

1. A particle is trapped in a harmonic oscillator potential. We know that at $t = 0$, the particle can be represented by the wavefunction

$$\Psi(x, 0) = A (2\psi_0(x) + 5\psi_2(x)) ,$$

where ψ_0 and ψ_2 are the stationary-state solutions for $n = 0$ and $n = 2$, respectively.

- (a) Normalize $\Psi(x, 0)$.

$$1 = \langle \Psi(x, 0) | \Psi(x, 0) \rangle = A^2 (4\langle \psi_0 | \psi_0 \rangle + 25\langle \psi_2 | \psi_2 \rangle + 20\langle \psi_0 | \psi_2 \rangle) = 29A^2.$$

Thus,

$$A = \sqrt{\frac{1}{29}}$$

- (b) Construct $\Psi(x, t)$ and then determine $|\Psi(x, t)|^2$. Will $\langle x \rangle$ depend on time?

The complete wavefunction is given by

$$\Psi(x, t) = \sqrt{\frac{1}{29}} (2\Psi_0 + 5\Psi_2) ,$$

where Ψ_n represents $\psi_n(x)\phi_n(t)$. Thus,

$$\begin{aligned} |\Psi(x, t)|^2 &= \frac{1}{29} (2\Psi_0^* + 5\Psi_2^*) (2\Psi_0 + 5\Psi_2) \\ &= \frac{1}{29} (4|\Psi_0|^2 + 2\Psi_0^*\Psi_2 + 5\Psi_2^*\Psi_0 + 25|\Psi_2|^2) \\ &= \frac{1}{29} (4|\psi_0|^2 + 20\psi_0\psi_2 (e^{-2i\omega t} + e^{2i\omega t}) + 25|\psi_2|^2) \\ &= \frac{1}{29} (4|\psi_0|^2 + 10\psi_0\psi_2 \cosh(2i\omega t) + 25|\psi_2|^2) \end{aligned}$$

2. Consider the stationary states of the harmonic oscillator. As straightforwardly as possible, compute the following quantities for the n th stationary state $\psi_n(x)$.

(a) $\langle x \rangle$.

Given

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_+ + \hat{a}_-).$$

Then,

$$\langle x \rangle = \langle n | \hat{x} | n \rangle = \sqrt{\frac{\hbar}{2m\omega}} (\langle n | \hat{a}_+ | n \rangle + \langle n | \hat{a}_- | n \rangle) = 0.$$

(b) $\langle x^2 \rangle$.

Given

$$\begin{aligned} \hat{x}^2 &= \hat{x}^2 \\ &= \frac{\hbar}{2m\omega} (\hat{a}_+^2 + \hat{a}_-^2 + \hat{a}_- \hat{a}_+ + \hat{a}_+ \hat{a}_-) \\ &= \frac{\hbar}{2m\omega} \left(\hat{a}_+^2 + \hat{a}_-^2 + \frac{1}{\hbar\omega} \left(\hat{H} + \frac{1}{2} \right) + \frac{1}{\hbar\omega} \left(\hat{H} - \frac{1}{2} \right) \right) \\ &= \frac{\hbar}{2m\omega} \left(\hat{a}_+^2 + \hat{a}_-^2 + \frac{2}{\hbar\omega} \hat{H} \right). \end{aligned}$$

Then,

$$\langle x^2 \rangle = \langle n | \hat{x}^2 | n \rangle = \frac{\hbar}{2m\omega} \left(\langle n | \hat{a}_+^2 | n \rangle + \langle n | \hat{a}_-^2 | n \rangle + \frac{2}{\hbar\omega} \langle n | \hat{H} | n \rangle \right) = \frac{E_n}{m\omega^2}.$$

(c) $\langle p \rangle$.

$$\langle p \rangle = m \frac{d}{dt} \langle x \rangle = m \frac{d}{dt} (0) = 0.$$

(d) $\langle p^2 \rangle$

Given

$$\langle H \rangle = \frac{\langle p^2 \rangle}{2m} = E_n.$$

Then,

$$\langle p^2 \rangle = 2m \langle H \rangle = 2mE_n.$$

(e) $\langle T \rangle$

Given

$$\langle H \rangle = \langle T \rangle + \langle V \rangle = \langle T \rangle + \frac{1}{2}m\omega^2 \langle x^2 \rangle.$$

Then,

$$\langle T \rangle = \langle H \rangle - \langle V \rangle = E_n - \frac{1}{2}m\omega^2 \frac{E_n}{m\omega^2} = \frac{1}{2}E_n.$$

(f) Is the Heisenberg uncertainty principle satisfied for all values of n ?

The uncertainty of x , σ_x , is given by

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{E_n}{m\omega^2}}.$$

The uncertainty of p , σ_p , is given by

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{2mE_n}.$$

The product of uncertainties is given by

$$\sigma_x \sigma_p = \sqrt{\frac{E_n}{m\omega^2}} \sqrt{2mE_n} = \frac{E_n}{\omega} \sqrt{2} = \hbar \left(n + \frac{1}{2} \right) \sqrt{2}.$$

The Heisenberg uncertainty principle requires that

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}.$$

Where a factor of $\hbar/2$ appears on both sides thus simplifying to

$$(2n + 1) \sqrt{2} \geq 1.$$

The left side increases with n so we check only the smallest n , that is $n = 0$.

$$\sqrt{2} \geq 1,$$

is indeed true. Whence, the uncertainty principle is satisfied for all n .

3. A particle in a harmonic oscillator potential is described by the normalized wave-function

$$|\Psi(x, 0)\rangle = \frac{1}{\sqrt{5}}|1\rangle + \frac{2}{\sqrt{5}}|2\rangle,$$

where $|n\rangle$ represents the n th stationary state.

- (a) What is $|\Psi(x, t)\rangle$?
- (b) What is the expectation value for energy?
- (c) What is $\langle x(t) \rangle$?