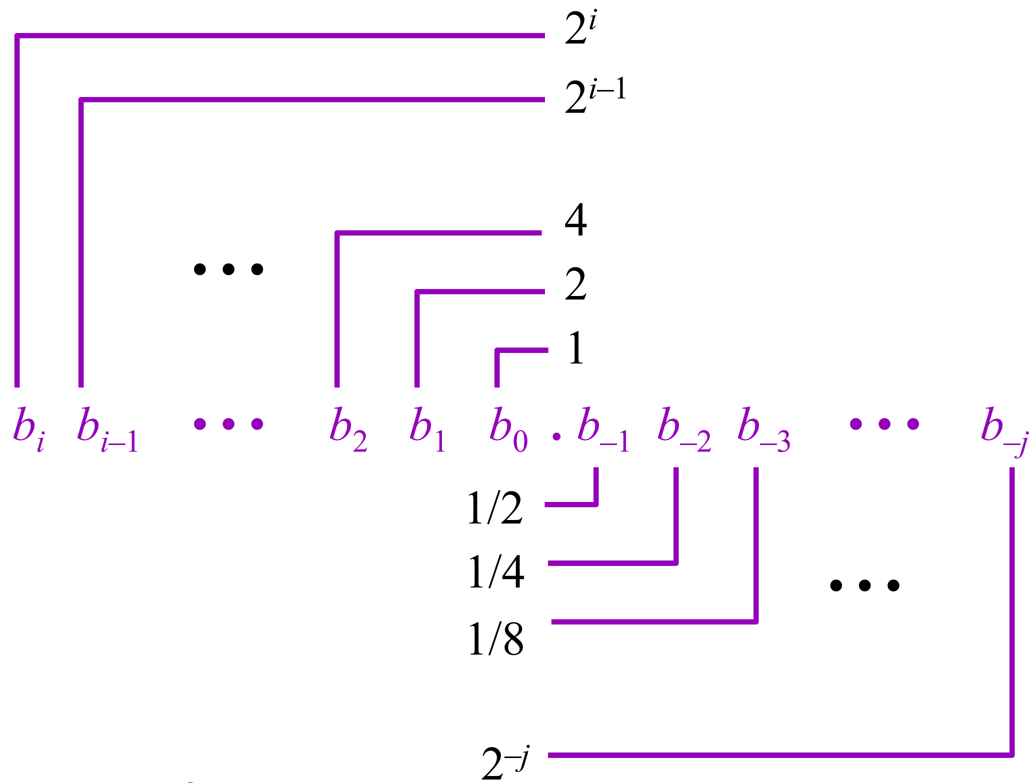
Floating Point

Chapter 2 of B&O

Some notes adopted from Bryant and O'Hallaron

Fractional Binary Numbers



- Representation
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i} b_k \cdot 2^k$

Frac. Binary Number Examples

- **Value** / **Representation**

Value	Representation
$5\ ^{3}/_{4}$	101.11_2
$2\ ^{7}/_{8}$	10.111_2
$^{63}/_{64}$	0.111111_2

- Observations
  - Divide by 2 by shifting right
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$

Representable Numbers

- Limitation
  - Can only exactly represent numbers of the form $x/2^k$
  - Other numbers have repeating bit representations
- Value | Representation

Value	Representation
1/3	$0.0101010101[01]\ldots_2$
1/5	$0.001100110011[0011]\ldots_2$
1/10	$0.0001100110011[0011]\ldots_2$

IEEE Floating Point

- IEEE Standard 754
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs
- Driven by Numerical Concerns
  - Nice standards for rounding, overflow, underflow
  - Hard to make go fast
    - Numerical analysts predominated over hardware types in defining standard

Floating Point Representation

- Numerical Form

  mantisa-fractional

  $-1^{s} M \ 2^{E}$

  - Sign bit s determines whether number is negative or positive
  - Significand M normally a fractional value in range [1.0,2.0).
  - Exponent E weights value by power of two

- Encoding

s	exp	frac

  – MSB is sign bit
  – `exp` field encodes E
  – `frac` field encodes M

Floating Point Precisions

- Encoding

s	exp	frac

  - MSB is sign bit
  - `exp` field encodes *E*
  - `frac` field encodes *M*
- Sizes
  - Single precision: 8 `exp` bits, 23 `frac` bits
    - 32 bits total
  - Double precision: 11 `exp` bits, 52 `frac` bits
    - 64 bits total

"Normalized" Numeric Values

If you wanted to raise some matisa by -1 then it would be E = 126
Subtracting from the bias does not change the range

- Condition
  - exp $\neq$ 000…0 and exp $\neq$ 111…1
- Exponent coded as biased value
  - E = Exp – Bias
    - Exp : unsigned value denoted by exp
    - Bias : Bias value
      - Single precision: 127 (Exp: 1…254, E: -126…127)
      - Double precision: 1023 (Exp: 1…2046, E: -1022…1023)
      - in general: Bias = $2^{e-1} - 1$, where e is number of exponent bits
- Significand coded with implied leading 1
  - M = $1.xxx…x_2$
    - xxx…x: bits of frac
    - Minimum when 000…0 (M = 1.0)
    - Maximum when 111…1 (M = 2.0 – ε)
    - Get extra leading bit for "free"

Normalized Encoding Example

- Value
  - Float F = 15213.0;
    $15213_{10} = 11101101101101_2 = 1.1101101101101_2 \times 2^{13}$
- Significand
  - M = 1.1101101101101_2
  - frac = $11011011011010000000000_2$
- Exponent
  - E= 13
  - Bias = 127
  - Exp = 140 = 10001100_2

Step 1: covert to 2's scientific notation
Step 2: you get an E
Step 3: You have your bias
Step 4: Get exponent and represent that in binary

Floating Point Representation:

Hex:	4	6	6	D	B	4	0	0
Binary:	0100	0110	0110	1101	1011	0100	0000	0000
140:	100	0110	0					
15213:			1110	1101	1011	01		

with the 1 it becomes the rest after the 1 is the mantisa

Denormalized Values

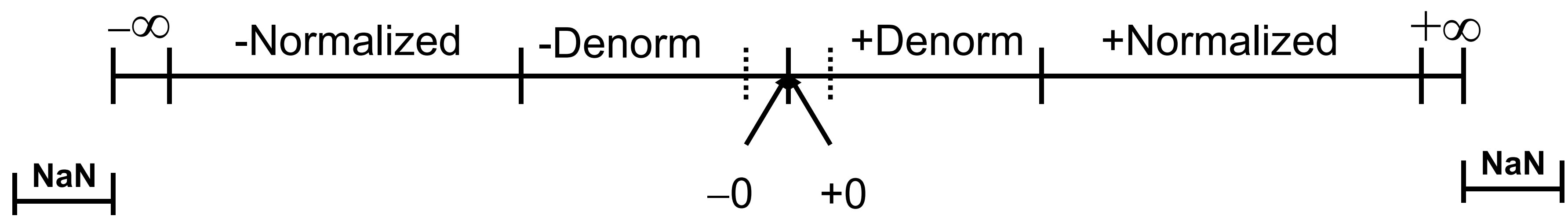
- Condition
  - exp = 000…0
- Value
  - Exponent value E = –Bias + 1
  - Significand value M = $0.xxx\ldots x_2$
    - xxx…x: bits of frac
- Cases
  - exp = 000…0, frac = 000…0
    - Represents value 0
    - Note that have distinct values +0 and –0
  - exp = 000…0, frac $\neq$ 000…0
    - Numbers very close to 0.0
    - "Gradual underflow"

01.11 x 2^-129
0.00111 x 2^-126
Frac 001110….
Exp 0…..0

Special Values

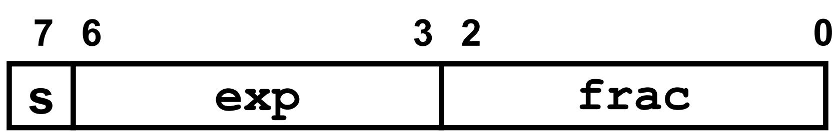
- Condition
  - exp = 111…1
- Cases
  - exp = 111…1, frac = 000…0
    - Represents value ∞ (infinity)
    - Operation that overflows
    - Both positive and negative
    - E.g., 1.0/0.0 = -1.0/-0.0 = $+\infty$, 1.0/-0.0 = ∞
  - exp = 111…1, frac $\neq$ 000…0
    - Not-a-Number (NaN)
    - Represents case when no numeric value can be determined
    - E.g., sqrt(–1), $\infty - \infty$

Floating Point Real Number Encodings



Tiny Floating Point Example

- 8-bit Floating Point Representation
  - the sign bit is in the most significant bit.
  - the next four bits are the exponent, with a bias of 7.
  - the last three bits are the `frac`
- Same General Form as IEEE Format
  - normalized, denormalized
  - representation of 0, NaN, infinity



Values Related to the Exponent

Exp	exp	E	2^E	
0	0000	-6	1/64	(denorms)
1	0001	-6	1/64	
2	0010	-5	1/32	
3	0011	-4	1/16	
4	0100	-3	1/8	
5	0101	-2	1/4	
6	0110	-1	1/2	
7	0111	0	1	
8	1000	+1	2	
9	1001	+2	4	
10	1010	+3	8	
11	1011	+4	16	
12	1100	+5	32	
13	1101	+6	64	
14	1110	+7	128	
15	1111	n/a		(inf, NaN)

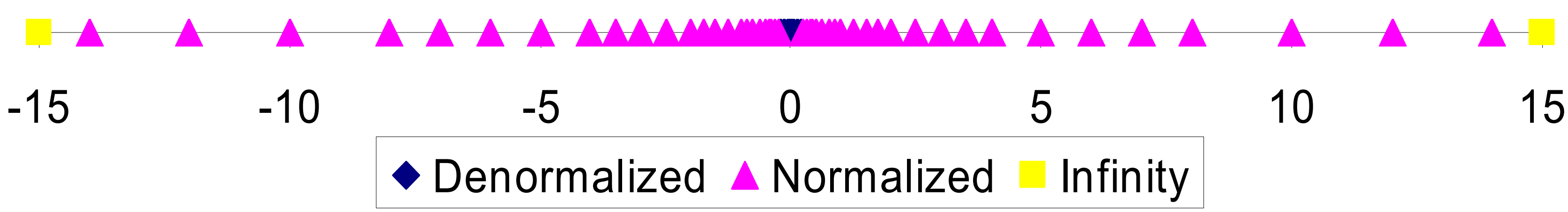
Dynamic Range

	s	exp	frac	*E*	Value	
	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	← closest to zero
Denormalized numbers	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	← largest denorm
	0	0001	000	-6	8/8*1/64 = 8/512	← smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	← closest to 1 below
Normalized numbers	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	← closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	← largest norm
	0	1111	000	n/a	inf	

Distribution of Values

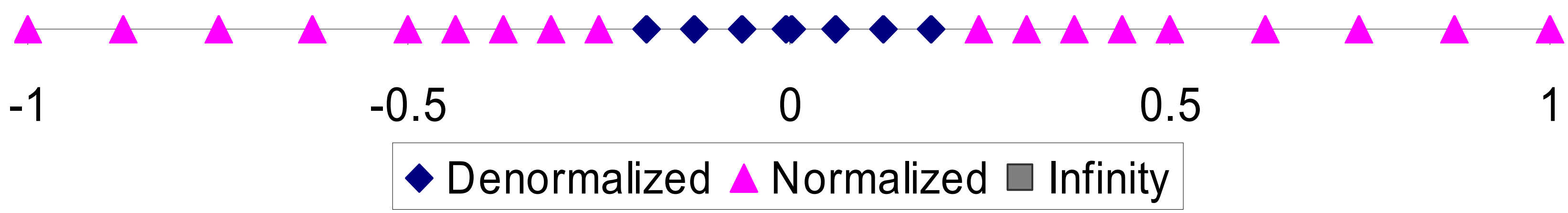
- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3

- Notice how the distribution gets denser toward zero.



Distribution of Values (close-up view)

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3



Interesting Numbers

Description	exp	frac	Num Val
Zero	00…00	00…00	0.0
Smallest +Denorm	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
– Single ≈ 1.4×10^{-45}			
– Double ≈ 4.9×10^{-324}			
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
– Single ≈ 1.18×10^{-38}			
– Double ≈ 2.2×10^{-308}			
Smallest Pos. Normalized	00…01	00…00	$1.0 \times 2^{-\{126,1022\}}$
– Just larger than largest denormalized			
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
– Single ≈ 3.4×10^{38}			
– Double ≈ 1.8×10^{308}			

Special Properties of Encoding

- FP Zero Same as Integer Zero
  - All bits = 0
- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider -0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

Floating Point Operations

- Conceptual View
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into `frac`
- Rounding Modes (illustrate with $ rounding)

	\$1.40	\$1.60	\$1.50	\$2.50	–\$1.50
– Zero	**\$1**	**\$1**	**\$1**	**\$2**	**–\$1**
– Round down (-∞)	**\$1**	**\$1**	**\$1**	**\$2**	**–\$2**
– Round up (+∞)	**\$2**	**\$2**	**\$2**	**\$3**	**–\$1**
– Nearest Even (default)	**\$1**	**\$2**	**\$2**	**\$2**	**–\$2**

Note:

1. **Round down: rounded result is close to but no greater than true result.**
2. **Round up: rounded result is close to but no less than true result.**

Closer Look at Round-To-Even

- Default Rounding Mode
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under- estimated
- Applying to Other Decimal Places / Bit Positions
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth
    - 1.2349999 1.23 (Less than half way)
    - 1.2350001 1.24 (Greater than half way)
    - 1.2350000 1.24 (Half way—round up)
    - 1.2450000 1.24 (Half way—round down)

Rounding Binary Numbers

- Binary Fractional Numbers
  - "Even" when least significant bit is `0`
  - Half way when bits to right of rounding position = $100\ldots_2$
- Examples
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
$2\ {}^{3}/_{32}$	10.00011_2	10.00_2	(<1/2—down)	2
$2\ {}^{3}/_{16}$	10.00110_2	10.01_2	(>1/2—up)	$2\ {}^{1}/_{4}$
$2\ {}^{7}/_{8}$	10.11100_2	11.00_2	(1/2—up)	3
$2\ {}^{5}/_{8}$	10.10100_2	10.10_2	(1/2—down)	$2\ {}^{1}/_{2}$

FP Multiplication

- Operands

  $(-1)^{s1}\ M1\ \ 2^{E1}$ * $(-1)^{s2}\ M2\ \ 2^{E2}$

- Exact Result

  $(-1)^{s}\ M\ \ 2^{E}$

  - Sign s: s1 ^ s2
  - Significand M: M1 * M2
  - Exponent E: E1 + E2

- Fixing
  - If M ≥ 2, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit frac precision

- Implementation
  - Biggest chore is multiplying significands

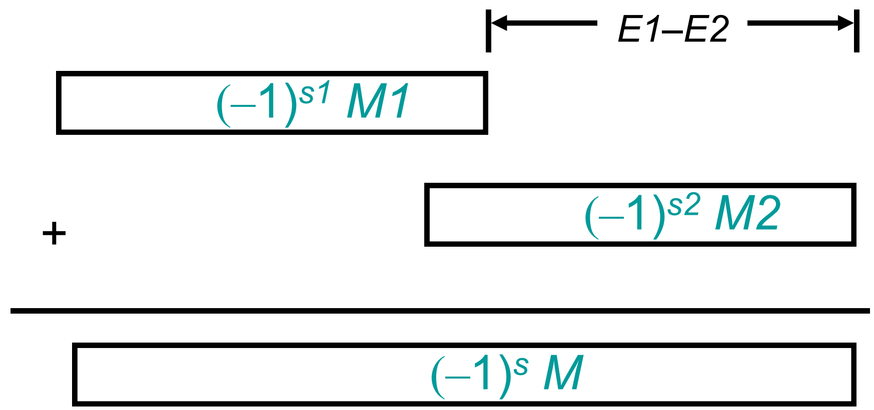
FP Addition

- Operands

  $(-1)^{s1}\ M1\ \ 2^{E1}$

  $(-1)^{s2}\ M2\ \ 2^{E2}$

  – Assume E1 > E2



- Exact Result

  $(-1)^{s}\ M\ \ 2^{E}$

  – Sign s, significand M:
    - Result of signed align & add

  – Exponent E: E1

2e^5 + 4.3e^-6 = you have to align them and then do the addition.
You need to shift them so they are the same order of amgnitude.

- Fixing
  - If $M \geq 2$, shift M right, increment E
  - if $M < 1$, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit frac precision

Mathematical Properties of FP Add

- Compare to those of Integers
  - Closed under addition? YES
    - But may generate infinity or NaN
  - Commutative? YES
  - Associative? NO
    - Overflow and inexactness of rounding
  - 0 is additive identity? YES
  - Every element has additive inverse ALMOST
    - Except for infinities & NaNs
- Monotonicity
  - $a \geq b \Rightarrow a+c \geq b+c$? ALMOST
    - Except for infinities & NaNs

Math. Properties of FP Mult

- Compare to Commutative Ring
  - Closed under multiplication? YES
    - But may generate infinity or NaN
  - Multiplication Commutative? YES
  - Multiplication is Associative? NO
    - Possibility of overflow, inexactness of rounding
  - 1 is multiplicative identity? YES
  - Multiplication distributes over addition? NO
    - Possibility of overflow, inexactness of rounding
- Monotonicity
  - $a \geq b \;\&\; c \geq 0 \Rightarrow a * c \geq b * c$? ALMOST
    - Except for infinities & NaNs

Floating Point in C

- C Guarantees Two Levels
  - float single precision
  - double double precision
- Conversions
  - Casting between int, float, and double changes numeric values
  - Double or float to int
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range
      - Generally saturates to TMin or TMax
  - int to double
    - Exact conversion, as long as int has ≤ 53 bit word size
  - int to float
    - Will round according to rounding mode

Floating Point Puzzles

– For each of the following C expressions, either:

- Argue that it is true for all argument values
- Explain why not true

```
int x = …;

float f = …;

double d = …;
```

Assume neither `d` nor `f` is NaN

- `x == (int)(float) x`
- `x == (int)(double) x`
- `f == (float)(double) f`
- `d == (float) d`
- `f == -(-f);`
- `2/3 == 2/3.0`
- `d < 0.0` $\Rightarrow$ `((d*2) < 0.0)`
- `d > f` $\Rightarrow$ `-f > -d`
- `d * d >= 0.0`
- `(d+f)-d == f`

Answers to Floating Point Puzzles

```
int x = …;

float f = …;

double d = …;
```

Assume neither `d` nor `f` is NAN

• `x == (int)(float) x`	**No: 24 bit significand**
• `x == (int)(double) x`	**Yes: 53 bit significand**
• `f == (float)(double) f`	**Yes: increases precision**
• `d == (float) d`	**No: loses precision**
• `f == -(-f);`	**Yes: Just change sign bit**
• `2/3 == 2/3.0`	**No: 2/3 == 0**
• `d < 0.0` ⇒ `((d*2) < 0.0)`	**Yes!**
• `d > f` ⇒ `-f > -d`	**Yes!**
• `d * d >= 0.0`	**Yes!**
• `(d+f)-d == f`	**No: Not associative**

Ariane 5

- Exploded 37 seconds after liftoff
- Cargo worth $500 million

- Why
  - Computed horizontal velocity as floating point number
  - Converted to 16-bit integer
  - Worked OK for Ariane 4
  - Overflowed for Ariane 5
    - Used same software

Summary

- IEEE Floating Point Has Clear Mathematical Properties
  - Represents numbers of form M **X** 2^E
  - Can reason about operations independent of implementation
    - As if computed with perfect precision and then rounded
  - Not the same as real arithmetic
    - Violates associativity/distributivity
    - Makes life difficult for compilers & serious numerical applications programmers