Problem 2.1: Suppose you flip four fair coins.

(a) Make a list of all possible outcomes.
The set of all possible outcomes, S, is

$$
\begin{aligned}
S_1 &= \{H, H, H, H\}, \\
S_2 &= \{H, H, H, T\}, \\
S_3 &= \{H, H, T, H\}, \\
S_4 &= \{H, T, H, H\}, \\
S_5 &= \{T, H, H, H\}, \\
S_6 &= \{H, H, T, T\}, \\
S_7 &= \{H, T, H, T\}, \\
S_8 &= \{H, T, T, H\}, \\
S_9 &= \{T, H, T, H\}, \\
S_{10} &= \{T, T, H, H\}, \\
S_{11} &= \{T, H, H, T\}, \\
S_{12} &= \{H, T, T, T\}, \\
S_{13} &= \{T, H, T, T\}, \\
S_{14} &= \{T, T, H, T\}, \\
S_{15} &= \{T, T, T, H\}, \\
S_{16} &= \{T, T, T, T\}.
\end{aligned}
$$

(b) Make a list of all the different "macrostates" and their probabilities. The set of all possible macrostates, p_n where p denotes the probability of finding n heads, is

$$p_0 = 1/16,$$
$$p_1 = 4/16,$$
$$p_2 = 6/16,$$
$$p_3 = 4/16,$$
$$p_4 = 1/16.$$

(c) Compute the multiplicity of each macrostate using the combinatorial formula 2.6, and check that these results agree with brute force counting. The combinatorial formula 2.6 states

$$\Omega(N, n) = \frac{N!}{n!(N-n)!},$$

where N is the number of objects, in this case $N = 4$ coins, and n is macrostate to count, in this case n is the number of heads. Then,

$$\Omega(4, 0) = 1,$$
$$\Omega(4, 1) = 4,$$
$$\Omega(4, 2) = 6,$$
$$\Omega(4, 3) = 4,$$
$$\Omega(4, 4) = 1.$$

The numerator of p_n is the number of microstates for a designated macrostates. With this in mind, the combinatorial formula agrees with brute force counting.

Problem 2.4: Calculate the number of possible five-card poker hands, dealt from a deck of 52 cards. A royal flush consists of the five highest-ranking cards of any one of the four suits. What is the probability of being dealt a royal flush?

The number of ways to combine 52 cards into groups of 5 without replacement is given by the permutation formula,

$$P(N, n) = \frac{N!}{(N-n)!},$$

where $N = 52$ and $n = 5$. Then,

$$P(52, 5) = \frac{52!}{(52-5)!} = 6.4974 \times 10^6.$$

For the probability of being dealt a royal flush we consider a probability tree. Suppose there are no other players, they would add complications at each step of the tree. Then, at the outset, there are 20 cards which correspond to a royal flush available to draw; that is, the probability of moving past node one is 20/52. Now, the player has drawn some card of a given suit and so is restricted to drawing cards from the same suit. There remain four cards of the same suit that the player must draw to obtain a royal flush. Thus nodes two, three, four, and five, have probabilities 4/51, 3/50, 2/49, 1/48 respectively. Here we see that the first node had probability 4×5, representing the four suits. So, the probability of drawing a royal flush from a specific suit is given by $5!/(52!/(52-5)!)$. The probability of drawing a royal flush of any suit can be found by adding each suits royal flush probability together. That is,

$$p = 4\frac{5!(52-5)!}{52!} \approx 1.539 \times 10^{-6}.$$

Notably, the probability of drawing a royal flush of a specific suit is a similar form as the inverse of the multiplicity formula.

Problem 2.8: Consider a system of two Einstein solids, A and B, each containing $N = 10$ oscillators, sharing a total of $q = 20$ units of energy. Assume the solids are weakly coupled, and that the total energy is fixed.

(a) How many different macrostates are available to the system?

There exist $q + 1 = 21$ different macrostates of the system.

(b) How many different microstates are available to the system?

By Eq. 2.9,

$$\Omega(N, q) = \frac{(N + q - 1)!}{q!(N - 1)!} \approx 6.9 \times 10^{10}.$$

(c) Assuming that this system is in thermal equilibrium, what is the probability of finding all the energy in solid A?

The configuration corresponding to finding all the energy in A has a number of microstates given by

$$\Omega(N_A, q)\Omega(N_B, 0).$$

Then the probability is given by the number of microstates for this configuration over the total number of microstates which are equally likely:

$$P = \frac{\Omega(N_A, q)\Omega(N_B, 0)}{\Omega(N, q)} \approx 1.5 \times 10^{-4}.$$

(d) What is the probability of finding exactly half of the energy in solid A?

By similar reasoning above, this configuration has multiplicity

$$\Omega(N_A, q/2)\Omega(N_B, q/2).$$

Then the probability is given similarly to be

$$P = \frac{\Omega(N_A, q/2)\Omega(N_B, q/2)}{\Omega(N, q)} \approx 0.12.$$

(e) Under what circumstances would this system exhibit irreversible behavior?

For this system, irreversible behavior corresponds to a continuous trend towards decreasing entropy. For example, should the system go from the state described in part **(d)** to the state described in part **(c)**, that would be described as irreversible behavior.

Problem 2.26: Consider an ideal monoatomic gas that lives in a two-dimensional universe, occupying an area A. Find a formula for the multiplicity of this gas, analogous to Equation 2.40.
We begin with Equation 2.29 expressed in terms of area rather than volume:

$$\Omega_1 = AA_p.$$

Using the same reasoning to arrive at Equation 2.30 we say our total energy U defines a momentum circle in 2-dimensional momentum space; that is,

$$U = \frac{1}{2m}\left(p_x^2 + p_y^2\right).$$

Thus, we have a circle of radius $\sqrt{2mU}$. Then the area of momentum space is really the circumference of this circle. We invoke the Heisenberg uncertainty principle to obtain the same result as in Equation 2.33 and thus

$$\Omega_1 = \frac{AA_p}{h^2}.$$

For a two particle system we find the circumference of a circle in 4-dimensional space such that

$$\Omega_2 = \frac{A^2}{h^4} \times (\text{circumference of momentum hypercircle}).$$

Then applying the indisinguishability correction,

$$\Omega_2 = \frac{1}{2}\frac{A^2}{h^4} \times (\text{circumference of momentum hypercircle}).$$

We generalize to N particles in the same way as in Equation 2.38 to obtain

$$\Omega_N = \frac{1}{N!}\left(\frac{A}{h^2}\right)^N \times (\text{circumference of momentum hypercircle}).$$

For this calculation we can take Equation 2.39 and substitute $d = 2N$ instead of $d = 3N$. This recovers the expected behavior for a single particle: the circumference of a circle. Then,

$$\Omega_N = \frac{1}{N!}\left(\frac{A}{h^2}\right)^N \frac{2\pi^{2N/2}}{\left(\frac{2N}{2} - 1\right)!}\left(2mU\right)^{\frac{2N-1}{2}}.$$

Reducing the fractions and assuming that $N \gg 1$,

$$\Omega_N \approx \frac{\pi^N}{(N!)^2}\frac{A^N}{h^{2N}}\left(2mU\right)^N.$$