The artifact I selected is a particular homework assignment from PHYS383 Electromagnetic Theory 3 taken Spring 2023. I selected this assignment not because it is my best preforming, I got a 29/30, or most challenging, but because I remember working on this assignment with joy. This is also around the time I felt that my time-consuming challenge of using LaTeX to prepare my homework was finally paying off and improving the way I did physics. Typesetting my homework has enabled me to use more explanation in my work which not only serves to demonstrate my knowledge but helped me understand the process of doing physics. Throughout this assignment I am utilizing Maxwell's equations, vector calculus, and a healthy dose of appropriate approximations to understand relativity in the context of Electromagnetic Theory. Problem 2 in particular is concerned with arriving at the precursor to relativity through an understanding of how information on electromagnetic systems takes time to propagate. I am also able to leverage this understanding of physical systems, and a healthy dose of approximations, in problem 3 to explore signal radiation. Most important to me, however, is the theoretical understanding explored in problem 1. The determination of gauge in an electromagnetic system reveals a lot of information about the system before a particular solution is found. The problem of determining gauge shows up frequently in electromagnetic theory and understanding how to work with a gauge is an important precursor to physics theory. While this is not my most perfect assignment nor most challenging I feel this represents my understanding of key concepts, my ability to apply them, and my desire to take my education further.

PHYS383

Homework 6

Due 2023 May 09

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1. **Gauge Transformation [10 points]**
   Say the potentials throughout space and time are $V(\mathbf{r},\mathbf{t}) = \mathbf{0}$ and $A(\mathbf{r},\mathbf{t}) = \mathbf{A_0}\sin(\mathbf{k(z-ct)})\hat{\mathbf{x}}$, where A_0 and k are given constants, and c is the speed of light.

   a) Find the $\mathbf{E}$ and $\mathbf{B}$ fields everywhere in space and time, and comment on the physics here; what have we got going on? [3 pnts]
   The electric field $\mathbf{E}$ is given by

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}.$$

The first piece, $\nabla\mathbf{V}$, is found by taking the gradient of the potential, which in this case is 0. The second piece, $\frac{\partial \mathbf{A}}{\partial t}$, is simply

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial t} &= \frac{\partial A_0 \sin(k(z-ct))\hat{x}}{\partial t}\\ &= A_0\hat{x}\frac{\partial \sin(k(z-ct))}{\partial t}\\ &= A_0\hat{x}(ck)(-\cos(k(z-ct)))\\ &= -A_0 ck\cos(k(z-ct))\hat{x}.\end{aligned}$$

The $\mathbf{E}$ field is then

$$\mathbf{E}(\mathbf{r},t) = A_0 ck\cos(k(z-ct))\hat{x}.$$

The magnetic field $\mathbf{B}$ is given by

$$\mathbf{B} = \nabla\times\mathbf{A}.$$

Where the only non-zero component of the curl is the partial derivative of the x-component with respect to z, which results in a $\hat{y}$ curl

$$\nabla\times\mathbf{A} = \frac{\partial A_0\sin(k(z-ct))}{\partial z} = A_0 k\cos(k(z-ct))\hat{y}.$$

The magnetic field is then

$$\mathbf{B}(\mathbf{r},t) = A_0 k\cos(k(z-ct))\hat{y}.$$

b) Are we in the Coulomb gauge, the Lorentz gauge, both, or neither? [2 pnts]

In order to determine which gauge we are in, we can take the divergence of $\mathbf{A}$, which can once again be simplified by discounting the 0 terms of the y and z components.

$$\nabla \cdot \mathbf{A} = \frac{\partial A_0 \sin(k(z - ct))}{\partial x}\hat{x} = 0.$$

$\nabla \cdot \mathbf{A} = 0$ corresponds to the Coulomb Gauge, which is most useful in magnetostatics. We cannot, however, rule out that we are not in the Lorenz gauge, as the Lorenz gauge states

$$\nabla \cdot \mathbf{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t},$$

where the partial derivative of our potential with respect to time is also 0. We are thus in both gauges.

c) Find ρ and $\mathbf{J}$ here. Does the answer agree with your intuition about the physics of this question? [3 pnts]

Since we have no potential, we can conclude that we have no charge distribution; this is further supported by being in the Coulomb Potential, which is used in magnetostatics.

$$\rho = 0.$$

$\mathbf{J}$ can be found by taking the Laplacian of $\mathbf{A}$. This can, however, be simplified since we know the divergence to be 0; the Laplacian is, in this case, equal to minus the curl of the curl, and we already found the curl of $\mathbf{A}$, its $\mathbf{B}$.

$$\begin{aligned}
\nabla^2 \mathbf{A} &= \nabla (\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \\
&= 0 - \nabla \times \mathbf{B} \\
&= -\nabla \times (A_0 k \cos(k(z - ct))\hat{y}) \\
&= -A_0 k (\nabla \times [\cos(k(z - ct))\hat{y}]) \\
&= A_0 k^2 \sin(k(z - ct))\hat{x}.
\end{aligned}$$

$$\mathbf{J} = A_0 k^2 \sin(k(z - ct))\hat{x}.$$

With $\mathbf{J}$ we can prove that ρ is static in time, which would justify using the potential to demonstrate $\rho = 0$, by taking the divergence of $\mathbf{J}$. The divergence of $\mathbf{J}$ is found in the same way as the divergence of $\mathbf{A}$ which we already know to be 0, $\nabla \cdot \mathbf{J} = 0$.

d) Is it possible, in principle, to find a different gauge for the problem in which $\mathbf{A}(\mathbf{r}, t) = 0$? (If so, find the gauge transformation. If not, why not?) [2 pnts]

No gauge transformation would yield $\mathbf{A} = 0$ because $\mathbf{A}$ represents a real physical quantity, a relationship to a real current density $\mathbf{J}$ and no gauge transformation changes the relationship between $\mathbf{J}$ and $\mathbf{A}$.

2. **Retarded Potentials [10 points]** Consider a problem similar in spirit to Griffiths Example 10.2 (which will be very helpful to study before working this one!) But instead of an infinite straight wire, we have an infinite sheet that lies in the xy-plane. The surface current $K(t) = 0$ for $t \leq 0$, but at $t = 0$, suddenly the surface current turns on, so it is a constant $K(t) = K_0\hat{x}$ instantly and everywhere in the plane. There is no charge density anywhere.

$$K(t) = \begin{cases} 0, & t \leq 0 \\ K_0, & t > 0 \end{cases}$$

Since there is no charge density anywhere the scalar potential is 0, and so is ∇V. For a point P located a distance z along the z-axis the amount of current which has had time to reach it is a circle on the infinite sheet of radius s which expands with time. The segment which has had time to reach P will look very similar to the example problem.

$$r \leq \sqrt{(ct)^2 - z^2},$$

the full area a which is able to affect P is

$$a = 2\pi r^2 = 2\pi \left((ct)^2 - z^2\right).$$

Since the area on the surface is an expanding circle and the point P is a distance z, cylindrical coordinates are a natural choice with the base of the cylinder on the xy-plane. The da term is then simply

$$\mathrm{d}a = r\mathrm{d}r\mathrm{d}\phi, \quad 0 \leq r \leq \sqrt{(ct)^2 - z^2}, \quad 0 \leq \phi \leq 2\pi.$$

The separation vector, $\mathscr{r}$, can also be expressed in terms of r and z

$$\mathscr{r} = \sqrt{r^2 + z^2}.$$

The vector potential $\mathbf{A}$ can be found by

$$\mathbf{A}(z,t) = -\frac{\mu_0}{4\pi}\hat{y}\int_{\phi=0}^{2\pi}\int_{r=0}^{\infty}\frac{K(t_r)}{\mathscr{r}}r\mathrm{d}r\mathrm{d}\phi.$$

The only non-zero contributions are from the regions which have had time to reach P which are the regions $r \leq \sqrt{(ct)^2 - z^2}$. The $-\hat{y}$ term was found using the right hand rule. Since none of the terms in $\mathbf{A}$ depend on ϕ that part of the integral can be evaluated to get an integral just in terms of r. Finally $\mathbf{A}$ can be evaluated as

$$\mathbf{A}(z,t) = -\frac{\mu_0 K_0}{2}\hat{y}\int_0^{\sqrt{(ct)^2 - z^2}}\frac{r}{\sqrt{r^2 + z^2}}\mathrm{d}r.$$

A z^2 term can be pulled out of the square root creating an opportunity for a u-sub

$$\mathbf{A}(z,t) = -\frac{\mu_0 K_0}{2z}\hat{y}\int_0^{\sqrt{(ct)^2-z^2}} \frac{r}{\sqrt{\frac{r^2}{z^2}+1}}\mathrm{d}r.$$

$$\begin{aligned} u &= \frac{r^2}{z^2}+1, \\ \mathrm{d}u &= \frac{2}{z^2}r\mathrm{d}r. \end{aligned}$$

Now $\mathbf{A}$ becomes

$$\begin{aligned} \mathbf{A}(z,t) &= -\frac{\mu_0 K_0 z}{4}\hat{y}\int_0^{u\left(\sqrt{(ct)^2-z^2}\right)} u^{-1/2}\mathrm{d}u \\ &= -\frac{\mu_0 K_0 z}{4}\hat{y}\left(2\sqrt{u}\right)_0^{u\left(\sqrt{(ct)^2-z^2}\right)} \\ &= -\frac{\mu_0 K_0 z}{2}\hat{y}\left(\sqrt{\frac{r^2}{z^2}+1}\right)_0^{\sqrt{(ct)^2-z^2}} \\ &= -\frac{\mu_0 K_0}{2}\hat{y}\left(\sqrt{r^2+z^2}\right)_0^{\sqrt{(ct)^2-z^2}} \\ &= -\frac{\mu_0 K_0}{2}\hat{y}\left(\sqrt{(ct)^2-z^2+z^2}-\sqrt{0+z^2}\right) \\ &= -\frac{\mu_0 K_0}{2}\hat{y}\left(\sqrt{(ct)^2}-z\right) \\ &= -\frac{\mu_0 K_0}{2}\left(ct-z\right)\hat{y} \end{aligned}$$

$\mathbf{E}$ is simply minus the partial derivative with respect to time of $\mathbf{A}$ because $\nabla V = 0$

$$\mathbf{E}(z,t) = -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 K_0 c}{2}\hat{y}.$$

The magnetic field $\mathbf{B}$ is simply the curl of $\mathbf{A}$ whose only non-zero term is the partial derivative of the y component with respect to z which is in the $\hat{x}$ direction,

$$\mathbf{B}(z,t) = -\frac{\mu_0 K_0}{2}\hat{x}.$$

3. **Radiation Intensity [10 points]** A pirate radio station has been set up on the top of Discovery Hall. At the top of a tall (height H) ham radio tower is a small electric dipole antenna, of length d, with its axis oriented vertically. They broadcast at a frequency ω, with total time-averaged radiated power P. A neighbor living nearby is a friend of yours. Although they don't want to turn the station into the authorities, your friend is worried that they are being "irradiated", with possible health effects. Knowing you're a physics major, they ask for your help:

   a) In terms of variable names given above, find a formula for the intensity of EM radiation at ground level a distance s away from the tower. What assumptions are you making in generating this formula? List them explicitly.

      - The source is a point source at the top of Discovery hall and we are standing far enough that the waves can be approximated to monochromatic plane waves propagating radially. $d << H$, $s >> d$.
      - The frequency is high enough such that we can just use the time-averaged radiated power P in the "human" time-scales.
      - The human is a perfect absorber and can be modeled as a cylinder of height $h = 2$ m and radius $R = 0.5$ m standing tall perpendicular to the ground.
      - The medium the waves are traveling through, air, can be approximated to vacuum. $\epsilon \approx \epsilon_0$, $\mu \approx \mu_0$.

      The intensity I is related to the Poynting vector S, and in turn to time-averaged power P, by

      $$I = \langle S \rangle = \frac{P}{A},$$

      where A is the area the wave is striking, or the area of a sphere at a radius r away from the point source,

      $$I = \frac{P}{4\pi r^2},$$

      where at ground level the source is a distance $r = \sqrt{s^2 + H^2}$ away,

      $$I(s) = \frac{P}{4\pi\left(s^2 + H^2\right)}.$$

b) Let's identify the "worst case" - where should you measure the intensity for it to be as large as it can get at ground level? Use your formula and rewrite it at this spot. Simplify as much as possible.

Supposing it is possible to stand directly under the antenna the distance would be minimized and the top of the cylinder would be perpendicular to incoming plane waves.

Caveat: in reality the dipole antenna would not radiate much directly downward, this is a result of the point source approximation.

The power absorbed by the human cylinder can be found by taking the dot product of the flux density $\langle \mathbf{S} \rangle$ with the surface area of the cylinder. When standing directly below the antenna the side of the cylinder absorbs no radiation because it is perpendicular to the plane waves, only the top of the cylinder absorbs radiation. Since we have taken the monochromatic plane waves approximation the Poynting vector has the same value at all points on the top of the cylinder and so the flux absorbed is simply the intensity times the area at the top of the cylinder.

$$I_{max} = \frac{P}{4\pi H^2}(\pi R^2) = P\frac{R^2}{H^2}$$

c) Let's put in some plausible numbers. Suppose the pirate station is broadcasting at $\omega = 100$ MHz (FM), and putting out a power of $P = 20$ kW. Their electric dipole antenna is $d = 5$ cm tall, and the tower is $H = 50$ m. According to an FCC information webpage, 100 mW/cm^2 is where bodies start to heat up, and in some circumstances, intensity levels down to $1 - 10$ mW/cm^2 are potentially harmful (e.g to eyeballs). What do you conclude - does your friend need to pull the plug on this thing? (Let's not get into ethical issues about pirate stations and focus only on EM exposures!)

$$I_{max} = 20 \times 10^3 \left(\frac{0.5}{50}\right)^2 \text{ W} = 2 \text{ W}.$$

Comparing this to the FCC intensity levels

$$100 \times 10^{-3} (100)^2 = 10 \text{ kW},$$

represents the range that bodes begin to heat up and $0.1 - 1$ kW represents when eyes would be damaged at this worst case position. Based on this the pirate tower is actually quite safe and need not be removed.