Problem 2.1: Suppose you flip four fair coins.

(a) Make a list of all possible outcomes.

The set of all possible outcomes, S, is

$$
\begin{aligned}
S_1 &= \{H,H,H,H\},\\
S_2 &= \{H,H,H,T\},\\
S_3 &= \{H,H,T,H\},\\
S_4 &= \{H,T,H,H\},\\
S_5 &= \{T,H,H,H\},\\
S_6 &= \{H,H,T,T\},\\
S_7 &= \{H,T,H,T\},\\
S_8 &= \{H,T,T,H\},\\
S_9 &= \{T,H,T,H\},\\
S_{10} &= \{T,T,H,H\},\\
S_{11} &= \{T,H,H,T\},\\
S_{12} &= \{H,T,T,T\},\\
S_{13} &= \{T,H,T,T\},\\
S_{14} &= \{T,T,H,T\},\\
S_{15} &= \{T,T,T,H\},\\
S_{16} &= \{T,T,T,T\}.
\end{aligned}
$$

(b) Make a list of all the different "macrostates" and their probabilities. The set of all possible macrostates, p_n where p denotes the probability of finding n heads, is

$$p_0 = 1/16,$$
$$p_1 = 4/16,$$
$$p_2 = 6/16,$$
$$p_3 = 4/16,$$
$$p_4 = 1/16.$$

(c) Compute the multiplicity of each macrostate using the combinatorial formula 2.6, and check that these results agree with brute force counting. The combinatorial formula 2.6 states

$$\Omega(N, n) = \frac{N!}{n!(N-n)!},$$

where N is the number of objects, in this case $N = 4$ coins, and n is macrostate to count, in this case n is the number of heads. Then,

$$\Omega(4, 0) = 1,$$
$$\Omega(4, 1) = 4,$$
$$\Omega(4, 2) = 6,$$
$$\Omega(4, 3) = 4,$$
$$\Omega(4, 4) = 1.$$

The numerator of p_n is the number of microstates for a designated macrostates. With this in mind, the combinatorial formula agrees with brute force counting.

Problem 2.4: Calculate the number of possible five-card poker hands, dealt from a deck of 52 cards. A royal flush consists of the five highest-ranking cards of any one of the four suits. What is the probability of being dealt a royal flush?

The number of ways to combine 52 cards into groups of 5 without replacement is given by the permutation formula,

$$P(N, n) = \frac{N!}{(N-n)!},$$

where $N = 52$ and $n = 5$. Then,

$$P(52, 5) = \frac{52!}{(52-5)!} = 6.4974 \times 10^6.$$

For the probability of being dealt a royal flush we consider a probability tree. Suppose there are no other players, they would add complications at each step of the tree. Then, at the outset, there are 20 cards which correspond to a royal flush available to draw; that is, the probability of moving past node one is 20/52. Now, the player has drawn some card of a given suit and so is restricted to drawing cards from the same suit. There remain four cards of the same suit that the player must draw to obtain a royal flush. Thus nodes two, three, four, and five, have probabilities 4/51, 3/50, 2/49, 1/48 respectively. Here we see that the first node had probability 4×5, representing the four suits. So, the probability of drawing a royal flush from a specific suit is given by $5!/(52!/(52-5)!)$. The probability of drawing a royal flush of any suit can be found by adding each suits royal flush probability together. That is,

$$p = 4\frac{5!(52-5)!}{52!} \approx 1.539 \times 10^{-6}.$$

Notably, the probability of drawing a royal flush of a specific suit is a similar form as the inverse of the multiplicity formula.

Problem 2.5: For an einstein solid with each of the following values of N and q, list all of the possible microstates, count them, and verify formula 2.9.

(a) $N = 3$, $q = 4$

(b) $N = 3$, $q = 5$

(c) $N = 3$, $q = 6$

(d) $N = 4$, $q = 2$

(e) $N = 4$, $q = 3$

(f) $N = 1$, $q = \text{anything}$

(g) $N = \text{anything}$, $q = 1$

Problem 2.8: Consider a system of two Einstein solids, A and B, each containing $N = 10$ oscillators, sharing a total of $q = 20$ units of energy. Assume the solids are weakly coupled, and that the total energy is fixed.

(a) How many different macrostates are available to the system?

(b) How many different microstates are available to the system?
By Eq. 2.9,

$$\Omega(N, q) =$$

(c) Assuming that this system is in thermal equilibrium, what is the probability of finding all the energy in solid A?

(d) What is the probability of finding exactly half of the energy in solid A?

(e) Under what circumstances would this system exhibit irreversible behavior?

Problem 2.26: Consider an ideal monoatomic gas that lives in a two-dimensional universe, occupying an area A. Find a formula for the multiplicity of this gas, analogous to Equation 2.40.