1. A non-relativistic particle with mass m moves in a three-dimensional potential, $V(r)$, which is spherically-symmetric and vanishes as $r \to \infty$. At a certain time, this particle is found in the state

$$\psi(r, \theta, \phi) = Cr^{\sqrt{3}} \exp\left[-\alpha r\right] \cos\left[\theta\right],$$

where C and α are constants. We have ignored spin.

(a) What is the orbital angular momentum of this state; that is, what are the quantum numbers l and m_l?

(b) What is the energy, E, of this state? The radial equation, $u = rR$, may be helpful here. Recall, $V(r) \to 0$ as $r \to \infty$.

(c) Now that the energy, E, is known from part (b), what is the potential, $V(r)$?

(a) Since the potential, $V(r)$, is spherically-symmetric, the wavefunction, ψ, may be separated into a radial and angular function:

$$\psi(r, \theta, \phi) = R_{n,\ell}(r) Y_\ell^{m_\ell}(\theta, \phi).$$

Let $C = C_R C_Y$, where C_R is a constant associated with the radial equation and C_Y is a constant associated with the angular equation. The information about orbital angular momentum will come from the radial part of the equation, which will be a spherical harmonic of the form

$$Y_\ell^{m_\ell} = C_Y \cos\left(\theta\right).$$

Griffiths Table 4.3 lists normalized spherical harmonics for particular values of ℓ and m_ℓ. The only spherical harmonic of a this form is given by $\ell = 1$ and $m_\ell = 0$; that is,

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\left(\theta\right).$$

Thus, $\ell = 1$ and $m_\ell = 0$.

(b) By Griffiths Eq. 4.37,

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2 u}{\mathrm{d}r^2}+\left(V+\frac{\hbar^2}{2m}\frac{\ell\left(\ell+1\right)}{r^2}\right)u=Eu,$$

where $u = rR$. Substituting $\ell = 1$ from part (a) yields

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2 u}{\mathrm{d}r^2}+\left(V+\frac{\hbar^2}{mr^2}\right)u=Eu.$$

See below for the evaluation of the second derivative of u with respect to r. The solution is given in terms of u. Then, multiplication of both sides by the inverse of u results in

$$E=-\frac{\hbar^2}{2m}\left(\alpha^2-2\alpha r^{-1}\left(1+\sqrt{3}\right)+r^{-2}\left(3+\sqrt{3}\right)\right)+V+\frac{\hbar^2}{mr^2}.$$

Since the energy, E, is constant we may take any r to evaluate E. Let $r \to \infty$. Then, $V \to 0$. Therefore,

$$E=-\frac{\hbar^2\alpha^2}{2m}.$$

(c) To find the potential, we rearrange the expression from part (b) before letting $r \to \infty$:

$$V=E-\frac{\hbar^2}{mr^2}+\frac{\hbar^2}{2m}\left(\alpha^2-2\alpha r^{-1}\left(1+\sqrt{3}\right)r^{-2}\left(3+\sqrt{3}\right)\right).$$

Then, we substitute E from part (b):

$$V=\frac{\hbar^2}{2m}\left(-\alpha^2-2r^{-2}+r^{-2}\left(3+\sqrt{3}\right)-2\alpha r^{-1}\left(1+\sqrt{3}\right)+\alpha^2\right).$$

Simplifying yields the potential energy, V, as a function of r:

$$V=\frac{\hbar^2}{2m}\left(1+\sqrt{3}\right)\left(r^{-2}-2\alpha r^{-1}\right).$$

The aforementioned derivative, where $u = rR$, given in terms of u:

$$\begin{aligned}
\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} &= \frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r r^{\sqrt{3}} e^{-\alpha r}\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^{\sqrt{3}+1} e^{-\alpha r}\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}r}\left[r^{\sqrt{3}+1}\frac{\mathrm{d}}{\mathrm{d}r}\left[e^{-\alpha r}\right] + e^{-\alpha r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^{\sqrt{3}+1}\right]\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}r}\left[r r^{\sqrt{3}(-\alpha)} e^{-\alpha r} + e^{-\alpha r}\left(\sqrt{3}+1\right) r^{\sqrt{3}}\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\sqrt{3}+1-\alpha r\right) R\right] \\
&= \left(\sqrt{3}+1-\alpha r\right)\frac{\mathrm{d}R}{\mathrm{d}r} + R\frac{\mathrm{d}}{\mathrm{d}r}\left[\sqrt{3}+1-\alpha r\right] \\
&= \left(\sqrt{3}+1-\alpha r\right)\frac{\mathrm{d}R}{\mathrm{d}r} + (-\alpha)R \\
&= \left(\sqrt{3}+1-\alpha r\right)\left(r^{\sqrt{3}}\frac{\mathrm{d}}{\mathrm{d}r}\left[e^{-\alpha r}\right] + e^{-\alpha r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^{\sqrt{3}}\right]\right) - \alpha R \\
&= \left(\sqrt{3}+1-\alpha r\right)\left(r^{\sqrt{3}}(-\alpha)e^{-\alpha r} + e^{-\alpha r}\sqrt{3} r^{\sqrt{3}-1}\right) - \alpha R \\
&= \left(\left(\sqrt{3}+1-\alpha r\right)\left(-\alpha + r^{-1}\sqrt{3}\right) - \alpha\right) R \\
&= \left(r\alpha^2 - 2\alpha\left(1+\sqrt{3}\right) + r^{-1}\left(3+\sqrt{3}\right)\right) R \\
&= \left(\alpha^2 - 2\alpha r^{-1}\left(1+\sqrt{3}\right) + r^{-2}\left(3+\sqrt{3}\right)\right) u.
\end{aligned}$$