1. An electron is in the spin state

$$\chi = A \begin{bmatrix} 3i \\ 4 \end{bmatrix}.$$

(a) Normalize χ to determine A.

We begin by recognizing that

$$\chi = A \begin{bmatrix} 3i \\ 4 \end{bmatrix} = A\left(3i\langle\uparrow| + 4|\downarrow\rangle\right).$$

Then,

$$1 = |\chi|^2 = A^* A\left((3i)(-3i)\langle\uparrow|\downarrow\rangle + (4)(4)\langle\downarrow|\downarrow\rangle + 0 + 0\right) = A^2\left(9 + 16\right),$$

which is satisfied for $A = 1/5$.

(b) Determine $\langle S_x\rangle$, $\langle S_y\rangle$, $\langle S_z\rangle$.

From Griffiths Eq. 4.145 and Eq. 4.147,

$$\widehat{S_x} = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \widehat{S_y} = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \widehat{S_z} = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then, in general,

$$\langle S_a\rangle = \langle\chi|\,\widehat{S_a}\,|\chi\rangle = \frac{1}{25}\begin{bmatrix} -3i & 4 \end{bmatrix}\widehat{S_a}\begin{bmatrix} 3i \\ 4 \end{bmatrix},$$

where a represents x, y, or z. Thus,

$$\langle S_x\rangle = \frac{1}{25}\frac{\hbar}{2}\begin{bmatrix} -3i & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3i \\ 4 \end{bmatrix} = 0,$$
$$\langle S_y\rangle = \frac{1}{25}\frac{\hbar}{2}\begin{bmatrix} -3i & 4 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 3i \\ 4 \end{bmatrix} = -\frac{24}{50}\hbar,$$
$$\langle S_z\rangle = \frac{1}{25}\frac{\hbar}{2}\begin{bmatrix} -3i & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3i \\ 4 \end{bmatrix} = -\frac{7}{50}\hbar.$$

(c) Determine the uncertainties σ_{S_x}, σ_{S_y}, and σ_{S_x}. In general,

$$\sigma_a = \sqrt{\langle a^2 \rangle - \langle a \rangle^2},$$

where a represents some observable. Our square spin operators are then

$$\widehat{S_x^2} = \frac{\hbar^2}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \widehat{S_y^2} = \frac{\hbar^2}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \widehat{S_z^2} = \frac{\hbar^2}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since the operators are equal, their expectation values on χ will be equal. We also notice that the matrix for each operator is the identity matrix. Thus,

$$\left\langle S_x^2 \right\rangle = \left\langle S_y^2 \right\rangle = \left\langle S_z^2 \right\rangle = \frac{\hbar^2}{4}\langle \chi | \widehat{S_x^2} | \chi \rangle = \frac{\hbar^2}{4}\langle \chi \,|\, \chi \rangle = \frac{\hbar^2}{4}.$$

Therefore,

$$\begin{aligned}
\sigma_{S_x} &= \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2} = \sqrt{\frac{\hbar^2}{4} - 0} = \frac{\hbar}{2}, \\
\sigma_{S_y} &= \sqrt{\left\langle S_y^2 \right\rangle - \langle S_y \rangle^2} = \sqrt{\frac{\hbar^2}{4} - \left(-\frac{12}{25}\hbar\right)^2} = \frac{7}{50}\hbar \\
\sigma_{S_z} &= \sqrt{\langle S_z^2 \rangle - \langle S_y \rangle^2} = \sqrt{\frac{\hbar^2}{4} - \left(-\frac{7}{50}\hbar\right)^2} = \frac{12}{25}\hbar.
\end{aligned}$$

(d) Confirm that the results are consistent with all three uncertainty principles; that is,

$$\sigma_{S_x}\sigma_{S_y} \geq \frac{\hbar}{2}\left|\langle S_z \rangle\right|,$$

and the cyclic permutations. Notice from part (c),

$$\sigma_{S_x} = \frac{\hbar}{2}, \quad \sigma_{S_y} = \frac{7}{50}\hbar = \left|\langle S_z \rangle\right|, \quad \sigma_{S_z} = \frac{12}{50}\hbar = \left|\langle S_y \rangle\right|.$$

Then,

$$\begin{aligned}
\sigma_{S_x}\sigma_{S_y} \geq \frac{\hbar}{2}\left|\langle S_z \rangle\right| &\Rightarrow \frac{\hbar}{2}\left|\langle S_z \rangle\right| \geq \frac{\hbar}{2}\left|\langle S_z \rangle\right|, \\
\sigma_{S_y}\sigma S_z \geq \frac{\hbar}{2}\left|\langle S_x \rangle\right| &\Rightarrow \left(\frac{7}{50}\hbar\right)\left(\frac{12}{50}\hbar\right) \geq 0, \\
\sigma_{S_z}\sigma S_x \geq \frac{\hbar}{2}\left|\langle S_y \rangle\right| &\Rightarrow \left|\langle S_y \rangle\right|\frac{\hbar}{2} \geq \frac{\hbar}{2}\left|\langle S_y \rangle\right|.
\end{aligned}$$

2. For a generalized spinor,

$$\chi = a\chi_{+,z} + b\chi_{-,z} = a\langle\uparrow| + b|\downarrow\rangle,$$

where $\chi_{+,z}$ is spin up and $\chi_{-,z}$ is spin down, compute $\langle S_x\rangle$, $\langle S_y\rangle$, $\langle S_z\rangle$, $\langle S_x^2\rangle$, $\left\langle S_y^2\right\rangle$, $\langle S_z^2\rangle$. Check that

$$\left\langle S_x^2\right\rangle + \left\langle S_y^2\right\rangle + \left\langle S_z^2\right\rangle = \left\langle S^2\right\rangle.$$

From Griffiths Eq. 4.145 and Eq. 4.147,

$$\widehat{S_x} = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \widehat{S_y} = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \widehat{S_z} = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then,

$$\widehat{S_x^2} = \widehat{S_y^2} = \widehat{S_z^2} = \frac{\hbar^2}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Consequently,

$$\widehat{S^2} = \widehat{S_x^2} + \widehat{S_y^2} + \widehat{S_z^2} = \hbar^2\frac{3}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then,

$$\langle S_x\rangle = \langle\chi|\,\widehat{S_x}\,|\chi\rangle = \frac{\hbar}{2}\begin{bmatrix} a^* & b^* \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar}{2}\left(a^*b + b^*a\right),$$

$$\langle S_y\rangle = \langle\chi|\,\widehat{S_y}\,|\chi\rangle = \frac{\hbar}{2}\begin{bmatrix} a^* & b^* \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = i\frac{\hbar}{2}\left(ab^* - a^*b\right),$$

$$\langle S_z\rangle = \langle\chi|\,\widehat{S_z}\,|\chi\rangle = \frac{\hbar}{2}\begin{bmatrix} a^* & b^* \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar}{2}\left(a^*a + b^*b\right).$$

Then,

$$\left\langle S_x^2\right\rangle = \left\langle S_y^2\right\rangle = \left\langle S_z^2\right\rangle = \frac{\hbar^2}{4}.$$

Finally,

$$\left\langle S^2\right\rangle = \left\langle S_x^2\right\rangle + \left\langle S_y^2\right\rangle + \left\langle S_z^2\right\rangle = \hbar^2\frac{3}{4}.$$