PHYS 474 - Quantum Mechanics **Winter 2024**

Homework #5

1. Consider a free particle that is described at $t = 0$ s by the wavefunction, $\Psi(x, 0) = Ae^{-a|x|}$ where both A and a are positive, real constants.

   (a) Normalize $\Psi(x, 0)$.
   (b) Determine $\phi(k)$.
   (c) Construct $\Psi(x, t)$ in the form of an integral.
   (d) Evaluate the integral for $\Psi(x, t)$ in the limiting cases of a very large and a very small.

2. As we discussed in class, the time-independent Schrödinger equation for the free particle has solutions that look like $Ae^{ikx} + Be^{-ikx}$ or like $C\cos(kx) + D\sin(kx)$. Show that these are equivalent solutions. Determine what the constants C and D are as a function of A and B and vice versa.

3. Consider a bead with mass m that slides frictionlessly around a circular wire ring with circumference L. We can think about this problem like a free particle assuming boundary condition of the form $\psi(x+L) = \psi(x)$. Determine the normalized stationary states and their corresponding energies. You should find two distinct solutions for each energy (so there is two-fold degeneracy in this system). These two states represent clockwise and counter-clockwise rotation.

4. Consider a particle interacting with a potential energy given by:

$$V(x) = \begin{cases} \infty, & x < 0 \\ -\alpha\delta(x-a), & x > 0 \end{cases} \tag{1}$$

   (a) Determine the bound-state solutions (assume that $E < 0$) to the time-independent Schrödinger equation in three different regions of x ($x < 0$, $0 \leq x \leq a$, and $x > a$). Define $K^2 = -\frac{2mE}{\hbar^2}$ in your solutions.

   (b) Demand continuity of the wavefunctions at $x = 0$ and $x = a$ to reduce the number of unknown constant parameters.

   (c) Show that the discontinuity in the derivative of the wavefunctions at $x = a$ is given by $\frac{d\psi}{dx}|_{a+\epsilon} - \frac{d\psi}{dx}|_{a-\epsilon} = -\frac{2m\alpha}{\hbar^2}\psi(a)$.

   (d) Using the boundary condition from part (c), show that $\frac{K\hbar^2}{m\alpha} = 1 - e^{-2Ka}$. This transcendental equation relates K with α. How many different K values will solve the equation? You might try to graph the left-hand and right-hand equations vs. K to see where they intersect.