PHYS 361: Lab 8, Solving ODE's Part 1

Background

This week, we learned some methods to do something really powerful: evolve differential equations. Many critical physical processes (weather, climate, chemical reactions, rocket flight, pandemics, etc.) can be modeled with differential equations that can't be solved analytically. We rely on computer models to solve them for us. Euler's and Runge-Kutta are the most commonly used methods, and many more advanced techniques differ slightly. For example, they might automatically determine an appropriate time step.

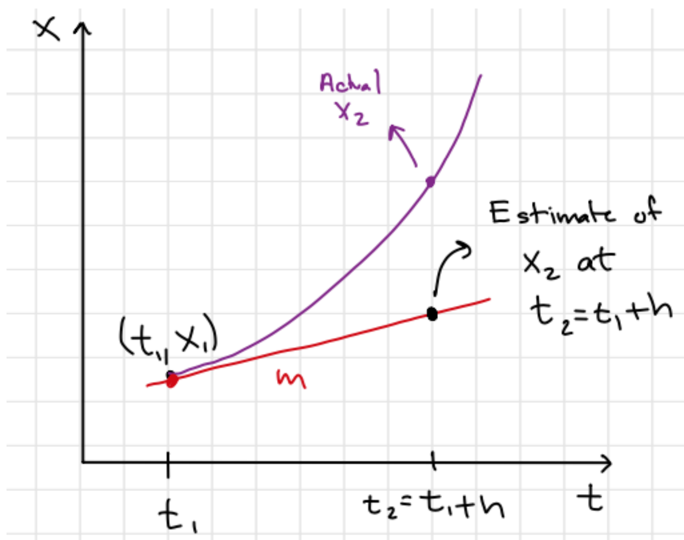

Figure 1: Euler's Method

Euler's Method

Euler's Method is the easiest method to understand. A first-order differential equation describes the rate-of-change of some system. It could describe how position changes with time (e.g., $\frac{dx}{dt}$), temperature changes with position (e.g., $\frac{dT}{dx}$), etc. On a graph, this rate of change is called the slope. In Euler's Method, we use the right-hand side of our differential equation to calculate the slope at some initial value. Then we assume that the rate of change is good for some small distance or time (or whatever we are differentiating with respect to), and we try to estimate the next value.

Let's define a general 1st-order differential equation,

$$\frac{dx}{dt} = f(t, x), \tag{1}$$

where $f(t, x)$ (or the rate of change of x with respect to t) is some to-be-defined function that can depend on both x and t. If we know an initial value of x at an initial t, we can estimate the value of x at some future time $t+h$ by assuming the rate of change or slope of the function is constant over the small period h.

$$x(t+h) = x(t) + f(t,x)h. \tag{2}$$

Now you can use x(t+h) and t+h to calculate the slope again using $f(t,x)$. You can keep taking these steps to estimate how x changes with time.

Steps

1. Use $f(t,x)$ to calculate the slope, m, at $x(t)$
2. Use the slope, m, to take a step, h, to estimate the value of the function at $t+h$

Algorithm

1. $m = f(t,x)$
2. $x(t+h) = x(t) + m \times h$

If your step size, h, is too big, assuming that the slope or rate of change is constant over that time step is invalid. In that case, your numerical estimate for how the x changes with time might diverge from the exact solution. A very small, h, will do much better but require more computing time.

2nd-Order Runge-Kutta

The 2nd-order Runge-Kutta method is a little more complex but more effective than Euler's. Euler's method only uses the slope at the initial value to determine the next value, $x(t+h)$. The 2nd-order Runge-Kutta method calculates the slope at two locations and uses the estimated slope in the middle of the step, h, to determine $x(t+h)$.

Steps

1. Use the slope at $x(t)$, k_1, to estimate the value of x at the middle of your step size h
2. Plug the middle point values, $x(t+h/2)$ and $t+h/2$, into $f(t,x)$ to estimate the slope at the mid-point, k_2.
3. Go back to $x(t)$ and use the mid-point slope, k_2, to take the full step.

Algorithm

1. $k_1 = f(t,x)$
2. $k_2 = f(t + \frac{1}{2}h, x + \frac{1}{2}hk_1)$
3. $x(t+h) = x(t) + hk_2$

Since there are more values to calculate each time you step the solution forward, this method is slightly more "computationally expensive" but is much more accurate than the Euler Method.

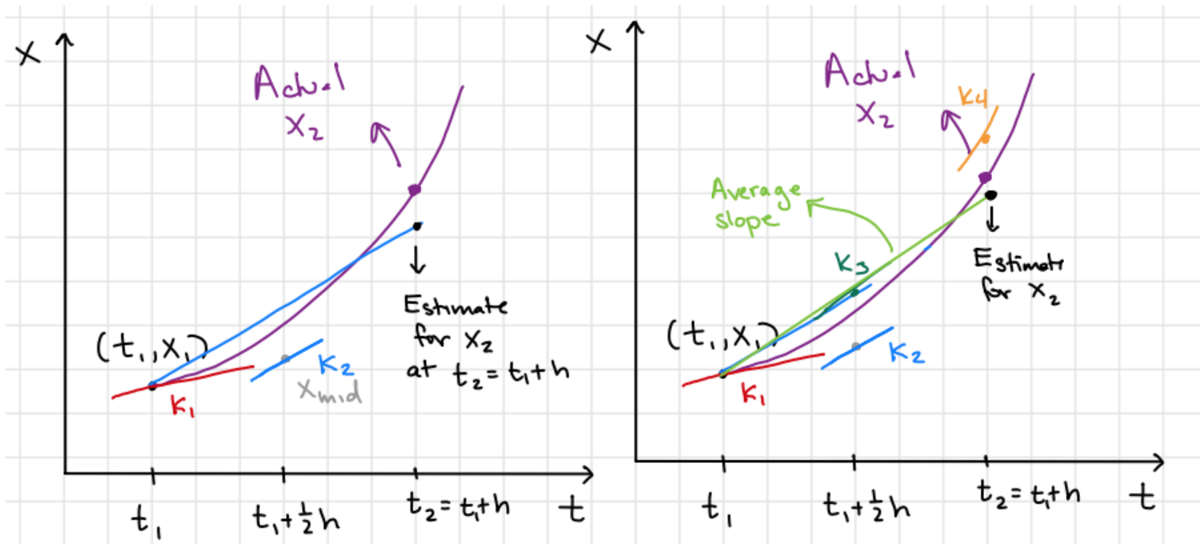

Figure 2: 2nd-order Runge-Kutta Method (left) and 4th-order Runge-Kutta Method (right)

4th-Order Runge-Kutta

The 4th-order Runge-Kutta method is even more accurate but requires roughly twice as much computing time. Instead of estimating the slope at the mid-point of h, it uses weighted estimates of the slope at four locations to take a step. The weighted slope is calculated from the slopes estimated at the beginning of the step or x(t), two estimates at the mid-point, and another at the end of the step.

Steps

1. Use the slope at $x(t)$, k_1, to estimate the value of x at the middle of your step size h.

2. Plug the middle point values, $x(t + h/2)$ and $t + h/2$, into $f(t, x)$ to estimate the slope at the mid-point, k_2.

3. Go back to $x(t)$ and use the mid-point slope, k_2, to another half step and find a new estimate of the value of x at the middle of your step size h

4. Plug the new middle point values, $x(t + h/2)$ and $t + h/2$, into $f(t, x)$ to estimate the slope at the mid-point again, k_3.

5. Go back to $x(t)$ and use the new mid-point slope, k_3, to take a full step and find a new estimate of the value of x at the end of the step size h

6. Plug the end point values, $x(t + h)$ and $t + h$, into $f(t, x)$ to estimate the slope at the end point, k_4.

7. Use a weighted average of the four slopes you calculated (k_1, k_2, k_3, and k_4) to estimate the average slope over the whole step. More heavily weigh the estimates at the mid-point.

8. Go back to $x(t)$ and use the weighted average slope to take the full step.

Luckily, the algorithm is a little less wordy.
Algorithm

1. $k_1 = f(t, x)$

2. $k_2 = f(t + \frac{1}{2}h, x + \frac{1}{2}hk_1)$

3. $k_3 = f(t + \frac{1}{2}h, x + \frac{1}{2}hk_2)$

4. $k_4 = f(t + h, x + hk_3)$

5. $x(t + h) = x(t) + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$

This weighted slope is a better estimate for the rate of change of the system over the step, h. This extra accuracy may be necessary depending on what you are trying to model. However, many models value computational speed and use the 2nd-order Runge-Kutta method instead.

Directions: You should write a single script (m-file) for each problem below. Put the number of the problem in the name of your script. Include all user-built functions at the bottom of your script. Comment on your code and follow the provided example script format (an example can be found on Canvas).

1. If you haven't already, solve the radioactive decay problem using your user-defined Euler's algorithm.

$$\frac{dN}{dt} = -\gamma N \tag{3}$$

   Make a single figure (with four panels) showing the exact solution ($N(t) = N(0)e^{-\gamma t}$) and the numerical solution for the following conditions:

   (a) N(0)=1, γ=0.05, h=1

   (b) N(0)=1, γ=0.05, h=5

   (c) N(0)=1, γ=0.05, h=10

   (d) N(0)=1, γ=0.05, h=50

2. An inductor and a nonlinear resistor with resistance $R = 500 + 250I^2\ \Omega$ are connected in series with a DC power source and a switch, as shown in the figure on the next page. The switch is initially open and then closed at $t = 0$ seconds. The current I in the circuit for $t > 0$ is determined from the solution of the equation:

$$\frac{dI}{dt} = \frac{V_0}{L} - \frac{R}{L}I. \tag{4}$$

   For $V_0 = 500\ V$ and $L = 15$ Henries, determine and plot the current as a function of time for $0 \leq t \leq 0.1\ s$. Solve the problem using your 2nd-order Runge-Kutta Function. Use a time step of 0.005 s.

3. An inductor $L = 15\ H$ and a resistor $R = 1000\ \Omega$ are connected in series with an AC power source providing voltage of $V = 10sin(2\pi ft)\ V$, where $f = 100\ kHz$, as shown in the figure on the next page. The current I in the circuit is determined from the solution of the equation:

$$\frac{dI}{dt} = \frac{10sin(2\pi ft)}{L} - \frac{R}{L}I. \tag{5}$$

   Solve the equation and plot the current as a function of time for $0 \leq t \leq 10^{-4}\ s$ with $I(0) = 0$. Solve the problem using your 4th-order Runge-Kutta Function. Use a time step of $1 \times 10^{-9}\ s$. (Note that the ODE depends on the current and time, and be careful with units in the sin function.)

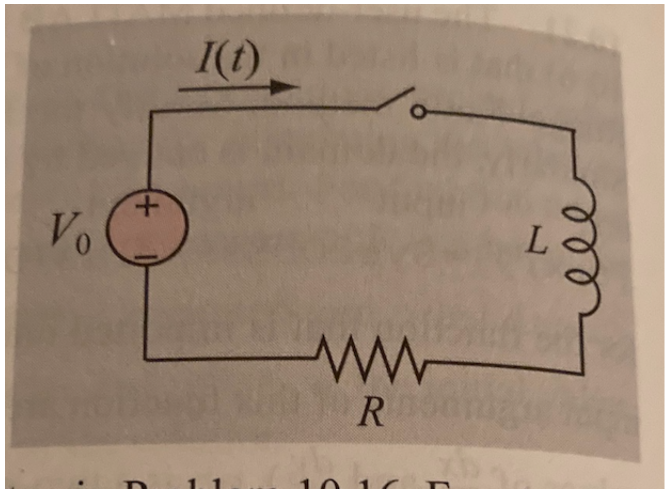

Figure 3: DC inductor-resistor circuit

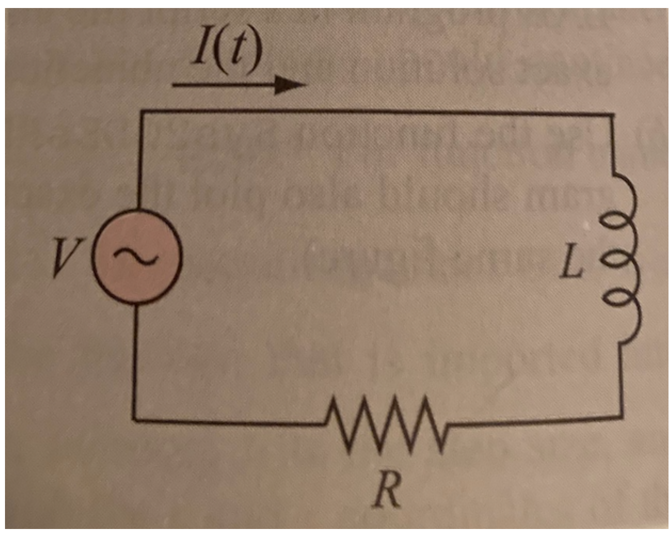

Figure 4: AC inductor-resistor circuit