1. Let the wavefunction $\Psi(x, t)$ be a solution to the time-dependent Schrödinger equation when the potential energy is given by $V(x)$. What is the solution to the Shrödinger equation if we now consider a potential of $V(x) + V_0$ where $V_0$ is a real positive constant.

2. A particle is observed in a quantum state described by the wavefunction

$$\Psi(x,t) = A \exp\left(-a\left(\frac{mx^2}{\hbar} + it\right)\right),$$

where $A$ and $a$ are real positive constants.

(a) Normalize $\Psi$.

As shown in Griffiths, if $\Psi$ is normalized at any time $t$ it is normalized at all times $t$. We will normalize $\Psi$ at $t = 0$.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \Psi(x, t=0)\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} A \exp\left(\frac{-amx^2}{\hbar}\right)\, \mathrm{d}x \\
&= A \int_{-\infty}^{\infty} \exp\left(\frac{-amx^2}{\hbar}\right)\, \mathrm{d}x \\
&= A\sqrt{\frac{\pi}{\left(\frac{am}{\hbar}\right)}} \\
&= A\sqrt{\frac{\pi\hbar}{am}}.
\end{aligned}$$

Thus, $A = \sqrt{am/(\pi\hbar)}$.

(b) What is the potential $V(x)$ that this particle finds itself within?

$\Psi$ is, by definition, a solution to the Schrödinger equation. Thus,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi.$$

Arranging for the potential energy function $V$ we get

$$V = \frac{1}{\Psi}\left(i\hbar\frac{\partial\Psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}\right).$$

To make these partial derivatives less threatening we can begin by writing $\Psi$ in the form $\Psi = A\psi(x)\phi(t)$.[1] The $A$ component is simply $A$ as solved for above during normalization. The exponential term in $\Psi$ can be separated into an $x$-dependent and $t$-dependent component;

$$\exp\left(\frac{-amx^2}{\hbar} - ait\right) = \exp\left(-\frac{amx^2}{\hbar}\right)\exp\left(-ait\right),$$

which become $\psi$ and $\phi$ respectively. We know

$$\phi(t) = \exp\left(-\frac{iEt}{\hbar}\right) = \exp\left(-ait\right),$$

thus $a = E/\hbar$. We can now write $V$ in a more approachable way with ordinary derivatives:

$$V = \frac{1}{A\psi\phi}\left(i\hbar A\psi\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\hbar^2}{2m}A\phi\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}\right).$$

The ordinary derivatives are

$$\begin{aligned}\frac{\mathrm{d}\psi}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(-ait)\right]\\ &= -ai\exp(-ait)\\ &= -ai\psi,\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\exp\left(-\frac{amx^2}{\hbar}\right)\right]\\ &= -\frac{2amx}{\hbar}\exp\left(-\frac{amx^2}{\hbar}\right)\\ &= -\frac{2amx}{\hbar}\psi.\end{aligned}$$

[1]Since separable equations are the only ones we know how to deal with $\Psi$ must be separable!

(c) Determine the expectation values $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$.

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$$\langle x \rangle = \int_{-\infty}^{\infty} x \, |\psi| \, \mathrm{d}x$$

(d) Determine the standard deviations for position, $\sigma_x$, and momentum, $\sigma_p$.

(e) Are your values for $\sigma_x$ and $\sigma_p$ consistent with the uncertainty principle?

3. An electron is trapped in a harmonic quadratic potential. Suppose the expectation value for its position is given by $\langle x \rangle = \frac{a}{2}\sin(\omega t)$. Here, $a$ is a real constant with units of length and $\omega$ is an angular frequency. What, if anything, can be concluded about the electron's momentum?