1. A particle is trapped in a harmonic oscillator potential. We know that at $t = 0$, the particle can be represented by the wavefunction

$$\Psi(x,0) = A\left(2\psi_0(x) + 5\psi_2(x)\right),$$

where ψ_0 and ψ_2 are the stationary-state solutions for $n = 0$ and $n = 0$, respectively.

(a) Normalize $\Psi(x,0)$.

$$1 = \langle \Psi(x,0) \,|\, \Psi x, 0\rangle = A^2\left(4\langle \psi_0 \,|\, \psi_0\rangle + 25\langle \psi_2 \,|\, \psi_2\rangle + 20\langle \psi_0 \,|\, \psi_2\rangle\right) = 29A^2.$$

Thus,

$$A = \sqrt{\frac{1}{29}}$$

(b) Construct $\Psi(x,t)$ and then determine $|\Psi(x,t)|^2$. Will $\langle x\rangle$ depend on time?

The complete wavefunction is given by

$$\Psi(x,t) = \sqrt{\frac{1}{29}}\left(2\Psi_0 + 5\Psi_2\right),$$

where Ψ_n represents $\psi_n(x)\phi_n(t)$. Thus,

$$\begin{aligned}
|\Psi(x,t)|^2 &= \frac{1}{29}\left(2\Psi_0^* + 5\Psi_2^*\right)\left(2\Psi_0 + 5\Psi_2\right)\\
&= \frac{1}{29}\left(4\left|\Psi_0\right|^2 + 2\Psi_0^*\Psi_2 + 5\Psi_2^*\Psi_0 + 25\left|\Psi_2\right|^2\right)\\
&= \frac{1}{29}\left(4\psi_0^*\psi_0 + 10\psi_0^*\psi_2 e^{-2i\omega t} + 10\psi_2^*\psi_0 e^{2i\omega t} + 25\psi_2^*\psi_2\right)\\
&= \frac{1}{29}\left(4\psi_0^2 + 10\psi_0\psi_2\left(e^{-2i\omega t} + e^{2i\omega t}\right) + 25\psi_2^2\right)\\
&= \frac{1}{29}\left(4\psi_0^2 + 20\psi_0\psi_2\cosh(2i\omega t) + 25\psi_2^2\right)\\
&= \frac{1}{29}\left(4\psi_0^2 + 20\psi_0\psi_2\cos(2\omega t) + 25\psi_2^2\right)
\end{aligned}$$

The expectation value of position will not depend on time. The expectation value of position is only time dependent if the two wavefunctions are within $\pm n$ of each other because the position operator includes a raising and lowering operator. As it stands now, applying the position operator and integrating over all space will result in an expectation position of 0.

2. Consider the stationary states of the harmonic oscillator. As straightforwardly as possible, compute the following quantities for the nth stationary state $\psi_n(x)$.

   (a) $\langle x \rangle$.

   Given

$$\widehat{x} = \sqrt{\frac{\hbar}{2m\omega}} \left(\widehat{a_+} + \widehat{a_-}\right).$$

   Then,

$$\langle x \rangle = \langle n | \widehat{x} | n \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\langle n | \widehat{a_+} | n \rangle + \langle n | \widehat{a_-} | n \rangle\right) = 0.$$

   (b) $\langle x^2 \rangle$.

   Given

$$\begin{aligned}
\widehat{x^2} &= \widehat{x}^2 \\
&= \frac{\hbar}{2m\omega} \left(\widehat{a_+}^2 + \widehat{a_-}^2 + \widehat{a_-}\widehat{a_+} + \widehat{a_+}\widehat{a_-}\right) \\
&= \frac{\hbar}{2m\omega} \left(\widehat{a_+}^2 + \widehat{a_-}^2 + \frac{1}{\hbar\omega}\left(\widehat{H} + \frac{1}{2}\right) + \frac{1}{\hbar\omega}\left(\widehat{H} - \frac{1}{2}\right)\right) \\
&= \frac{\hbar}{2m\omega} \left(\widehat{a_+}^2 + \widehat{a_-}^2 + \frac{2}{\hbar\omega}\widehat{H}\right).
\end{aligned}$$

   Then,

$$\left\langle x^2 \right\rangle = \langle n | \widehat{x^2} | n \rangle = \frac{\hbar}{2m\omega} \left(\langle n | \widehat{a_+}^2 | n \rangle + \langle n | \widehat{a_-}^2 | n \rangle + \frac{2}{\hbar\omega} \langle n | \widehat{H} | n \rangle\right) = \frac{E_n}{m\omega^2}.$$

   (c) $\langle p \rangle$.

$$\langle p \rangle = m \frac{\mathrm{d}}{\mathrm{d}t} \langle x \rangle = m \frac{\mathrm{d}}{\mathrm{d}t}(0) = 0.$$

   (d) $\langle p^2 \rangle$

   Given

$$\langle H \rangle = \frac{\langle p^2 \rangle}{2m} = E_n.$$

   Then,

$$\left\langle p^2 \right\rangle = 2m \langle H \rangle = 2mE_n.$$

   (e) $\langle T \rangle$

   Given

$$\langle H \rangle = \langle T \rangle + \langle V \rangle = \langle T \rangle + \frac{1}{2} m\omega^2 \left\langle x^2 \right\rangle.$$

   Then,

$$\langle T \rangle = \langle H \rangle - \langle V \rangle = E_n - \frac{1}{2} m\omega^2 \frac{E_n}{m\omega^2} = \frac{1}{2} E_n.$$

(f) Is the Heisenberg uncertainty principle satisfied for all values of n?

The uncertainty of x, σ_x, is given by

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{E_n}{m\omega^2}}.$$

The uncertainty of p, σ_p, is given by

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{2mE_n}.$$

The product of uncertainties is given by

$$\sigma_x \sigma_p = \sqrt{\frac{E_n}{m\omega^2}} \sqrt{2mE_n} = \frac{E_n}{\omega}\sqrt{2} = \hbar \left(n + \frac{1}{2} \right) \sqrt{2}.$$

The Heisenberg uncertainty principle requires that

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}.$$

Where a factor of $\hbar/2$ appears on both sides thus simplifying to

$$(2n+1)\sqrt{2} \geq 1.$$

The left side increases with n so we check only the smallest n, that is $n = 0$.

$$\sqrt{2} \geq 1,$$

is indeed true. Hence, the uncertainty principle is satisfied for all n.

3. A particle in a harmonic oscillator potential is described by the normalized wavefunction

$$|\Psi(x,0)\rangle = \frac{1}{\sqrt{5}}|1\rangle + \frac{2}{\sqrt{5}}|2\rangle,$$

where $|n\rangle$ represents the nth stationary state.

(a) What is $|\Psi(x,t)\rangle$? $|\Psi(x,t)\rangle$ can be found by adding the time-dependent term to $|\Psi(x,0)\rangle$,

$$|\Psi(x,t)\rangle = \frac{1}{\sqrt{5}}\left(|1\rangle \exp\left(-\frac{3i\omega}{2}\right) + 2|2\rangle \exp\left(-\frac{5i\omega}{2}\right)\right).$$

(b) What is the expectation value for energy?

The expectation value for energy is given by

$$\begin{aligned}
\langle H\rangle &= \langle\Psi|\widehat{H}|\Psi\rangle \\
&= \frac{1}{\sqrt{5}}\left(\langle 1|\phi_1^* + 2\langle 2|\phi_2^*\right)\widehat{H}\frac{1}{\sqrt{5}}\left(|1\rangle\phi_1 + 2|2\rangle\phi_2\right) \\
&= \frac{1}{5}\left(\langle 1|\phi_1^* + 2\langle 2|\phi_2^*\right)\left(\widehat{H}|1\rangle\phi_1 + 2\widehat{H}|2\rangle\phi_2\right) \\
&= \frac{1}{5}\left(\langle 1|\widehat{H}|2\rangle\phi_1^*\phi_1 + \langle 1|\widehat{H}|2\rangle\phi_1^*\phi_2 + 2\langle 2|\widehat{H}|1\rangle\phi_2^*\phi_1 + 4\langle 2|\widehat{H}|2\rangle\phi_2^*\phi_2\right) \\
&= \frac{1}{5}\left(E_1 + 0 + 0 + 4E_2\right) \\
&= \frac{23}{10}\hbar\omega.
\end{aligned}$$

(c) What is $\langle x(t)\rangle$?

The expectation value of position as a function of time is given by

$$
\begin{aligned}
\langle x(t)\rangle &= \langle\Psi|\widehat{x}|\Psi\rangle \\
&= \frac{1}{5}\sqrt{\frac{\hbar}{2m\omega}}\Bigg(\langle 1|\left[\widehat{a_+}+\widehat{a_-}\right]|1\rangle \\
&\qquad\qquad + 4\langle 2|\left[\widehat{a_+}+\widehat{a_-}\right]|2\rangle \\
&\qquad\qquad + 2\langle 1|\left[\widehat{a_+}+\widehat{a_-}\right]|2\rangle\phi_1^*\phi_2 \\
&\qquad\qquad + 2\langle 2|\left[\widehat{a_+}+\widehat{a_-}\right]|1\rangle\phi_2^*\phi_1\Bigg) \\
&= \frac{1}{5}\sqrt{\frac{\hbar}{2m\omega}}\left(0+0+2\langle 1|\widehat{a_-}|2\rangle\phi_1^*\phi_2+0+2\langle 2|\widehat{a_+}|1\rangle\phi_2^*\phi_1+0\right) \\
&= \frac{2}{5}\sqrt{\frac{\hbar}{2m\omega}}\left(\langle 1|\widehat{a_-}|2\rangle\phi_1^*\phi_2+\langle 2|\widehat{a_+}|1\rangle\phi_2^*\phi_1\right) \\
&= \frac{2}{5}\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{2}\phi_1^*\phi_2+\sqrt{2}\phi_2^*\phi_1\right) \\
&= \frac{2}{5}\sqrt{\frac{\hbar}{m\omega}}\left(\phi_1^*\phi_2+\phi_2^*\phi_1\right) \\
&= \frac{4}{5}\sqrt{\frac{\hbar}{m\omega}}\left(\frac{\exp(-i\omega t)+\exp(i\omega t)}{2}\right) \\
&= \frac{4}{5}\sqrt{\frac{\hbar}{m\omega}}\cosh(i\omega t) \\
&= \frac{4}{5}\sqrt{\frac{\hbar}{m\omega}}\cos(\omega t).
\end{aligned}
$$

The cos popping out to remove the imaginary components is really cool!