1. A particle is trapped in a harmonic oscillator potential. We know that at $t = 0$, the particle can be represented by the wavefunction

$$\Psi(x, 0) = A\left(2\psi_0(x) + 5\psi_2(x)\right),$$

where $\psi_0$ and $\psi_2$ are the stationary-state solutions for $n = 0$ and $n = 0$, respectively.

(a) Normalize $\Psi(x, 0)$.

$$1 = \langle \Psi(x,0) \mid \Psi x, 0\rangle = A^2 \left(4\langle \psi_0 \mid \psi_0\rangle + 25\langle \psi_2 \mid \psi_2\rangle + 20\langle \psi_0 \mid \psi_2\rangle\right) = 29A^2.$$

Thus,

$$A = \sqrt{\frac{1}{29}}$$

(b) Construct $\Psi(x, t)$ and then determine $|\Psi(x, t)|^2$. Will $\langle x\rangle$ depend on time?

The complete wavefunction is given by

$$\Psi(x, t) = \sqrt{\frac{1}{29}}\left(2\Psi_0 + 5\Psi_2\right),$$

where $\Psi_n$ represents $\psi_n(x)\phi_n(t)$. Thus,

$$\begin{aligned}
|\Psi(x,t)|^2 &= \frac{1}{29}\left(2\Psi_0^* + 5\Psi_2^*\right)\left(2\Psi_0 + 5\Psi_2\right)\\
&= \frac{1}{29}\left(4\left|\Psi_0\right|^2 + 2\Psi_0^*\Psi_2 + 5\Psi_2^*\Psi_0 + 25\left|\Psi_2\right|^2\right)\\
&= \frac{1}{29}\left(4\left|\psi_0\right|^2 + 20\psi_0\psi_2\left(e^{-2i\omega t} + e^{2i\omega t}\right) + 25\left|\psi_2\right|^2\right)\\
&= \frac{1}{29}\left(4\left|\psi_0\right|^2 + 10\psi_0\psi_2\cosh(2i\omega t) + 25\left|\psi_2\right|^2\right)\\
&= \frac{1}{29}\left(4\left|\psi_0\right|^2 + 10\psi_0\psi_2\cos(2\omega t) + 25\left|\psi_2\right|^2\right)\\
&= \frac{1}{29}\left(4\left|\psi_0\right|^2 + \frac{10}{\sqrt{2}}\psi_0\widehat{a_+}^2\psi_0\cos(2\omega t) + 25\left|\psi_0\right|^2\right)
\end{aligned}$$

$\langle x\rangle$ will depend on time!

2. Consider the stationary states of the harmonic oscillator. As straightforwardly as possible, compute the following quantities for the $n$th stationary state $\psi_n(x)$.

   (a) $\langle x \rangle$.

   Given
$$\widehat{x} = \sqrt{\frac{\hbar}{2m\omega}} \left( \widehat{a_+} + \widehat{a_-} \right).$$
   Then,
$$\langle x \rangle = \langle n | \widehat{x} | n \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left( \langle n | \widehat{a_+} | n \rangle + \langle n | \widehat{a_-} | n \rangle \right) = 0.$$

   (b) $\langle x^2 \rangle$.

   Given
$$\begin{aligned}
\widehat{x^2} &= \widehat{x}^2 \\
&= \frac{\hbar}{2m\omega} \left( \widehat{a_+}^2 + \widehat{a_-}^2 + \widehat{a_-}\widehat{a_+} + \widehat{a_+}\widehat{a_-} \right) \\
&= \frac{\hbar}{2m\omega} \left( \widehat{a_+}^2 + \widehat{a_-}^2 + \frac{1}{\hbar\omega}\left( \widehat{H} + \frac{1}{2} \right) + \frac{1}{\hbar\omega}\left( \widehat{H} - \frac{1}{2} \right) \right) \\
&= \frac{\hbar}{2m\omega} \left( \widehat{a_+}^2 + \widehat{a_-}^2 + \frac{2}{\hbar\omega}\widehat{H} \right).
\end{aligned}$$
   Then,
$$\left\langle x^2 \right\rangle = \langle n | \widehat{x^2} | n \rangle = \frac{\hbar}{2m\omega} \left( \langle n | \widehat{a_+}^2 | n \rangle + \langle n | \widehat{a_-}^2 | n \rangle + \frac{2}{\hbar\omega} \langle n | \widehat{H} | n \rangle \right) = \frac{E_n}{m\omega^2}.$$

   (c) $\langle p \rangle$.
$$\langle p \rangle = m \frac{\mathrm{d}}{\mathrm{d}t} \langle x \rangle = m \frac{\mathrm{d}}{\mathrm{d}t}(0) = 0.$$

   (d) $\langle p^2 \rangle$

   Given
$$\langle H \rangle = \frac{\langle p^2 \rangle}{2m} = E_n.$$
   Then,
$$\left\langle p^2 \right\rangle = 2m \langle H \rangle = 2mE_n.$$

   (e) $\langle T \rangle$

   Given
$$\langle H \rangle = \langle T \rangle + \langle V \rangle = \langle T \rangle + \frac{1}{2} m\omega^2 \left\langle x^2 \right\rangle.$$
   Then,
$$\langle T \rangle = \langle H \rangle - \langle V \rangle = E_n - \frac{1}{2} m\omega^2 \frac{E_n}{m\omega^2} = \frac{1}{2} E_n.$$

(f) Is the Heisenberg uncertainty principle satisfied for all values of $n$?

The uncertainty of $x$, $\sigma_x$, is given by

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{E_n}{m\omega^2}}.$$

The uncertainty of $p$, $\sigma_p$, is given by

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{2mE_n}.$$

The product of uncertainties is given by

$$\sigma_x \sigma_p = \sqrt{\frac{E_n}{m\omega^2}} \sqrt{2mE_n} = \frac{E_n}{\omega}\sqrt{2} = \hbar \left( n + \frac{1}{2} \right) \sqrt{2}.$$

The Heisenberg uncertainty principle requires that

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}.$$

Where a factor of $\hbar/2$ appears on both sides thus simplifying to

$$(2n+1)\sqrt{2} \geq 1.$$

The left side increases with $n$ so we check only the smallest $n$, that is $n = 0$.

$$\sqrt{2} \geq 1,$$

is indeed true. Whence, the uncertainty principle is satisfied for all $n$.

3. A particle in a harmonic oscillator potential is described by the normalized wavefunction

$$|\Psi(x,0)\rangle = \frac{1}{\sqrt{5}}|1\rangle + \frac{2}{\sqrt{5}}|2\rangle,$$

where $|n\rangle$ represents the $n$th stationary state.

   (a) What is $|\Psi(x,t)\rangle$?

   (b) What is the expectation value for energy?

   (c) What is $\langle x(t)\rangle$?