Note that integrals have been evaluated by the provided integral table.

1. Consider the continuous Gaussian distribution, $\rho(x) = Ae^{-\lambda(x-a)^2}$, where A, a, and λ are positive, real constants. Note that this is not a wavefunction, but rather a distribution.

   (a) Normalize the distribution to determine A.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \rho(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2}\,\mathrm{d}x \\
&= A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}\,\mathrm{d}x \\
&= A\int_{-\infty}^{\infty} e^{-(\lambda x^2-2\lambda ax+\lambda a^2)}\,\mathrm{d}x \\
&= A\sqrt{\frac{\pi}{\lambda}}\exp\left(\frac{(-2\lambda a)^2-4\lambda^2a^2}{4\lambda}\right) \\
&= A\sqrt{\frac{\pi}{\lambda}}.
\end{aligned}$$

   Thus, $A = \sqrt{\lambda/\pi}$.

   (b) Determine $\langle x\rangle$, $\langle x^2\rangle$, and σ.

   The average value of x, or expectation value, is given by

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} x\rho(x)\,\mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} x\exp\left(-\lambda(x-a)^2\right)\mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}}a\sqrt{\frac{\pi}{\lambda}} \\
&= a.
\end{aligned}$$

The average of the squares of x is given by

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 \rho(x) \, \mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\lambda(x-a)^2\right) \, \mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\lambda x^2 + 2\lambda a x - \lambda a^2\right) \, \mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\lambda x^2 + 2\lambda a x\right) \exp\left(-\lambda a^2\right) \, \mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \exp\left(-\lambda a^2\right) \int_{-\infty}^{\infty} x^2 \exp\left(-\lambda x^2 + 2\lambda a x\right) \, \mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \exp\left(-\lambda a^2\right) \frac{\sqrt{\pi}(2\lambda + (2\lambda a)^2)}{4\lambda^{5/2}} \exp\left(\frac{(2\lambda a)^2}{4\lambda}\right) \\
&= \frac{2\lambda + (2\lambda a)^2}{4\lambda^2} \\
&= \frac{1 + 2\lambda a^2}{2\lambda}.
\end{aligned}$$

The standard deviation, σ, of ρ is given by

$$\begin{aligned}
\sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{1 + 2\lambda a^2}{2\lambda} - a^2} \\
&= \frac{1}{\sqrt{2\lambda}}.
\end{aligned}$$

2. At time $t = 0\,\mathrm{s}$, an electron is represented by the wave function,

$$\Psi(x,0) = \begin{cases} A\frac{x}{a}, & 0 \le x \le a \\ A\frac{(b-x)}{(b-a)}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

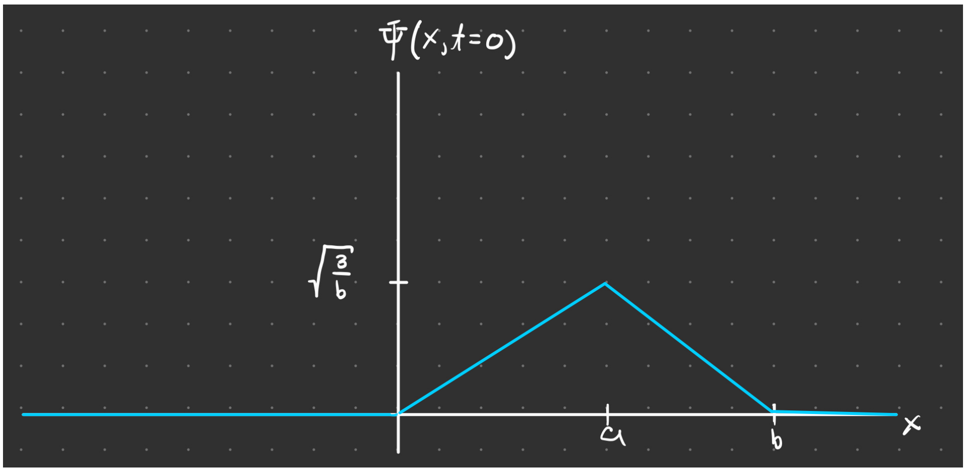
where A, a, and b are constants.

(a) Normalize Ψ.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\Psi|^2 \, \mathrm{d}x \\ &= \int_{-\infty}^{0} (0)^2 \, \mathrm{d}x + \int_0^a \left(A\frac{x}{a}\right)^2 \mathrm{d}x + \int_a^b \left(A\frac{(b-x)}{(b-a)}\right)^2 \mathrm{d}x + \int_b^{\infty} (0)^2 \, \mathrm{d}x \\ &= 0 + \frac{A^2}{a^2}\int_0^a x^2 \, \mathrm{d}x + \frac{A^2}{(b-a)^2}\int_a^b (b-x)^2 \, \mathrm{d}x + 0 \\ &= \frac{A^2}{a^2}\left[\frac{1}{3}x^3\right]_0^a + \frac{A^2}{(b-a)^2}\left[-\frac{a^3}{3} + a^2 b - ab^2 + \frac{b^3}{3}\right]_a^b \\ &= \frac{A^2}{a^2}\frac{a^3}{3} + \frac{A^2}{(b-a)^2}\frac{(b-a)^3}{3} \\ &= A^2\left(\frac{a}{3} + \frac{b-a}{3}\right) \\ &= A^2\frac{b}{3}. \end{aligned}$$

Thus, $A = \sqrt{3/b}$.

(b) Sketch $\Psi(x,0)$ as a function of x.



Note, $\Psi = 0$ when $x \geq b$ contrary to the sketch.

(c) Where is the electron most likely to be found at $t = 0\,\mathrm{s}$?

The most likely position of the electron is given by solving

$$\frac{\mathrm{d}\,|\Psi|^2}{\mathrm{d}x} = 0$$

for maximum values. We begin on the interval $0 \leq x \leq a$.

$$\begin{aligned}
\frac{\mathrm{d}\left(A\frac{x}{a}\right)^2}{\mathrm{d}x} &= 0, \\
\frac{3}{ba^2}\frac{\mathrm{d}x^2}{\mathrm{d}x} &= 0, \\
\frac{6}{ba^2}x &= 0, \\
x &= 0.
\end{aligned}$$

The probability of finding the electron at $x = 0$ is given by

$$\begin{aligned}
|\Psi(x = 0, t = 0)|^2 &= \frac{3}{ba^2}(0)^2 \\
&= 0.
\end{aligned}$$

We now check the interval $a \leq x \leq b$.

$$\begin{aligned}
\frac{\mathrm{d}\left(A\frac{b-x}{b-a}\right)^2}{\mathrm{d}x} &= 0, \\
\frac{3}{ba^2(b-a)^2}\frac{\mathrm{d}(b-x)^2}{\mathrm{d}x} &= 0, \\
\frac{-6}{ba^2(b-a)^2}(b-x) &= 0, \\
b - x &= 0, \\
x &= b.
\end{aligned}$$

The probability of finding the electron at $x = b$ is given by

$$\begin{aligned}
|\Psi(x = b, t = 0)|^2 &= \frac{3}{ba^2(b-a)^2}(b-b)^2 \\
&= 0.
\end{aligned}$$

We now check the probability of finding the electron at the boundaries, that is, $x = a$.

$$\begin{aligned}
|\Psi(x = a, t = 0)|^2 &= \frac{3}{ba^2}a^2 \\
&= \frac{3}{b}.
\end{aligned}$$

Thus, the electron is most likely to be found at $x = a$. In hindsight, this could have been simpler since a valid wave-function is continuous. At the boundaries $x = 0$ and $x = b$ we could have simply observed the "otherwise" case and seen that, to be continuous, $\Psi(x = 0, t = 0)$ and $\Psi(x = b, t = 0)$ must be equal to 0. Thus, we would not have needed to calculate the probability of finding the electron at $x = 0$ or $x = b$.

(d) What is the probability the electron will be found in the region $x \leq a$? Check your result in the limiting case where $b = a$ and $b = 2a$.

The probability that the electron will be found in the region $x \leq a$ is given by

$$\begin{aligned}
\int_{-\infty}^{a} |Psi|^2 \, \mathrm{d}x &= \int_{0}^{a} \left(\frac{Ax}{a}\right)^2 \, \mathrm{d}x \\
&= \frac{3}{ba^2} \int_{0}^{a} x^2 \, \mathrm{d}x \\
&= \frac{3}{ba^2} \left[\frac{1}{3}x^3\right]_0^a \\
&= \frac{3}{ba^2} \frac{a^3}{3} \\
&= \frac{a}{b}.
\end{aligned}$$

If $b = a$ the wave-function takes on non-zero values only in the interval $0 \leq x \leq a$ meaning the electron can only be found on the interval $0 \leq x \leq a$. Thus the probability of finding the electron on the interval $0 \leq x \leq a$ should be 1:

$$\frac{a}{b} = \frac{a}{a} = 1.$$

If $b = 2a$ the probability of finding the electron on the interval $0 \leq x \leq a$ should be less than 1:

$$\frac{a}{b} = \frac{a}{2a} = \frac{1}{2}.$$

(e) Determine $\langle x\rangle$.

$$
\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} x\,|\Psi|^2\ \mathrm{d}x \\
&= \int_0^a x\left(\sqrt{\frac{3}{b}}\frac{x}{a}\right)^2\ \mathrm{d}x + \int_a^b x\left(\sqrt{\frac{3}{b}}\frac{(b-x)}{(b-a)}\right)^2\ \mathrm{d}x \\
&= \frac{3}{ba^2}\int_0^a x^3\ \mathrm{d}x + \frac{3}{b}\frac{1}{(b-a)^2}\int_a^b (b^2x - 2bx^2 + x^3)\ \mathrm{d}x \\
&= \frac{3}{ba^2}\left[\frac{1}{4}x^4\right]_0^a + \frac{3}{b}\frac{1}{(b-a)^2}\left[\frac{b^2}{2}x^2 - \frac{2b}{3}x^3 + \frac{1}{4}x^4\right]_a^b \\
&= \frac{3}{4}\frac{a^2}{b} + 3\frac{1}{b(b-a)^2}\left(\frac{b^4}{2} - \frac{2b^4}{3} + \frac{b^4}{4} - \frac{a^2b^2}{2} + \frac{2ba^3}{3} - \frac{a^4}{4}\right) \\
&= \frac{3}{4}\frac{a^2}{b} + 3\frac{1}{b(b-a)^2}\frac{(b-a)^3(3a+b)}{12} \\
&= \frac{3}{4}\frac{a^2}{b} + \frac{1}{4}\frac{(b-a)(3a+b)}{b} \\
&= \frac{3a^2 + (b-a)(3a+b)}{4b} \\
&= \frac{1}{4}(2a+b).
\end{aligned}
$$