My chosen artifact for this goal is homework assignment 4 from PHYS451 Analytical Mechanics 2[1]. This was my second attempt at the homework for which I got a 39/40. On this first attempt I had neglected the very nice notation for central force problems opting instead for a far more complicated analysis of problem 1. My first solution was in more absolute coordinates and lacked an insightful interpretation. I was fortunate to have submitted the homework early and received the feedback in time to correct my solution prior to the deadline. My second attempt, using central forces, yielded the very insightful solution that the system as a whole falls like a single solid object on Earth; that is, it accelerates downwards at g. The spring element of the system then *predictably* introduces a simple harmonic oscillation of the masses independent of their falling! I feel here the power of analytical methods clicked through brute force. With the right technique a solution can be reached that allows a deep understanding of the system in terms of simpler systems. The remainder of the assignment is dedicated to orbital type problems which as an astronomy minor I found quite enjoyable.

[1]Need I say more? Yes.

1. Two masses, m and M are joined by a spring with spring constant k in a uniform gravitational field. The mass m is held in place while mass M is suspended freely. Mass m is then launched vertically.

   Starting with

   $$\mathcal{L} = \frac{1}{2}(m+M)\left(\dot{\vec{R}}\right)^2 + \frac{1}{2}\frac{mM}{m+M}\left(\dot{\vec{r}}\right)^2 - U(r) - U(R),$$

   where $\vec{R}$ and $\vec{r}$ point only in the $\hat{z}$-direction so the vector notation can be dropped. The potential $U(r)$ is the potential from the spring,

   $$U(r) = -\frac{1}{2}k(r-r_0)^2,$$

   where r_0 is the rest length of the spring. The potential $U(R)$ is the potential from the uniform gravitational field the masses are in,

   $$U(R) = (M+m)gR.$$

   The position of the masses are given by

   $$r_m = R - \frac{m}{m+M}r, \quad r_M = R + \frac{M}{m+M}r.$$

   The full Lagrangian is then

   $$\mathcal{L} = \frac{1}{2}(m+M)\dot{R}^2 + \frac{1}{2}\left(\frac{mM}{m+M}\right)\dot{r}^2 + \frac{1}{2}k(r-r_0)^2 - (M+m)gR.$$

   The Euler-Lagrange equation in the R coordinate is then

   $$\begin{aligned} 0 &= \partial_R\mathcal{L} - d_t\partial_{\dot{R}}\mathcal{L}, \\ &= -(m+M)g - (m+M)\ddot{R}, \\ \ddot{R} &= g. \end{aligned}$$

   The Euler-Lagrange equation in the r coordinate is then

   $$\begin{aligned} 0 &= \partial_r\mathcal{L} - d_t\partial_{\dot{r}}\mathcal{L}, \\ &= -k(r-r_0) - \left(\frac{mM}{m+M}\right)\ddot{r}, \\ \ddot{r} &= -k\left(\frac{m+M}{mM}\right)(r-r_0). \end{aligned}$$

   Thus, the sculpture, as a whole, falls smoothly under the effect of gravity in the same way a ball tossed would. During the fall, the two components act as a simple harmonic oscillator.

2. Consider a binary star system orbiting a common center of mass. Star A is a white dwarf star of mass $M_A = 2.20M_\odot$ and star B is a white dwarf of mass $M_B = 0.987M_\odot$, where $M_\odot = 1.989 * 10^{30}$kg. The orbital period of the two objects is $\tau = 50.09$ years.

   A. What is the length of the semi-major axis a of their orbit?
   Given Kepler's Third Law:

$$\tau^2 = \left(\frac{\mu}{\gamma}\right) 4\pi^2 a^3,$$

   where τ is the orbital period, a is the semi-major axis, μ is the reduced mass $M_A M_B/(M_A + M_B)$, and γ is the gravitational factor GM_AM_B. The length of the semi-major axis a is given by

$$a = \left(\tau^2 \left(\frac{\mu}{\gamma}4\pi^2\right)^{-1}\right)^{1/3}.$$

   Substituting relationships for μ and γ:

$$a = \left(\frac{G\tau^2}{4\pi^2}(M_A + M_B)\right)^{1/3}.$$

   Substituting given values, and converting τ to SI units of seconds:

$$a = 2.99 \times 10^{12}\text{m},$$

$$a = 20.0\text{au}.$$

B. Given the distance of closest approach is $r_{\text{min}} = 7.989\text{au}$, where $1\text{au} = 1.496 \times 10^{11}\text{m}$. Find the orbital parameters c and ϵ.

Expressions for ϵ and c can be found from

$$\begin{aligned} r_{\text{min}} &= \frac{c}{1+\epsilon}, \\ a - r_{\text{min}} &= a\epsilon. \end{aligned}$$

These expressions become

$$\begin{aligned} \epsilon &= \frac{a - r_{\text{min}}}{a}, \\ c &= r_{\text{min}} \left(1 + \frac{a - r_{\text{min}}}{a} \right), \end{aligned}$$

which are evaluated to

$$\begin{aligned} \epsilon &= 0.600, \\ c &= 1.91 \times 10^{12}\text{m}, \\ c &= 12.8\text{au}. \end{aligned}$$

C. Find the radial distance r_A and r_B from the center of mass to each star in terms of r.

From the derivation of the central forces Lagrangian the separation vector $\vec{r}$ was defined as

$$\vec{r} = \vec{r}_1 - \vec{r}_2,$$

and $\vec{r}_1$ was defined as

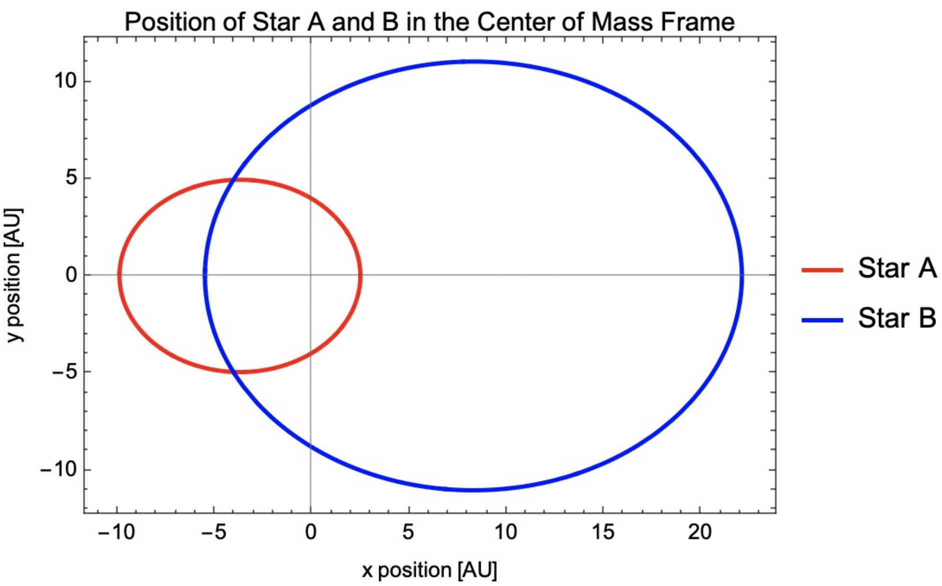
$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2}\vec{r},$$

where we can choose a coordinate system based on $\vec{R}$ such that $\vec{R} = 0$. Assuming we have done this

$$\vec{r}_1 = \frac{m_2}{m_1 + m_2}\vec{r},$$
$$\vec{r}_2 = -\frac{m_1}{m_1 + m_2}\vec{r}.$$

Substituting the quantities from the binary star system we get

$$\vec{r}_A = \frac{M_B}{M_A + M_B}\vec{r},$$
$$\vec{r}_B = -\frac{M_A}{M_A + M_B}\vec{r}.$$



The stars orbital paths are shown in the above figure as cartesian positions in astronomical units.

3. Consider a class-M star located 500 light-years away from Earth. The transit takes 3.37 hours: $t = 3.37\text{h}$. The brightness of the star is reduced by a factor 6.7×10^{-4}. By observing multiple transits the orbital period is found to be 22.4 days: $\tau = 22.4\text{d}$.

   A. What is the ratio R_p/R_s between the radius of the planet and the radius of the star?

   Assume the star emits light as a uniform disc and the exoplanet acts as a perfectly opaque disc for the wavelengths observed. The amount of light emitted by the star normally can be given by

$$P = cA = c\pi R_s^2,$$

   where c is a proportionality constant which describes how emitted light relates to visible area. The amount of light seen during transit is then given by

$$P_t = c(A - A_p) = c\pi(R_s^2 - R_p^2).$$

   It is given that, during transit,

$$P_t = (1 - 6.7 \times 10^{-4})P.$$

   Substituting expressions for P and P_t in terms of R_s and R_p yields

$$c\pi(R_s^2 - R_p^2) = (1 - 6.7 \times 10^{-4})c\pi R_s^2,$$

   which can be re-arranged for R_p/R_s:

$$\frac{R_p}{R_s} = \sqrt{6.7 \times 10^{-4}}.$$

   B. Given $R_s = 0.524R_\odot$, where $R_\odot = 6.96 \times 10^8\text{m}$ is the radius of the sun. What is the radius of the planet R_p in units of the radius of earth, $R_\oplus = 6.37 \times 10^6\text{m}$.

$$\begin{aligned} R_p &= R_s\sqrt{6.7 \times 10^{-4}}, \\ &= 0.524\sqrt{6.7 \times 10^{-4}}R_\odot, \\ &= 0.524\sqrt{6.7 \times 10^{-4}}\frac{R_\odot}{R_\oplus}R_\oplus, \\ &= 1.48R_\oplus. \end{aligned}$$

   Given that $2R_\oplus$ is roughly the cutoff between rocky worlds, $R < 2R_\oplus$ and gas giants $R > 2R_\oplus$, this planet is likely a rocky planet.

C. The star is likely to be $M_s = 0.54 M_\odot$, where $M_\odot$ is the mass of sun. Determine the semi-major axis of the planet's orbit, assuming it is elliptical with an eccentricity of $\epsilon = 0.2$. Determine aphelion and perihelion distances.

Given, from Keppler's Laws,

$$\begin{aligned}\tau^2 &= \frac{\mu}{\gamma}\pi^2 a^3, \\ &= \frac{\frac{M_s M_p}{M_s + M_p}}{G M_s M_p}\pi^2 a^3, \\ &= \frac{4\pi^2 a^3}{G(M_s + M_p)},\end{aligned}$$

where $M_s >> M_p$ is a good approximation for a sun-like star and an earth-like planet

$$\tau^2 \approx \frac{4\pi^2 a^3}{G M_s}.$$

This can be re-arranged for a:

$$\begin{aligned}a &= \left(\frac{\tau^2 G M_s}{4\pi^2}\right), \\ &= 1.9 \times 10^{10}\mathrm{m}, \\ &= 0.13\mathrm{au}.\end{aligned}$$

The perihelion and aphelion are given respectively by

$$\begin{aligned}r_{\min} &= \frac{c}{1+\epsilon} = \frac{a(1-\epsilon)^2}{1+\epsilon} = 0.10\mathrm{au} \\ r_{\max} &= \frac{c}{1-\epsilon} = \frac{a(1-\epsilon)^2}{1-\epsilon} = 0.15\mathrm{au}.\end{aligned}$$

D. The habitable zone of the sun is given by

$$0.8\mathrm{au} < R < 3.0\mathrm{au}.$$

Assuming the habitable zone radius scales with the square root of luminosity, the scaled radius for the observed star is then approximately

$$0.16\mathrm{au} < R < 0.61\mathrm{au}.$$

The exo-planet observed falls too close to its star during the course of its orbit to be within the habitable zone.

4. Consider a hypothetical outer planet Nemesis with orbital period $\tau = 27$Myr. The Oort cloud, which Nemesis is proposed to intersect, is a distance 5×10^5au.

   A. Assuming that Nemesis passes through the Oort cloud at perigee, what is the minimum eccentricity, ϵ, of the orbit?
   As in the above problem, since $M_\odot >> M_{\text{Nemesis}}$,

$$\begin{aligned} a &= \left(\frac{\tau^2 G M_\odot}{4\pi^2}\right)^{1/3}, \\ &= 1.3 \times 10^{16}\text{m}, \\ &= 90\text{kau}. \end{aligned}$$

   The minimum eccentricity can then be found by re-arranging r_{min}:

$$r_{\text{min}} = \frac{c}{1+\epsilon}.$$

   Substituting $c = a(1-\epsilon)^2$ yields

$$r_{\text{min}} = \frac{a(1-\epsilon)^2}{1+\epsilon}.$$

   This can be re-arranged as a quadratic in terms of epsilon:

$$0 = \epsilon^2 - \left(2 + \frac{R_{\text{Oort}}}{a}\right)\epsilon + \left(1 - \frac{R_{\text{Oort}}}{a}\right).$$

   There is only one valid solution satisfying an elliptical orbit $(0 \leq \epsilon < 1)$,

$$\epsilon = 0.19.$$

   B. Assuming Nemesis was at perihelion 11Myr ago how long will it take from today to reach aphelion and at what distance from the sun will it be?
   It takes half an orbit to go from periapsis to apoapsis so given that Nemesis was at perihelion 11Myr ago the time remaining to reach aphelion is given by

$$t = \frac{27\text{Myr}}{2} - 11\text{Myr} = 2.5\text{Myr}.$$

   At aphelion Nemesis will be a distance r_{max} from the sun

$$r_{\text{max}} = \frac{a(1-\epsilon)^2}{1-\epsilon} = a(1-\epsilon) = 73\text{kau}.$$