§2.6 problems.

2. Prove Theorem 1: For all integers, a, b, c, x and y, if $a|b$ and $b|c$, then $a|(bx + cy)$.

$$\forall a, b, c, x, y \in \mathbb{Z}\left((a|b \wedge a|c) \rightarrow a|(bx + cy)\right).$$

Proof. Let a, b, c, x, and y be given as arbitrary integers. Assume that $a|b$ and $a|c$. By the definition of divides there exist $w, u \in \mathbb{Z}$ such that

$$\begin{aligned} aw &= b, \\ au &= c. \end{aligned}$$

Let $k \in \mathbb{Z}$ be defined as

$$k = wx + uy.$$

Then

$$\begin{aligned} bx + cy &= awx + auy, \\ &= a(wx + uy), \\ &= ak. \end{aligned}$$

Thus, by definition, $a|(bx + cy)$. Therefore, for $a, b, c, x, y \in \mathbb{Z}$, if $a|b$ and $b|c$, then $a|(bx + cy)$. □

4. Let $m, n \in \mathbb{Z}, n \neq 0$. Prove that if $n^2x^2 - 2mnx + m^2 = n^2$, then x is irrational.

$$\forall m, n \in \mathbb{Z}\left(\left((n \neq 0) \wedge (n^2x^2 - 2mnx + m^2 = n^2)\right) \rightarrow \left(\forall a \in \mathbb{Z}, b \in \mathbb{Z}^+\, x \neq \frac{a}{b}\right)\right).$$

I think I stumped myself by trying to write the symbolic version.

6. Use the contrapositive to prove the following statement. For all $x \in \mathbb{R}^+$ if x is irrational, then $\sqrt{x}$ is irrational. You will need to use the following consequence of the Closure Properties for the Rational Numbers: If x is rational, then x^2 is rational.

$$\forall\, x \in \mathbb{R}^+ \left((x \text{ is irrational}) \to (\sqrt{x} \text{ is irrational})\right).$$

Proof. Let $x \in \mathbb{R}^+$ be given. We proceed with a proof by the contrapositive:

$$(\sqrt{x} \text{ is rational}) \to (x \text{ is rational}).$$

Assume $\sqrt{x}$ is rational, that is, there exist $a, b \in \mathbb{Z}$ such that

$$\sqrt{x} = \frac{a}{b}.$$

Then

$$\begin{aligned} \sqrt{x} &= \frac{a}{b}, \\ x &= \frac{a^2}{b^2}. \end{aligned}$$

By the Closure Properties of the Rational Numbers a^2/b^2 is rational, that is, there exist $c, d \in \mathbb{Z}$ such that

$$\frac{a^2}{b^2} = \frac{c}{d}.$$

This is a contradiction, I think (check this later, it just feeeeeels like one). Thus our assumption must be wrong; $\sqrt{x}$ is irrational. Therefore if x is irrational then $\sqrt{x}$ is irrational. $\square$

9. Prove or disprove the following statement. For all $x, y \in \mathbb{R}$, if x and y are irrational, then xy is irrational.

$$\forall x, y \in \mathbb{R}\left(((x \text{ is irrational}) \wedge (y \text{ is irrational})) \rightarrow (xy \text{ is irrational})\right).$$

Counterexample. Suppose $x = \sqrt{2}$ and $y = \sqrt{2}$. As proven in class, $\sqrt{2}$ is irrational. Then

$$\begin{aligned} xy &= \sqrt{2} \cdot \sqrt{2}, \\ &= \sqrt{2 \cdot 2}, \\ &= \sqrt{4}, \\ &= 2. \end{aligned}$$

Thus for two particular irrational numbers, x and y, their product, xy is rational. This disproves the statement

$$\forall x, y \in \mathbb{R}\left(((x \text{ is irrational}) \wedge (y \text{ is irrational})) \rightarrow (xy \text{ is irrational})\right).$$

11. Prove, for all integers x and y, $14x + 36y \neq 51$.

$$\forall x, y \in \mathbb{Z}\ 14x + 36y \neq 51.$$

Proof. We proceed with a proof by contradiction:

$$\exists\, x, y \in \mathbb{Z}\, 14x + 36y = 51.$$

If there are two integers, x and y such that $14x + 36y = 51$ then they can be found by re-arrangement. That is

$$\begin{aligned} 14x + 36y &= 51, \\ 14x &= 51 - 36y, \\ x &= \frac{51 - 36y}{14}, \\ &= \frac{51}{14} - \frac{36}{14}y. \end{aligned}$$

This is a contradiction since $x, y \in \mathbb{Z}$ but integers are not closed under division. Thus,

$$\forall x, y \in \mathbb{Z}\ 14x + 36y \neq 51.$$

$\square$

16 a) Use the contrapositive to prove, for all $x \in \mathbb{Z}$, that if $3|x^2$, then $3|x$. There will be two cases, namely $x \bmod 3 = 1$ and $x \bmod 3 = 2$.

Proof. Let $x \in \mathbb{Z}$ be given. We use the contrapositive, that is, if $3 \not| x$ then $3 \not| x^2$. We proceed with a proof by cases:

(i) $x \bmod 3 = 0$,

(ii) $x \bmod 3 = 1$,

(iii) $x \bmod 3 = 2$.

Case (i). Suppose $x \bmod 3 = 0$. Then $x = 3q + 1$ for some $q \in \mathbb{Z}$. Clearly, $x|3$ by the same q. Thus we have vacuous truth.
Case (ii). Suppose $x \bmod 3 = 1$. Then $x = 3q + 1$ for some $q \in \mathbb{Z}$. Clearly, $x \not| 3$. Then

$$\begin{aligned} x^2 &= (3q+1)^2, \\ &= 9q^2 + 6q + 2, \\ &= 3q(3q+2) + 2. \end{aligned}$$

That is, $x^2 \bmod 3 = 2$. Thus $x^2 \not| 3$.
Case (iii). Suppose $x \bmod 3 = 2$. Then $x = 3q + 2$ for some $q \in \mathbb{Z}$. Clearly $x \not| 3$. Then

$$\begin{aligned} x^2 &= (3q+2)^2, \\ &= 9q^2 + 12q + 4, \\ &= 3(3q^2 + 4q) + 4. \end{aligned}$$

That is, $x^2 \bmod 3 = 4$. Thus $x^2 \not| 3$. Therefore, for all $x \in \mathbb{Z}$ if $3 \not| x$ then $3 \not| x^2$. By the contrapositive, for all $x \in \mathbb{Z}$ if $3|x$ then $3|x^2$. $\square$

b) Use part a) of this exercise to prove that the square root of 3, $\sqrt{3}$, is irrational.

§2.7 problems.

2. Prove $\log_{32} 16$ is irrational.

3. Prove $\log_{10} 7$ is irrational.

4. Prove $\log_4 5$ is irrational.

§3.2 problems.

2. Let A, B, and C be sets, x an object, and p and q statements. For each expression given below, determine whether it makes sense (yes) or it does not make sense (no). If your answer is yes, state whether the expression is a statement or a set. If the answer is no, briefly explain why.

   a) $x \in A$.

      Yes. This is a statement.

   b) $p \in \mathbb{Z}$.

      No. A statement does not belong to the set of integers.

   c) $A \in q$.

      No. A set does not belong to a statement.

   d) $\neg A$.

      No. There is no meaning to the negation of a set.

   e) $x \in A \wedge B$.

      No. While $x \in A$ is a statement that makes sense B is not a statement.

   f) $x \in A \vee q$.

      Yes. This is a statement.

   g) $(A \cup B) \cap C$.

      Yes. This is a set.

   h) $(A \cup B) \subseteq C$.

      Yes. This is a statement.

   i) $(x \in A)^c$.

      No. $x \in A$ is a statement and there is no meaning to the complement of a statement.

   j) $x \in A^c$.

      Yes. This is a statement.

   k) $\neg(x \in A) \cap B$.

      No. $\neg(x \in A)$ is a statement and there is no meaning to the intersect of a statement and a set.

   l) $\neg(x \in (A \cap B))$.

      Yes. This is a statement.

3. Let $S = \{x \in \mathbb{Z} : \exists\, k \in \mathbb{Z}, x = 2k - 1\}$ and $T = \{x \in \mathbb{Z} : x \text{ is odd}\}$. Prove that $S = T$.

Proof. Let $x \in \mathbb{Z}$ be given. We proceed with a proof by cases

(i). $x \in S$,

(ii). $x \in T$.

Case (i). Suppose $x \in S$, that is, $x = 2k - 1$ for some $k \in \mathbb{Z}$.
Case (ii). Suppose $x \in T$, that is, x is odd. Then, by definition, there exists $q \in \mathbb{Z}$ such that $x = 2q + 1$. $\square$

5. Let $S = \{x \in \mathbb{Z} : x \bmod 12 = 8\}$ and $T = \{x \in \mathbb{Z} : 4|x\}$. Prove that $S \subseteq T$, but $T \nsubseteq S$.

9. Let $S = \{x \in \mathbb{Z} : \exists\, r, s \in \mathbb{Z}, x = 9r + 6s\}$ and $T = \{x \in \mathbb{Z} : 3|x\}$.

    a) Prove that $S \subseteq T$.

    b) Prove that $T \subseteq S$.

10. Let A, B, and C be subsets of a universal set $\mathcal{U}$. Prove each of the following set theory theorems using a sequence of logically equivalent compound forms.

    d. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

    k. $A \cap A^c = \emptyset$.

14. Prove that, if $A \cap B^c = \emptyset$, then $A \subseteq B$.

24. Find the mistake in the "proof" of the following "proposition." Is this "proposition" true? If not, find a counterexample.
    "**Proposition.**" Let A, B, and C be sets and suppose that $A \subseteq (B \cup C)$. Then $A \subseteq B$ or $A \subseteq C$.
    "*proof.*" Let x be any object and suppose that $x \in A$. Then $x \in (B \cup C)$ since $A \subseteq (B \cup C)$. Thus, by the definition of union, $x \in B$ or $x \in C$. Therefore, for all objects x, if $x \in A$, then $x \in B$ or, for all objects x, if $x \in A$, then $x \in C$.
    Hence, by the definition of subset, $A \subseteq B$ or $A \subseteq C$.