

1. Determine the eigenfunctions and eigenvalues of the operator

$$\hat{Q} = \frac{d^2}{d\phi^2}$$

where ϕ is the angle in polar coordinates. Due to the rotational symmetry of the problem, your eigenfunctions, f , should satisfy the boundary condition

$$f(\phi) = f(\phi + 2\pi).$$

The eigenvalue equation

$$\hat{Q}f = qf,$$

where q is the eigenvalue, has the general solution

$$f = Ae^{q^b \phi}.$$

Then,

$$\frac{d^2 f}{d\phi^2} = (q^b)^2 Ae^{q^b \phi} = (q^b)^2 f,$$

where $(q^b)^2$ is the eigenvalue yielding the condition

$$(q^b)^2 = q.$$

Thus, $b = 1/2$. Thusly the eigenfunction f is given by

$$f = A \exp(\sqrt{q}\phi).$$

We apply the boundary condition by letting $\phi = 0$. Thus, having neglected the amplitudes which are constant,

$$1 = \exp(\sqrt{q}2\pi).$$

This is satisfied for

$$\sqrt{q}2\pi = in2\pi.$$

Thus,

$$q = -n^2.$$

Is the spectrum of eigenvalues degenerate or non-degenerate?

The spectrum of eigenvalues, $n \in \mathbb{Z}$, is degenerate for $n \neq 0$.

2. Suppose that $f(x)$ and $g(x)$ are both eigenfunctions of an operator \hat{Q} . The spectrum is degenerate such that both $f(x)$ and $g(x)$ have the same eigenvalue, q .

(a.) Prove that any linear combination of $f(x)$ and $g(x)$ is also an eigenfunction of Q . What is its eigenvalue?

Proof. Let $f(x)$ and $g(x)$ be defined as eigenfunctions of an operator \hat{Q} such that

$$\hat{Q}f = qf, \quad \hat{Q}g = qg.$$

Furthermore let $a, b \in \mathbb{R}$ be given. Then, let $j(x)$ be a function defined as a linear combination of f and g such that

$$j(x) = af(x) + bg(x).$$

Then,

$$\begin{aligned}\hat{Q}j &= \hat{Q}[af + bg] \\ &= \hat{Q}af + \hat{Q}bg \\ &= a\hat{Q}f + b\hat{Q}g \\ &= aqf + bqg \\ &= q(af + bg) \\ &= qj.\end{aligned}$$

That is, a linear combination of eigenfunctions of \hat{Q} , each with eigenvalue q , is also an eigenfunction of \hat{Q} with eigenvalue q . \square

(b.) An anti-hermitian operator obeys the following condition:

$$\hat{Q}^\dagger = -\hat{Q}.$$

Show that the expectation value of an anti-hermitian operator is imaginary.

$$\begin{aligned}\langle Q \rangle &= \langle f | \hat{Q} | f \rangle \\ &= \langle \hat{Q}^\dagger f | f \rangle \\ &= -\langle \hat{Q} f | f \rangle \\ &= -\left(\langle f | \hat{Q} f \rangle \right)^* \\ &= -\langle Q \rangle^*.\end{aligned}$$

(c.) Show that the commutator of two hermitian operators is anti-hermitian.

Let \hat{A} and \hat{B} be hermitian operators. The commutator of \hat{A} and \hat{B} is given by

$$\begin{aligned} [\hat{A}, \hat{B}] &= \hat{A}\hat{B} - \hat{B}\hat{A} \\ &= \hat{A}^\dagger \hat{B}^\dagger - \hat{B}^\dagger \hat{A}^\dagger \\ &= (\hat{B}\hat{A})^\dagger - (\hat{A}\hat{B})^\dagger \\ &= -(\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger \\ &= -[\hat{A}, \hat{B}]^\dagger. \end{aligned}$$

(d.) Show that the commutator of two anti-hermitian operators is also anti-hermitian.

Let \hat{A} and \hat{B} be anti-hermitian operators. The commutator of \hat{A} and \hat{B} is given by

$$\begin{aligned} [\hat{A}, \hat{B}] &= \hat{A}\hat{B} - \hat{B}\hat{A} \\ &= \hat{A}^\dagger \hat{B}^\dagger - \hat{B}^\dagger \hat{A}^\dagger \\ &= (\hat{B}\hat{A})^\dagger - (\hat{A}\hat{B})^\dagger \\ &= -(\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger \\ &= -[\hat{A}, \hat{B}]^\dagger. \end{aligned}$$

Because the commutator is the product of two operators if both are anti-hermitian the minus-sign has no effect. Thus this has the same form as in part (c.).

3. We have two operators \hat{A} and \hat{B} each with two eigenstates. The eigenstates and corresponding eigenvalues are characterized by the equations

$$\hat{A}\psi_1 = a_1\psi_1,$$

$$\hat{A}\psi_2 = a_2\psi_2,$$

$$\hat{B}\phi_1 = b_1\phi_1,$$

$$\hat{B}\phi_2 = b_2\phi_2.$$

Suppose we know that the eigenstates for each operator are related by

$$\psi_1 = \frac{1}{5}(3\phi_1 + 4\phi_2),$$

$$\psi_2 = \frac{1}{5}(4\phi_1 - 3\phi_2).$$

- (a.) If observable A is measured and we obtain a value of a_1 , what is the state of the system in the instant after the measurement was made?

To measure a_1 the system must be in state ψ_1 . Therefore, a measurement of a_1 puts the system in state ψ_1 .

- (b.) If B is now measured following the measurement in part (a.), what are the possible results and what are their associated probabilities?

Before measurement the system is in state ψ_1 which is a linear combination of the states ϕ_1 and ϕ_2 . The probability of measuring each of these states, $P(\phi_n)$, is given by the square of their amplitudes. Thus,

$$P(\phi_1) = \left| \frac{3}{5} \right|^2 = \frac{9}{25},$$

$$P(\phi_2) = \left| \frac{4}{5} \right|^2 = \frac{16}{25}.$$

- (c.) If we measure A again immediately following the measurement of B in part (b.), what is the probability of obtaining a_1 ? This is tricky because we do not know what value of B we obtained in part (b.).

Expressing ϕ_1 and ϕ_2 as linear combinations of ψ_1 and ψ_2 we have

$$\phi_1 = \frac{3}{5}\psi_1 + \frac{4}{5}\psi_2,$$

$$\phi_2 = \frac{4}{5}\psi_1 - \frac{3}{5}\psi_2.$$

Futhermore, we can define a new quantum state, ξ , based on the unknown outcome of measuring the observable B . This quantum state will be a superposition of outcomes of a measurement of B with amplitudes given by the root of the probability of measuring that quantum state. That is,

$$\begin{aligned}\xi &= \frac{3}{5}\phi_1 + \frac{4}{5}\phi_2 \\ &= \frac{3}{5} \left(\frac{3}{5}\psi_1 + \frac{4}{5}\psi_2 \right) + \frac{4}{5} \left(\frac{4}{5}\psi_1 - \frac{3}{5}\psi_2 \right) \\ &= \psi_1.\end{aligned}$$

This, however, is wrong¹. For as cool of a result as this would be it neglects the fact that measurement of B changes the wavefunction so we cannot construct ξ in this way. I kept this in the response because I thought it was neat. We instead must consider the probability of measuring ψ_1 given ϕ_1 or ϕ_2 , $P(\psi_1 | \phi_n)$. This is simply the product of the probability of making the first measurement, ϕ_n , and the probability of making the measurement ψ_1 .

$$\begin{aligned}P(\psi_1 | \phi_1) &= \frac{9}{25} \left| \frac{3}{5} \right|^2 = \frac{81}{625}, \\ P(\psi_1 | \phi_2) &= \frac{16}{25} \left| \frac{4}{5} \right|^2 = \frac{256}{625}.\end{aligned}$$

We then sum these two probabilities to find the probability of measuring ψ_1 after making some measurement of observable B :

$$P(\psi_1) = \frac{81}{625} + \frac{256}{625} = \frac{337}{625}.$$

¹Realization that this was wrong required a lot of “this cant be right” type thoughts followed by a probability tree from my prior stats class. I feel like there is some interpretation to be had here given that I returned ψ_1 which was the wavefunction I started with.