1. Use

$$\vec{B}_{\text{int}} = \frac{1}{4\pi\epsilon_0}\frac{e}{m_e c^2 r^3}\vec{L},$$

where e is the elementary charge, to estimate the internal magnetic field in a hydrogen atom. This value characterizes the boundary between the strong and weak field limit.

The magnitude of the internal magnetic field is given by

$$\vec{B}_{\text{int}} \cdot \vec{B}_{\text{int}} = \left|\vec{B}_{\text{int}}\right| = \left(\frac{1}{4\pi\epsilon_0}\frac{e}{m_e c^2}\right)^2 \left(\frac{1}{r^3}\right)^2 \left|\vec{L}^2\right|.$$

Let β be defined as the pre-factors,

$$\beta \equiv \frac{1}{4\pi\epsilon_0}\frac{e}{m_e c^2}.$$

Then, the magnitude of the internal magnetic field may be represented in terms of operators

$$\left|\vec{B}_{\text{int}}\right| = \hat{B}_{\text{int}} = \beta^2 \frac{1}{\hat{r}^3}\frac{1}{\hat{r}^3}\hat{L}^2.$$

The average internal magnetic field can then be found as

$$\langle\Psi|\hat{B}_{\text{int}}|\Psi\rangle = \beta^2 \langle\Psi| \left(\frac{1}{\hat{r}^3}\frac{1}{\hat{r}^3}\hat{L}^2\right)|\Psi\rangle = \beta^2 \left\langle \frac{1}{r^3}\frac{1}{r^3}L^2 \right\rangle.$$

We may insert the identity matrix between each of the three terms of the expectation value. The identity matrix may be expressed as the outer-product of wavefunctions. This allows us to express the expectation value of the internal magnetic field as

$$\langle B_{\text{int}}\rangle = \beta^2 \left\langle \frac{1}{r^3}\right\rangle \left\langle \frac{1}{r^3}\right\rangle \left\langle L^2\right\rangle.$$

These expectation values are known for our chosen basis:

$$\left\langle \frac{1}{r^3}\right\rangle = \frac{1}{l\left(1+\frac{1}{2}\right)(l+1)n^3a^3}, \quad \left\langle L^2\right\rangle = \hbar^2 l(l+1).$$

Then,

$$\langle B_{\text{int}}\rangle = \beta^2 \frac{\hbar^2 l(l+1)}{\left[l\left(l+\frac{1}{2}\right)(l+1)n^3a^3\right]^2} = \beta^2 \frac{\hbar^2}{l(l+1)\left[\left(l+\frac{1}{2}\right)n^3a^3\right]^2}.$$

This implies that the strength of the internal magnetic field is strongest at low values of n and l, which makes sense by the classical picture. Let $n = 1$. Then, $l \in \{0, 1\}$. While $l = 0$ appears problematic, we recall that for $l = 0$ $\langle L^2 \rangle = 0$ and thus $\langle B_{\text{int}} \rangle = 0$. Therefore, let $l = 1$. Then, substituting known values,

$$\langle B_{\text{int}} \rangle \approx 34\,\text{T}.$$

Therefore, $|B_{\text{ext}}| \ll 30\,\text{T}$ is a weak field while $|B_{\text{ext}}| \gg 30\,\text{T}$ is a strong field.

2. Consider the eight $n = 2$ states for the hydrogen atom, $\langle 2, l, j, m_j|$. Determine the energy of each state under weak-field Zeeman splitting and construct a diagram like the one in Figure 6.11 of Griffiths to show how the energies evolve as a function of B_{ext}. Label each line clearly and indicate the slope of each line on the graph.

The total energy of each of the eight states is given by the sum of Equation 7.69, the Bohr energy and fine-structure correction, and Equation 7.79, the Zeeman effect energy. Equation 7.79 expresses the Zeeman effect energy as

$$E_Z^1 = \mu_B B_{\text{ext}} g_J m_j.$$

The Landé g-factor, g_J, is given as part of Equation 7.78 as

$$g_J = \left(1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)}\right).$$

For electrons, $s = 1/2$. Then, the Landé g-factor may be computed and multiplied by the allowed values of m_j for all eight states.

$\|\Psi\rangle$	n	ℓ	j	m_j	g_J	E_Z^1
$\|1\rangle$	2	0	1/2	1/2	2	$\mu_B B_{\text{ext}}$
$\|2\rangle$	2	0	1/2	-1/2	2	$-\mu_B B_{\text{ext}}$
$\|3\rangle$	2	1	1/2	1/2	2/3	$\frac{1}{3}\mu_B B_{\text{ext}}$
$\|4\rangle$	2	1	1/2	-1/2	2/3	$-\frac{1}{3}\mu_B B_{\text{ext}}$
$\|5\rangle$	2	1	3/2	3/2	4/3	$2\mu_B B_{\text{ext}}$
$\|6\rangle$	2	1	3/2	1/2	4/3	$\frac{2}{3}\mu_B B_{\text{ext}}$
$\|7\rangle$	2	1	3/2	-1/2	4/3	$-\frac{2}{3}\mu_B B_{\text{ext}}$
$\|8\rangle$	2	1	3/2	-3/2	4/3	$-2\mu_B B_{\text{ext}}$

For plotting purposes, we are interested in the factor in front of $\mu_B B_{\text{ext}}$, the slope. Then, the quantities needed for plotting are given by the table below.

$\lvert\Psi\rangle$	j	$\frac{E_Z^1}{\mu_B B_{\text{ext}}}$	E_{nj}
$\lvert 1\rangle$	$1/2$	1	$-\frac{13.6\,\text{eV}}{4}\left[1+\frac{5}{16}\alpha^2\right]$
$\lvert 2\rangle$	$1/2$	-1	$-\frac{13.6\,\text{eV}}{4}\left[1+\frac{5}{16}\alpha^2\right]$
$\lvert 3\rangle$	$1/2$	$\frac{1}{3}$	$-\frac{13.6\,\text{eV}}{4}\left[1+\frac{5}{16}\alpha^2\right]$
$\lvert 4\rangle$	$1/2$	$-\frac{1}{3}$	$-\frac{13.6\,\text{eV}}{4}\left[1+\frac{5}{16}\alpha^2\right]$
$\lvert 5\rangle$	$3/2$	2	$-\frac{13.6\,\text{eV}}{4}\left[1+\frac{1}{16}\alpha^2\right]$
$\lvert 6\rangle$	$3/2$	$\frac{2}{3}$	$-\frac{13.6\,\text{eV}}{4}\left[1+\frac{1}{16}\alpha^2\right]$
$\lvert 7\rangle$	$3/2$	$-\frac{2}{3}$	$-\frac{13.6\,\text{eV}}{4}\left[1+\frac{1}{16}\alpha^2\right]$
$\lvert 8\rangle$	$3/2$	-2	$-\frac{13.6\,\text{eV}}{4}\left[1+\frac{1}{16}\alpha^2\right]$