1. Consider a free particle that is described at $t = 0\,\text{s}$ by the wavefunction, $\Psi(x, 0) = Ae^{-a|x|}$ where both A and a are positive, real constants.

   (a) Normalize $\Psi(x, 0)$.

   We begin by expressing Ψ as a piecewise function:

   $$\Psi(x, 0) = \begin{cases} Ae^{-ax}, & x > 0 \\ Ae^{ax}, & x < 0 \end{cases}$$

   Now, Ψ can be normalized by the typical method:

   $$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\Psi|^2 \ \mathrm{d}x \\ &= A^2 \int_{-\infty}^{0} e^{2ax} \ \mathrm{d} + A^2 \int_{0}^{\infty} e^{-2ax} \ \mathrm{d}x \\ &= A^2 \left(\left[\frac{1}{2a} e^{2ax} \right]_{-\infty}^{0} + \left[-\frac{1}{2a} e^{-2ax} \right]_{0}^{\infty} \right) \\ &= A^2 \left(\frac{1}{2a} - 0 + 0 + \frac{1}{2a} \right) \\ &= A^2 \frac{1}{a}. \end{aligned}$$

   Thus, $A = \sqrt{a}$.

   (b) Determine $\phi(k)$.

   We begin with the typical $\phi(k)$ and exploit Euler's formula:

   $$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x, 0) e^{-ikx} \ \mathrm{d}x = \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \left(\cos(kx) - i\sin(kx) \right) \ \mathrm{d}x.$$

   Ψ has not been expressed in piecewise form because we can simplify here. Ψ is even, cos is even, and sin is odd. Our limits of integration are symmetric so the sin term integrates to 0. We are left with only even functions over symmetric limits so we can further simplify to

   $$\phi(k) = \sqrt{\frac{2a}{\pi}} \int_{0}^{\infty} e^{-ax} \cos(kx) \ \mathrm{d}x.$$

   The cos function can be expressed in terms of imaginary exponentials:

   $$\phi(k) = \frac{1}{2} \sqrt{\frac{2a}{\pi}} \int_{0}^{\infty} e^{-ax} \left(e^{ikx} + e^{-ikx} \right) \ \mathrm{d}x.$$

These represent waves moving in the $+x$ and $-x$ direction. Then,

$$\phi(k) = \sqrt{\frac{a}{2\pi}} \left(\int_0^\infty e^{ikx-ax}\, dx + \int_0^\infty e^{-ikx-ax}\, dx \right).$$

These are deceptively simple integrals and evaluate to

$$\phi(k) = \sqrt{\frac{a}{2\pi}} \left[e^{-ax} \left(\frac{e^{ikx}}{ik-a} + \frac{e^{-ikx}}{ik+a} \right) \right]_0^\infty .$$

$$= \sqrt{\frac{a}{2\pi}} \left(0 - \frac{1}{ik-a} - \frac{1}{ik+a} \right)$$

$$= -\sqrt{\frac{a}{2\pi}} \left(\frac{(ik+a) - (ik-a)}{(ik-a)(ik+1)} \right)$$

$$= -\sqrt{\frac{a}{2\pi}} \frac{2a}{(ik-a)(ik+a)}$$

$$= \frac{1}{k^2+a^2} \sqrt{\frac{2a^3}{\pi}}.$$

(c) Construct $\Psi(x,t)$ in the form of an integral.

The full wavefunction is then given by

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \phi(k) e^{i(kx-\frac{\hbar k^2}{2m}t)}\, dk$$

$$= \frac{a^{3/2}}{\pi} \int_{-\infty}^\infty \frac{\exp\left[i\left(kx - \frac{\hbar k^2}{2m}t \right) \right]}{k^2+a^2}\, dk.$$

(d) Evaluate the integral for $\Psi(x,t)$ in the limiting cases of a very large a and a very small a.

2. As we discussed in class, the time-independent Schrödinger equation for the free particle has solutions that look like $Ae^{ikx} + Be^{-ikx}$ or like $C\cos(kx) + D\sin(kx)$. Show that these are equivalent solutions. Determine what the constants C and D are as a function of A and B and vice versa.

We begin with the Euler formula:

$$Ae^{ikx} + Be^{-ikx} = A\left(\cos(kx) + i\sin(kx)\right) + B\left(\cos(-kx) + i\sin(-kx)\right)$$
$$= A\cos(kx) + B\cos(kx) + Ai\sin(kx) - Bi\sin(kx)$$
$$= (A + B)\cos(kx) + (A - B)i\sin(kx).$$

Thus,
$$C = A + B, \quad D = i(A - B).$$

Some simple arrangement yields
$$A = \frac{C - D}{2}, \quad B = \frac{C + D}{2}.$$

3. Consider a bead with mass m that slides frictionlessly around a circular wire ring with circumference L. We can think about this problem like a free particle assuming boundary condition of the form $\psi(x + L) = \psi(x)$. Determine the normalized stationary states and their corresponding energies. You should find two distinct solutions for each energy (so there is a two-fold degeneracy in this system). These two states represent clockwise and counter-clockwise rotation.

We begin with the TISE with $V(x) = 0$, $k^2 = 2mE/\hbar^2$, and some typical rearrangement has been done:
$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = -k^2\psi.$$
This differential equation has general solutions of the form

$$\psi(x) = Ae^{ikx} + Be^{-ikx}.$$

We apply the boundary condition $\psi(x + L) = \psi(x)$:

$$A\exp\left(ik(x + L)\right) + B\exp\left(-ik(x + L)\right) = A\exp\left(ikx\right) + B\exp\left(-ikx\right).$$

This is arranged to the (hopefully useful) form

$$A\exp(ikx)\left(\exp(ikL) - 1\right) = B\exp(-ikx)\left(1 - \exp(-ikL)\right).$$

To find the coefficients A and B we exploit the fact that this is valid for all x. Let $x = 0$; then,

$$A \left(\exp(ikL) - 1 \right) = B \left(1 - \exp(-ikL) \right),$$

which on its own does not tell us much. Since we are trapped on a circular ring a displacement L corresponds to 2π radians. Thus, $kL = 2\pi$. As cool as this is to notice substituting $x = L$ and applying the relation $kL = 2\pi$ yields the extremely insightful expression $0 = 0$. We instead let $x = L/2$; then,

$$Ae^{i\pi} \left(e^{i\pi} - 1 \right) = Be^{-i\pi} \left(1 - e^{-i\pi} \right) \Rightarrow A = -B.$$

Then,

$$
\begin{aligned}
0 &= A \left(\exp(ikL) - 1 \right) + B \left(\exp(-ikL) - 1 \right) \\
&= A \left(\exp(ikL) - 1 \right) - A \left(\exp(-ikL) - 1 \right) \\
&= A \left(e^{ikL} - e^{-ikL} \right) \\
&= i2A \sin(kL).
\end{aligned}
$$

Iff $A \neq 0$, $kL = n\pi$ where $n \in \mathbb{Z}^+$. From the earlier definition of k we find

$$E = \frac{k^2 \hbar^2}{2m} = \frac{n^2 \pi^2 \hbar^2}{2mL^2}.$$

4. Consider a particle interacting with a potential energy given by

$$V(x) \begin{cases} \infty, & x < 0 \\ -\alpha\delta(x-b), & x > 0 \end{cases}$$

(a) Determine the bound-state solutions (assume $E < 0$) to the time-independent Schrödinger equation in three different regions of x: $x < 0$, $0 \le x \le b$, and $x > b$. Define $K^2 = -2mE/\hbar^2$ in your solutions.

I'm going to argue that solutions should be independent of b (a in the original problem) since b only moves the Dirac-delta well. Assuming that's true we can take steps from §2.6 The Finite Square Well in the limit $a \to 0$. Thus,

$$\psi(x) = \begin{cases} Fe^{-kx}, & x > b, \\ D\cos(lx), & 0 < x < b, \\ 0, & x < 0. \end{cases}$$

(b) Demand continuity of the wavefunctions at $x = 0$ and $x = a$ to reduce the number of unknown constant parameters.

(c) Show that the discontinuity in the derivative of the wavefunctions at $x = a$ is given by

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}x}\right|_{a+\epsilon} - \left.\frac{\mathrm{d}\psi}{\mathrm{d}x}\right|_{a-\epsilon} = -\frac{2m\alpha}{\hbar^2}\psi(a).$$

(d) Using the boundary condition from part (c), show that

$$\frac{K\hbar^2}{m\alpha} = 1 - e^{-2Ka}.$$

This transcendental equation relates K with α. How many different K values will solve the equation? You might try to graph the left-hand and right-hand equations vs. K to see where they intersect.