

1. Consider a free particle that is described at $t = 0$ s by the wavefunction, $\Psi(x, 0) = Ae^{-a|x|}$ where both A and a are positive, real constants.

- (a) Normalize $\Psi(x, 0)$.

We begin by expressing Ψ as a piecewise function:

$$\Psi(x, 0) = \begin{cases} Ae^{-ax}, & x > 0 \\ Ae^{ax}, & x < 0 \end{cases}$$

Now, Ψ can be normalized by the typical method:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\Psi|^2 dx \\ &= A^2 \int_{-\infty}^0 e^{2ax} dx + A^2 \int_0^{\infty} e^{-2ax} dx \\ &= A^2 \left(\left[\frac{1}{2a} e^{2ax} \right]_{-\infty}^0 + \left[-\frac{1}{2a} e^{-2ax} \right]_0^{\infty} \right) \\ &= A^2 \left(\frac{1}{2a} - 0 + 0 + \frac{1}{2a} \right) \\ &= A^2 \frac{1}{a}. \end{aligned}$$

Thus, $A = \sqrt{a}$.

- (b) Determine $\phi(k)$.

We begin with the typical $\phi(k)$ and exploit Euler's formula:

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x, 0) e^{-ikx} dx = \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} (\cos(kx) - i \sin(kx)) dx.$$

Ψ has not been expressed in piecewise form because we can simplify here. Ψ is even, \cos is even, and \sin is odd. Our limits of integration are symmetric so the \sin term integrates to 0. We are left with only even functions over symmetric limits so we can further simplify to

$$\phi(k) = \sqrt{\frac{2a}{\pi}} \int_0^{\infty} e^{-ax} \cos(kx) dx.$$

The \cos function can be expressed in terms of imaginary exponentials:

$$\phi(k) = \frac{1}{2} \sqrt{\frac{2a}{\pi}} \int_0^{\infty} e^{-ax} (e^{ikx} + e^{-ikx}) dx.$$

These represent waves moving in the $+x$ and $-x$ direction. Then,

$$\phi(k) = \sqrt{\frac{a}{2\pi}} \left(\int_0^\infty e^{ikx-ax} dx + \int_0^\infty e^{-ikx-ax} dx \right).$$

These are deceptively simple integrals and evaluate to

$$\begin{aligned} \phi(k) &= \sqrt{\frac{a}{2\pi}} \left[e^{-ax} \left(\frac{e^{ikx}}{ik-a} + \frac{e^{-ikx}}{ik+a} \right) \right]_0^\infty \\ &= \sqrt{\frac{a}{2\pi}} \left(0 - \frac{1}{ik-a} - \frac{1}{ik+a} \right) \\ &= -\sqrt{\frac{a}{2\pi}} \left(\frac{(ik+a) - (ik-a)}{(ik-a)(ik+a)} \right) \\ &= -\sqrt{\frac{a}{2\pi}} \frac{2a}{(ik-a)(ik+a)} \\ &= \frac{1}{k^2+a^2} \sqrt{\frac{2a^3}{\pi}}. \end{aligned}$$

- (c) Construct $\Psi(x, t)$ in the form of an integral.

The full wavefunction is then given by

$$\begin{aligned} \Psi(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \phi(k) e^{i(kx - \frac{\hbar k^2}{2m}t)} dk \\ &= \frac{a^{3/2}}{\pi} \int_{-\infty}^\infty \frac{\exp \left[i \left(kx - \frac{\hbar k^2}{2m}t \right) \right]}{k^2 + a^2} dk. \end{aligned}$$

- (d) Evaluate the integral for $\Psi(x, t)$ in the limiting cases of a very large a and a very small a .

2. As we discussed in class, the time-independent Schrödinger equation for the free particle has solutions that look like $Ae^{ikx} + Be^{-ikx}$ or like $C \cos(kx) + D \sin(kx)$. Show that these are equivalent solutions. Determine what the constants C and D are as a function of A and B and vice versa.

We begin with the Euler formula:

$$\begin{aligned} Ae^{ikx} + Be^{-ikx} &= A(\cos(kx) + i \sin(kx)) + B(\cos(-kx) + i \sin(-kx)) \\ &= A \cos(kx) + B \cos(kx) + Ai \sin(kx) - Bi \sin(kx) \\ &= (A + B) \cos(kx) + (A - B)i \sin(kx). \end{aligned}$$

Thus,

$$C = A + B, \quad D = i(A - B).$$

Some simple arrangement yields

$$A = \frac{C - D}{2}, \quad B = \frac{C + D}{2}.$$

3. Consider a bead with mass m that slides frictionlessly around a circular wire ring with circumference L . We can think about this problem like a free particle assuming boundary condition of the form $\psi(x + L) = \psi(x)$. Determine the normalized stationary states and their corresponding energies. You should find two distinct solutions for each energy (so there is a two-fold degeneracy in this system). These two states represent clockwise and counter-clockwise rotation.

4. Consider a particle interacting with a potential energy given by

$$V(x) \begin{cases} \infty, & x < 0 \\ -\alpha\delta(x-a), & x > 0 \end{cases}$$

- (a) Determine the bound-state solutions (assume $E < 0$) to the time-independent Schrödinger equation in three different regions of x : $x < 0$, $0 \leq x \leq a$, and $x > a$. Define $K^2 = -2mE/\hbar^2$ in your solutions.
- (b) Demand continuity of the wavefunctions at $x = 0$ and $x = a$ to reduce the number of unknown constant parameters.
- (c) Show that the discontinuity in the derivative of the wavefunctions at $x = a$ is given by

$$\left. \frac{d\psi}{dx} \right|_{a+\epsilon} - \left. \frac{d\psi}{dx} \right|_{a-\epsilon} = -\frac{2m\alpha}{\hbar^2} \psi(a).$$

- (d) Using the boundary condition from part (c), show that

$$\frac{K\hbar^2}{m\alpha} = 1 - e^{-2Ka}.$$

This transcendental equation relates K with α . How many different K values will solve the equation? You might try to graph the left-hand and right-hand equations vs. K to see where they intersect.