1. Let the wavefunction $\Psi(x, t)$ be a solution to the time-dependent Schrödinger equation when the potential energy is given by $V(x)$. What is the solution to the Shrödinger equation if we now consider a potential of $V(x) + V_0$ where V_0 is a real positive constant.

2. A particle is observed in a quantum state described by the wavefunction

$$\Psi(x,t) = A \exp\left(-a\left(\frac{mx^2}{\hbar} + it\right)\right),$$

where A and a are real positive constants.

(a) Normalize Ψ.

As shown in Griffiths, if Ψ is normalized at any time t it is normalized at all times t. We will normalize Ψ at $t = 0$.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \Psi(x, t=0)\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} A \exp\left(\frac{-amx^2}{\hbar} - (ai \times 0)\right)\, \mathrm{d}x \\
&= A \int_{-\infty}^{\infty} \exp\left(\frac{-amx^2}{\hbar}\right)\, \mathrm{d}x \\
&= A\sqrt{\frac{\pi}{\left(\frac{am}{\hbar}\right)}} \\
&= A\sqrt{\frac{\pi\hbar}{am}}.
\end{aligned}$$

Thus, $A = \sqrt{am/(\pi\hbar)}$.

(b) What is the potential $V(x)$ that this particle finds itself within?

Ψ is, by definition, a solution to the Schrödinger equation. Thus,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi.$$

Arranging for the potential energy function V we get

$$V = \frac{1}{\Psi}\left(i\hbar\frac{\partial\Psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}\right).$$

To make these partial derivatives less threatening, we can begin by writing Ψ in the form $\Psi = A\psi(x)\phi(t)$. The A component is simply A, as solved for above during normalization. The exponential term in Ψ can be separated into an x-dependent and t-dependent component;

$$\exp\left(\frac{-amx^2}{\hbar} - ait\right) = \exp\left(-\frac{amx^2}{\hbar}\right)\exp\left(-ait\right),$$

which become ψ and ϕ respectively. We know

$$\phi(t) = \exp\left(-\frac{iEt}{\hbar}\right) = \exp\left(-ait\right),$$

thus $a = E/\hbar$. We can now write V in a more approachable way with ordinary derivatives:

$$V = \frac{1}{A\psi\phi}\left(i\hbar A\psi\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\hbar^2}{2m}A\phi\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}\right).$$

The ordinary derivatives are

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(-ait)\right] = -ai\exp(-ait) = -ai\phi,$$

and

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\exp\left(-\frac{amx^2}{\hbar}\right)\right] = -\frac{2amx}{\hbar}\exp\left(-\frac{amx^2}{\hbar}\right) = -\frac{2amx}{\hbar}\psi.$$

Thus the potential energy function V is given by

$$\begin{aligned} V &= \frac{1}{A\psi\phi}\left(i\hbar A\psi\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\hbar^2}{2m}A\phi\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}\right) \\ &= \frac{1}{A\psi\phi}\left(-i\hbar A\psi\frac{E}{\hbar}i\phi + \frac{\hbar^2}{2m}A\phi\frac{-2Emx}{\hbar^2}\psi\right) \\ &= E - Ex \\ &= E(1-x). \end{aligned}$$

(c) Determine the expectation values $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$.

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$$\langle x \rangle = \int_{-\infty}^{\infty} x \, |\psi| \; \mathrm{d}x$$

(d) Determine the standard deviations for position, σ_x, and momentum, σ_p.

(e) Are your values for σ_x and σ_p consistent with the uncertainty principle?

3. An electron is trapped in a harmonic quadratic potential. Suppose the expectation value for its position is given by $\langle x \rangle = \frac{a}{2} \sin(\omega t)$. Here, a is a real constant with units of length and ω is an angular frequency. What, if anything, can be concluded about the electron's momentum?