Note that integrals have been evaluated by the provided integral table.

1. Consider the continuous Gaussian distribution, $\rho(x) = Ae^{-\lambda(x-a)^2}$, where $A$, $a$, and $\lambda$ are positive, real constants. Note that this is not a wavefunction, but rather a distribution.

   (a) Normalize the distribution to determine $A$.

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} \rho(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2}\,\mathrm{d}x \\
&= A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}\,\mathrm{d}x \\
&= A\int_{-\infty}^{\infty} e^{-(\lambda x^2-2\lambda ax+\lambda a^2)}\,\mathrm{d}x \\
&= A\sqrt{\frac{\pi}{\lambda}}\exp\left(\frac{(-2\lambda a)^2-4\lambda^2a^2}{4\lambda}\right) \\
&= A\sqrt{\frac{\pi}{\lambda}}.
\end{aligned}
$$

   Thus, $A = \sqrt{\lambda/\pi}$.

   (b) Determine $\langle x\rangle$, $\langle x^2\rangle$, and $\sigma$.

   The average value of $x$, or expectation value, is given by

$$
\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} x\rho(x)\,\mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} x\exp\left(-\lambda(x-a)^2\right)\mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}}a\sqrt{\frac{\pi}{\lambda}} \\
&= a.
\end{aligned}
$$

The average of the squares of $x$ is given by

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 \rho(x) \,\mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\lambda(x-a)^2\right) \,\mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\lambda x^2 + 2\lambda a x - \lambda a^2\right) \,\mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\lambda x^2 + 2\lambda a x\right) \exp\left(-\lambda a^2\right) \,\mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \exp\left(-\lambda a^2\right) \int_{-\infty}^{\infty} x^2 \exp\left(-\lambda x^2 + 2\lambda a x\right) \,\mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \exp\left(-\lambda a^2\right) \frac{\sqrt{\pi}(2\lambda + (2\lambda a)^2)}{4\lambda^{5/2}} \exp\left(\frac{(2\lambda a)^2}{4\lambda}\right) \\
&= \frac{2\lambda + (2\lambda a)^2}{4\lambda^2} \\
&= \frac{1 + 2\lambda a^2}{2\lambda}.
\end{aligned}$$

The standard deviation, $\sigma$, of $\rho$ is given by

$$\begin{aligned}
\sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{1 + 2\lambda a^2}{2\lambda} - a^2} \\
&= \frac{1}{\sqrt{2\lambda}}.
\end{aligned}$$

2. At time $t = 0\,\mathrm{s}$, an electron is represented by the wave function,

$$\Psi(x,0) = \begin{cases} A\frac{x}{a}, & 0 \le x \le a \\ A\frac{(b-x)}{(b-a)}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

where $A$, $a$, and $b$ are constants.

(a) Normalize $\Psi$.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\Psi|^2 \,\mathrm{d}x \\ &= \int_{-\infty}^{0} (0)^2 \,\mathrm{d}x + \int_{0}^{a} \left(A\frac{x}{a}\right)^2 \,\mathrm{d}x + \int_{a}^{b} \left(A\frac{(b-x)}{(b-a)}\right)^2 \,\mathrm{d}x + \int_{b}^{\infty} (0)^2 \,\mathrm{d}x \end{aligned}$$

Thus, $A = \sqrt{3/b}$.

(b) Sketch $\Psi(x,0)$ as a function of $x$.

(c) Where is the electron most likely to be found at $t = 0\,\mathrm{s}$?

(d) What is the probability the electron will be found in the region $x \le a$? Check your result in the limiting case where $b = a$ and $b = 2a$.

(e) Determine $\langle x \rangle$.