Problem 4.1. Recall Problem 1.34, which concerned an ideal diatomic gas taken around a rectangular cycle on a PV diagram. Suppose now that this system is used as a heat engine, to convert the heat added into mechanical work.

(a) Evaluate the efficiency of this engine for the case $V_2 = 3V_1$, and $P_2 = 2P_1$.

(b) Calculate the efficiency of an "ideal" engine operating between the same temperature extremes.

(a) The efficiency, e, of this engine is given by Equation 4.1,

$$e \equiv \frac{W}{Q_h}.$$

The work done by the engine can be found by the area enclosed,

$$W = (P_2 - P_1)(V_2 - V_1) = 2P_1V_1.$$

The heat absorbed in step A can be found by arranging Equation 1.41,

$$C = \frac{Q}{\Delta T},$$

for Q in terms of the heat capacity at constant volume, C_V,

$$Q = C_V \Delta T.$$

Equation 1.46 gives the heat capacity at constant volume for an ideal gas as

$$C_V = \frac{Nfk}{2}.$$

Air is nearly entirely composed of oxygen and nitrogen gas which are diatomic molecules allowing for three spatial and two rotation degrees of freedom, $f = 5$. Then, the heat absorbed in part A is given by

$$Q_A = \frac{5}{2}Nk(T_i - T_f).$$

The ideal gas law states

$$PV = NkT.$$

We recognize by Figure 1.10b that $V_i = V_f = V_1$. Then,

$$Q_A = \frac{5}{2}V_1(P_2 - P_1).$$

For step B we use Equation 1.48 for the heat capacity at constant pressure, C_P,

$$C_P = C_V + Nk.$$

Then, by Equation 1.46 and Equation 1.41,

$$Q_B = \frac{7}{2}Nk(T_i - T_f).$$

We recognize by Figure 1.10b that $P_i = P_f = P_2$. By the ideal gas law,

$$Q_B = \frac{7}{2}P_2(V_2 - V_1).$$

Then, the total heat absorbed is given by

$$Q_H = Q_A + Q_B = \frac{33}{2}P_1V_1.$$

Then,

$$e = \frac{|W|}{Q_h} = \frac{2P_1V_1}{\left(\frac{33}{2}\right)P_1V_1} = \frac{4}{33} = 12\,\%.$$

Problem 4.4. It has been proposed to use the thermal gradient of the ocean to drive a heat engine. Suppose that at a certain location the water temperature is $22\,°\mathrm{C}$ at the ocean surface and $4\,°\mathrm{C}$ at the ocean floor.

(a) What is the maximum possible efficiency of an engine operating between these two temperatures?

(b) If the engine is to produce $1\,\mathrm{GW}$ of electrical power, what minimum volume of water must be processed (to suck out the heat) in every second?

(a) The maximum possible efficiency of an engine operating between two temperatures is given by Equation 4.5,

$$e = 1 - \frac{T_c}{T_h},$$

where T_c is the temperature at the ocean floor, $T_c = 277\,\mathrm{K}$, and T_h is the temperature at the ocean surface, $T_h = 295\,\mathrm{K}$. Then,

$$e = 1 - \frac{277\,\mathrm{K}}{295\,\mathrm{K}} = 0.06.$$

(b) Suppose the size of our reservoirs is so large so as to render the temperatures constant. Then, in a perfect world, we may extract

$$\Delta T = 295\,\mathrm{K} - 277\,\mathrm{K} = 18\,\mathrm{K}.$$

The heat energy obtained by extracting $18\,\mathrm{K}$ is given by

$$Q = cm\Delta T,$$

where $c = 4186\,\mathrm{J\,kg^{-1}\,K^{-1}}$ and $Q = 1\,\mathrm{GW}$. Then,

$$m = \frac{Q}{m\Delta T} = \frac{10^9\,\mathrm{J\,s^{-1}}}{4186\,\mathrm{J\,kg^{-1}\,K^{-1}}18\,\mathrm{K}} = 13 \times 10^3\,\mathrm{kg\,s^{-1}}.$$

But that would be at $100\,\%$ efficiency. Operating at a mere $e = 6\,\%$ efficiency,

$$m = \frac{1}{e}\frac{Q}{m\Delta T} = 217 \times 10^3\,\mathrm{kg\,s^{-1}}.$$

Problem 4.5. Prove directly (by calculating the heat taken in and the heat expelled) that a Carnot engine using an ideal gas as the working substance has an efficiency of $1 - T_c/T_h$.

Q_h and Q_c are given by finding the work done along the isotherms in step $1 \to 2$ and step $3 \to 4$. Since we are on an isotherm, $\Delta T = 0$ and $\Delta U = 0$. Then,

$$Q_h = |W_{12}| = \int_{V_1}^{V_2} P\mathrm{d}V = NkT_h \ln\left(\frac{V_2}{V_1}\right),$$
$$Q_c = |W_{34}| = \int_{V_4}^{V_3} P\mathrm{d}V = NkT_c \ln\left(\frac{V_3}{V_4}\right).$$

The relationship between volume and temperature on an adiabat is given by Equation 1.39,

$$V_i T_i^{f/2} = V_f T_f^{f/2}.$$

On step $2 \to 3$,

$$V_2 T_h^{f/2} = V_3 T_c^{f/2} \quad \Rightarrow \quad \frac{V_2}{V_3} = \frac{T_c^{f/2}}{T_c^{f/2}}.$$

On step $4 \to 1$,

$$V_4 T_c^{f/2} = V_1 T_h^{f/2} \quad \Rightarrow \quad \frac{V_4}{V_1} = \frac{T_h^{f/2}}{T_c^{f/2}}.$$

Then, joining the two equations by the common temperature ratio yields

$$\frac{V_2}{V_3} = \frac{V_1}{V_4} \quad \Rightarrow \quad \frac{V_2}{V_1} = \frac{V_3}{V_4}.$$

Finally, since the temperature ratios are equal,

$$e = 1 - \frac{T_c}{T_h}.$$

Problem 4.8. Can you cool off your kitchen by leaving the refrigerator door open?

A room cannot be cooled by leaving the refrigerator open. The refrigerator cools the space inside the fridge by pumping heat into the surrounding environment. The surrounding environment is then heated and the space inside the fridge is cooled. By leaving the door open, a small temperature gradient is then created as the air inside is cooled and the air outside is heated. In the best case scenario where this requires no energy the room is left at the same overall temperature but there exists a separation. In reality, this whole process requires some extra work which is cast into the room as waste heat. Then, the fridge is simply acting as a poor space heater.

Problem 4.9. Estimate the maximum possible coefficient of performance of a household air conditioner. Use any reasonable values for the reservoir temperatures.

The maximum coefficient of performance, c, is given by Equation 4.9,

$$c = \frac{T_c}{T_h - T_c}.$$

Suppose the household air conditioner is designed for a cool summer day in Ellensburg. Let the temperature outside the house be $T_h = 40\,^\circ\mathrm{C} = 313\,\mathrm{K}$. Let the desired temperature inside the house be room temperature, $T_c = 20\,^\circ\mathrm{C} = 293\,\mathrm{K}$. Then,

$$c = \frac{293\,\mathrm{K}}{313\,\mathrm{K} - 293\,\mathrm{K}} = 14.65.$$

We interpret this to mean that for each joule of electrical energy consumed, 14.65 joules of heat energy can be removed from inside the house.

Problem 4.14. A heat pump is an electrical device that heats a building by pumping heat in from the cold outside. In other words, it's the same as a refrigerator, but its purpose is to warm the hot reservoir rather than to cool the cold reservoir (even though it does both). Let us define the following standard symbols, all taken to be positive by convention:

$$\begin{aligned} T_h &= \text{temperature inside the building} \\ T_c &= \text{temperature outside} \\ Q_h &= \text{heat pumped into building in 1 day} \\ Q_c &= \text{heat taken from outdoors in 1 day} \\ W &= \text{electrical energy used by heat pump in 1 day} \end{aligned}$$

(a) Explain why the coefficient of performance (COP) for a heat pump should be defined as Q_h/W.

(b) What relation among Q_h, Q_c, and W is implied by energy conservation alone? Will energy conservation permit the COP to be greater than 1?

(c) Use the second law of thermodynamics to derive an upper limit on the COP, in terms of the temperatures T_h and T_c alone.

(d) Explain why a heat pump is better than an electric furnace, which simply converts electrical work directly into heat.

(a) The desireable affect of a heat pump is the transfer of heat energy to the hot reservoir, Q_h. The cost of running a heat pump is the work required, W.

(b) The work may be defined in terms of the heat flow, $W = Q_h - Q_c$. Then,

$$c = \frac{Q_h}{Q_h - Q_c} = \frac{1}{1 - \frac{Q_c}{Q_h}}.$$

Since $Q_h > Q_c$, $c > 1$.

(c) The second law of thermodynamics is given by Equation 4.8,

$$\frac{Q_h}{T_h} \geq \frac{Q_c}{T_c} \quad \Rightarrow \quad \frac{Q_h}{Q_c} \geq \frac{T_h}{T_c} \quad \Rightarrow \quad \frac{Q_c}{Q_h} \leq \frac{T_c}{T_h}.$$

Substituting into our expression for c yields

$$c \leq \frac{1}{1 - \frac{T_c}{T_h}}.$$

(d) A heat pump leverages an existing temperature gradient which could, in principle, be large. An electric furnace, however, generates a temperature gradient by converting work into heat. The best case for an electric furnace is that all work is converted to heat, a coefficient of performance of 1. As demonstrated in part **(a)**, the coefficient of performance for a heat pump is always greater than 1. Thus, the heat pump is always going to be more efficient than the electric furnace but only functions if there exists a temperature gradient.

Problem 4.18*. Derive Equation 4.10 for the efficiency of the Otto cycle.

Equation 4.10 gives the efficiency of the Otto cycle in terms of the compression ratio as

$$e = 1 - \left(\frac{V_2}{V_1}\right)^{\gamma-1}.$$

We begin with the most general form of the efficiency of an engine is given by Equation 4.3,

$$e = 1 - \frac{Q_c}{Q_h}.$$

Figure 4.5 shows the Otto cycle on a PV diagram where we can see that heat is added, Q_h, on step $2 \to 3$ and heat is removed, Q_c, on step $4 \to 1$. These are the adiabatic steps where no work is preformed. Then, by the first law of thermodynamics, at each adiabatic step,

$$\Delta U = Q.$$

For step $2 \to 3$,

$$Q_h = U_3 - U_2.$$

By the equipartition theorem,

$$Q_h = \frac{f}{2}NkT_3 - \frac{f}{2}NkT_2.$$

The Ideal Gas Law states

$$PV = NkT.$$

Then,

$$Q_h = \frac{f}{2}V_3P_3 - \frac{f}{2}V_2P_2 = \frac{f}{2}V_2(P_3 - P_2),$$

where we recognized by Figure 4.5, $V_3 = V_2$. By a similar chain of reason,

$$Q_c = \frac{f}{2}V_1(P_4 - P_1).$$

The efficiency of our engine can then be expressed as

$$e = 1 - \frac{V_1}{V_2}\frac{(P_4 - P_1)}{(P_3 - P_2)}.$$

Equation 1.40 gives a conserved quantity for adiabatic processes as

$$V^{\gamma}P = \text{constant}.$$

Then, for the two adiabatic phases,

$$(3 \to 4) : V_2^\gamma P_3 = V_1^\gamma P_4, \qquad (1 \to 2) : V_1^\gamma P_1 = V_2^\gamma P_2.$$

These equations can be arranged for P_3 and P_2:

$$P_3 = \left(\frac{V_1}{V_2}\right)^\gamma P_4, \qquad P_2 = \left(\frac{V_1}{V_2}\right)^\gamma P_1.$$

Then,

$$P_3 - P_2 = \left(\frac{V_1}{V_2}\right)^\gamma P_4 - \left(\frac{V_1}{V_2}\right)^\gamma P_1 = (P_4 - P_1)\left(\frac{V_1}{V_2}\right)^\gamma.$$

Substituting into our expression for engine efficiency yields

$$e = 1 - \frac{V_1}{V_2}\frac{(P_4 - P_1)}{(P_4 - P_1)}\frac{1}{\left(\frac{V_1}{V_2}\right)^\gamma} = 1 - \frac{V_1}{V_2}\left(\frac{V_2}{V_1}\right)^\gamma.$$

Finally,

$$e = 1 - \left(\frac{V_2}{V_1}\right)^{\gamma - 1},$$

as desired.

Problem 4.19. The amount of work done by each stroke of an automobile engine is controlled by the amount of fuel injected into the cylinder: the more fuel, the higher the temperature and pressure at points 3 and 4 in the cycle. But according to Equation 4.10, the efficiency of the cycle depends only on the compression ratio which is always the same for any particular engine, not on the amount of fuel consumed. Do you think this conclusion still holds when various other effects such as friction are taken into account? Would you expect a real engine to be most efficient when operating at high power or at low power?

This conclusion probably represents an average efficiency over the range of performance of the engine. The energy loss due to friction is relatively constant at different operating levels because it would depend on the piston displacement, which is given to be constant. At high power levels, with correspondingly high temperature and fuel injection, conductive heat loss and incomplete combustion render the engine less efficient. A real engine is tuned to some general operating efficiency, which in most internal combustion engine cars is related to the tachometer by the transmission. During the oil crisis in particular, engines and transmissions were tuned to be maximally efficient at the highest gear when doing 55 mph, the typical highway speed. Then the transmission would be engineered such that other common speeds would associate with a particular gear and result in the engine rpm being similar to the engine rpm at the ideal tuning. This way engines can maintain efficiency during long coasting periods, only dipping into high power cycles during acceleration. The exception to this is an electric generator engine present in modern hybrid vehicles. These engines are run at a near constant rpm against a near constant load, the converter, and are able to maintain near maximum efficiency at all vehicle speeds.