Calvin Sprouse | Topology Problem Set 1 | 2024 April 02

Complete the following from Exercises 1.1: 1a, 2a, 7, 5(i), 5(iii).

1. Let $X = \{a, b, c, d, e, f\}$. Determine whether or not each of the following collections of subsets of X is a topology on X:

   (a) $\tau_1 = \{X, \emptyset, \{a\}, \{a, f\}, \{b, f\}, \{a, b, f\}\}$.

   τ_1 is not a topology on X. Notice, $\{b, f\} \cap \{a, f\} = \{f\}$ and $\{f\} \notin \tau_1$. Thus, τ_1 does not satisfy Definition 1.1.1 (iii).

2. Let $X = \{a, b, c, d, e, f\}$. Which of the following collections of subsets of X is a topology on X?

   (a) $\tau_1 = \{X, \emptyset, \{c\}, \{b, d, e\}, \{b, c, d, e\}, \{b\}\}$.

   τ_1 is not a topology on X. Notice, $\{c\} \cup \{b\} = \{c, b\}$ and $\{c, b\} \notin \tau_1$. Thus, τ_1 does not satisfy Definition 1.1.1 (ii).

7. List all possible topologies on the following sets:

(a) $X = \{a, b\}$;

i. $\{X, \emptyset\}$,

ii. $\{X, \emptyset, \{a\}\}$,

iii. $\{X, \emptyset, \{b\}\}$,

iv. $\{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$.

(b) $Y = \{a, b, c\}$.

i. $\{X, \emptyset\}$,

ii. $\{X, \emptyset, \{a\}\}$,

iii. $\{X, \emptyset, \{b\}\}$,

iv. $\{X, \emptyset, \{c\}\}$,

v. $\{X, \emptyset, \{a, c\}\}$,

vi. $\{X, \emptyset, \{b, c\}\}$,

vii. $\{X, \emptyset, \{a, b\}\}$,

viii. $\{X, \emptyset, \{a\}, \{a, c\}\}$,

ix. $\{X, \emptyset, \{b\}, \{a, c\}\}$,

x. $\{X, \emptyset, \{c\}, \{a, c\}\}$,

xi. $\{X, \emptyset, \{a\}, \{b, c\}\}$,

xii. $\{X, \emptyset, \{b\}, \{b, c\}\}$,

xiii. $\{X, \emptyset, \{c\}, \{b, c\}\}$,

xiv. $\{X, \emptyset, \{a\}, \{a, b\}\}$,

xv. $\{X, \emptyset, \{b\}, \{a, b\}\}$,

xvi. $\{X, \emptyset, \{c\}, \{a, b\}\}$,

xvii. $\{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$,

xviii. $\{X, \emptyset, \{b\}, \{a, b\}, \{b, c\}\}$,

xix. $\{X, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$,

xx. $\{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$,

xxi. $\{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$,

xxii. $\{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$,

xxiii. $\{X, \emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$,

xxiv. $\{X, \emptyset, \{a\}, \{c\}, \{a, c\}, \{b, a\}\}$,

xxv. $\{X, \emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b\}\}$,

xxvi. $\{X, \emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$,

xxvii. $\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$,

xxviii. $\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$,

xxix. $\{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

5. Let $\mathbb{R}$ be the set of all real numbers. Prove that each of the following collections of subsets of $\mathbb{R}$ is a topology.

(i) τ_1 consists of $\mathbb{R}$, $\emptyset$, and every interval $(-n, n)$, for $n \in \mathbb{Z}^+$, where $(-n, n)$ denotes the set $\{x \in \mathbb{R} : -n < x < n\}$;

Proof. Let $k, l \in \mathbb{Z}^+$ be given. By definition, the sets $(-k, k)$ and $(-l, l)$ are in τ_1. We now proceed to check the three possible relations between k and l; that is, (i.) $k < l$, (ii.) $k = l$, (iii.) $k > l$.

(i.) Suppose $k < l$. Then, $(-k, k) \cup (-l, l) = (-l, l)$ which is in τ_1. Also notice, $(-k, k) \cap (-l, l) = (-k, k)$ which is in τ_1.

(ii.) Suppose $k = l$. Then, $(-k, k) \cup (-l, l) = (-k, k)$ which is in τ_1. Also notice, $(-k, k) \cap (-l, l) = (-k, k)$ which is in τ_1.

(iii.) Suppose $k > l$. Then, $(-k, k) \cup (-l, l) = (-k, k)$ which is in τ_1. Also notice, $(-k, k) \cap (-l, l) = (-l, l)$ which is in τ_1.

We also notice $\bigcup_{n \in \mathbb{Z}^+} (-n, n) = \mathbb{R}$. Thus, we satisfy Definition 1.1.1 (i) by including $\mathbb{R}$ and $\emptyset$, Definition 1.1.1 (ii) by considering any combination of finite unions and infinite unions, and Definition 1.1.1 (iii) by considering any two intersections. Therefore, τ_1 is a topology on $\mathbb{R}$. $\square$

(iii) τ_3 consists of $\mathbb{R}$, $\emptyset$, and every interval $[n, \infty)$, for $n \in \mathbb{Z}^+$, where $[n, \infty)$ denotes the set $\{x \in \mathbb{R} : n \leq x\}$.

Proof. Let $k, l \in \mathbb{Z}^+$ be given. By definition, the sets $[k, \infty)$ and $[l, \infty)$ are in τ_3. We now proceed to check the three possible relations between k and l; that is, (i.) $k < l$, (ii.) $k = l$, (iii.) $k > l$.

(i.) Suppose $k < l$. Then, $[k, \infty) \cup [l, \infty) = [k, \infty)$ which is in τ_3. Also notice, $[k, \infty) \cap [l, \infty) = [l, \infty)$ which is in τ_3.

(ii.) Suppose $k = l$. Then, $[k, \infty) \cup [l, \infty) = [k, \infty)$ which is in τ_3. Also notice, $[k, \infty) \cap [l, \infty) = [k, \infty)$ which is in τ_3.

(iii.) Suppose $k > l$. Then, $[k, \infty) \cup [l, \infty) = [l, \infty)$ which is in τ_3. Also notice, $[k, \infty) \cap [l, \infty) = [k, \infty)$ which is in τ_3.

We also notice $\bigcup_{n \in \mathbb{Z}^+} = [1, \infty)$ which is in τ_3. Thus, we satisfy Definition 1.1.1 (i) by including $\mathbb{R}$ and $\emptyset$, Definition 1.1.1 (ii) by considering any combination of finite unions and infinite unions, and Definition 1.1.1 (iii) by considering any two intersections. Therefore, τ_3 is a topology on $\mathbb{R}$. $\square$