1. Use induction to prove that $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{Z}^+$.

$$\forall\, n \in \mathbb{Z}^+ \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Proof. Let $P(n)$ be the statement

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4},$$

for all $n \in \mathbb{Z}^+$. We proceed with a proof by induction on n.

Base case: $P(1)$ is the statement

$$\sum_{k=1}^{1} k^3 = \frac{1^2(1+1)^2}{4}.$$

Then

$$\sum_{k=1}^{1} k^3 = 1,$$

and

$$\begin{aligned} \frac{1^2(1+1)^2}{4} &= \frac{1(2)^2}{4}, \\ &= \frac{4}{4}, \\ &= 1. \end{aligned}$$

Thus, $P(1)$ is true.

Induction step: Suppose $P(n)$, that is,

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Then

$$\begin{aligned}
\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3, \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3, \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4}, \\
&= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4}, \\
&= \frac{(n+1)^2(n+2)^2}{4}, \\
&= \frac{(n+1)^2(n+1+1)^2}{4}.
\end{aligned}$$

Thus, from $P(n)$ we have proven $P(n+1)$. Therefore, by induction, $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{Z}^+$. $\square$

2. Prove that $4|(6^n - 2^n)$ for all $n \in \mathbb{Z}$.

$$\forall\, n \in \mathbb{Z}^+\, 4|(6^n - 2^n).$$

Proof. Let $P(n)$ be the statement

$$4|(6^n - 2^n),$$

for all $n \in \mathbb{Z}^+$. We proceed with a proof by induction on n.
Base case: $P(1)$ is the statement

$$4|(6^1 - 2^1).$$

Let $k \in \mathbb{Z}$ be defined as

$$k = 1.$$

Then

$$\begin{aligned} 6^1 - 2^1 &= 6 - 2, \\ &= 4, \\ &= 4(1), \\ &= 4k. \end{aligned}$$

Thus, $4|(6^1 - 2^1)$, $P(1)$ is true.
Induction step: Suppose $P(n)$, that is

$$4|(6^n - 2^n),$$

that there exists $k \in \mathbb{Z}$ such that

$$4k = 6^n - 2^n.$$

Then $P(n+1)$ is the statement

$$4|(6^{n+1} - 2^{n+1}).$$

Let $q \in \mathbb{Z}$ be defined as

$$q = 6^n + 2k.$$

Then

$$\begin{aligned}
6^{n+1} - 2^{n+1} &= 6 \cdot 6^n - 2 \cdot 2^n, \\
&= (4+2)6^n - 2 \cdot 2^n, \\
&= 4 \cdot 6^n + 2 \cdot 6^n - 2 \cdot 2^n, \\
&= 4 \cdot 6^n + 2(6^n - 2^n), \\
&= 4 \cdot 6^n + 2(4k), \\
&= 4(6^n + 2k), \\
&= 4q.
\end{aligned}$$

Thus, $4|(6^{n+1} - 2^{n+1})$. Therefore, from $P(n)$ we have proven $P(n+1)$. Thusly, by induction, $4|(6^n - 2^n)$ for all $n \in \mathbb{Z}^+$. $\square$

3. Let $r \neq 1$ be a real number. Use induction to show that for any $m \in \mathbb{Z}^+$, $\sum_{k=m}^{n} r^k = \frac{r^m - r^{m+1}}{1-r}$ for all $n \in \mathbb{Z}^+$ with $n \geq m$.

$$\forall\, r \in \mathbb{R} \left((r \neq 1) \to \forall\, n, m \in \mathbb{Z}^+ \left((n \geq m) \to \left(\sum_{k=m}^{n} r^k = \frac{r^m - r^{n+1}}{1-r} \right) \right) \right).$$

Proof. Let $r \in \mathbb{R}$ be given such that $r \neq 1$. We proceed with a proof by induction on n.

Base case: We will prove $P(1)$, that is $n = 1$. There is only one $m \in \mathbb{Z}^+$ such that $n \geq m$, that is $m = 1$. Then, $P(1)$ is the statement

$$\sum_{k=1}^{1} r^k = \frac{r^1 - r^{1+1}}{1-r}.$$

Then

$$\sum_{k=1}^{1} r^k = r,$$

and

$$\begin{aligned} \frac{r^1 - r^{1+1}}{1-r} &= \frac{r(1-r)}{1-r}, \\ &= r. \end{aligned}$$

Thus, we have proven $P(1)$.

Induction step: Suppose $P(n)$, that is

$$\sum_{k=m}^{n} r^k = \frac{r^m - r^{n+1}}{1-r}.$$

Then $P(n+1)$ is the statement

$$\sum_{k=m}^{n+1} r^k = \frac{r^m - r^{n+2}}{1-r}.$$

Then

$$
\begin{aligned}
\sum_{k=m}^{n+1} r^k &= \sum k = m^n r^k + r^{n+1}, \\
&= \frac{r^m - r^{n+1}}{1-r} + r^{n+1}, \\
&= \frac{r^m - r^{n+1}}{1-r} + \frac{(1-r)}{(1-r)} r^{n+1}, \\
&= \frac{r^m - r^{n+1} + (1-r) r^{n+1}}{1-r}, \\
&= \frac{r^m - r^{n+1} + r^{n+1} - r \cdot r^{n+1}}{1-r}, \\
&= \frac{r^m - r^{n+1+1}}{1-r}, \\
&= \frac{r^m - r^{n+2}}{1-r}.
\end{aligned}
$$

Thus we have proven $P(n+1)$ by supposing $P(n)$. Therefore, by induction, we prove

$$
\forall\, r \in \mathbb{R} \left((r \neq 1) \to \forall\, n, m \in \mathbb{Z}^+ \left((n \geq m) \to \left(\sum_{k=m}^{n} r^k = \frac{r^m - r^{n+1}}{1-r} \right) \right) \right).
$$

$\square$

I couldn't think of an elegant way to restate it.

4. Let $A = \{x \in \mathbb{Z} : x \bmod 15 = 10\}$ and $B = \{x \in \mathbb{Z} : x \bmod 3 = 1\}$.

   (a) Prove $A \subseteq B$.

$$\forall\, x \in \mathbb{Z}\ (x \in A \to x \in B)\,.$$

   Proof. Let $x \in \mathbb{Z}$ be given. Suppose $x \in A$, that is, $x \bmod 15 = 10$. Then, by definition, there exists $q \in \mathbb{Z}$ such that

$$x = 15q + 10.$$

   Let $k \in \mathbb{Z}$ be defined as

$$k = 5q + 9.$$

   Then

$$\begin{aligned} x &= 15q + 10, \\ &= 15q + 9 + 1, \\ &= 3(5q + 9) + 1, \\ &= 3k + 1. \end{aligned}$$

   Thus, $x \bmod 3 = 1$. Therefore, if $x \in A$ then $x \in B$. Thus, $A \subseteq B$. □

   (b) Either show that $B \subseteq A$, or explain why $B \not\subseteq A$.
   $B \not\subseteq A$. Suppose $x = 1$. Then $1 \bmod 3 = 1$ but $1 \bmod 15 \neq 10$. Thus, $\forall x \in \mathbb{Z}\,(x \in B \to x \in A)$ is not true, so $B \not\subseteq A$.

5. Suppose A and B are subsets of a universe $\mathcal{U}$. Show that if $A \subseteq B^c$, then $A \cap B = \emptyset$.

Proof. Let A and B be subsets of a universe $\mathcal{U}$. We will use the contrapositive, that is, if $A \cap B \neq \emptyset$ then $A \not\subseteq B^c$. Suppose $A \cap B \neq \emptyset$. Then, there exists some $x \in \mathcal{U}$ such that $x \in A \cap B$. Thus, $x \in A$ and $x \in B$. Equivalently, $x \in A$ and $x \notin B^c$. Therefore, if $A \cap B \neq \emptyset$ then $A \not\subseteq B^c$. By the contrapositive, if $A \subseteq B^c$ then $A \cap B = \emptyset$. □