1 Setup

Fix G, a directed erdos-renyi graph, let Dom_G be the set of graphs that have the same degree sequence as G. Typically, we will use H to be a member of this set. Additionally, let $p(H)$ be profit vector of H. Throughtout this analysis all parameters are fixed. $\mathbb{P}\diagdown$**Primary Questions**

1. What is $\|H + H^T\|$(What norm is the right norm is a good question)

2. Let q(H) be an altered price vector, where we fix the norms terms. What is the distribution of q(H)

3. What is probability distribution of these price vectors in the doman H.

We want to resolve the question of what is $Pr[v(H)_i = k]$

$$\begin{aligned}
\mathbb{P}\diagdown[v(H)_i =] &= \mathbb{P}\diagdown[\frac{a-c}{2} + \frac{a-c}{2}\frac{\rho}{\|H+H^T}K(H+H^T, \frac{\rho}{\|H+H^T\|}) = k] \\
&= \mathbb{P}\diagdown[\frac{1}{\|H+H^T\|}K(H^T+H, \frac{\rho}{\|H+H^T}) = \underbrace{(\frac{2}{\rho(a-c)}[k - \frac{a-c}{2}])}_{k'}]
\end{aligned}$$

1.1 Number of kth neighbors

We will define two generating functions.

The first will be $g_0(z) = \sum_{k=0} p_k z^k$ where p_k is the probabality of a given node having degree k. We know that $p_k \approx \frac{e^{-\lambda}\lambda^k}{k!}$ where $\lambda = (n-1)p$. Therefore $g_0(z) = \sum_{k=0} \frac{e^{-\lambda}\lambda^k}{k!} z^k = e^{-\lambda}e^{z\lambda}$.

The next generating function will be of the distribution of the degree size of a neighboring vertices. We analyzed this in networks class. We will call if

$g_1 = (z)$ and the probability distribution of this neighbor $p_k^{(2)}$. We have

$$\begin{aligned} g_1(z) &= \sum_{k=0} p_k^{(k)} z^k \\ &= \sum_{k=0} \frac{kp_k}{E[d]} z^k \\ &= \frac{1}{E[d]} \sum_{k=0} kp_k z^k \\ &= \frac{1}{E[d]} zD_z \sum_{k=0} p_k z^k \\ &= \frac{1}{E[d]} zD_z (e^{-\lambda} e^{z\lambda}) \\ &= \frac{ze^{-\lambda}\lambda e^{z\lambda}}{E[d]} \end{aligned}$$

To see why these are helpful, we will now derive the size of 2 distance neighbors from a node which we will call d_2.

$$\begin{aligned} g_2(k) &= \sum p_m^{(2)} * z^m \qquad \underbrace{P_2(k|m)}_{\text{Distribution of 2 neighbors given starting node had degree m}} \\ &= \sum_m p_m^{(2)} \left(\sum_{x_1+x_2+\cdots+x_m=k} \prod_j Pr[q_{x_j}] \right) \\ &= \sum_m p_m^{(2)} (\sum q_i z^i)^m \\ &= \sum_m p_m^{(2)} (d_1(z))^m \\ &= g_0(g_1(z)) \end{aligned}$$

To make this transition clear, consider what $(\sum_{i=0}^{\infty} q_i z^i)(\sum_{i=0}^{\infty} q_i z^i)$ organized by terms of z. The term of z^k will have coefficient $(q_0 q_k + q_1 q_{k-1} + \cdots + q_{k-1} q_1 + q_k q_0) = \sum_{i+j=k} q_i q_j$.

For us this is interesting because $g_0(z)$ is completely fixed by the true graph G(or said alternatively is exactly the same for all the H graphs with the same degree sequence) but g_1 was not fixed directly by H. d_1 is a random function dependent on the graph it is based on. I'm not sure how this g_1 correlates with the first degree information which we fixed. Importantly, is the g_1 functions the same for our graphs vs all erdos-renyi graphs with the same n and p or does this also depend indirectly on the information we fixed. Ideally it is the first case and if so we would have shown that walk sizes are a function of the true graph's information applied to a random variable which if things are nice only depends on n and p.

We can continue this for higher distance walks. Now we will use $P_3(k|m)$ to be the probability that a node has k 3 distance given it has m 2-distance neighbors. The three step generating function, $g_3(z)$, can be expressed as

$$\begin{aligned}
g_3(z) &= \sum_k \sum_m p_m^2 P_3(k|m) z^k \\
&= \sum_m p_m^2 \sum_{x_1+x_2+\cdots x_m=k} z^k \prod_j^m Pr[q_{x_j}] \\
&= \sum_m p_m^2 g_1(z)^m \\
&= g_2(g_1(z)) = g_0(g_1(g_1(z)))
\end{aligned}$$

In general, the generating function representing the walks of length ℓ is $g_0(g_1^{\ell-1}(z))$

These have been very helpful and contains much of this analysis. https://people.cs.clemson.edu/ isafr https://static.squarespace.com/static/5436e695e4b07f1e91b30155/t/5445263ee4b0d3d410795e1f/14138 graphs-with-arbitrary-degree-distributions-and-their-applications.pdf

2 Norms

Currently unsure how I can bound the norms more tightly than what was done in the price discrimination paper.