition the equivalence class of $(-1)^n dx_1 \dots dx_{n-1}$ for $n \geq 2$ and -1 for $n = 1$; the sign $(-1)^n$ is needed to make Stokes' theorem sign-free. In general for M an oriented manifold with boundary, we define the *induced orientation* $[\partial M]$ on ∂M by the following requirement: if ϕ is an orientation-preserving diffeomorphism of some open set U in M into the upper half space $\mathbb{H}^n$, then

$$\phi^*[\partial \mathbb{H}^n] = [\partial M]\,|_{\partial U},$$

where $\partial U = (\partial M) \cap U$ (see Figure 3.1).

Stokes' Theorem

A basic result in the theory of integration is

Theorem 3.5 (Stokes' Theorem). *If ω is an $(n-1)$-form with compact support on an oriented manifold M of dimension n and if ∂M is given the induced* *orientation, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

We first examine two special cases.

Special Case 1 ($\mathbb{R}^n$). By the linearity of the integrand we may take ω to be $f\,dx_1 \dots dx_{n-1}$. Then $d\omega = \pm\,\partial f/\partial x_n\,dx_1 \dots dx_n$. By Fubini's theorem,

$$\int_{\mathbb{R}^n} d\omega = \pm \int \left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x_n}\,dx_n \right) dx_1 \dots dx_{n-1}.$$

But $\int_{-\infty}^{\infty} \partial f/\partial x_n\,dx_n = f(x_1, \dots, x_{n-1}, \infty) - f(x_1, \dots, x_{n-1}, -\infty) = 0$ because f has compact support. Since $\mathbb{R}^n$ has no boundary, this proves Stokes' theorem for $\mathbb{R}^n$.

Special Case 2 (The upper half plane). In this case (see Figure 3.2)

$$\omega = f(x, y)\,dx + g(x, y)\,dy$$

and

$$d\omega = \left(-\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} \right) dx\,dy.$$

Note that

$$\int_{\mathbb{H}^2} \frac{\partial g}{\partial x}\,dx\,dy = \int_0^{\infty} \left(\int_{-\infty}^{\infty} \frac{\partial g}{\partial x}\,dx \right) dy = \int g(\infty, y) - g(-\infty, y)\,dy = 0,$$

since g has compact support. Therefore,

$$\begin{aligned} \int_{\mathbb{H}^2} d\omega &= -\int_{\mathbb{H}^2} \frac{\partial f}{\partial y}\,dx\,dy = -\int_{-\infty}^{\infty} \left(\int_0^{\infty} \frac{\partial f}{\partial y}\,dy \right) dx \\ &= -\int_{-\infty}^{\infty} (f(x, \infty) - f(x, 0))\,dx \\ &= \int_{-\infty}^{\infty} f(x, 0)\,dx = \int_{\partial \mathbb{H}^2} \omega \end{aligned}$$

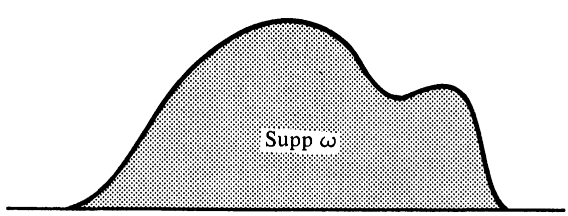


Figure 3.2

where the last equality holds because the restriction of $g(x, y)dy$ to $\partial\mathbb{H}^2$ is 0. So Stokes' theorem holds for the upper half plane.

The case of the upper half space in $\mathbb{R}^n$ is entirely analogous.

Exercise 3.6. Prove Stokes' theorem for the upper half space.

We now consider the general case of a manifold of dimension n. Let $\{U_\alpha\}$ be an oriented atlas for M and $\{\rho_\alpha\}$ a partition of unity subordinate to $\{U_\alpha\}$. Write $\omega = \sum \rho_\alpha \omega$. Since Stokes' theorem $\int_M d\omega = \int_{\partial M} \omega$ is linear in ω, we need to prove it only for $\rho_\alpha \omega$, which has the virtue that its support is contained entirely in U_α. Furthermore, $\rho_\alpha \omega$ has compact support because

$$\operatorname{Supp} \rho_\alpha \omega \subset \operatorname{Supp} \rho_\alpha \cap \operatorname{Supp} \omega$$

is a closed subset of a compact set. Since U_α is diffeomorphic to either $\mathbb{R}^n$ or the upper half space $\mathbb{H}^n$, by the computations above Stokes' theorem holds for U_α. Consequently

$$\int_M d\,\rho_\alpha \omega = \int_{U_\alpha} d\,\rho_\alpha \omega = \int_{\partial U_\alpha} \rho_\alpha \omega = \int_{\partial M} \rho_\alpha \omega .$$

This concludes the proof of Stokes' theorem in general.

§4 Poincaré Lemmas

The Poincaré Lemma for de Rham Cohomology

In this section we compute the ordinary cohomology and the compactly supported cohomology of $\mathbb{R}^n$. Let $\pi : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$ be the projection on the first factor and $s : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^1$ the zero section.

$$\begin{tikzcd} \mathbb{R}^n \times \mathbb{R}^1 \arrow[d, shift left, "\pi"] & \Omega^*(\mathbb{R}^n \times \mathbb{R}^1) \arrow[d, shift right, "s^*"'] & \pi(x, t) = x \\ \mathbb{R}^n \arrow[u, shift left, "s"] & \Omega^*(\mathbb{R}^n) \arrow[u, shift right, "\pi^*"'] & s(x) = (x, 0) \end{tikzcd}$$

We will show that these maps induce inverse isomorphisms in cohomology and therefore $H^*(\mathbb{R}^{n+1}) \simeq H^*(\mathbb{R}^n)$. As a matter of convention all maps are assumed to be C^∞ unless otherwise specified.

Since $\pi \circ s = 1$, we have trivially $s^* \circ \pi^* = 1$. However $s \circ \pi \neq 1$ and correspondingly $\pi^* \circ s^* \neq 1$ on the level of forms. For example, $\pi^* \circ s^*$ sends the function $f(x, t)$ to $f(x, 0)$, a function which is constant along every fiber. To show that $\pi^* \circ s^*$ is the identity in cohomology, it is enough to find a map K on $\Omega^*(\mathbb{R}^n \times \mathbb{R}^1)$ such that

$$1 - \pi^* \circ s^* = \pm(dK \pm Kd),$$

for $dK \pm Kd$ maps closed forms to exact forms and therefore induces zero in cohomology. Such a K is called a *homotopy operator*; if it exists, we say that $\pi^* \circ s^*$ is *chain homotopic* to the identity. Note that the homotopy operator K decreases the degree by 1.

Every form on $\mathbb{R}^n \times \mathbb{R}$ is uniquely a linear combination of the following two types of forms:

$$\text{(I)}\ (\pi^*\phi)f(x, t),$$
$$\text{(II)}\ (\pi^*\phi)f(x, t)\, dt,$$

where ϕ is a form on the base $\mathbb{R}^n$. We define $K : \Omega^q(\mathbb{R}^n \times \mathbb{R}) \to \Omega^{q-1}(\mathbb{R}^n \times \mathbb{R})$ by

$$\text{(I)}\ (\pi^*\phi)f(x, t) \mapsto 0,$$
$$\text{(II)}\ (\pi^*\phi)f(x, t)\, dt \mapsto (\pi^*\phi) \textstyle\int_0^t f.$$

Let's check that K is indeed a homotopy operator. We will use the simplified notation $\partial f/\partial x\, dx$ for $\sum \partial f/\partial x_i\, dx_i$, and $\int g$ for $\int g(x, t)\, dt$. On forms of type (I),

$$\omega = (\pi^*\phi) \cdot f(x, t), \qquad \deg \omega = q,$$
$$(1 - \pi^*s^*)\omega = (\pi^*\phi) \cdot f(x, t) - \pi^*\phi \cdot f(x, 0),$$
$$(dK - Kd)\omega = -Kd\omega = -K\left((d\pi^*\phi)f + (-1)^q\pi^*\phi \left(\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial t}\, dt\right)\right)$$
$$= (-1)^{q-1}\pi^*\phi \int_0^t \frac{\partial f}{\partial t} = (-1)^{q-1}\pi^*\phi[f(x, t) - f(x, 0)].$$

Thus,

$$(1 - \pi^*s^*)\omega = (-1)^{q-1}(dK - Kd)\omega.$$

On forms of type (II),

$$\omega = (\pi^*\phi)f\, dt, \qquad \deg \omega = q,$$
$$d\omega = (\pi^*\, d\phi)f\, dt + (-1)^{q-1}(\pi^*\phi)\frac{\partial f}{\partial x}\, dx\, dt.$$

$(1 - \pi^*s^*)\omega = \omega$ because $s^*(dt) = d(s^*t) = d(0) = 0$.

$$Kd\omega = (\pi^*\, d\phi)\int_0^t f + (-1)^{q-1}(\pi^*\phi)\, dx \int_0^t \frac{\partial f}{\partial x},$$
$$dK\omega = (\pi^*\, d\phi)\int_0^t f + (-1)^{q-1}(\pi^*\phi)\left[dx\left(\int_0^t \frac{\partial f}{\partial x}\right) + f\, dt\right].$$

Thus

$$(dK - Kd)\omega = (-1)^{q-1}\omega.$$

In either case,

$$1 - \pi^* \circ s^* = (-1)^{q-1}(dK - Kd) \qquad \text{on} \quad \Omega^q(\mathbb{R}^n \times \mathbb{R}).$$

This proves

Proposition 4.1. *The maps* $H^*(\mathbb{R}^n \times \mathbb{R}^1) \underset{s^*}{\overset{\pi^*}{\leftrightarrows}} H^*(\mathbb{R}^n)$ *are isomorphisms.*

By induction, we obtain the cohomology of $\mathbb{R}^n$.

Corollary 4.1.1 (Poincaré Lemma).

$$H^*(\mathbb{R}^n) = H^*(\textit{point}) = \begin{cases} \mathbb{R} & \textit{in dimension } 0 \\ 0 & \textit{elsewhere.} \end{cases}$$

Consider more generally

$$\begin{tikzcd} M \times \mathbb{R}^1 \arrow[d, shift right, "\pi"'] \\ M \arrow[u, shift right, "s"'] \end{tikzcd} .$$

If $\{U_\alpha\}$ is an atlas for M, then $\{U_\alpha \times \mathbb{R}^1\}$ is an atlas for $M \times \mathbb{R}^1$. Again every form on $M \times \mathbb{R}^1$ is a linear combination of the two types of forms (I) and (II). We can define the homotopy operator K as before and the proof carries over word for word to show that $H^*(M \times \mathbb{R}^1) \simeq H^*(M)$ is an isomorphism via π^* and s^*.

Corollary 4.1.2 (Homotopy Axiom for de Rham Cohomology). *Homotopic maps induce the same map in cohomology.*

Proof. Recall that a *homotopy* between two maps f and g from M to N is a map $F : M \times \mathbb{R}^1 \to N$ such that

$$\begin{cases} F(x, t) = f(x) & \text{for} \quad t \geq 1 \\ F(x, t) = g(x) & \text{for} \quad t \leq 0. \end{cases}$$

Equivalently if s_0 and $s_1 : M \to M \times \mathbb{R}^1$ are the 0-section and 1-section respectively, i.e., $s_1(x) = (x, 1)$, then

$$f = F \circ s_1,$$
$$g = F \circ s_0.$$

Thus

$$f^* = (F \circ s_1)^* = s_1^* \circ F^*,$$

$$g^* = (F \circ s_0)^* = s_0^* \circ F^*.$$

Since s_1^* and s_0^* both invert π^*, they are equal. Hence,

$$f^* = g^*. \qquad \square$$

Two manifolds M and N are said to have the same *homotopy type in the C^∞ sense* if there are C^∞ maps $f : M \to N$ and $g : N \to M$ such that $g \circ f$ and $f \circ g$ are C^∞ homotopic to the identity on M and N respectively.* A manifold having the homotopy type of a point is said to be *contractible.*

Corollary 4.1.2.1. *Two manifolds with the same homotopy type have the same de Rham cohomology.*

If $i : A \subset M$ is the inclusion and $r : M \to A$ is a map which restricts to the identity on A, then r is called a *retraction* of M onto A. Equivalently, $r \circ i : A \to A$ is the identity. If in addition $i \circ r : M \to M$ is *homotopic* to the identity on M, then r is said to be a *deformation retraction* of M onto A. In this case A and M have the same homotopy type.

Corollary 4.1.2.2. *If A is a deformation retract of M, then A and M have the same de Rham cohomology.*

Exercise 4.2. Show that $r : \mathbb{R}^2 - \{0\} \to S^1$ given by $r(x) = x/\| x \|$ is a deformation retraction.

Exercise 4.3. The cohomology of the n-sphere S^n. Cover S^n by two open sets U and V where U is slightly larger than the northern hemisphere and V slightly larger than the southern hemisphere (Figure 4.1). Then $U \cap V$ is diffeomorphic to $S^{n-1} \times \mathbb{R}^1$ where S^{n-1} is the equator. Using the Mayer-Vietoris sequence, show that

$$H^*(S^n) = \begin{cases} \mathbb{R} & \text{in dimensions } 0, n \\ 0 & \text{otherwise.} \end{cases}$$

We saw previously that a generator of $H^1(S^1)$ is a bump 1-form on S^1 which gives the isomorphism $H^1(S^1) \simeq \mathbb{R}^1$ under integration (see Figure

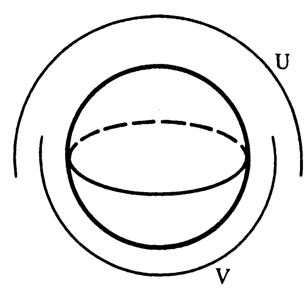


Figure 4.1

* In fact two manifolds have the same homotopy type in the C^∞ sense if and only if they have the same homotopy type in the usual (continuous) sense. This is because every continuous map between two manifolds is continuously homotopic to a C^∞ map (see Proposition 17.8).

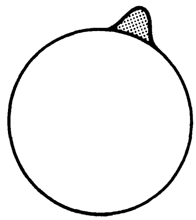


Figure 4.2

4.2). This bump 1-form propagates by the boundary map of the Mayer-Vietoris sequence to a bump 2-form on S^2, which represents a generator of $H^2(S^2)$. In general a generator of $H^n(S^n)$ can be taken to be a bump n-form on S^n.

Exercise 4.3.1 Volume form on a sphere. Let $S^n(r)$ be the sphere of radius r

$$x_1^2 + \cdots + x_{n+1}^2 = r^2$$

in $\mathbb{R}^{n+1}$, and let

$$\omega = \frac{1}{r} \sum_{i=1}^{n+1} (-1)^{i-1} x_i \, dx_1 \cdots \widehat{dx_i} \cdots dx_{n+1}.$$

(a) Write S^n for the unit sphere $S^n(1)$. Compute the integral $\int_{S^n} \omega$ and conclude that ω is not exact.

(b) Regarding r as a function on $\mathbb{R}^{n+1} - 0$, show that $(dr) \cdot \omega = dx_1 \cdots dx_{n+1}$. Thus ω is the Euclidean volume form on the sphere $S^n(r)$.

From (a) we obtain an explicit formula for the generator of the top cohomology of S^n (although not as a bump form). For example, the generator of $H^2(S^2)$ is represented by

$$\sigma = \frac{1}{4\pi}(x_1 \, dx_2 \, dx_3 - x_2 \, dx_1 \, dx_3 + x_3 \, dx_1 \, dx_2).$$

The Poincaré Lemma for Compactly Supported Cohomology

The computation of the compactly supported cohomology $H_c^*(\mathbb{R}^n)$ is again by induction; we will show that there is an isomorphism

$$H_c^{*+1}(\mathbb{R}^n \times \mathbb{R}^1) \simeq H_c^*(\mathbb{R}^n).$$

Note that here, unlike the previous case, the dimension is shifted by one.

More generally consider the projection $\pi : M \times \mathbb{R}^1 \to M$. Since the pullback of a form on M to a form on $M \times \mathbb{R}^1$ necessarily has noncompact support, the pullback map π^* does not send $\Omega_c^*(M)$ to $\Omega_c^*(M \times \mathbb{R}^1)$. However, there is a push-forward map $\pi_* : \Omega_c^*(M \times \mathbb{R}^1) \to \Omega_c^{*-1}(M)$, called *integration along the fiber*, defined as follows. First note that a compactly supported form on $M \times \mathbb{R}^1$ is a linear combination of two types of forms:

$$\text{(I) } \pi^*\phi \cdot f(x, t),$$
$$\text{(II) } \pi^*\phi \cdot f(x, t)\, dt,$$

where ϕ is a form on the base (not necessarily with compact support), and $f(x, t)$ is a function with compact support. We define π_* by

$$\text{(4.4)} \qquad \begin{aligned} &\text{(I) } \pi^*\phi \cdot f(x, t) \mapsto 0, \\ &\text{(II) } \pi^*\phi \cdot f(x, t)\, dt \mapsto \phi \int_{-\infty}^{\infty} f(x, t)\, dt. \end{aligned}$$

Exercise 4.5. Show that $d\pi_* = \pi_* d$; in other words, $\pi_* : \Omega_c^*(M \times \mathbb{R}^1) \to \Omega_c^{*-1}(M)$ is a chain map.

By this exercise π_* induces a map in cohomology $\pi_* : H_c^* \to H_c^{*-1}$. To produce a map in the reverse direction, let $e = e(t)\, dt$ be a compactly supported 1-form on $\mathbb{R}^1$ with total integral 1 and define

$$e_* : \Omega_c^*(M) \to \Omega_c^{*+1}(M \times \mathbb{R}^1)$$

by

$$\phi \mapsto (\pi^*\phi) \wedge e.$$

The map e_* clearly commutes with d, so it also induces a map in cohomology. It follows directly from the definition that $\pi_* \circ e_* = 1$ on $\Omega_c^*(\mathbb{R}^n)$. Although $e_* \circ \pi_* \neq 1$ on the level of forms, we shall produce a homotopy operator K between 1 and $e_* \circ \pi_*$; it will then follow that $e_* \circ \pi_* = 1$ in cohomology.

To streamline the notation, write $\phi \cdot f$ for $\pi^*\phi \cdot f(x, t)$ and $\int f$ for $\int f(x, t)\, dt$. The homotopy operator $K : \Omega_c^*(M \times \mathbb{R}^1) \to \Omega_c^{*-1}(M \times \mathbb{R}^1)$ is defined by

$$\text{(I) } \phi \cdot f \mapsto 0,$$
$$\text{(II) } \phi \cdot f\, dt \mapsto \phi \int_{-\infty}^{t} f - \phi A(t) \int_{-\infty}^{\infty} f \qquad \text{where } A(t) = \int_{-\infty}^{t} e.$$

Proposition 4.6. $1 - e_*\pi_* = (-1)^{q-1}(dK - Kd)$ *on* $\Omega_c^q(M \times \mathbb{R}^1)$.

PROOF. On forms of type (I), assuming $\deg \phi = q$, we have

$$(1 - e_*\pi_*)\phi \cdot f = \phi \cdot f,$$

$$\begin{aligned} (dK - Kd)\phi \cdot f &= -K\left(d\phi \cdot f + (-1)^q \phi \frac{\partial f}{\partial x}\, dx + (-1)^q\, \phi \frac{\partial f}{\partial t}\, dt\right) \\ &= (-1)^{q-1}\left(\phi \int_{-\infty}^{t} \frac{\partial f}{\partial t} - \phi A(t) \int_{-\infty}^{\infty} \frac{\partial f}{\partial t}\right) \\ &= (-1)^{q-1}\phi f. \quad \left[\text{Here } \int_{-\infty}^{\infty} \frac{\partial f}{\partial t} = f(x, \infty) - f(x, -\infty) = 0.\right] \end{aligned}$$

So

$$1 - e_* \pi_* = (-1)^{q-1}(dK - Kd).$$

On forms of type (II), now assuming deg $\phi = q - 1$, we have

$$(1 - e_* \pi_*)\phi f\, dt = \phi f\, dt - \phi\left(\int_{-\infty}^{\infty} f\right) \wedge e,$$

$$\begin{aligned}(dK)(\phi f\, dt) &= (d\phi)\int_{-\infty}^{t} f + (-1)^{q-1}\phi\left(\int_{-\infty}^{t} \frac{\partial f}{\partial x}\right) dx + (-1)^{q-1}\phi f\, dt \\ &\quad - (d\phi)A(t)\int_{-\infty}^{\infty} f - (-1)^{q-1}\phi\left[e\int_{-\infty}^{\infty} f + A(t)\left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x}\right) dx\right]\end{aligned}$$

$$\begin{aligned}(Kd)(\phi f\, dt) &= K\left((d\phi)\cdot f\, dt + (-1)^{q-1}\phi \frac{\partial f}{\partial x}\, dx\, dt\right) \\ &= (d\phi)\int_{-\infty}^{t} f - (d\phi)A(t)\int_{-\infty}^{\infty} f \\ &\quad + (-1)^{q-1}\left[\phi\left(\int_{-\infty}^{t} \frac{\partial f}{\partial x}\right) dx - \phi A(t)\left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x}\right) dx\right].\end{aligned}$$

So

$$(dK - Kd)\phi f\, dt = (-1)^{q-1}\left[\phi f\, dt - \phi\left(\int_{-\infty}^{\infty} f\right)e\right]$$

and the formula again holds. □

This concludes the proof of the following

Proposition 4.7. *The maps*

$$H_c^*(M \times \mathbb{R}^1) \underset{e_*}{\overset{\pi_*}{\rightleftarrows}} H_c^{*-1}(M)$$

are isomorphisms.

Corollary 4.7.1 (Poincaré Lemma for Compact Supports).

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \textit{in dimension } n \\ 0 & \textit{otherwise.} \end{cases}$$

Here the isomorphism $H_c^n(\mathbb{R}^n) \xrightarrow{\sim} \mathbb{R}$ is given by iterated π_*, i.e., by integration over $\mathbb{R}^n$.

To determine a generator for $H_c^n(\mathbb{R}^n)$, we start with the constant function 1 on a point and iterate with e_*. This gives $e(x_1)\, dx_1\, e(x_2)\, dx_2 \ldots e(x_n)\, dx_n$.

So a generator for $H_c^n(\mathbb{R}^n)$ is a bump n-form $\alpha(x)\,dx_1 \ldots dx_n$ with

$$\int_{\mathbb{R}^n} \alpha(x)\,dx_1 \ldots dx_n = 1.$$

The support of α can be made as small as we like.

REMARK. This Poincaré lemma shows that the compactly supported cohomology is not invariant under homotopy equivalence, although it is of course invariant under diffeomorphisms.

Exercise 4.8. Compute the cohomology groups $H^*(M)$ and $H_c^*(M)$ of the open Möbius strip M, i.e., the Möbius strip without the bounding edge (Figure 4.3). [*Hint*: Apply the Mayer-Vietoris sequences.]

The Degree of a Proper Map

As an application of the Poincaré lemma for compact supports we introduce here a C^∞ invariant of a proper map between two Euclidean spaces of the same dimension. Later, after Poincaré duality, this will be generalized to a proper map between any two oriented manifolds; for compact manifolds the properness assumption is of course redundant.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a proper map. Then the pullback $f^* : H_c^n(\mathbb{R}^n) \to H_c^n(\mathbb{R}^n)$ is defined. It carries a *generator* of $H_c^n(\mathbb{R}^n)$, i.e., a compactly supported closed form with total integral one, to some multiple of the generator. This multiple is defined to be the *degree* of f. If α is a generator of $H_c^n(\mathbb{R}^n)$, then

$$\deg f = \int_{\mathbb{R}^n} f^*\alpha.$$

A priori the degree of a proper map is a real number; surprisingly, it turns out to be an integer. To see this, we need Sard's theorem. Recall that a *critical point* of a smooth map $f\colon \mathbb{R}^m \to \mathbb{R}^m$ is a point p where the differential $(f_*)_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^n$ is not surjective, and a *critical value* is the image of a critical point. A point of $\mathbb{R}^n$ which is not a critical value is called a *regular value*. According to this definition any point of $\mathbb{R}^n$ which is not in the image of f is a regular value so that the inverse image of a regular value may be empty.

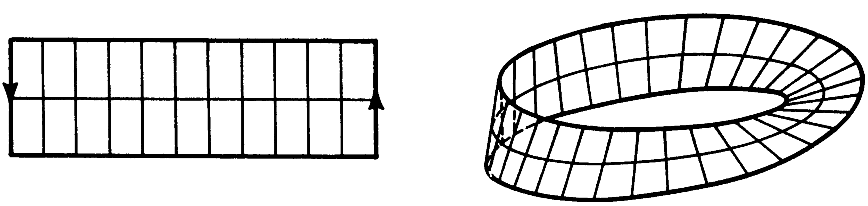


Figure 4.3

Theorem 4.9 (Sard's Theorem for $\mathbb{R}^n$). *The set of critical values of a smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$ has measure zero in $\mathbb{R}^n$ for any integers m and n.*

This means that given any $\varepsilon > 0$, the set of critical values can be covered by cubes with total volume less than ε. Important special cases of this theorem were first published by A. P. Morse [1]. Sard's proof of the general case may be found in Sard [1].

Proposition 4.10 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a proper map. If f is not surjective, then it has degree 0.*

PROOF. Since the image of a proper map is closed (why?), if f misses a point q, it must miss some neighborhood U of q. Choose a bump n-form α whose support lies in U. Then $f^*\alpha \equiv 0$ so that $\deg f = 0$. □

Exercise 4.10.1. Prove that the image of a proper map is closed.

So to show that the degree is an integer we only need to look at surjective proper maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. By Sard's theorem, almost all points in the image of such a map are regular values. Pick one regular value, say q. By hypothesis the inverse image of q is nonempty. Since in our case the two Euclidean spaces have the same dimension, the differential f_* is surjective if and only if it is an isomorphism. So by the inverse function theorem, around any point in the pre-image of q, f is a local diffeomorphism. It follows that $f^{-1}(q)$ is a discrete set of points. Since f is proper, $f^{-1}(q)$ is in fact a finite set of points. Choose a generator α of $H_c^n(\mathbb{R}^n)$ whose support is localized near q. Then $f^*\alpha$ is an n-form whose support is localized near the points of $f^{-1}(q)$ (see Figure 4.4). As noted earlier, a diffeomorphism preserves an integral only up to sign, so the integral of $f^*\alpha$ near each point of $f^{-1}(q)$ is ± 1. Thus

$$\int_{\mathbb{R}^n} f^*\alpha = \sum_{f^{-1}(q)} \pm 1.$$

This proves that *the degree of a proper map between two Euclidean spaces of the same dimension is an integer*. More precisely, it shows that *the number of*

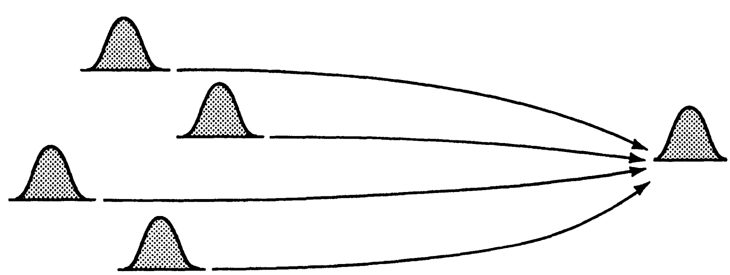


Figure 4.4

points, counted with multiplicity ± 1, *in the inverse image of any regular value is the same for all regular values and that this number is equal to the degree of the map.*

Sard's theorem for $\mathbb{R}^n$, a key ingredient of this discussion, has a natural extension to manifolds. We take this opportunity to state Sard's theorem in general. A subset S of a manifold M is said to have *measure zero* if it can be covered by countably many coordinate open sets U_i such that $\phi_i(S \cap U_i)$ has measure zero in $\mathbb{R}^n$; here ϕ_i is the trivialization on U_i. A *critical point* of a smooth map $f : M \to N$ between two manifolds is a point p in M where the differential $(f_*)_p : T_p M \to T_{f(p)}N$ is not surjective, and a *critical value* is the image of a critical point.

Theorem 4.11 (Sard's Theorem). *The set of critical values of a smooth map* $f : M \to N$ *has measure zero.*

Exercise 4.11.1. Prove Theorem 4.11 from Sard's theorem for $\mathbb{R}^n$.

§5 The Mayer-Vietoris Argument

The Mayer-Vietoris sequence relates the cohomology of a union to those of the subsets. Together with the Five Lemma, this gives a method of proof which proceeds by induction on the cardinality of an open cover, called the *Mayer-Vietoris argument.* As evidence of its power and versatility, we derive from it the finite dimensionality of the de Rham cohomology, Poincaré duality, the Künneth formula, the Leray-Hirsch theorem, and the Thom isomorphism, all for manifolds with finite good covers.

Existence of a Good Cover

Let M be a manifold of dimension n. An open cover $\mathfrak{U} = \{U_\alpha\}$ of M is called a *good cover* if all nonempty finite intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ are diffeomorphic to $\mathbb{R}^n$. A manifold which has a finite good cover is said to be *of finite type*.

Theorem 5.1. *Every manifold has a good cover. If the manifold is compact, then the cover may be chosen to be finite.*

To prove this theorem we will need a little differential geometry. A *Riemannian structure* on a manifold M is a smoothly varying metric $\langle\ ,\ \rangle$ on the tangent space of M at each point; it is smoothly varying in the following sense: if X and Y are two smooth vector fields on M, then $\langle X, Y\rangle$ is a smooth function on M. Every manifold can be given a Riemannian structure by the following splicing procedure. Let $\{U_\alpha\}$ be a coordinate open cover of M, $\langle\ ,\ \rangle_\alpha$ a Riemannian metric on U_α, and $\{\rho_\alpha\}$ a partition of unity subordinate to $\{U_\alpha\}$. Then $\langle\ ,\ \rangle = \sum \rho_\alpha \langle\ ,\ \rangle_\alpha$ is a Riemannian metric on M.

PROOF OF THEOREM 5.1. Endow M with a Riemannian structure. Now we quote the theorem in differential geometry that every point in a Riemannian manifold has a geodesically convex neighborhood (Spivak [1, Ex. 32(f), p. 491]). The intersection of any two such neighborhoods is again geodesically convex. Since a geodesically convex neighborhood in a Riemannian manifold of dimension n is diffeomorphic to $\mathbb{R}^n$, an open cover consisting of geodesically convex neighborhoods will be a good cover. □

Given two covers $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ and $\mathfrak{V} = \{V_\beta\}_{\beta \in J}$, if every V_β is contained in some U_α, we say that $\mathfrak{V}$ is a *refinement* of $\mathfrak{U}$ and write $\mathfrak{U} < \mathfrak{V}$. To be more precise we specify a refinement by a map $\phi: J \to I$ such that $V_\beta \subset U_{\phi(\beta)}$. By a slight modification of the above proof we can show that *every open cover on a manifold has a refinement which is a good cover*: simply take the geodesically convex neighborhoods around each point to be inside some open set of the given cover.

A *directed set* is a set I with a relation $<$ satisfying

(a) (reflexivity) $a < a$ for all $a \in I$.
(b) (transitivity) if $a < b$ and $b < c$, then $a < c$.
(c) (upper bound) for any $a, b \in I$, there is an element c in I such that $a < c$ and $b < c$.

The set of open covers on a manifold is a directed set, since any two open covers always have a common refinement. A subset J of a directed set I is *cofinal* in I if for every i in I there is a j in J such that $i < j$. It is clear that J is also a directed set.

Corollary 5.2. *The good covers are cofinal in the set of all covers of a manifold M.*

Finite Dimensionality of de Rham Cohomology

Proposition 5.3.1. *If the manifold M has a finite good cover, then its cohomology is finite dimensional.*

PROOF. From the Mayer-Vietoris sequence

$$\cdots \to H^{q-1}(U \cap V) \xrightarrow{d^*} H^q(U \cup V) \xrightarrow{r} H^q(U) \oplus H^q(V) \to \cdots$$

we get

$$H^q(U \cup V) \simeq \ker r \oplus \operatorname{im} r \simeq \operatorname{im} d^* \oplus \operatorname{im} r.$$

Thus,
(*) *if $H^q(U)$, $H^q(V)$ and $H^{q-1}(U \cap V)$ are finite-dimensional, then so is $H^q(U \cup V)$.*

For a manifold which is diffeomorphic to $\mathbb{R}^n$, the finite dimensionality of $H^*(M)$ follows from the Poincaré lemma (4.1.1). We now proceed by induction on the cardinality of a good cover. Suppose the cohomology of any manifold having a good cover with at most p open sets is finite dimensional. Consider a manifold having a good cover $\{U_0, \ldots, U_p\}$ with $p + 1$ open sets. Now $(U_0 \cup \ldots \cup U_{p-1}) \cap U_p$ has a good cover with p open sets, namely $\{U_{0p}, U_{1p}, \ldots, U_{p-1,p}\}$. By hypothesis, the qth cohomology of $U_0 \cup \ldots \cup U_{p-1}$, U_p and $(U_0 \cup \ldots \cup U_{p-1}) \cap U_p$ are finite dimensional; from Remark (*), so is the qth cohomology of $U_0 \cup \ldots \cup U_p$. This completes the induction. □

Similarly,

Proposition 5.3.2. *If the manifold M has a finite good cover, then its compact cohomology is finite dimensional.*

Poincaré Duality on an Orientable Manifold

A pairing between two finite-dimensional vector spaces

$$\langle\ ,\ \rangle : V \otimes W \to \mathbb{R}$$

is said to be *nondegenerate* if $\langle v, w\rangle = 0$ for all $w \in W$ implies $v = 0$ and $\langle v, w\rangle = 0$ for all $v \in V$ implies $w = 0$; equivalently, the map $v \mapsto \langle v,\ \rangle$ should define an injection $V \hookrightarrow W^*$ and the map $w \mapsto \langle\ , w\rangle$ also defines an injection $W \hookrightarrow V^*$.

Lemma. *Let V and W be finite-dimensional vector spaces. The pairing $\langle\ ,\ \rangle : V \otimes W \to \mathbb{R}$ is nondegenerate if and only if the map $v \mapsto \langle v,\ \rangle$ defines an isomorphism $V \xrightarrow{\sim} W^*$.*

Proof. ($\Rightarrow$) Since $V \hookrightarrow W^*$ and $W \hookrightarrow V^*$ are injective,

$$\dim V \le \dim W^* = \dim W \le \dim V^* = \dim V;$$

hence, $\dim V = \dim W^*$ and $V \hookrightarrow W^*$ must be an isomorphism.

($\Leftarrow$) is left to the reader. □

Because the wedge product is an antiderivation, it descends to cohomology; by Stokes' theorem, integration also descends to cohomology. So for an oriented manifold M there is a pairing

$$\int : H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R}$$

given by the integral of the wedge product of two forms. Our first version of Poincaré duality asserts that *this pairing is nondegenerate whenever M is orientable and has a finite good cover*; equivalently,

$$H^q(M) \simeq (H_c^{n-q}(M))^*. \tag{5.4}$$

Note that by (5.3.1) and (5.3.2) both $H^q(M)$ and $H_c^{n-q}(M)$ are finite-dimensional.

A couple of lemmas will be needed in the proof of Poincaré duality.

Exercise 5.5. Prove the Five Lemma: given a commutative diagram of Abelian groups and group homomorphisms

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C & \xrightarrow{f_3} & D & \xrightarrow{f_4} & E & \longrightarrow & \cdots \\
 & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\delta}\downarrow & & {\scriptstyle\varepsilon}\downarrow & & \\
\cdots & \longrightarrow & A' & \xrightarrow[f_1']{} & B' & \xrightarrow[f_2']{} & C' & \xrightarrow[f_3']{} & D' & \xrightarrow[f_4']{} & E' & \longrightarrow & \cdots
\end{array}$$

in which the rows are exact, if the maps α, β, δ and ε are isomorphisms, then so is the middle one γ.

Lemma 5.6. *The two Mayer-Vietoris sequences* (2.4) *and* (2.8) *may be paired together to form a sign-commutative diagram*

$$\begin{array}{ccccccc}
\cdots \to H^q(U \cup V) & \xrightarrow{\text{restriction}} & H^q(U) \oplus H^q(V) & \xrightarrow{\text{difference}} & H^q(U \cap V) & \xrightarrow{d^*} & H^{q+1}(U \cup V) \to \cdots \\
\otimes & & \otimes & & \otimes & & \otimes \\
\cdots \leftarrow H_c^{n-q}(U \cup V) & \xleftarrow{\text{sum}} & H_c^{n-q}(U) \oplus H_c^{n-q}(V) & \longleftarrow & H_c^{n-q}(U \cap V) & \xleftarrow{d_*} & H_c^{n-q-1}(U \cup V) \\
\downarrow \int_{U \cup V} & & \downarrow \int_U + \int_V & & \downarrow \int_{U \cap V} & & \downarrow \int_{U \cup V} \\
\mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R}
\end{array}$$

Here sign-commutativity means, for instance, that

$$\int_{U \cap V} \omega \wedge d_* \tau = \pm \int_{U \cup V} (d^*\omega) \wedge \tau,$$

for $\omega \in H^q(U \cap V)$, $\tau \in H_c^{n-q-1}(U \cup V)$. This lemma is equivalent to saying that the pairing induces a map from the upper exact sequence to the dual of the lower exact sequence such that the following diagram is sign-commutative:

$$\begin{array}{ccccccc}
\to & H^q(U \cup V) & \to & H^q(U) \oplus H^q(V) & \to & H^q(U \cap V) & \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to & H_c^{n-q}(U \cup V)^* & \to & H_c^{n-q}(U)^* \oplus H_c^{n-q}(V)^* & \to & H_c^{n-q}(U \cap V)^* & \to .
\end{array}$$

PROOF. The first two squares are in fact commutative as is straightforward to check. We will show the sign-commutativity of the third square.

Recall from (2.5) and (2.7) that $d^*\omega$ is a form in $H^{q+1}(U \cup V)$ such that

$$d^*\omega\,|_U = -d(\rho_V \omega)$$
$$d^*\omega\,|_V = d(\rho_U \omega),$$

and $d_* \tau$ is a form in $H_c^{n-q}(U \cap V)$ such that

$$(-(\text{extension by } 0 \text{ of } d_* \tau \text{ to } U), (\text{extension by } 0 \text{ of } d_* \tau \text{ to } V)) = (d(\rho_U \tau), d(\rho_V \tau)).$$

Note that $d(\rho_V \tau) = (d\rho_V)\tau$ because τ is closed; similarly, $d(\rho_V \omega) = (d\rho_V)\omega$.

$$\int_{U \cap V} \omega \wedge d_* \tau = \int_{U \cap V} \omega \wedge (d\rho_V)\tau = (-1)^{\deg \omega} \int_{U \cap V} (d\rho_V)\omega \wedge \tau.$$

Since $d^*\omega$ has support in $U \cap V$,

$$\int_{U \cup V} d^*\omega \wedge \tau = -\int_{U \cap V} (d\rho_V)\omega \wedge \tau.$$

Therefore,

$$\int_{U \cap V} \omega \wedge d_* \tau = (-1)^{\deg \omega + 1} \int_{U \cup V} d^* \omega \wedge \tau. \qquad \square$$

By the Five Lemma if Poincaré duality holds for U, V, and $U \cap V$, then it holds for $U \cup V$. We now proceed by induction on the cardinality of a good cover. For M diffeomorphic to $\mathbb{R}^n$, Poincaré duality follows from the two Poincaré lemmas

$$H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } 0 \\ 0 & \text{elsewhere} \end{cases}$$

and

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } n \\ 0 & \text{elsewhere.} \end{cases}$$

Next suppose Poincaré duality holds for any manifold having a good cover with at most p open sets, and consider a manifold having a good cover $\{U_0, \ldots, U_p\}$ with $p + 1$ open sets. Now $(U_0 \cup \cdots \cup U_{p-1}) \cap U_p$ has a good cover with p open sets, namely $\{U_{0p}, U_{1p}, \ldots, U_{p-1,p}\}$. By hypothesis Poincaré duality holds for $U_0 \cup \ldots \cup U_{p-1}$, U_p, and $(U_0 \cup \ldots \cup U_{p-1}) \cap U_p$, so it holds for $U_0 \cup \ldots \cup U_{p-1} \cup U_p$ as well. This induction argument proves Poincaré duality for any orientable manifold having a finite good cover. $\square$

REMARK 5.7. The finiteness assumption on the good cover is in fact not necessary. By a closer analysis of the topology of a manifold, the Mayer-Vietoris argument above can be extended to any orientable manifold (Greub, Halperin, and Vanstone [1, p. 198 and p. 14]). The statement is as follows: *if M is an orientable manifold of dimension n, whose cohomology is not necessarily finite dimensional, then*

$$H^q(M) \simeq (H_c^{n-q}(M))^* \quad , \quad \textit{for any integer } q.$$

However, the reverse implication $H_c^q(M) \simeq (H^{n-q}(M))^*$ is not always true. The asymmetry comes from the fact that the dual of a direct sum is a direct product, but the dual of a direct product is not a direct sum. For example, consider the infinite disjoint union

$$M = \coprod_{i=1}^{\infty} M_i,$$

where the M_i's are all manifolds of finite type of the same dimension n. Then the de Rham cohomology is a direct product

$$H^q(M) = \prod_i H^q(M_i), \tag{5.7.1}$$

but the compact cohomology is a direct sum

$$H_c^q(M) = \bigoplus_i H_c^q(M_i). \tag{5.7.2}$$

Taking the dual of the compact cohomology $H_c^q(M)$ gives a direct product

$$(H_c^q(M))^* = \prod_i H_c^q(M_i). \tag{5.7.3}$$

So by (5.7.1) and (5.7.3), it follows from Poincaré duality for the manifolds of finite type M_i, that

$$H^q(M) = (H_c^{n-q}(M))^*.$$

Corollary 5.8. *If M is a connected oriented manifold of dimension n, then $H_c^n(M) \simeq \mathbb{R}$. In particular if M is compact oriented and connected, $H^n(M) \simeq \mathbb{R}$.*

Let $f : M \to N$ be a map between two compact oriented manifolds of dimension n. Then there is an induced map in cohomology

$$f^* : H^n(N) \to H^n(M).$$

The *degree* of f is defined to be $\int_M f^*\omega$, where ω is the generator of $H^n(N)$. By the same argument as for the degree of a proper map between two Euclidean spaces, the degree of a map between two compact oriented manifolds is an integer and is equal to the number of points, counted with multiplicity ± 1, in the inverse image of any regular point in N.

The Künneth Formula and the Leray-Hirsch Theorem

The Künneth formula states that the cohomology of the product of two manifolds M and F is the tensor product

$$H^*(M \times F) = H^*(M) \otimes H^*(F). \tag{5.9}$$

This means

$$H^n(M \times F) = \bigoplus_{p+q=n} H^p(M) \otimes H^q(F) \qquad \text{for every nonnegative integer } n.$$

More generally we are interested in the cohomology of a *fiber bundle*.

Definition. Let G be a topological group which acts effectively on a space F on the left. A surjection $\pi : E \to B$ between topological spaces is a *fiber bundle with fiber F and structure group G* if B has an open cover $\{U_\alpha\}$ such that there are fiber-preserving homeomorphisms

$$\phi_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times F$$

and the transitions functions are continuous functions with values in G:

$$g_{\alpha\beta}(x) = \phi_\alpha \phi_\beta^{-1}\big|_{\{x\}\times F} \in G.$$

Sometimes the *total space* E is referred to as the fiber bundle. A fiber bundle with structure group G is also called a *G-bundle.* If $x \in B$, the set $E_x = \pi^{-1}(x)$ is called the *fiber* at x.

Since we are working with de Rham theory, the spaces E, B, and F will be assumed to be C^∞ manifolds and the maps C^∞ maps. We may also speak of a fiber bundle without mentioning its structure group; in that case, the group is understood to be the group of diffeomorphisms of F, denoted Diff(F).

REMARK. The action of a group G on a space F is said to be *effective* if the only element of G which acts trivially on F is the identity, i.e., if $g \cdot y = y$ for all y in F, then $g = 1 \in G$. In the C^∞ case, this is equivalent to saying that the kernel of the natural map $G \to \mathrm{Diff}(F)$ is the identity or that G is a subgroup of Diff(F), the group of diffeomorphisms of F. In the definition of a fiber bundle the action of G on F is required to be effective in order that the diffeomorphism

$$\phi_\alpha \phi_\beta^{-1}\big|_{\{x\}\times F}$$

of F can be identified unambiguously with an element of G.

The transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ satisfy the *cocycle condition:*

$$g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}.$$

Given a cocycle $\{g_{\alpha\beta}\}$ with values in G we can construct a fiber bundle E having $\{g_{\alpha\beta}\}$ as its transition functions by setting

$$E = \left(\coprod U_\alpha \times F\right)/(x, y) \sim \left(x, g_{\alpha\beta}(x)y\right) \tag{5.10}$$

for (x, y) in $U_\beta \times F$ and $(x, g_{\alpha\beta}(x)y)$ in $U_\alpha \times F$.

The following proof of the Künneth formula assumes that M has a finite good cover. This assumption is necessary for the induction argument.

The two natural projections

$$\begin{array}{ccc} M \times F & \xrightarrow{\rho} & F \\ \downarrow{\scriptstyle \pi} & & \\ M & & \end{array}$$

give rise to a map on forms

$$\omega \otimes \phi \mapsto \pi^*\omega \wedge \rho^*\phi$$

which induces a map in cohomology (exercise)

$$\psi : H^*(M) \otimes H^*(F) \to H^*(M \times F).$$

We will show that ψ is an isomorphism.

If $M = \mathbb{R}^m$, this is simply the Poincaré lemma.

In the following we will regard $M \times F$ as a product bundle over M. Let U and V be open sets in M and n a fixed integer. From the Mayer-Vietoris sequence

$$\cdots \to H^p(U \cup V) \to H^p(U) \oplus H^p(V) \to H^p(U \cap V) \cdots$$

we get an exact sequence by tensoring with $H^{n-p}(F)$

$$\cdots \to H^p(U \cup V) \otimes H^{n-p}(F) \to (H^p(U) \otimes H^{n-p}(F)) \oplus (H^p(V) \otimes H^{n-p}(F)) \\ \to H^p(U \cap V) \otimes H^{n-p}(F) \to \cdots$$

since tensoring with a vector space preserves exactness. Summing over all integers p yields the exact sequence

$$\begin{aligned} \cdots \to & \bigoplus_{p=0}^{n} H^p(U \cup V) \otimes H^{n-p}(F) \\ \to & \bigoplus_{p=0}^{n} (H^p(U) \otimes H^{n-p}(F)) \oplus (H^p(V) \otimes H^{n-p}(F)) \\ \to & \bigoplus_{p=0}^{n} H^p(U \cap V) \otimes H^{n-p}(F) \to \cdots. \end{aligned}$$

The following diagram is commutative

$$\begin{tikzcd} \bigoplus_{p=0}^{n} H^p(U \cup V) \otimes H^{n-p}(F) \arrow[r] \arrow[d, "\psi"] & \bigoplus_{p=0}^{n} (H^p(U) \otimes H^{n-p}(F)) \oplus (H^p(V) \otimes H^{n-p}(F)) \arrow[r] \arrow[d, "\psi"] & \bigoplus_{p=0}^{n} H^p(U \cap V) \otimes H^{n-p}(F) \arrow[d, "\psi"] \\ H^n((U \cup V) \times F) \arrow[r] & H^n(U \times F) \oplus H^n(V \times F) \arrow[r] & H^n((U \cap V) \times F) \end{tikzcd}$$

The commutativity is clear except possibly for the square

$$\begin{tikzcd} \bigoplus (H^p(U \cap V) \otimes H^{n-p}(F)) \arrow[r, "d^*"] \arrow[d, "\psi"'] & \bigoplus H^{p+1}(U \cup V) \otimes H^{n-p}(F) \arrow[d, "\psi"] \\ H^n((U \cap V) \times F) \arrow[r, "d^*"] & H^{n+1}((U \cup V) \times F), \end{tikzcd}$$

which we now check. Let $\omega \otimes \phi$ be in $H^p(U \cap V) \otimes H^{n-p}(F)$. Then

$$\psi d^*(\omega \otimes \phi) = \pi^*(d^*\omega) \wedge \rho^*\phi$$

$$d^*\psi(\omega \otimes \phi) = d^*(\pi^*\omega \wedge \rho^*\phi).$$

Recall from (2.5) that if $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to $\{U, V\}$ then

$$d^*\omega = \begin{cases} -d(\rho_V \omega) & \text{on } U \\ d(\rho_U \omega) & \text{on } V. \end{cases}$$

Since the pullback functions $\{\pi^*\rho_U, \pi^*\rho_V\}$ form a partition of unity on $(U \cup V) \times F$ subordinate to the cover $\{U \times F, V \times F\}$, on $(U \cap V) \times F$

$$\begin{aligned} d^*(\pi^*\omega \wedge \rho^*\phi) &= d((\pi^*\rho_U)\pi^*\omega \wedge \rho^*\phi) \\ &= (d\pi^*(\rho_U \omega)) \wedge \rho^*\phi \qquad \text{since } \phi \text{ is closed} \\ &= \pi^*(d^*\omega) \wedge \rho^*\phi. \end{aligned}$$

So the diagram is commutative.

By the Five Lemma if the theorem is true for U, V, and $U \cap V$, then it is also true for $U \cup V$. The Künneth formula now follows by induction on the cardinality of a good cover, as in the proof of Poincaré duality. □

Let $\pi : E \to M$ be a fiber bundle with fiber F. Suppose there are cohomology classes $e_1, \ldots, e_r$ on E which restrict to a basis of the cohomology of each fiber. Then we can define a map

$$\psi : H^*(M) \otimes \mathbb{R}\{e_1, \ldots, e_r\} \to H^*(E).$$

The same argument as the Künneth formula gives

Theorem 5.11 (Leray-Hirsch). *Let E be a fiber bundle over M with fiber F. Suppose M has a finite good cover. If there are global cohomology classes $e_1, \ldots, e_r$ on E which when restricted to each fiber freely generate the cohomology of the fiber, then $H^*(E)$ is a free module over $H^*(M)$ with basis $\{e_1, \ldots, e_r\}$*, i.e.

$$H^*(E) \simeq H^*(M) \otimes \mathbb{R}\{e_1, \ldots, e_r\} \simeq H^*(M) \otimes H^*(F).$$

Exercise 5.12 Künneth formula for compact cohomology. The Künneth formula for compact cohomology states that for any manifolds M and N having a finite good cover.

$$H_c^*(M \times N) = H_c^*(M) \otimes H_c^*(N).$$

(a) In case M and N are orientable, show that this is a consequence of Poincaré duality and the Künneth formula for de Rham cohomology.

(b) Using the Mayer-Vietoris argument prove the Künneth formula for compact cohomology for any M and N having a finite good cover.

The Poincaré Dual of a Closed Oriented Submanifold

Let M be an oriented manifold of dimension n and S a closed oriented submanifold of dimension k; here by "closed" we mean as a subspace of M. Figure 5.1 is a closed submanifold of $\mathbb{R}^2 - \{0\}$, but Figure 5-2 is not. To every closed oriented submanifold $i : S \hookrightarrow M$ of dimension k, one can associate a unique cohomology class $[\eta_S]$ in $H^{n-k}(M)$, called its *Poincaré dual*, as follows. Let ω be a closed k-form with compact support on M. Since S is

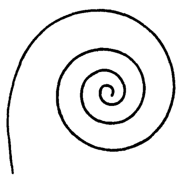


Figure 5.1

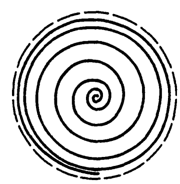


Figure 5.2

closed in M, $\text{Supp}(\omega|_S)$ is closed not only in S, but also in M. Now because $\text{Supp}(\omega|_S) \subset (\text{Supp } \omega) \cap S$ is a closed subset of a compact set, $i^*\omega$ also has compact support on S, so the integral $\int_S i^*\omega$ is defined. By Stokes's theorem integration over S induces a linear functional on $H_c^k(M)$. It follows by Poincaré duality: $(H_c^k(M))^* \simeq H^{n-k}(M)$, that integration over S corresponds to a unique cohomology class $[\eta_S]$ in $H^{n-k}(M)$. We will often call both the cohomology class $[\eta_S]$ and a form representing it the *Poincaré dual* of S. By definition the Poincaré dual η_S is the unique cohomology class in $H^{n-k}(M)$ satisfying

$$\int_S i^*\omega = \int_M \omega \wedge \eta_S \tag{5.13}$$

for any ω in $H_c^k(M)$.

Now suppose S is a *compact* oriented submanifold of dimension k in M. Since a compact subset of a Hausdorff space is closed, S is also a closed oriented submanifold and hence has a Poincaré dual $\eta_S \in H^{n-k}(M)$. This η_S we will call the *closed Poincaré dual* of S, to distinguish it from the *compact Poincaré dual* to be defined below. Because S is compact, one can in fact integrate over S not only k-forms with compact support on M, but *any* k-form on M. In this way S defines a linear functional on $H^k(M)$ and so by Poincaré duality corresponds to a unique cohomology class $[\eta'_S]$ in $H_c^{n-k}(M)$, the *compact Poincaré dual* of S. We must assume here that M has a finite good cover; otherwise, the duality $(H^k(M))^* \simeq H_c^{n-k}(M)$ does not hold. The compact Poincaré dual $[\eta'_S]$ is uniquely characterized by

$$\int_S i^*\omega = \int_M \omega \wedge \eta'_S, \tag{5.14}$$

for any $\omega \in H^k(M)$. If (5.14) holds for any closed k-form ω, then it certainly holds for any closed k-form ω with compact support. So *as a form*, η'_S is also the closed Poincaré dual of S, i.e., the natural map $H_c^{n-k}(M) \to H^{n-k}(M)$ sends the compact Poincaré dual to the closed Poincaré dual. Therefore we can in fact demand the closed Poincaré dual of a compact oriented submanifold to have compact support. However, as cohomology classes, $[\eta_S] \in H^{n-k}(M)$ and $[\eta'_S] \in H_c^{n-k}(M)$ could be quite different, as the following examples demonstrate.

Example 5.15 (The Poincaré duals of a point P on $\mathbb{R}^n$). Since $H^n(\mathbb{R}^n) = 0$, the closed Poincaré dual η_P is trivial and can be represented by any closed n-form on R^n, but the compact Poincaré dual is the nontrivial class in $H_c^n(\mathbb{R}^n)$ represented by a bump form with total integral 1.

Example-Exercise 5.16 (The ray and the circle in $\mathbb{R}^2 - \{0\}$). Let x, y be the standard coordinates and r, θ the polar coordinates on $\mathbb{R}^2 - \{0\}$.

(a) Show that the Poincaré dual of the ray $\{(x, 0) \mid x > 0\}$ in $\mathbb{R}^2 - \{0\}$ is $d\theta/2\pi$ in $H^1(\mathbb{R}^2 - \{0\})$.

(b) Show that the closed Poincaré dual of the unit circle in $H^1(\mathbb{R}^2 - \{0\})$ is 0, but the compact Poincaré dual is the nontrivial generator $\rho(r)dr$ in $H_c^1(\mathbb{R}^2 - \{0\})$ where $\rho(r)$ is a bump function with total integral 1. (By a bump function we mean a smooth function whose support is contained in some disc and whose graph looks like a "bump".)

Thus the generator of $H^1(\mathbb{R}^2 - \{0\})$ is represented by the ray and the generator of $H_c^1(\mathbb{R}^2 - \{0\})$ by the circle (see Figure 5.3).

Remark 5.17. The two Poincaré duals of a compact oriented submanifold correspond to the two homology theories—closed homology and compact homology. Closed homology has now fallen into disuse, while compact homology is known these days as the homology of singular chains. In Example-Exercise 5.16, the generator of $H_{1,\,\text{closed}}(\mathbb{R}^2 - \{0\})$ is the ray, while the generator of $H_{1,\,\text{compact}}(\mathbb{R}^2 - \{0\})$ is the circle. (The circle is a boundary in closed homology since the punctured closed disk is a closed 2-chain in $\mathbb{R}^2 - \{0\}$.) In general Poincaré duality sets up an isomorphism between closed homology and de Rham cohomology, and between compact homology and compact de Rham cohomology.

Let S be a compact oriented submanifold of dimension k in M. If $W \subset M$ is an open subset containing S, then the compact Poincaré dual of S in W, $\eta'_{S,W} \in H_c^{n-k}(W)$, extends by 0 to a form η'_S in $H_c^{n-k}(M)$. η'_S is clearly the compact Poincaré dual of S in M because

$$\int_S i^*\omega = \int_W \omega \wedge \eta'_{S,W} = \int_M \omega \wedge \eta'_S.$$

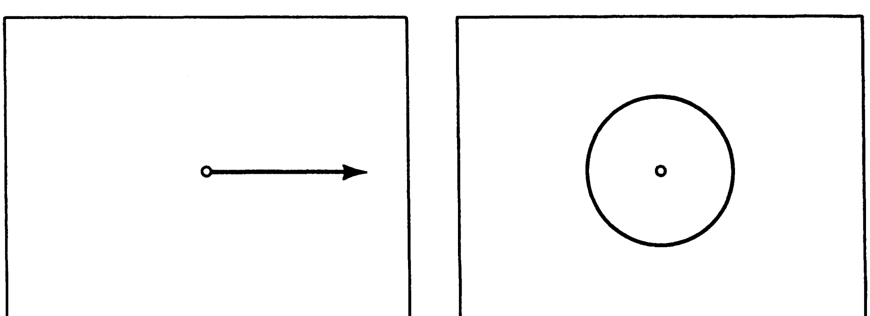


Figure 5.3

Thus, *the support of the compact Poincaré dual of S in M may be shrunk into any open neighborhood of S*. This is called the *localization principle.* For a noncompact closed oriented submanifold S the localization principle also holds. We will take it up in Proposition 6.25.

In this book we will mean by the Poincaré dual the *closed* Poincaré dual. However, as we have seen, if the submanifold is compact, we can demand that its closed Poincaré dual have compact support, even as a cohomology class in $H^{n-k}(M)$. Of course, on a compact manifold M, there is no distinction between the closed and the compact Poincaré duals.

§6 The Thom Isomorphism

So far we have encountered two kinds of C^∞ invariants of a manifold, de Rham cohomology and compactly supported cohomology. For vector bundles there is another invariant, namely, cohomology with compact support *in the vertical direction.* The Thom isomorphism is a statement about this last-named cohomology. In this section we use the Mayer-Vietoris argument to prove the Thom isomorphism for an orientable vector bundle. We then explain why the Poincaré dual and the Thom class are in fact one and the same thing. Using the interpretation of the Poincaré dual of a submanifold as the Thom class of the normal bundle, it is easy to write down explicitly the Poincaré dual, at least when the normal bundle is trivial. Next we give an explicit construction of the Thom class for an oriented rank 2 bundle, introducing along the way the global angular form and the Euler class. The higher-rank analogues will be taken up in Sections 11 and 12. We conclude this section with a brief discussion of the relative de Rham theory, citing the Thom class as an example of a relative class.

Vector Bundles and the Reduction of Structure Groups

Let $\pi : E \to M$ be a surjective map of manifolds whose fiber $\pi^{-1}(x)$ is a vector space for every x in M. The map π is a C^∞ *real vector bundle* of rank n if there is an open cover $\{U_\alpha\}$ of M and fiber-preserving diffeomorphisms

$$\phi_\alpha : E|_{U_\alpha} = \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$$

which are linear isomorphisms on each fiber. The maps

$$\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

are vector-space automorphisms of $\mathbb{R}^n$ in each fiber and hence give rise to maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$$

$$g_{\alpha\beta}(x) = \phi_\alpha \phi_{\beta^{-1}} |_{\{x\} \times \mathbb{R}^n} .$$

In the terminology of Section 5 a vector bundle of rank n is a fiber bundle with fiber $\mathbb{R}^n$ and structure group $GL(n, \mathbb{R})$. If the fiber is $\mathbb{C}^n$ and the structure group is $GL(n, \mathbb{C})$, the vector bundle is a *complex vector bundle.* Unless otherwise stated, by a vector bundle we mean a C^∞ real vector bundle.

Let U be an open set in M. A map $s : U \to E$ is a *section* of the vector bundle E over U if $\pi \circ s$ is the identity on U. The space of all sections over U is written $\Gamma(U, E)$. Note that every vector bundle has a well-defined global zero section. A collection of sections $s_1, \ldots, s_n$ over an open set U in M is a *frame* on U if for every point x in U, $s_1(x), \ldots, s_n(x)$ form a basis of the vector space $E_x = \pi^{-1}(x)$.

The transition functions $\{g_{\alpha\beta}\}$ of a vector bundle satisfy the *cocycle condition*

$$g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma.$$

The cocycle $\{g_{\alpha\beta}\}$ depends on the choice of the trivialization.

Lemma 6.1. *If the cocycle $\{g'_{\alpha\beta}\}$ comes from another trivialization $\{\phi'_\alpha\}$, then there exist maps $\lambda_\alpha : U_\alpha \to GL(n, \mathbb{R})$ such that*

$$g_{\alpha\beta} = \lambda_\alpha g'_{\alpha\beta} \lambda_\beta^{-1} \quad \textit{on} \quad U_\alpha \cap U_\beta.$$

Proof. The two trivializations differ by a nonsingular transformation of $\mathbb{R}^n$ at each point:

$$\phi_\alpha = \lambda_\alpha \phi'_\alpha \quad , \quad \lambda_\alpha : U_\alpha \to GL(n, \mathbb{R}).$$

Therefore,

$$g_{\alpha\beta} = \phi_\alpha \phi_\beta^{-1} = \lambda_\alpha \phi'_\alpha \phi'^{-1}_\beta \lambda_\beta^{-1} = \lambda_\alpha g'_{\alpha\beta} \lambda_\beta^{-1}. \qquad \square$$

Two cocycles related in this way are said to be *equivalent*.

Given a cocycle $\{g_{\alpha\beta}\}$ with values in $GL(n, \mathbb{R})$ we can construct a vector bundle E having $\{g_{\alpha\beta}\}$ as its cocycle as in (5.10). A homomorphism between two vector bundles, called a *bundle map*, is a fiber-preserving smooth map $f : E \to E'$ which is linear on corresponding fibers.

Exercise 6.2. Show that two vector bundles on M are isomorphic if and only if their cocycles relative to some open cover are equivalent.

Given a vector bundle with cocycle $\{g_{\alpha\beta}\}$, if it is possible to find an equivalent cocycle with values in a subgroup H of $GL(n, \mathbb{R})$, we say that *the structure group of E may be reduced to H*. A vector bundle is *orientable* if its structure group may be reduced to $GL^+(n, \mathbb{R})$, the linear transformations of $\mathbb{R}^n$ with positive determinant. A trivialization $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ on E is said to be *oriented* if for every α and β in I, the transition function $g_{\alpha\beta}$ has positive determinant. Two oriented trivializations $\{(U_\alpha, \phi_\alpha)\}$, $\{(V_\beta, \psi_\beta)\}$ are *equivalent* if for every x in $U_\alpha \cap V_\beta$, $\phi_\alpha \circ (\psi_\beta)^{-1}(x) : \mathbb{R}^n \to \mathbb{R}^n$ has positive determinant. It is easily checked that this is an equivalence relation and that on a connected manifold M it partitions all the oriented trivializations of the vector bundle E into two equivalence classes. Either equivalence class is called an *orientation* on the vector bundle E.

Example 6.3 (The tangent bundle). **By attaching to each point x in a manifold M, the tangent space to M at x, we obtain the *tangent bundle* of M:**

$$T_M = \bigcup_{x \in M} T_x M.$$

Let $\{(U_\alpha, \psi_\alpha)\}$ be an atlas for M. The diffeomorphism

$$\psi_\alpha : U_\alpha \xrightarrow{\sim} \mathbb{R}^n$$

induces a map

$$(\psi_\alpha)_* : T_{U_\alpha} \xrightarrow{\sim} T_{\mathbb{R}^n},$$

which gives a local trivialization of the tangent bundle T_M. From this we see that the transition functions of T_M are the Jacobians of the transition functions of M. Therefore *M is orientable as a manifold if and only if its tangent bundle is orientable as a bundle.* (However, the total space of the tangent bundle is always orientable as a manifold.) If $\psi_\alpha = (x_1, \ldots, x_n)$, then $\partial/\partial x_1, \ldots, \partial/\partial x_n$ is a frame for T_M over U_α. In the language of bundles a smooth vector field on U_α is a smooth section of the tangent bundle over U_α.

We now show that the structure group of every real vector bundle E may be reduced to the orthogonal group. First, we can endow E with a Riemannian structure—a smoothly varying positive definite symmetric bilinear form on each fiber—as follows. Let $\{U_\alpha\}$ be an open cover of M which trivializes E. On each U_α, choose a frame for $E|_{U_\alpha}$ and declare it to be orthonormal. This defines a Riemannian structure on $E|_{U_\alpha}$. Let $\langle\ ,\ \rangle_\alpha$ denote this inner product on $E|_{U_\alpha}$. Now use a partition of unity $\{\rho_\alpha\}$ to splice them together, i.e., form

$$\langle\ ,\ \rangle = \sum \rho_\alpha \langle\ ,\ \rangle_\alpha.$$

This will be an inner product over all of M.

As trivializations of E, we take only those maps ϕ_α that send orthonormal frames of E (relative to the global metric $\langle\ ,\ \rangle$) to orthonormal frames of $\mathbb{R}^n$—such maps exist by the Gram-Schmidt process. Then the transition functions $g_{\alpha\beta}$ will preserve orthonormal frames and hence take values in the orthogonal group $O(n)$. If the determinant of $g_{\alpha\beta}$ is positive, $g_{\alpha\beta}$ will actually be in the special orthogonal group $SO(n)$. Thus

Proposition 6.4. *The structure group of a real vector bundle of rank n can always be reduced to $O(n)$; it can be reduced to $SO(n)$ if and only if the vector bundle is orientable.*

Exercise 6.5. (a) Show that there is a direct product decomposition

$$GL(n, \mathbb{R}) = O(n) \times \{\text{positive definite symmetric matrices}\}.$$

(b) Use (a) to show that the structure group of any real vector bundle may be reduced to $O(n)$ by finding the λ_α's of Lemma 6.1.

Operations on Vector Bundles

Apart from introducing the functorial operations on vector bundles, our main purpose here is to establish the triviality of a vector bundle over a contractible manifold, a fact needed in the proof of the Thom isomorphism.

Functorial operations on vector spaces carry over to vector bundles. For instance, if E and E' are vector bundles over M of rank n and m respectively, their *direct sum* $E \oplus E'$ is the vector bundle over M whose fiber at the point x in M is $E_x \oplus E'_x$. The local trivializations $\{\phi_\alpha\}$ and $\{\phi'_\alpha\}$ for E and E' induce a local trivialization for $E \oplus E'$:

$$\phi_\alpha \oplus \phi_\alpha : E \oplus E'|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times (\mathbb{R}^n \oplus \mathbb{R}^m).$$

Hence the transition matrices for $E \oplus E'$ are

$$\begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & g'_{\alpha\beta} \end{pmatrix}.$$

Similarly we can define the tensor product $E \otimes E'$, the dual E^*, and $\mathrm{Hom}(E, E')$. Note that $\mathrm{Hom}(E, E')$ is isomorphic to $E^* \otimes E'$. The tensor product $E \otimes E'$ clearly has transition matrices $\{g_{\alpha\beta} \otimes g'_{\alpha\beta}\}$, but the transition matrices for the dual E^* are not so immediate. Recall that the dual V^* of a real vector space V is the space of all linear functionals on V, i.e., $V^* \simeq \mathrm{Hom}(V, \mathbb{R})$, and that a linear map $f: V \to W$ induces a map $f^t: W^* \to V^*$ represented by the transpose of the matrix of f. If

$$\phi_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$$

is a trivialization for E, then

$$(\phi_\alpha^t)^{-1} : E^*|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times (\mathbb{R}^n)^*$$

is a trivialization for E^*. Therefore the transition functions of E^* are

$$(\phi_\alpha^t)^{-1}\phi_\beta^t = ((\phi_\alpha \phi_\beta^{-1})^t)^{-1} = (g_{\alpha\beta}^t)^{-1}. \tag{6.6}$$

Let M and N be manifolds and $\pi : E \to M$ a vector bundle over M. Any map $f : N \to M$ induces a vector bundle $f^{-1}E$ on N, called the *pullback of E by f*. This bundle $f^{-1}E$ is defined to be the subset of $N \times E$ given by

$$\{(n, e) \mid f(n) = \pi(e)\}.$$

It is the unique maximal subset of $N \times E$ which makes the following diagram commutative

$$\begin{array}{ccc} \subset N \times E & & \\ f^{-1}E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \pi \\ N & \underset{f}{\longrightarrow} & M. \end{array}$$

The fiber of $f^{-1}E$ over a point y in N is isomorphic to $E_{f(y)}$. Since a product bundle pulls back to a product bundle we see that $f^{-1}E$ is locally trivial, and is therefore a vector bundle. Furthermore, if we have a composition

$$M'' \xrightarrow{g} M' \xrightarrow{f} M,$$

then

$$(f \circ g)^{-1}E = g^{-1}(f^{-1}E).$$

Let $\text{Vect}_k(M)$ be the isomorphism classes of rank k real vector bundles over M. It is a pointed set with base point the isomorphism class of the product bundle over M. If $f : M \to N$ is a map between two manifolds, let $\text{Vect}_k(f) = f^{-1}$ be the pullback map on bundles. In this way, for each integer k, $\text{Vect}_k(\ )$ becomes a functor from the category of manifolds and smooth maps to the category of pointed sets and base point preserving maps.

REMARK 6.7 Let $\{U_\alpha\}$ be a trivializing open cover for E and $g_{\alpha\beta}$ the transition functions. Then $\{f^{-1}U_\alpha\}$ is a trivializing open cover for $f^{-1}E$ over N and $(f^{-1}E)|_{f^{-1}U_\alpha} \simeq f^{-1}(E|_{U_\alpha})$. Therefore the transition functions for $f^{-1}E$ are the pullback functions $f^*g_{\alpha\beta}$.

A basic property of the pullback is the following.

Theorem 6.8 (Homotopy Property of Vector Bundles). *Assume Y to be a compact manifold. If f_0 and f_1 are homotopic maps from Y to a manifold X and E is a vector bundle on X, then $f_0^{-1}E$ is isomorphic to $f_1^{-1}E$, i.e., homotopic maps induce isomorphic bundles.*

PROOF. The problem of constructing an isomorphism between two vector bundles V and W of rank k over a space B may be turned into a problem in cross-sectioning a fiber bundle over B, as follows. Recall that $\text{Hom}(V, W) = V^* \otimes W$ is a vector bundle over B whose fiber at each point p consists of all the linear maps from V_p to W_p. Define $\text{Iso}(V, W)$ to be the subset of Hom(V, W) whose fiber at each point consists of all the *isomorphisms* from V_p to W_p. (This is like looking at the complement of the zero section of a line bundle.) Iso(V, W) inherits a topology from Hom(V, W), and is a fiber bundle with fiber $GL(n, \mathbb{R})$. An isomorphism between V and W is simply a section of Iso(V, W).

Let $f : Y \times I \to X$ be a homotopy between f_0 and f_1, and let $\pi : Y \times I \to Y$ be the projection. Suppose for some t_0 in I, $f_{t_0}^{-1}E$ is isomorphic to some vector bundle F on Y. We will show that for all t near t_0, $f_t^{-1}E \simeq F$. By the compactness and connectedness of the unit interval I it will then follow that $f_t^{-1}E \simeq F$ for all t in I.

Over $Y \times I$ there are two pullback bundles, $f^{-1}E$ and $\pi^{-1}F$. Since $f_{t_0}^{-1}E \simeq F$, Iso($f^{-1}E$, $\pi^{-1}F$) has a section over $Y \times t_0$, which a priori is also a section of Hom($f^{-1}E$, $\pi^{-1}F$). Since Y is compact, $Y \times t_0$ may be covered with a finite number of trivializing open sets for Hom($f^{-1}E$, $\pi^{-1}F$) (see Figure 6.1). As the fibers of Hom($f^{-1}E$, $\pi^{-1}F$) are Euclidean spaces, the section over $Y \times t_0$ may be extended to a section of Hom($f^{-1}E$, $\pi^{-1}F$) over the union of these open sets. Now any linear map near an isomorphism remains an isomorphism; thus we can extend the given section of Iso($f^{-1}E$, $\pi^{-1}F$) to a strip containing $Y \times t_0$. This proves that $f_t^{-1}E \simeq F$ for t near t_0. We now cover $Y \times I$ with a finite number of such strips. Hence $f_0^{-1}E \simeq F \simeq f_1^{-1}E$. □

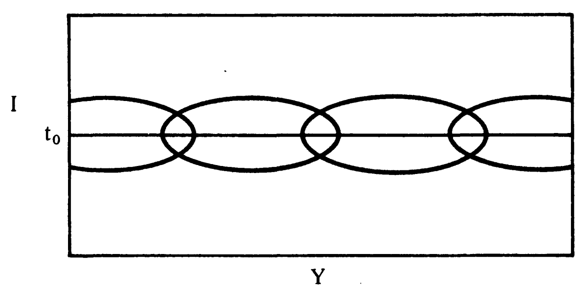


Figure 6.1

REMARK. If Y is not compact, we may not be able to find a strip of constant width over which Iso($f^{-1}E$, $\pi^{-1}F$) has a section; for example the strip may look like Figure 6.2.

But the same argument can be refined to give the theorem for Y a *paracompact space*. See, for instance, Husemoller [1, Theorem 4.7, p. 29]. Recall that Y is said to be *paracompact* if every open cover $\mathfrak{U}$ of Y has a *locally finite* open refinement $\mathfrak{U}'$, that is, every point in Y has a neighborhood which meets only finitely many open sets in $\mathfrak{U}'$. A compact space or a discrete space are clearly paracompact. By a theorem of A. H. Stone, so is every metric space (Dugundji [1, p. 186]). More importantly for us, every manifold is paracompact (Spivak [1, Ch. 2, Th. 13, p. 66]). Thus the homotopy

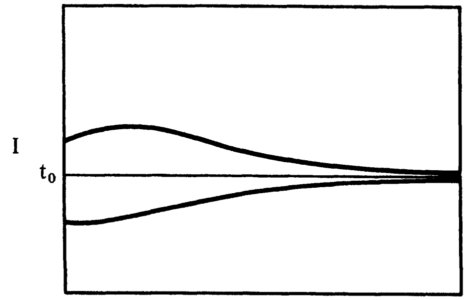


Figure 6.2

property of vector bundles (Theorem 6.8) actually holds over any manifold Y, compact or not.

Corollary 6.9. *A vector bundle over a contractible manifold is trivial.*

PROOF. Let E be a vector bundle over M and let f and g be maps

$$M \underset{g}{\overset{f}{\rightleftarrows}} \text{point}$$

such that $g \circ f$ is homotopic to the identity 1_M. By the homotopy property of vector bundles

$$E \simeq (g \circ f)^{-1}E \simeq f^{-1}(g^{-1}E).$$

Since $g^{-1}E$ is a vector bundle on a point, it is trivial, hence so is $f^{-1}(g^{-1}E)$. □

So for a contractible manifold M, $\mathrm{Vect}_k(M)$ is a single point.

REMARK. Although all the results in this subsection are stated in the differentiable category of manifolds and smooth maps, the corresponding statements with "manifold" replaced by "space" also hold in the continuous category of topological spaces and continuous maps, the only exception being Corollary 6.9, in which the space should be assumed paracompact.

Exercise 6.10. Compute $\mathrm{Vect}_k(S^1)$.

Compact Cohomology of a Vector Bundle

The Poincaré lemmas

$$H^*(M \times \mathbb{R}^n) = H^*(M)$$

$$H_c^*(M \times \mathbb{R}^n) = H_c^{*-n}(M)$$

may be viewed as results on the cohomology of the trivial bundle $M \times \mathbb{R}^n$ over M. More generally let E be a vector bundle of rank n over M. The zero section of E, $s : x \mapsto (x, 0)$, embeds M diffeomorphically in E. Since $M \times \{0\}$ is a deformation retract of E, it follows from the homotopy axiom for de Rham cohomology (Corollary 4.1.2.2) that

$$H^*(E) \simeq H^*(M).$$

For cohomology with compact support one may suspect that

$$H_c^*(E) \simeq H_c^{*-n}(M). \tag{6.11}$$

This is in general not true; the open Möbius strip, considered as a vector bundle over S^1, provides a counterexample, since the compact cohomology of the Möbius strip is identically zero (Exercise 4.8). However, if E and M are orientable manifolds of finite type, then formula (6.11) holds. The proof is based on Poincaré duality, as follows. Let m be the dimension of M. Then

$$\begin{aligned} H_c^*(E) &\simeq (H^{m+n-*}(E))^* && \text{by Poincaré duality on } E\\ &\simeq (H^{m+n-*}(M))^* && \text{by the homotopy axiom for de Rham cohomology}\\ &\simeq H_c^{*-n}(M) && \text{by Poincaré duality on } M. \end{aligned}$$

Lemma 6.12. *An orientable vector bundle E over an orientable manifold M is an orientable manifold.*

Proof. This follows from the fact that if $\{(U_\alpha, \psi_\alpha)\}$ is an oriented atlas for M with transition functions $h_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1}$ and

$$\phi_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$$

is a local trivialization for E with transition functions $g_{\alpha\beta}$, then the composition

$$E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^m \times \mathbb{R}^n$$

gives an atlas for E. The typical transition function of this atlas,

$$(\psi_\alpha \times 1) \circ \phi_\alpha \phi_\beta^{-1} \circ (\psi_\beta^{-1} \times 1) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$$

sends (x, y) to $(h_{\alpha\beta}(x), g_{\alpha\beta}(\psi_\alpha^{-1}(x))y)$ and has Jacobian matrix

$$\begin{pmatrix} D(h_{\alpha\beta}) & * \\ 0 & g_{\alpha\beta}(\psi_\alpha^{-1}(x)) \end{pmatrix}, \tag{6.12.1}$$

where $D(h_{\alpha\beta})$ is the Jacobian matrix of $h_{\alpha\beta}$. The determinant of the matrix (6.12.1) is clearly positive. □

Thus,

Proposition 6.13. *If $\pi : E \to M$ is an orientable vector bundle and M is orientable of finite type, then $H_c^*(E) \simeq H_c^{*-n}(M)$.*

REMARK 6.13.1. Actually the orientability assumption on M is superfluous. See Exercise 6.20.

REMARK 6.13.2. Let M be an oriented manifold with oriented atlas $\{(U_\alpha, \psi_\alpha)\}$ and $\pi : E \to M$ an oriented vector bundle over M with an oriented trivialization $\{(U_\alpha, \phi_\alpha)\}$ determining the orientation on the vector bundle (terminology on pp. 54–55). Then E can be made into an oriented manifold with orientation given by the oriented atlas

$$\{\pi^{-1}(U_\alpha), (\psi_\alpha \times 1) \circ \phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n\}.$$

This is called the *local product orientation* on E.

Compact Vertical Cohomology and Integration along the Fiber

As mentioned earlier, for vector bundles there is a third kind of cohomology. Instead of $\Omega_c^*(E)$, the complex of forms with compact support, we consider $\Omega_{cv}^*(E)$, the complex of forms with compact support *in the vertical direction*, defined as follows: a smooth n-form ω on E is in $\Omega_{cv}^n(E)$ if and only if for every compact set K in M, $\pi^{-1}(K) \cap \operatorname{Supp}\omega$ is compact. If $\omega \in \Omega_{cv}^n(E)$, then since $\operatorname{Supp}(\omega|_{\pi^{-1}(x)}) \subset \pi^{-1}(x) \cap \operatorname{Supp}\omega$ is a closed subset of a compact set, $\operatorname{Supp}(\omega|_{\pi^{-1}(x)})$ is compact. Thus, although a form in $\Omega_{cv}^*(E)$ need not have compact support in E, its restriction to each fiber has compact support. The cohomology of this complex, denoted $H_{cv}^*(E)$, is called *the cohomology of E with compact support in the vertical direction*, or *compact vertical cohomology*.

Let E be oriented as a rank n vector bundle. The formulas in (4.4) extend to this situation to give integration along the fiber, $\pi_* : \Omega_{cv}^*(E) \to \Omega^{*-n}(M)$, as follows. First consider the case of a trivial bundle $E = M \times \mathbb{R}^n$. Let $t_1, \dots, t_n$ be the coordinates on the fiber $\mathbb{R}^n$. A form on E is a real linear combination of two types of forms: the type (I) forms are those which do not contain as a factor the n-form $dt_1 \dots dt_n$ and the type (II) forms are those which do. The map π_* is defined by

$$\text{(I)}\ (\pi^*\phi)f(x, t_1, \dots, t_n)\, dt_{i_1} \dots dt_{i_r} \mapsto 0 \quad , \qquad r < n$$
$$\text{(II)}\ (\pi^*\phi)f(x, t_1, \dots, t_n)\, dt_1 \dots dt_n \mapsto \phi \int_{\mathbb{R}^n} f(x, t_1, \dots, t_n)\, dt_1 \dots dt_n,$$

where f has compact support for each fixed x in M and ϕ is a form on M. Next suppose E is an arbitrary oriented vector bundle, with oriented trivialization $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$. Let $x_1, \dots, x_m$ and $y_1, \dots, y_m$ be the coordinate functions on U_α and U_β, and $t_1, \dots, t_n$, $u_1, \dots, u_n$ the fiber coordinates on $E|_{U_\alpha}$ and $E|_{U_\beta}$ given by ϕ_α and ϕ_β respectively. Because $\{(U_\alpha, \phi_\alpha)\}$ is an oriented trivialization for E, the two sets of fiber coordinates $t_1, \dots, t_n$ and $u_1 \dots, u_n$ are related by an element of $GL^+(n, \mathbb{R})$ at each point of $U_\alpha \cap U_\beta$. Again a form ω in $\Omega_{cv}^*(E)$ is locally of type (I) or (II). The map π_* is defined to be zero on type (I) forms. To define π_* on type (II) forms, write ω_α for $\omega|_{\pi^{-1}(U_\alpha)}$. Then

$$\omega_\alpha = (\pi^*\phi)f(x_1, \dots, x_m, t_1, \dots, t_n)\, dt_1 \dots dt_n$$

and

$$\omega_\beta = (\pi^*\tau)g(y_1, \dots, y_m, u_1, \dots, u_n)\, du_1 \dots du_n.$$

Define

$$\pi_* \omega_\alpha = \phi \int_{\mathbb{R}^n} f(x, t)\, dt_1 \ldots dt_n.$$

Exercise 6.14. Show that if E is an oriented vector bundle, then $\pi_* \omega_\alpha = \pi_* \omega_\beta$. Hence $\{\pi_* \omega_\alpha\}_{\alpha \in I}$ piece together to give a global form $\pi_* \omega$ on M. Furthermore, this definition is independent of the choice of the oriented trivialization for E.

Proposition 6.14.1. *Integration along the fiber π_* commutes with exterior differentiation d.*

PROOF. Let $\{(U_\alpha, \phi_\alpha)\}$ be a trivialization for E, $\{\rho_\alpha\}$ a partition of unity subordinate to $\{U_\alpha\}$, and ω a form in $\Omega^*_{cv}(E)$. Since $\omega = \sum \rho_\alpha \omega$, and both π_* and d are linear, it suffices to prove the proposition for $\rho_\alpha \omega$, that is, $\pi_* d(\rho_\alpha \omega) = d\pi_*(\rho_\alpha \omega)$. Thus from the outset we may assume E to be the product bundle $M \times \mathbb{R}^n$. If $\omega = (\pi^* \phi) f(x, t)\, dt_1 \ldots dt_n$ is a type (II) form,

$$\begin{aligned} d\pi_* \omega &= d\left(\phi \int f(x, t)\, dt_1 \ldots dt_n\right) \\ &= (d\phi) \int f(x, t)\, dt_1 \ldots dt_n + (-1)^{\deg \phi}\, \phi \sum_i dx_i \int \frac{\partial f}{\partial x_i}(x, t)\, dt_1 \ldots dt_n \end{aligned}$$

and

$$\begin{aligned} \pi_* d\omega &= \pi_*\left((\pi^* d\phi) f\, dt_1 \ldots dt_n + (-1)^{\deg \phi}\, \pi^* \phi \sum \frac{\partial f}{\partial x_i} dx_i\, dt_1 \ldots dt_n\right) \\ &= (d\phi) \int f\, dt_1 \ldots dt_n + (-1)^{\deg \phi} \sum_i \phi\, dx_i \int \frac{\partial f}{\partial x_i} dt_1 \ldots dt_n. \end{aligned}$$

So $d\pi_* \omega = \pi_* d\omega$ for a type (II) form. Next let $\omega = (\pi^* \phi) f(x, t)\, dt_{i_1} \ldots dt_{i_r}$, $r < n$, be a type (I) form. Then

$$d\pi_* \omega = 0$$

and

$$\begin{aligned} \pi_* d\omega &= (-1)^{\deg \phi} \sum_i \pi_*\left((\pi^* \phi) \frac{\partial f}{\partial t_i}(x, t)\, dt_i\, dt_{i_1} \ldots dt_{i_r}\right) \\ &= 0 \quad \text{if} \quad dt_i\, dt_{i_1} \ldots dt_{i_r} \neq \pm dt_1 \ldots dt_n. \end{aligned}$$

If $dt_i\, dt_{i_1} \ldots dt_{i_r} = \pm dt_1 \ldots dt_n$, then $\int \partial f / \partial t_i(x, t)\, dt_i\, dt_{i_1} \ldots dt_{i_r}$ is again 0: because f has compact support,

$$\int_{-\infty}^{\infty} \frac{\partial f}{\partial t_i}(x, t)\, dt_i = f(\ldots, \infty, \ldots) - f(\ldots, -\infty, \ldots) = 0. \qquad \square$$

Note that integration along the fiber, $\pi_* : \Omega^*_{cv}(E) \to \Omega^{*-n}(M)$ lowers the degree of a form by the fiber dimension.

Proposition 6.15 (Projection Formula). (a) *Let $\pi : E \to M$ be an oriented rank n vector bundle, τ a form on M and ω a form on E with compact support along the fiber. Then*

$$\pi_*((\pi^*\tau) \cdot \omega) = \tau \cdot \pi_* \omega.$$

(b) *Suppose in addition that M is oriented of dimension m, $\omega \in \Omega^q_{cv}(E)$,* and *$\tau \in \Omega_c^{m+n-q}(M)$. Then with the local product orientation on E*

$$\int_E (\pi^*\tau) \wedge \omega = \int_M \tau \wedge \pi_* \omega.$$

Proof. (a) Since two forms are the same if and only if they are the same locally, we may assume that E is the product bundle $M \times \mathbb{R}^n$. If ω is a form of type (I), say $\omega = \pi^*\phi \cdot f(x, t)\, dt_{i_1} \dots dt_{i_r}$, where $r < n$, then

$$\pi_*((\pi^*\tau) \cdot \omega) = \pi_*(\pi^*(\tau\ \phi) \cdot f(x, t)\, dt_{i_1} \dots dt_{i_r})) = 0 = \tau \cdot \pi_*\omega.$$

If ω is a form of type (II), say $\omega = \pi^*\phi \cdot f(x, t)\, dt_1 \dots dt_n$, then

$$\pi_*((\pi^*\tau) \cdot \omega) = \tau\ \phi \int_{\mathbb{R}^n} f(x, t)\, dt_1 \dots dt_n = \tau \cdot \pi_* \omega.$$

(b) Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in J}$ be an oriented trivialization for E and $\{\rho_\alpha\}_{\alpha \in J}$ a partition of unity subordinate to $\{U_\alpha\}$. Writing $\omega = \sum \rho_\alpha \omega$, where $\rho_\alpha \omega$ has support in U_α, we have

$$\int_E (\pi^*\tau) \wedge \omega = \sum_\alpha \int_{E|_{U_\alpha}} (\pi^*\tau) \wedge (\rho_\alpha \omega)$$

and

$$\int_M \tau \wedge \pi_* \omega = \sum_\alpha \int_{U_\alpha} \tau \wedge \pi_*(\rho_\alpha \omega).$$

Here $\tau \wedge \pi_*(\rho_\alpha \omega)$ has compact support because its support is a closed subset of the compact set Supp τ; similarly, $(\pi^*\tau) \wedge (\rho_\alpha \omega)$ also has compact support. Therefore, it is enough to prove the proposition for $M = U_\alpha$ and E trivial. The rest of the proof proceeds as in (a). □

The proof of the Poincaré lemma for compact supports (4.7) carries over verbatim to give

Proposition 6.16 (Poincaré Lemma for Compact Vertical Supports). *Integration along the fiber defines an isomorphism*

$$\pi_* : H^*_{cv}(M \times \mathbb{R}^n) \to H^{*-n}(M).$$

This is a special case of

Theorem 6.17 (Thom Isomorphism). *If the vector bundle $\pi : E \to M$ over a manifold M of finite type is orientable, then*

$$H^*_{cv}(E) \simeq H^{*-n}(M)$$

where n is the rank of E.

PROOF. Let U and V be open subsets of M. Using a partition of unity from the base M we see that

$$0 \longrightarrow \Omega_{cv}^*(E|_{U \cup V}) \longrightarrow \Omega_{cv}^*(E|_U) \oplus \Omega_{cv}^*(E|_V) \longrightarrow \Omega_{cv}^*(E|_{U \cap V}) \longrightarrow 0$$

is exact, as in (2.3). So we have the diagram of Mayer-Vietoris sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H_{cv}^*(E|_{U\cup V}) \arrow[r] \arrow[d, "\pi_*"] & H_{cv}^*(E|_U) \oplus H_{cv}^*(E|_V) \arrow[r] \arrow[d, "\pi_*"] & H_{cv}^*(E|_{U\cap V}) \arrow[r, "d^*"] \arrow[d, "\pi_*"] & H_{cv}^{*+1}(E|_{U\cup V}) \arrow[r] \arrow[d, "\pi_*"] & \cdots \\
\cdots \arrow[r] & H^{*-n}(U \cup V) \arrow[r] & H^{*-n}(U) \oplus H^{*-n}(V) \arrow[r] & H^{*-n}(U \cap V) \arrow[r, "d^*"'] & H^{*+1-n}(U \cup V) \arrow[r] & \cdots
\end{tikzcd}$$

The commutativity of this diagram is trivial for the first two squares; we will check that of the third. Recalling from (2.5) the explicit formula for the coboundary operator d^*, we have by the projection formula (6.15)

$$\pi_* \, d^*\omega = \pi_*((\pi^* \, d\rho_U) \cdot \omega) = (d\rho_U) \cdot \pi_* \omega = d^*\pi_* \omega.$$

So the diagram in question is commutative.

By (6.9) if U is diffeomorphic to $\mathbb{R}^n$, then $E|_U$ is trivial, so that in this case the Thom isomorphism reduces to the Poincaré lemma for compact vertical supports (6.16). Hence in the diagram above, π_* is an isomorphism for contractible open sets. By the Five Lemma if the Thom isomorphism holds for U, V, and $U \cap V$, then it holds for $U \cup V$. The proof now proceeds by induction on the cardinality of a good cover for the base, as in the proof of Poincaré duality. This gives the Thom isomorphism for any manifold M having a finite good cover. $\square$

REMARK 6.17.1. Although the proof above works only for a manifold of finite type, the theorem is actually true for any base space. We will reprove the theorem for an arbitrary manifold in (12.2.2).

Under the Thom isomorphism $\mathscr{T} : H^*(M) \xrightarrow{\sim} H_{cv}^{*+n}(E)$, the image of 1 in $H^0(M)$ determines a cohomology class Φ in $H_{cv}^n(E)$, called the *Thom class* of the oriented vector bundle E. Because $\pi_* \Phi = 1$, by the projection formula (6.15)

$$\pi_*(\pi^*\omega \wedge \Phi) = \omega \wedge \pi_* \Phi = \omega.$$

So the Thom isomorphism, which is inverse to π_*, is given by

$$\mathscr{T}(\ \ ) = \pi^*(\ \ ) \wedge \Phi.$$

Proposition 6.18. *The Thom class Φ on a rank n oriented vector bundle E can be uniquely characterized as the cohomology class in $H_{cv}^n(E)$ which restricts to the generator of $H_c^n(F)$ on each fiber F.*

PROOF. Since $\pi_* \Phi = 1$, $\Phi|_{\text{fiber}}$ is a bump form on the fiber with total integral 1. Conversely if Φ' in $H_{cv}^n(E)$ restricts to a generator on each fiber, then

$$\pi_*((\pi^*\omega) \wedge \Phi') = \omega \wedge \pi_*\Phi' = \omega.$$

Hence $\pi^*(\quad)\wedge\Phi'$ must be the Thom isomorphism $\mathscr{T}$ and $\Phi' = \mathscr{T}(1)$ is the Thom class. □

Proposition 6.19. *If E and F are two oriented vector bundles over a manifold M, and π_1 and π_2 are the projections*

$$\begin{tikzcd} & E \oplus F \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ E & & F \end{tikzcd},$$

then the Thom class of $E \oplus F$ is $\Phi(E \oplus F) = \pi_1^*\Phi(E) \wedge \pi_2^*\Phi(F)$.

PROOF. Let $m = \operatorname{rank} E$ and $n = \operatorname{rank} F$. Then $\pi_1^*\Phi(E)\wedge\pi_2^*\Phi(F)$ is a class in $H_{cv}^{m+n}(E \oplus F)$ whose restriction to each fiber is a generator of the compact cohomology of the fiber, since the isomorphism

$$H_c^{m+n}(\mathbb{R}^m \times \mathbb{R}^n) \simeq H_c^m(\mathbb{R}^m) \otimes H_c^n(\mathbb{R}^n)$$

is given by the wedge product of the generators. □

Exercise 6.20. Using a Mayer-Vietoris argument as in the proof of the Thom isomorphism (Theorem 6.17), show that if $\pi : E \to M$ is an orientable rank n bundle over a manifold M of finite type, then

$$H_c^*(E) \simeq H_c^{*-n}(M).$$

Note that this is Proposition 6.13 with the orientability assumption on M removed.

Poincaré Duality and the Thom Class

Let S be a closed oriented submanifold of dimension k in an oriented manifold M of dimension n. Recall from (5.13) that the Poincaré dual of S is the cohomology class of the closed $(n-k)$-form η_S characterized by the property

$$\int_S \omega = \int_M \omega \wedge \eta_S \tag{6.21}$$

for any closed k-form with compact support on M. In this section we will explain how the Poincaré dual of a submanifold relates to the Thom class of a bundle (Proposition 6.24). To this end we first introduce the notion of a *tubular neighborhood* of S in M; this is by definition an open neighborhood of S in M diffeomorphic to a vector bundle of rank n-k over S such that S is diffeomorphic to the zero section. Now a sequence of vector bundles over M,

$$0 \to E \to E' \to E'' \to 0,$$

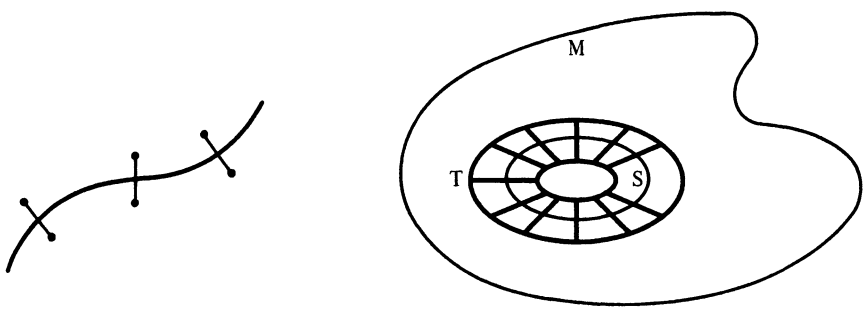
is said to be *exact* if at each point p in M, the sequence of vector spaces

$$0 \to E_p \to E'_p \to E''_p \to 0$$

is exact, where E_p is the fiber of E at p. If S is a submanifold in M, the *normal bundle* $N = N_{S/M}$ of S in M is the vector bundle on S defined by the exact sequence

$$(6.22) \qquad 0 \longrightarrow T_S \longrightarrow T_M|_S \longrightarrow N \longrightarrow 0,$$

where $T_M|_S$ is the restriction of the tangent bundle of M to S. The tubular neighborhood theorem states that every submanifold S in M has a tubular neighborhood T, and that in fact T is diffeomorphic to the normal bundle of S in M (see Spivak [1, p. 465] or Guillemin and Pollack [1, p. 76]). For example, if S is a curve in $\mathbb{R}^3$, then a tubular neighborhood of S may be constructed using the metric in $\mathbb{R}^3$ by attaching to each point of S an open disc of sufficiently small radius $\varepsilon > 0$ perpendicular to S at the center. The union of all these discs is a tubular neighborhood of S (Figure 6.3(a)).



(a) (b)

Figure 6.3

In general if A and B are two oriented vector bundles with oriented trivializations $\{(U_\alpha, \phi_\alpha)\}$ and $\{(U_\alpha, \psi_\alpha)\}$, respectively, then the *direct sum orientation* on $A \oplus B$ is given by the oriented trivialization $\{(U_\alpha, \phi_\alpha \oplus \psi_\alpha)\}$. Returning to our submanifold S in M, we let $j : T \hookrightarrow M$ be the inclusion of a tubular neighborhood T of S in M (see Figure 6.3(b)). Since S and M are orientable, the normal bundle N_S, being the quotient of $T_M|_S$ by T_S, is also orientable. By convention it is oriented in such a way that

$$N_S \oplus T_S = T_M|_S$$

has the direct sum orientation. So the Thom isomorphism theorem applies to the normal bundle $T = N_S$ over S and we have the sequence of maps

$$H^*(S) \overset{\wedge\Phi}{\cong} H_{cv}^{*+n-k}(T) \xrightarrow{j_*} H^{*+n-k}(M)$$

where Φ is the Thom class of the tube T and j_* is extension by 0; here j_* is defined because we are only concerned with forms on the tubular neighborhood T which vanish near the boundary of T. We claim that *the Poincaré* *dual of S is the Thom class of the normal bundle of S*; more precisely

$$\eta_S = j_*(\Phi \wedge 1) = j_* \Phi \quad \text{in} \quad H^{n-k}(M). \tag{6.23}$$

To prove this we merely have to show that $j_* \Phi$ satisfies the defining property (5.13) of the Poincaré dual η_S. Let ω be any closed k-form with compact support on M, and $i: S \to T$ the inclusion, regarded as the zero section of the bundle $\pi: T \to S$. Since π is a deformation retraction of T onto S, π^* and i^* are inverse isomorphisms in cohomology. Therefore on the level of forms, ω and $\pi^* i^* \omega$ differ by an exact form: $\omega = \pi^* i^* \omega + d\tau$.

$$
\begin{aligned}
\int_M \omega \wedge j_* \Phi \\
&= \int_T \omega \wedge \Phi && \text{because } j_* \Phi \text{ has support in } T \\
&= \int_T (\pi^* i^* \omega + d\tau) \wedge \Phi \\
&= \int_T (\pi^* i^* \omega) \wedge \Phi && \text{since } \int_T (d\tau) \wedge \Phi = \int_T d(\tau \wedge \Phi) = 0 \text{ by Stokes' theorem} \\
&= \int_S i^* \omega \wedge \pi_* \Phi && \text{by the projection formula (6.15)} \\
&= \int_S i^* \omega && \text{because } \pi_* \Phi = 1.
\end{aligned}
$$

This concludes the proof of the claim. Note that if S is compact, then its Poincaré dual $\eta_S = j_* \Phi$ has compact support.

Conversely, suppose E is an oriented vector bundle over an oriented manifold M. Then M is diffeomorphically embedded as the zero section in E and there is an exact sequence

$$0 \to T_M \to (T_E)|_M \to E \to 0,$$

i.e., *the normal bundle of M in E is E itself.* By (6.23), the Poincaré dual of M in E is the Thom class of E. In summary,

Proposition 6.24. (a) *The Poincaré dual of a closed oriented submanifold S in an oriented manifold M and the Thom class of the normal bundle of S can be represented by the same forms.*

(b) *The Thom class of an oriented vector bundle $\pi: E \to M$ over an oriented manifold M and the Poincaré dual of the zero section of E can be represented by the same form.*

Because the normal bundle of the submanifold S in M is diffeomorphic to any tubular neighborhood of S, we have the following proposition.

Proposition 6.25 (Localization Principle). *The support of the Poincaré dual of a submanifold S can be shrunk into any given tubular neighborhood of S.*

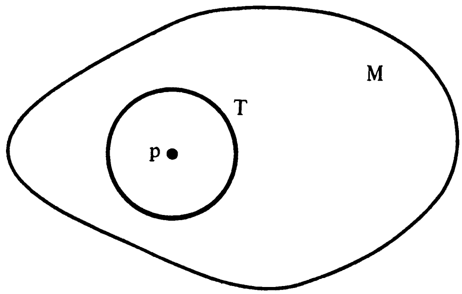


Figure 6.4

EXAMPLE 6.26.

(a) *The Poincaré dual of a point p in M.*
A tubular neighborhood T of p is simply an open ball around p (Figure 6.4). A generator of $H^n_{cv}(T)$ is a bump n-form with total integral 1. So the Poincaré dual of a point is a bump n-form on M. The form need not have support at p since all bump n-forms on a connected manifold are cohomologous. Here the dual of p is taken in $H^n_c(M)$, and not in $H^n(M)$.

(b) *The Poincaré dual of M.*
Here the tubular neighborhood T is M itself, and $H^*_{cv}(T) = H^*(M)$. So the Poincaré dual of M is the constant function 1.

(c) *The Poincaré dual of a circle on a torus.*

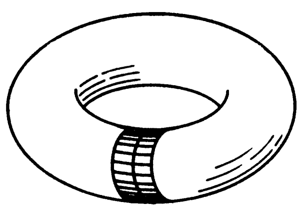


Figure 6.5

The Poincaré dual is a bump 1-form with support in a tubular neighborhood of the circle and with total integral 1 on each fiber of the tubular neighborhood (Figure 6.5). In the usual representation of the torus as a square, if the circle is a vertical segment, then its Poincaré dual is $\rho(x)\,dx$ where ρ is a bump function with total integral 1 (Figure 6.6).

Using the explicit construction of the Poincaré dual $\eta_S = j_*\Phi$ as the Thom class of the normal bundle, we now prove two basic properties of Poincaré duality. Two submanifolds R and S in M are said to *intersect transversally* if and only if

$$T_x R + T_x S = T_x M \tag{6.27}$$

at all points x in the intersection $R \cap S$ (Guillemin and Pollack [1, pp.

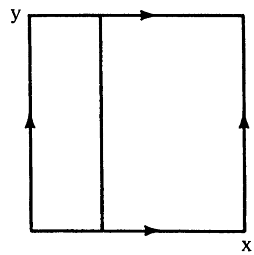


Figure 6.6

27–32]). For such a transversal intersection the codimension in M is additive:

$$\text{codim } R \cap S = \text{codim } R + \text{codim } S. \tag{6.28}$$

This implies that the normal bundle of $R \cap S$ in M is

$$N_{R \cap S} = N_R \oplus N_S. \tag{6.29}$$

Assume M to be an oriented manifold, and R and S to be closed oriented submanifolds. Denoting the Thom class of an oriented vector bundle E by $\Phi(E)$, we have by (6.19)

$$\Phi(N_{R \cap S}) = \Phi(N_R \oplus N_S) = \Phi(N_R) \wedge \Phi(N_S). \tag{6.30}$$

Therefore,

$$\eta_{R \cap S} = \eta_R \wedge \eta_S\,; \tag{6.31}$$

i.e., *under Poincaré duality the transversal intersection of closed oriented submanifolds corresponds to the wedge product of forms.*

More generally, a smooth map $f: M' \to M$ is said to be *transversal* to a submanifold $S \subset M$ if for every $x \in f^{-1}(S)$, $f_*(T_x M') + T_{f(x)}S = T_{f(x)}M$. If $f: M' \to M$ is an orientation-preserving map of oriented manifolds, T is a sufficiently small tubular neighborhood of the closed oriented submanifold S in M, and f is transversal to S and T, then $f^{-1}T$ is a tubular neighborhood of $f^{-1}S$ in M'. From the commutative diagram

$$\begin{array}{ccccc}
H^*(S) & \xrightarrow{\Phi(T)} & H^{*+k}_{cv}(T) & \xrightarrow{j_*} & H^*(M) \\
{\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow \\
H^*(f^{-1}S) & \xrightarrow{\Phi(f^{-1}T)} & H^{*+k}_{cv}(f^{-1}T) & \xrightarrow{j_*} & H^*(M'),
\end{array}$$

we see that if ω is the cohomology class on M representing the submanifold S in M, then $f^*\omega$ is the cohomology class on M' representing $f^{-1}(S)$, i.e., *under Poincaré duality the induced map on cohomology corresponds to the pre-image in geometry, i.e.,* $\eta_{f^{-1}(S)} = f^*\eta_S$. By the Transversality Homotopy Theorem, the transversality hypothesis on f is in fact not necessary (Guillemin and Pollack [1, p. 70]).

The Global Angular Form, the Euler Class, and the Thom Class

In this subsection we will construct explicitly the Thom class of an oriented rank 2 vector bundle $\pi : E \to M$, using such data as a partition of unity on M and the transition functions of E. The higher-rank case is similar but more involved, and will be taken up in (11.11) and (12.3). The construction is best understood as the vector-bundle analogue of the procedure for going from a generator of $H^{n-1}(S^{n-1}) = H^{n-1}(\mathbb{R}^n - \{0\})$ to a generator of $H_c^n(\mathbb{R}^n)$. So let us first try to understand the situation in $\mathbb{R}^n$.

We will call a top form on an oriented manifold M *positive* if it is in the orientation class of M. The standard orientation on the unit sphere S^{n-1} in $\mathbb{R}^n$ is by convention the following one: if σ is a generator of $H^{n-1}(S^{n-1})$ and $\pi : \mathbb{R}^n - \{0\} \to S^{n-1}$ is a deformation retraction, then σ is positive on S^{n-1} if and only if $dr \cdot \pi^*\sigma$ is positive on $\mathbb{R}^n - \{0\}$.

Exercise 6.32. (a) Show that if θ is the standard angle function on $\mathbb{R}^2$, measured in the counterclockwise direction, then $d\theta$ is positive on the circle S^1.

(b) Show that if ϕ and θ are the spherical coordinates on $\mathbb{R}^3$ as in Figure 6.7, then $d\phi \wedge d\theta$ is positive on the 2-sphere S^2.

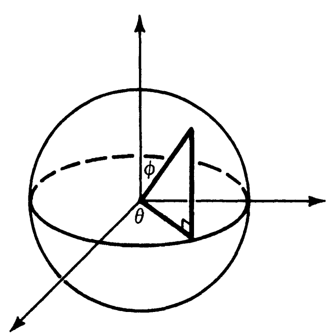


Figure 6.7

Let σ be the positive generator of $H^{n-1}(S^{n-1})$ and $\psi = \pi^*\sigma$ the corresponding generator of $H^{n-1}(\mathbb{R}^n - \{0\})$; ψ is called the *angular form* on $\mathbb{R}^n - \{0\}$. If $\rho(r)$ is the function of the radius shown in Figure 6.8, then $d\rho = \rho'(r)dr$ is a bump form on $\mathbb{R}^1$ with total integral 1 (Figure 6.9). Therefore $(d\rho) \cdot \psi$ is a compactly supported form on $\mathbb{R}^n$ with total integral 1, i.e., $(d\rho) \cdot \psi$ is the generator of $H_c^n(\mathbb{R}^n)$. Note that because ψ is closed, we can write

$$(d\rho) \cdot \psi = d(\rho \cdot \psi). \tag{6.33}$$

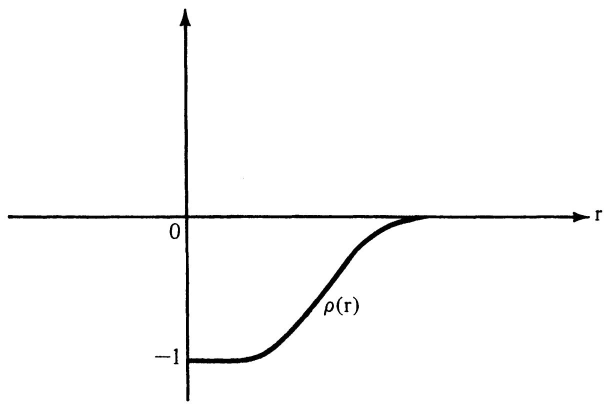


Figure 6.8

Now let E be an oriented rank n vector bundle over M, and E^0 the complement of the zero section in E. Endow E with a Riemannian structure as in (6.4) so that the radius function r makes sense on E. We begin our construction of the Thom class by finding a global form ψ on E^0 whose restriction to each fiber is the angular form on $\mathbb{R}^n - \{0\}$. ψ is called the *global angular form*. Once we have the angular form ψ, it is then easy to check that $\Phi = d(\rho \cdot \psi)$ is the Thom class.

Now suppose the rank of E is 2, and $\{U_\alpha\}$ is a coordinate open cover of M that trivializes E. Since E has a Riemannian structure, over each U_α we can choose an orthonormal frame. This defines on $E^0|_{U_\alpha}$ polar coordinates r_α and θ_α; if $x_1, \ldots, x_n$ are coordinates on U_α, then $\pi^*x_1, \ldots, \pi^*x_n, r_\alpha, \theta_\alpha$ are coordinates on $E^0|U_\alpha$. On the overlap $U_\alpha \cap U_\beta$, the radii r_α and r_β are equal but the angular coordinates θ_α and θ_β differ by a rotation. By the orientability of E, it makes sense to speak of the "counterclockwise direction" in each fiber. This allows

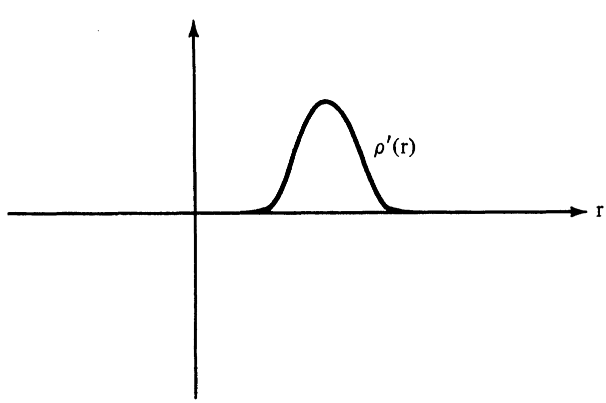


Figure 6.9

us to define unambiguously $\varphi_{\alpha\beta}$ (up to a constant multiple of 2π) as the angle of rotation in the counterclockwise direction from the α-coordinate system to the β-coordinate system:

(6.34) $$\theta_\beta = \theta_\alpha + \pi^*\varphi_{\alpha\beta}, \quad \varphi_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathbb{R}.$$

Although rotating from α to β and then from β to γ is the same as rotating from α to γ, it is not true that $\varphi_{\alpha\beta} + \varphi_{\beta\gamma} - \varphi_{\alpha\gamma} = 0$; indeed all that one can say is

$$\varphi_{\alpha\beta} + \varphi_{\beta\gamma} - \varphi_{\alpha\gamma} \in 2\pi\mathbb{Z}.$$

ASIDE. To each triple intersection we can associate an integer

(6.35) $$\varepsilon_{\alpha\beta\gamma} = \frac{1}{2\pi}(\varphi_{\alpha\beta} - \varphi_{\alpha\gamma} + \varphi_{\beta\gamma}).$$

The collection of integers $\{\varepsilon_{\alpha\beta\gamma}\}$ measures the extent to which $\{\varphi_{\alpha\beta}\}$ fails to be a cocyle. We will give another interpretation of $\{\varepsilon_{\alpha\beta\gamma}\}$ in Section 11.

Unlike the functions $\{\varphi_{\alpha\beta}\}$, the 1-forms $\{d\varphi_{\alpha\beta}\}$ satisfy the cocycle condition.

Exercise 6.36. There exist 1-forms ξ_α on U_α such that

$$\frac{1}{2\pi} d\varphi_{\alpha\beta} = \xi_\beta - \xi_\alpha .$$

[*Hint:* Take $\xi_\alpha = (1/2\pi)\sum_\gamma \rho_\gamma \, d\varphi_{\gamma\alpha}$, where $\{\rho_\gamma\}$ is a partition of unity subordinate to $\{U_\gamma\}$.]

It follows from Exercise 6.36 that $d\xi_\alpha = d\xi_\beta$ on $U_\alpha \cap U_\beta$. Hence the $d\xi_\alpha$ piece together to give a global 2-form e on M. This global form e is clearly closed. It is not necessarily exact since the ξ_α do not usually piece together to give a global 1-form. The cohomology class of e in $H^2(M)$ is called the *Euler class* of the oriented vector bundle E. We sometimes write $e(E)$ instead of e.

Claim. *The cohomology class of e is independent of the choice of ξ in our construction.*

PROOF OF CLAIM. If $\{\bar{\xi}_\alpha\}$ is a different choice of 1-forms such that

$$\frac{1}{2\pi} d\varphi_{\alpha\beta} = \bar{\xi}_\beta - \bar{\xi}_\alpha = \xi_\beta - \xi_\alpha,$$

then

$$\bar{\xi}_\beta - \xi_\beta = \bar{\xi}_\alpha - \xi_\alpha = \xi$$

is a global form. So $d\bar{\xi}_\alpha$ and $d\xi_\alpha$ differ by an exact global form. ☐

By (6.34) and (6.36), on $E^0|_{U_\alpha \cap U_\beta}$,

$$\frac{d\theta_\alpha}{2\pi} - \pi^* \xi_\alpha = \frac{d\theta_\beta}{2\pi} - \pi^* \xi_\beta . \tag{6.36.1}$$

These forms then piece together to give a global 1-form ψ on E^0, the *global angular form*, whose restriction to each fiber is the angular form $(1/2\pi)\, d\theta$, i.e., if $\iota_p : \mathbb{R}^2 \to E$ is the orthogonal inclusion of a fiber over p, then $\iota_p{}^*\psi = (1/2\pi)\, d\theta$. The global angular form is not closed:

$$d\psi = d\left(\frac{d\theta_\alpha}{2\pi} - \pi^*\xi_\alpha\right) = -\pi^* d\xi_\alpha = -\pi^* d\xi_\beta .$$

Therefore,

$$d\psi = -\pi^* e . \tag{6.37}$$

When E is a product, ψ could be taken to be the pullback of $(1/2\pi)\, d\theta$ under the projection $E^0 = M \times (\mathbb{R}^2 - 0) \to \mathbb{R}^2 - 0$. In this case ψ is closed and e is 0. The Euler class is in this sense a measure of the twisting of the oriented vector bundle E.

The Euler class of an oriented rank 2 vector bundle may be given in terms of the transition functions, as follows. Let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(2)$ be the transition functions of E. By identifying SO(2) with the unit circle in the complex plane via $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = e^{i\theta}$, $g_{\alpha\beta}$ may be thought of as complex-valued functions. In this context the angle from the β-coordinate system to the α-coordinate system is $(1/i)\log g_{\alpha\beta}$. Thus

$$\theta_\alpha - \theta_\beta = \pi^*(1/i)\log g_{\alpha\beta},$$

and

$$\pi^*\varphi_{\alpha\beta} = -\pi^*(1/i)\log g_{\alpha\beta}.$$

Since the projection π has maximal rank (i.e., π_* is onto), π^* is injective, so that

$$\varphi_{\alpha\beta} = -(1/i)\log g_{\alpha\beta}.$$

Let $\{\rho_\gamma\}$ be a partition of unity subordinate to $\{U_\gamma\}$. Then

$$\frac{1}{2\pi}\, d\varphi_{\alpha\beta} = \xi_\beta - \xi_\alpha,$$

where

$$\xi_\alpha = \frac{1}{2\pi} \sum_\gamma \rho_\gamma\, d\varphi_{\gamma\alpha} = -\frac{1}{2\pi i} \sum_\gamma \rho_\gamma\, d \log g_{\gamma\alpha}. \tag{6.37.1}$$

Therefore,

$$e(E) = -\frac{1}{2\pi i} \sum_\gamma d(\rho_\gamma\, d \log g_{\gamma\alpha}) \qquad \text{on } U_\alpha . \tag{6.38}$$

Proposition 6.39. *The Euler class is functorial, i.e., if $f: N \to M$ is a C^∞ map and E is a rank 2 oriented vector bundle over M, then*

$$e(f^{-1}E) = f^* e(E)\ .$$

PROOF. Since the transition functions of $f^{-1}E$ are $f^*g_{\alpha\beta}$, the proposition is an immediate consequence of (6.38). □

We claim that just as in the untwisted case (6.33), the Thom class is the cohomology class of

$$\Phi = d(\rho(r) \cdot \psi) = d\rho(r) \cdot \psi - \rho(r)\pi^* e\ . \tag{6.40}$$

In this formula although $\rho(r) \cdot \psi$ is defined only outside the zero section of E, the form Φ is a global form on E since $d\rho \equiv 0$ near the zero section. Φ has the following properties:

(*a*) compact support in the vertical direction;
(*b*) closed: $d\Phi = -d\rho(r) \cdot d\psi - d\rho(r)\pi^* e = 0$;
(*c*) restriction to each fiber has total integral 1:

$$\pi_* \iota_p^* \Phi = \int_0^\infty \int_0^{2\pi} d\rho(r) \cdot \frac{d\theta}{2\pi} = \rho(\infty) - \rho(0) = 1\ ,$$

where $\iota_p: E_p \to E$ is the inclusion of the fiber E_p into E;
(*d*) the cohomology class of Φ is independent of the choice of $\rho(r)$. Suppose $\bar\rho(r)$ is another function of r which is -1 near 0 and 0 near infinity, and which defines $\bar\Phi$. Then

$$\Phi - \bar\Phi = d((\rho(r) - \bar\rho(r)) \cdot \psi)$$

where $(\rho(r) - \bar\rho(r)) \cdot \psi$ is a global form on E because $\rho(r) - \bar\rho(r)$ vanishes near the zero section.

Therefore Φ indeed defines the Thom class. Furthermore, if $s: M \to E$ is the zero section of E, then

$$s^*\Phi = d(\rho(0)) \cdot s^*\psi - \rho(0)s^*\pi^* e = e\ .$$

This proves

Proposition 6.41. *The pullback of the Thom class to M by the zero section is the Euler class.*

Let $\{U_\alpha\}$ be a trivializing cover for E, $\{\rho_\alpha\}$ a partition of unity subordinate to $\{U_\alpha\}$, and $g_{\alpha\beta}$ the transition functions for E. Since

$$\psi = \frac{d\theta_\alpha}{2\pi} - \pi^*\xi_\alpha$$
$$= \frac{d\theta_\alpha}{2\pi} + \frac{1}{2\pi i}\pi^* \sum_\gamma \rho_\gamma \, d \log g_{\gamma\alpha}.$$

(cf. (6.36.1) and (6.37.1)), we have by (6.40),

$$\Phi = d\left(\rho(r)\frac{d\theta_\alpha}{2\pi}\right) + \frac{1}{2\pi i} d\left(\rho(r)\pi^* \sum_\gamma \rho_\gamma \, d \log g_{\gamma\alpha}\right). \tag{6.42}$$

This is the explicit formula for the Thom class.

Exercise 6.43. Let $\pi : E \to M$ be an oriented rank 2 bundle. As we saw in the proof of the Thom isomorphism, wedging with the Thom class is an isomorphism $\wedge\Phi : H^*(M) \simeq H_{cv}^{*+2}(E)$. Therefore every cohomology class on E is the wedge product of Φ with the pullback of a cohomology class on M. Find the class u on M such that

$$\Phi^2 = \Phi \wedge \pi^* u \text{ in } H_{cv}^*(E).$$

Exercise 6.44. The *complex projective space* $\mathbb{C}P^n$ is the space of all lines through the origin in $\mathbb{C}^{n+1}$, topologized as the quotient of $\mathbb{C}^{n+1}$ by the equivalence relation

$$z \sim \lambda z \quad \text{for} \quad z \in \mathbb{C}^{n+1}, \quad \lambda \text{ a nonzero complex number.}$$

Let $z_0, \ldots, z_n$ be the complex coordinates on $\mathbb{C}^{n+1}$. These give a set of *homogeneous coordinates* $[z_0, \ldots, z_n]$ on $\mathbb{C}P^n$, determined up to multiplication by a nonzero complex number λ. Define U_i to be the open subset of $\mathbb{C}P^n$ given by $z_i \neq 0$. $\{U_0, \ldots, U_n\}$ is called the *standard open cover* of $\mathbb{C}P^n$.

(*a*) Show that $\mathbb{C}P^n$ is a manifold.

(*b*) Find the transition functions of the normal bundle $N_{\mathbb{C}P^1/\mathbb{C}P^2}$ relative to the standard open cover of $\mathbb{C}P^1$.

EXAMPLE 6.44.1. (The Euler class of the normal bundle of $\mathbb{C}P^1$ in $\mathbb{C}P^2$). Let $N = N_{\mathbb{C}P^1/\mathbb{C}P^2}$ be the normal bundle of $\mathbb{C}P^1$ in $\mathbb{C}P^2$. Since $\mathbb{C}P^1$ is a compact oriented manifold of real dimension 2, its top-dimensional cohomology is $H^2(\mathbb{C}P^1) = \mathbb{R}$. We will find the Euler class $e(N)$ as a multiple of the generator in $H^2(\mathbb{C}P^1)$.

By Exercise 6.44 the transition function of N relative to the standard open cover is $g_{01} = z_1/z_0$ at the point $[z_0, z_1]$. Let $z = z_1/z_0$ be the coordinate of U_0, which we identify with the complex plane $\mathbb{C}$. Let $w = z_0/z_1 = 1/z$

be the coordinate on $U_1 \simeq \mathbb{C}$. Then $g_{01} = z = 1/w$ on $U_0 \cap U_1$. The Euler class of N is given by

$$e(N) = -\frac{1}{2\pi i} d\left(\rho_0 d \log \frac{1}{w}\right) \quad \text{on } U_1 \qquad \text{(by (6.38))}$$

$$= -\frac{1}{2\pi i} d(\rho_0 d \log z) \quad \text{on } U_0 \cap U_1,$$

where ρ_0 is 1 in a neighborhood of the origin, and 0 in a neighborhood of infinity in the complex z-plane $U_0 \simeq \mathbb{C}$.

Fix a circle C in the complex plane with so large a radius that Supp ρ_0 is contained inside C. Let A_r be the annulus centered at the origin whose outer circle is C and whose inner circle B_r has radius r (Figure 6.10). Note that as the boundary of A_r, the circle C is oriented counterclockwise while B is oriented clockwise.

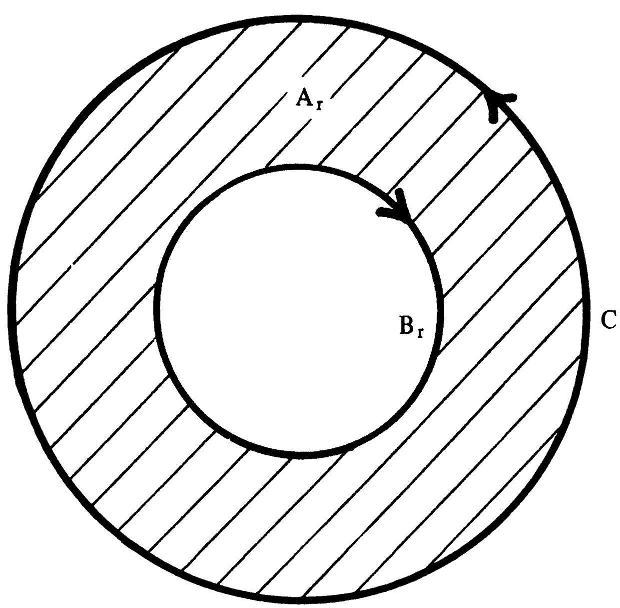


Figure 6.10

Now

$$\int_{\mathbb{C}P^1} e(N) = -\frac{1}{2\pi i} \int_C d\rho_0 \, d \log z,$$

and

$$\begin{aligned}
\int_{\mathbb{C}} d(\rho_0\, dz/z) &= \lim_{r\to 0} \int_{A_r} d(\rho_0\, dz/z) \\
&= \lim_{r\to 0} \int_C \rho_0\, dz/z + \int_{B_r} \rho_0\, dz/z \qquad \text{by Stokes' theorem} \\
&= \lim_{r\to 0} \int_{B_r} dz/z \\
&= -2\pi i,
\end{aligned}$$

where the minus sign is due to the clockwise orientation on B_r. Therefore,

$$\int_{\mathbb{C}P^1} e(N) = -\frac{1}{2\pi i}(-2\pi i) = 1.$$

Exercise 6.45. On the complex projective space $\mathbb{C}P^n$ there is a tautological line bundle S, called the *universal subbundle*; it is the subbundle of the product bundle $\mathbb{C}P^n \times \mathbb{C}^{n+1}$ given by

$$S = \{(\ell, z) \mid z \in \ell\}.$$

Above each point ℓ in $\mathbb{C}P^n$, the fiber of S is the line represented by ℓ. Find the transition functions of the universal subbundle S of $\mathbb{C}P^1$ relative to the standard open cover and compute its Euler class.

Exercise 6.46. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and i the antipodal map on S^n:

$$i : (x_1, \ldots, x_{n+1}) \to (-x_1, \ldots, -x_{n+1}).$$

The *real projective space* $\mathbb{R}P^n$ is the quotient of S^n by the equivalence relation

$$x \sim i(x), \qquad \text{for} \qquad x \in \mathbb{R}^{n+1}.$$

(*a*) An *invariant form* on S^n is a form ω such that $i^*\omega = \omega$. The vector space of invariant forms on S^n, denoted $\Omega^*(S^n)^I$, is a differential complex, and so the invariant cohomology $H^*(S^n)^I$ of S^n is defined. Show that $H^*(\mathbb{R}P^n) \simeq H^*(S^n)^I$.

(*b*) Show that the natural map $H^*(S^n)^I \to H^*(S^n)$ is injective. [*Hint:* If ω is an invariant form and $\omega = d\tau$ for some form τ on S^n, then $\omega = d(\tau + i^*\tau)/2$.]

(*c*) Give S^n its standard orientation (p. 70). Show that the antipodal map $i : S^n \to S^n$ is orientation-preserving for n odd and orientation-reversing for n even. Hence, if $[\sigma]$ is a generator of $H^n(S^n)$, then $[\sigma]$ is a nontrivial invariant cohomology class if and only if n is odd.

(*d*) Show that the de Rham cohomology of $\mathbb{R}P^n$ is

$$H^q(\mathbb{R}P^n) = \begin{cases} \mathbb{R} & \text{for } q = 0, \\ 0 & \text{for } 0 < q < n, \\ \mathbb{R} & \text{for } q = n \text{ odd}, \\ 0 & \text{for } q = n \text{ even}. \end{cases}$$

Relative de Rham Theory

The Thom class of an oriented vector bundle may be viewed as a *relative* cohomology class, which we now define. Let $f : S \to M$ be a map between two manifolds. Define a complex $\Omega^*(f) = \oplus_{q \geq 0} \Omega^q(f)$ by

$$\Omega^q(f) = \Omega^q(M) \oplus \Omega^{q-1}(S),$$
$$d(\omega, \theta) = (d\omega, f^*\omega - d\theta).$$

It is easily verified that $d^2 = 0$. Note that a cohomology class in $\Omega^*(f)$ is represented by a closed form ω on M which becomes exact when pulled back to S.

By definition we have the exact sequence

$$0 \longrightarrow \Omega^{q-1}(S) \xrightarrow{\alpha} \Omega^q(f) \xrightarrow{\beta} \Omega^q(M) \longrightarrow 0$$

with the obvious maps α and β : $\alpha(\theta) = (0, \theta)$ and $\beta(\omega, \theta) = \omega$. Clearly β is a chain map but α is not quite a chain map; in fact it anticommutes with d, $\alpha d = -d\alpha$. In any case there is still a long exact sequence in cohomology

$$\cdots \to H^{q-1}(S) \xrightarrow{\alpha^*} H^q(f) \xrightarrow{\beta^*} H^q(M) \xrightarrow{\delta^*} H^q(S) \to \cdots \tag{6.47}$$

Claim 6.48. $\delta^* = f^*$.

PROOF OF CLAIM. Consider the diagram

$$\begin{array}{ccccccccc} 0 \to & \Omega^q(S) & \to & \Omega^{q+1}(f) & \to & \Omega^{q+1}(M) & \to 0 \\ & d\uparrow & & d\uparrow & & d\uparrow & \\ 0 \to & \Omega^{q-1}(S) & \to & \Omega^q(f) & \to & \Omega^q(M) & \to 0 \\ & & & \Psi & & \Psi & \\ & & & (\omega, \theta) & & \omega & \end{array}$$

Let $\omega \in \Omega^q(M)$ be a closed form and (ω, θ) any element of $\Omega^q(f)$ which maps to ω. Then $d(\omega, \theta) = (0, f^*\omega - d\theta)$. So $\delta^*[\omega] = [f^*\omega - d\theta] = [f^*\omega]$. $\square$

Combining (6.47) and (6.48) we have

Proposition 6.49. *Let $f: S \to M$ be a differentiable map between two manifolds. Then there is an exact sequence*

$$\cdots \to H^q(f) \xrightarrow{\beta^*} H^q(M) \xrightarrow{f^*} H^q(S) \xrightarrow{\alpha^*} H^{q+1}(f) \to \cdots.$$

Exercise 6.50. If $f, g: S \to M$ are homotopic maps, show that $H^*(f)$ and $H^*(g)$ are isomorphic algebras.

If S is a submanifold of M and $i: S \to M$ is the inclusion map, we define the *relative de Rham cohomology* $H^q(M, S)$ to be $H^q(i)$.

We now turn to the Thom class. Recall that if $\pi: E \to M$ is a rank 2 oriented vector bundle and E^0 is the complement of the zero section, then there is a global angular form ψ on E^0 such that $d\psi = -\pi^*e$, where e represents the Euler class of E (6.37). Furthermore, if $s: M \to E$ is the zero section, then $e = s^*\Phi$ (Proposition 6.41). Hence, $(s \circ \pi)^*\Phi = -d\psi$, where $s \circ \pi: E^0 \to E$. This shows that $(\Phi, -\psi)$ is closed in the complex $\Omega^*(s \circ \pi)$ and so represents a class in $H^2(s \circ \pi)$. Since the map $s \circ \pi: E^0 \to E$ is clearly homotopic to the inclusion $i: E^0 \to E$, by Exercise 6.50, $H^2(s \circ \pi) = H^2(i)$. Hence, $(\Phi, -\psi)$ represents a class in the relative cohomology $H^2(E, E^0)$. The rank n case is entirely analogous and will be taken up in Section 12.

§7 The Nonorientable Case

Since the integral of a differential form on $\mathbb{R}^n$ is not invariant under the whole group of diffeomorphisms of $\mathbb{R}^n$, but only under the subgroup of orientation-preserving diffeomorphisms, a differential form cannot be integrated over a nonorientable manifold. However, by modifying a differential form we obtain something called a *density*, which can be integrated over any manifold, orientable or not. This will give us a version of Poincaré duality for nonorientable manifolds and of the Thom isomorphism for nonorientable vector bundles.

The Twisted de Rham Complex

Let M be a manifold and E a vector space. The space of *differential forms on M with values in E*, denoted $\Omega^*(M, E)$, is by definition the vector space spanned by $\omega \otimes v$, where $\omega \in \Omega^*(M)$, $v \in E$, and the tensor product is over $\mathbb{R}$. This space can be made naturally into a differential complex if we let the differential be

$$d(\omega \otimes v) = (d\omega) \otimes v.$$

So the cohomology $H^*(M, E)$ is defined. Indeed, if E is a vector space of dimension n, then $H^*(M, E)$ is isomorphic to n copies of $H^*_{DR}(M)$.

Now let E be a vector bundle. We define the space of *E-valued q-forms*, $\Omega^q(M, E)$, to be the global sections of the vector bundle $(\Lambda^q T_M^*) \otimes E$. Locally such a q-form can be written as $\sum \omega_i \otimes e_i$, where ω_i are q-forms and e_i are sections of E over some open set U in M, and the tensor product is over the C^∞ functions on U. For these vector-valued differential forms, no natural extension of the de Rham complex is possible, unless one is first given a way of differentiating the sections of E.

Suppose the vector bundle E has a trivialization $\{(U_\alpha, \phi_\alpha)\}$ relative to which the transition functions are locally constant. Such a vector bundle is called a *flat vector bundle* and the trivialization a *locally constant trivialization.* For a flat vector bundle E a differential operator on $\Omega^*(M, E)$ may be defined as follows. Let $e_\alpha^1, \ldots, e_\alpha^n$ be the sections of E over U_α corresponding to the standard basis under the trivialization $\phi_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$. We declare these to be the *standard locally constant sections,* i.e., $de_\alpha^i = 0$. Over U_α an E-valued q-form s in $\Omega^q(M, E)$ can be written as $\sum \omega_i \otimes e_\alpha^i$, where the ω_i are q-forms over U_α. We define the exterior derivative ds over U_α by linearity and the Leibnitz rule:

$$d\left(\sum \omega_i \otimes e_\alpha^i\right) = \sum (d\omega_i) \otimes e_\alpha^i.$$

It is easy to show that, because the transition functions of E relative to $\{(U_\alpha, \phi_\alpha)\}$ are locally constant, this definition of exterior differentiation is independent of the open sets U_α. More precisely, on the overlap $U_\alpha \cap U_\beta$, if

$$s = \sum \omega_i \otimes e_\alpha^i = \sum \tau_j \otimes e_\beta^j$$

and $e_\alpha^i = \sum c_{ij} e_\beta^j$, where the c_{ij} are locally constant functions, then

$$\tau_j = \sum c_{ij} \omega_i$$

and

$$\begin{aligned} d\left(\sum \tau_j \otimes e_\beta^j\right) &= \sum (d\tau_j) \otimes e_\beta^j \\ &= \sum (c_{ij}\, d\omega_i) \otimes e_\beta^j \\ &= \sum (d\omega_i) \otimes e_\alpha^i \\ &= d\left(\sum \omega_i \otimes e_\alpha^i\right). \end{aligned}$$

Hence ds is globally defined and is an element of $\Omega^{q+1}(M, E)$. Because d^2 is clearly zero, $\Omega^*(M, E)$ is a differential complex and the cohomology $H^*(M, E)$ makes sense. As defined, d very definitely depends on the trivialization $\{(U_\alpha, \phi_\alpha)\}$, for it is through the trivialization that the locally constant sections are given. Hence, d, $\Omega^*(M, E)$, and $H^*(M, E)$ are more properly denoted as d_ϕ, $\Omega_\phi^*(M, E)$, and $H_\phi^*(M, E)$.

EXAMPLE 7.1 (Two trivializations of a vector bundle E which give rise to distinct cohomology groups $H^*(M, E)$).

Let M be the circle S^1 and E the trivial line bundle $S^1 \times \mathbb{R}^1$ over the circle. If E is given the usual constant trivialization ϕ:

$$\phi(x, r) = r \quad \text{for} \quad x \in S^1 \qquad \text{and} \qquad r \in \mathbb{R}^1,$$

then the cohomology $H^0_\phi(S^1, E) = \mathbb{R}$.

However, we can define another locally constant trivialization ψ for E as follows. Cover S^1 with two open sets U and V as indicated in Figure 7.1.

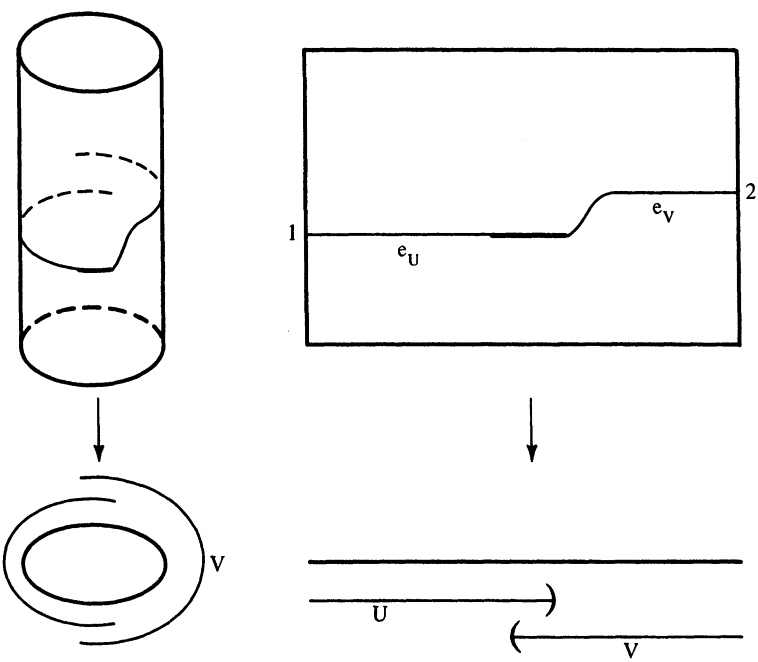


Figure 7.1

Let $\rho(x)$ be the real-valued function on V whose graph is as in Figure 7.2. The trivialization ψ is given by

$$\psi(x, r) = \begin{cases} r & \text{for } x \in U,\ r \in \mathbb{R}^1, \\ \rho(x)r & \text{for } x \in V,\ r \in \mathbb{R}^1. \end{cases}$$

The standard locally constant sections over U and V are $e_U(x) = (x, 1)$ and $e_V(x) = (x, 1/\rho(x))$ respectively. Relative to the trivialization ψ, the cohomology $H^0_\psi(S^1, E) = 0$, since the locally constant sections over U and V do not piece together to form a global section (except for the zero section).

It is natural to ask: to what extent is the twisted cohomology $H^*_\phi(M, E)$ independent of the trivialization ϕ for E?

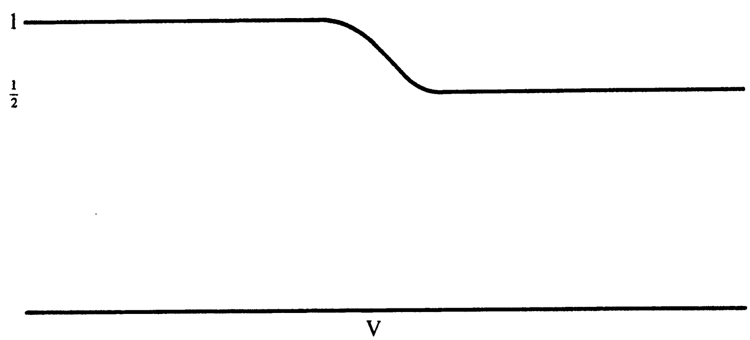


Figure 7.2

Proposition 7.2. *The twisted cohomology is invariant under the refinement of open covers. More precisely, let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ be a locally constant trivialization for E. Suppose $\{V_\beta\}_{\beta \in J}$ is a refinement of $\{U_\alpha\}_{\alpha \in I}$ and the coordinates maps ψ_β on $V_\beta \subset U_\alpha$ are the restrictions of ϕ_α. Then the two twisted complexes $\Omega_\phi^*(M, E)$ and $\Omega_\psi^*(M, E)$ are identical and so are their cohomology:*

$$H_\phi^*(M, E) = H_\psi^*(M, E).$$

PROOF. Since the definition of the differential operator on a twisted complex is local, and ϕ and ψ agree on the open cover $\{V_\beta\}$, we have $d_\phi = d_\psi$. Therefore the two complexes $\Omega_\phi^*(M, E)$ and $\Omega_\psi^*(M, E)$ are identical. □

Still assuming E to be a flat vector bundle, suppose $\{(U_\alpha, \phi_\alpha)\}$ and $\{(U_\alpha, \psi_\alpha)\}$ are two locally constant trivializations which *differ by a locally constant comparison* 0-*cochain*, i.e., if e_α^i and f_α^j are the standard locally constant sections over U_α relative to the trivializations ϕ and ψ respectively, then

$$e_\alpha^i = \sum_j a_\alpha^{ij} f_\alpha^j$$

for some locally constant function

$$a_\alpha = (a_\alpha^{ij}) \colon U_\alpha \to \mathrm{GL}(n, \mathbb{R}).$$

In this case there is an obvious isomorphism

$$F \colon \Omega_\phi^q(M, E) \to \Omega_\psi^q(M, E)$$

given by

$$e_\alpha^i \mapsto \sum_j a_\alpha^{ij} f_\alpha^j.$$

It is easily checked that the diagram

$$\begin{array}{ccc} \Omega_\psi^*(M, E) & \xrightarrow{d_\phi} & \Omega_\phi^{*+1}(M, E) \\ \downarrow F & & \downarrow F \\ \Omega_\psi^*(M, E) & \xrightarrow[d_\psi]{} & \Omega_\psi^{*+1}(M, E) \end{array}$$

commutes. Hence F induces an isomorphism in cohomology. Next, suppose we are given two locally constant trivializations $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for E, with possibly different open covers. By taking a common refinement, which does not affect the twisted cohomology (Proposition 7.2), we may assume that the two open covers are identical. The discussion above therefore proves the following.

Proposition 7.3. (a) *Let E be a flat vector bundle over M, and $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ two locally constant trivializations for E. Suppose after a common refinement the two trivializations differ by a locally constant comparison 0-cochain. Then there are isomorphisms*

$$\Omega_\phi^*(M, E) \simeq \Omega_\psi^*(M, E)$$

and

$$H_\phi^*(M, E) \simeq H_\psi^*(M, E).$$

This proposition may also be stated in terms of the transition functions for E.

Proposition 7.3. (b) *Let E be a flat vector bundle of rank n and $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$ the transition functions for E relative to two locally constant trivializations ϕ and ψ with the same open cover. If there exist locally constant functions*

$$\lambda_\alpha\colon U_\alpha \to \mathrm{GL}(n, \mathbb{R})$$

such that

$$g_{\alpha\beta} = \lambda_\alpha h_{\alpha\beta} \lambda_\beta^{-1},$$

then there are isomorphisms as in 7.3(a).

Proposition 7.4. *If E is a trivial rank n vector bundle over a manifold M, with ϕ a trivialization of E given by n global sections, then*

$$H_\phi^*(M, E) = H^*(M, \mathbb{R}^n) = \bigoplus_{i=1}^{n} H^*(M).$$

PROOF. Let $e_1, \ldots, e_n$ be the n global sections corresponding to the standard basis of $\mathbb{R}^n$. Then every element in $\Omega^*(M, E)$ can be written uniquely as $\sum \omega_i \otimes e_i$, where $\omega_i \in \Omega^*(M)$ and the tensor product is over the C^∞ functions on M. The map

$$\sum \omega_i \otimes e_i \mapsto (\omega_1, \ldots, \omega_n)$$

gives an isomorphism of the complexes $\Omega_\phi^*(M, E)$ and $\Omega^*(M, \mathbb{R}^n)$. $\square$

Now let $\{(U_\alpha, \phi_\alpha)\}$ be a coordinate open cover for the manifold M, with transition functions $g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$. Define the *sign function* on $\mathbb{R}^1$ to be

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{for } x \text{ positive} \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x \text{ negative.} \end{cases}$$

The *orientation bundle* of M is the line bundle L on M given by transition functions sgn $J(g_{\alpha\beta})$, where $J(g_{\alpha\beta})$ is the Jacobian determinant of the matrix of partial derivatives of $g_{\alpha\beta}$. It follows directly from the definition that *M is orientable if and only if its orientation bundle is trivial.*

Relative to the atlas $\{(U_\alpha, \phi_\alpha)\}$ for M with transition functions $g_{\alpha\beta}$, the orientation bundle is by definition the quotient

$$(U_\alpha \times \mathbb{R}^1)/(x, v) \sim (x, \operatorname{sgn} J(g_{\alpha\beta}(x))v),$$

where $(x, v) \in U_\alpha \times \mathbb{R}^1$ and $(x, \operatorname{sgn} J(g_{\alpha\beta}(x))v) \in U_\beta \times \mathbb{R}^1$. By construction there is a natural trivialization ϕ' on L,

$$\phi'_\alpha \colon L|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^1,$$

which we call *the trivialization induced from the atlas $\{(U_\alpha, \phi_\alpha)\}$ on M.* Because sgn $J(g_{\alpha\beta})$ are locally constant functions on M, the locally constant sections of L relative to this trivialization are the equivalence classes of $\{(x, v) \mid x \in U_\alpha\}$ for v fixed in $\mathbb{R}^1$.

Proposition 7.5. *If ϕ' and ψ' are two trivializations for L induced from two atlases ϕ and ψ on M, then the two twisted complexes $\Omega^*_{\phi'}(M, L)$ and $\Omega^*_{\psi'}(M, L)$ are isomorphic and so are their cohomology $H^*_{\phi'}(M, L)$ and $H^*_{\psi'}(M, L)$.*

PROOF. By going to a common refinement we may assume that the two atlases ϕ and ψ have the same open cover. Thus on each U_α there are two sets of coordinate functions, ϕ_α and ψ_α (Figure 7.3.).

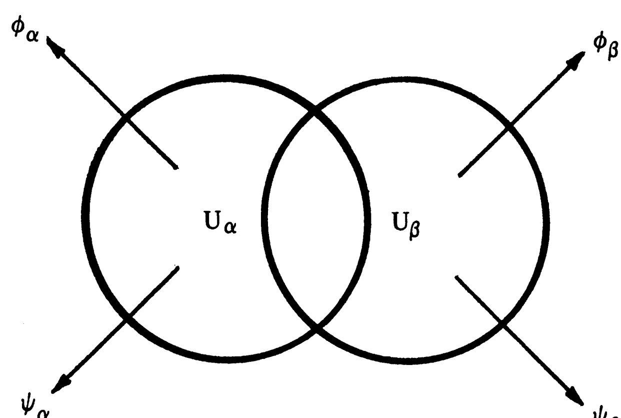


Figure 7.3

The transition functions $g_{\alpha\beta}$ and $h_{\alpha\beta}$ for the two atlases ϕ and ψ respectively are related by

$$\begin{aligned} g_{\alpha\beta} &= \phi_\alpha \circ \phi_\beta^{-1} \\ &= \phi_\alpha \circ \psi_\alpha^{-1} \circ \psi_\alpha \circ \psi_\beta^{-1} \circ \psi_\beta \circ \phi_\beta^{-1} \\ &= \mu_\alpha \circ h_{\alpha\beta} \circ \mu_\beta^{-1}, \end{aligned}$$

where $\mu_\alpha := \phi_\alpha \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha) \to \phi_\alpha(U_\alpha)$. It follows that

$$\operatorname{sgn} J(g_{\alpha\beta}) = \operatorname{sgn} J(\mu_\alpha) \cdot \operatorname{sgn} J(h_{\alpha\beta}) \cdot \operatorname{sgn} J(\mu_\beta)^{-1}.$$

Define a 0-chain $\lambda_\alpha : U_\alpha \to \mathrm{GL}(1, \mathbb{R})$ by $\lambda_\alpha(x) = \operatorname{sgn} J(\mu_\alpha)(\psi_\alpha(x))$ for $x \in U_\alpha$. Since $\lambda_\alpha(x) = \pm 1$, by Proposition 7.3(b)

$$\Omega^*_{\phi'}(M, L) \simeq \Omega^*_{\psi'}(M, L). \qquad \square$$

We define the *twisted de Rham complex* $\Omega^*(M, L)$ and *the twisted de Rham cohomology* $H^*(M, L)$ to be $\Omega^*_{\phi'}(M, L)$ and $H^*_{\phi'}(M, L)$ for any trivialization ϕ' on L which is induced from M. Similarly one also has the *twisted de Rham cohomology with compact support*, $H^*_c(M, L)$.

REMARK. If a trivialization ψ on L is not induced from M, then $H^*_\psi(M, L)$ may not be equal to the twisted de Rham cohomology $H^*(M, L)$.

The following statement is an immediate consequence of Proposition 7.4 and the triviality of L on an orientable manifold.

Proposition 7.6. *On an orientable manifold M the twisted de Rham cohomology $H^*(M, L)$ is the same as the ordinary de Rham cohomology.*

Integration of Densities, Poincaré Duality, and the Thom Isomorphism

Let M be a manifold of dimension n with coordinate open cover $\{(U_\alpha, \phi_\alpha)\}$ and transition functions $g_{\alpha\beta}$. A *density* on M is an element of $\Omega^n(M, L)$, or equivalently, a section of the *density bundle* $(\Lambda^n T^*_M) \otimes L$. One may think of a density as a top-dimensional differential form twisted by the orientation bundle. Since the transition function for the exterior power $\Lambda^n T^*_M$ is $1/J(g_{\alpha\beta})$, the transition function for the density bundle is

$$\frac{1}{J(g_{\alpha\beta})} \cdot \operatorname{sgn} J(g_{\alpha\beta}) = \frac{1}{|J(g_{\alpha\beta})|}.$$

Let e_α be the section of $L|_{U_\alpha}$ corresponding to 1 under the trivialization of L induced from the atlas $\{(U_\alpha, \phi_\alpha)\}$. If $\phi_\alpha = (x_1, \ldots, x_n)$ are the coordinates on U_α, we define the density $|dx_1 \cdots dx_n|$ in $\Gamma(U_\alpha, (\Lambda^n T_M^*) \otimes L))$ to be

$$|dx_1 \cdots dx_n| = e_\alpha \, dx_1 \cdots dx_n.$$

Locally we may then write a density as $g(x_1, \ldots, x_n)|dx_1 \cdots dx_n|$ for some smooth function g.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism of $\mathbb{R}^n$ with coordinates $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively. If $\omega = g|dy_1 \ldots dy_n|$ is a density on $\mathbb{R}^n$, the pullback of ω by T is

$$\begin{aligned} T^*\omega &= (g \circ T)|d(y_1 \circ T) \ldots d(y_n \circ T)| \\ &= (g \circ T)|J(T)||dx_1 \ldots dx_n|. \end{aligned}$$

The density $g|dy_1 \ldots dy_n|$ is said to have compact support on $\mathbb{R}^n$ if g has compact support, and the integral of such a density over $\mathbb{R}^n$ is defined to be the corresponding Riemann integral. Then

$$\begin{aligned} \int_{\mathbb{R}^n} T^*\omega &= \int_{\mathbb{R}^n} (g \circ T)|J(T)||dx_1 \ldots dx_n| \\ &= \int_{\mathbb{R}^n} g|dy_1 \ldots dy_n| \quad \text{by the change of variable formula} \\ &= \int_{\mathbb{R}^n} \omega. \end{aligned}$$

Thus the integration of a density is invariant under the group of all diffeomorphisms on $\mathbb{R}^n$. This means we can globalize the integration of a density to a manifold. If $\{\rho_\alpha\}$ is a partition of unity subordinate to the open cover $\{(U_\alpha, \phi_\alpha)\}$ and $\omega \in \Omega_c^n(M, L)$, define

$$\int_M \omega = \sum_\alpha \int_{\mathbb{R}^n} (\phi_\alpha^{-1})^* (\rho_\alpha \omega).$$

It is easy to check that this definition is independent of the choices involved.

Just as for differential forms there is a Stokes' theorem for densities. We state below only the weak version that we need.

Theorem 7.7 (Stokes' Theorem for Densities). *On any manifold M of dimension n, orientable or not, if $\omega \in \Omega_c^{n-1}(M, L)$, then*

$$\int_M d\omega = 0.$$

The proof is essentially the same as (3.5).

It follows from this Stokes' theorem that the pairings

$$\Omega^q(M) \otimes \Omega_c^{n-q}(M, L) \to \mathbb{R}$$

and

$$\Omega_c^q(M) \otimes \Omega^{n-q}(M, L) \to \mathbb{R}$$

given by

$$\omega \wedge \tau \mapsto \int_M \omega \wedge \tau$$

descend to cohomology.

Theorem 7.8 (Poincaré Duality). *On a manifold M of dimension n with a finite good cover, there are nondegenerate pairings*

$$H^q(M) \underset{\mathbb{R}}{\otimes} H_c^{n-q}(M, L) \to \mathbb{R}$$

and

$$H_c^q(M) \underset{\mathbb{R}}{\otimes} H^{n-q}(M, L) \to \mathbb{R}.$$

PROOF. By tensoring the Mayer-Vietoris sequences (2.2) and (2.7) with $\Gamma(M, L)$ we obtain the corresponding Mayer-Vietoris sequences for twisted cohomology. The Mayer-Vietoris argument for Poincaré duality on an orientable manifold then carries over word for word. □

Corollary 7.8.1. *Let M be a connected manifold of dimension n having a finite good cover. Then*

$$H^n(M) = \begin{cases} \mathbb{R} & \textit{if } M \textit{ is compact orientable} \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. By Poincaré duality, $H^n(M) = H_c^0(M, L)$. Let $\{U_\alpha\}$ be a coordinate open cover for M. An element of $H_c^0(M, L)$ is given by a collection of constants f_α on U_α satisfying

$$f_\alpha = (\operatorname{sgn} J(g_{\alpha\beta}))\, f_\beta.$$

If $f_\alpha = 0$ for some α, then by the connectedness of M, we have $f_\alpha = 0$ for all α. It follows that a nonzero element of $H_c^0(M, L)$ is nowhere vanishing. Thus, $H_c^0(M, L) \neq 0$ if and only if M is compact and L has a nowhere-vanishing section, i.e., M is compact orientable. In that case,

$$H_c^0(M, L) = H_c^0(M) = \mathbb{R}.$$ □

Exercise 7.9. Let M be a manifold of dimension n. Compute the cohomology groups $H_c^n(M)$, $H^n(M, L)$, and $H_c^n(M, L)$ for each of the following four cases: M compact orientable, noncompact orientable, compact nonorientable, noncompact nonorientable.

Finally, we state but do not prove the Thom isomorphism theorem in all orientational generality. Let E be a rank n vector bundle over a manifold

M, and let $\{(U_\alpha, \phi_\alpha)\}$ and $g_{\alpha\beta}$ be a trivialization and transition functions for E. Neither E nor M is assumed to be orientable. The *orientation bundle* of E, denoted $o(E)$, is the line bundle over M with transition functions sgn $J(g_{\alpha\beta})$. With this terminology, the orientation bundle of M is simply the orientation bundle of its tangent bundle T_M. It is easy to see that when E is not orientable, integration along the fiber of a form in $\Omega^*_{cv}(E)$ does not yield a global form on M, but an element of the twisted complex $\Omega^*(M, o(E))$.

Theorem 7.10 (Nonorientable Thom Isomorphism). *Under the hypothesis above, integration along the fiber gives an isomorphism*

$$\pi_* : H^{*+n}_{cv}(E) \xrightarrow{\sim} H^*(M, o(E)).$$

Exercise 7.11. Compute the twisted de Rham cohomology $H^*(\mathbb{R}P^n, L)$.