1. Consider a free particle that is described at $t = 0\,\mathrm{s}$ by the wavefunction, $\Psi(x, 0) = Ae^{-a|x|}$ where both A and a are positive, real constants.

   (a) Normalize $\Psi(x, 0)$.

   We begin by expressing Ψ as a piecewise function:

   $$\Psi(x, 0) = \begin{cases} Ae^{-ax}, & x > 0 \\ Ae^{ax}, & x < 0 \end{cases}$$

   Now, Ψ can be normalized by the typical method:

   $$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\Psi|^2 \, \mathrm{d}x \\ &= A^2 \int_{-\infty}^{0} e^{2ax} \, \mathrm{d} + A^2 \int_{0}^{\infty} e^{-2ax} \, \mathrm{d}x \\ &= A^2 \left(\left[\frac{1}{2a} e^{2ax} \right]_{-\infty}^{0} + \left[-\frac{1}{2a} e^{-2ax} \right]_{0}^{\infty} \right) \\ &= A^2 \left(\frac{1}{2a} - 0 + 0 + \frac{1}{2a} \right) \\ &= A^2 \frac{1}{a}. \end{aligned}$$

   Thus, $A = \sqrt{a}$.

   (b) Determine $\phi(k)$.

   (c) Construct $\Psi(x, t)$ in the form of an integral.

   (d) Evaluate the integral for $\Psi(x, t)$ in the limiting cases of a very large a and a very small a.

2. As we discussed in class, the time-independent Schrödinger equation for the free particle has solutions that look like $Ae^{ikx} + Be^{-ikx}$ or like $C\cos(kx) + D\sin(kx)$. Show that these are equivalent solutions. Determine what the constants C and D are as a function of A and B and vice versa.

3. Consider a bead with mass m that slides frictionlessly around a circular wire ring with circumference L. We can think about this problem like a free particle assuming boundary condition of the form $\psi(x + L) = \psi(x)$. Determine the normalized stationary states and their corresponding energies. You should find two distinct solutions for each energy (so there is a two-fold degeneracy in this system). These two states represent clockwise and counter-clockwise rotation.

4. Consider a particle interacting with a potential energy given by

$$V(x) \begin{cases} \infty, & x < 0 \\ -\alpha\delta(x-a), & x > 0 \end{cases}$$

(a) Determine the bound-state solutions (assume $E < 0$) to the time-independent Schrödinger equation in three different regions of x: $x < 0$, $0 \le x \le a$, and $x > a$. Define $K^2 = -2mE/\hbar^2$ in your solutions.

(b) Demand continuity of the wavefunctions at $x = 0$ and $x = a$ to reduce the number of unknown constant parameters.

(c) Show that the discontinuity in the derivative of the wavefunctions at $x = a$ is given by

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}x}\right|_{a+\epsilon} - \left.\frac{\mathrm{d}\psi}{\mathrm{d}x}\right|_{a-\epsilon} = -\frac{2m\alpha}{\hbar^2}\psi(a).$$

(d) Using the boundary condition from part (c), show that

$$\frac{K\hbar^2}{m\alpha} = 1 - e^{-2Ka}.$$

This transcendental equation relates K with α. How many different K values will solve the equation? You might try to graph the left-hand and right-hand equations vs. K to see where they intersect.