1. Let the wavefunction $\Psi(x,t)$ be a solution to the time-dependent Schrödinger equation when the potential energy is given by $V(x)$. What is the solution to the Shrödinger equation if we now consider a potential of $V(x) + V_0$ where V_0 is a real positive constant.

   Disclaimer: it is late, I'm very tired and have spend the last couple hours looking at this problem so type-setting seems very daunting.

   Let Ψ' be our wavefunction for our new potential. If we do separation of variables as described in Griffiths §2.1 we arrive at Eq. 2.3:

$$i\hbar \frac{1}{\phi'} \frac{\mathrm{d}\phi'}{\mathrm{d}t} = -\frac{\hbar^2}{2m} \frac{1}{\psi} \frac{\mathrm{d}^2\psi'}{\mathrm{d}x^2} + V + V_0.$$

   Here is where the magic happens, a process I still have no justification for other than it works and not doing results in nonsense (or hours at a whiteboard). By subtracting V_0 to the "time-side" of the equation and proceeding to Eq. 2.4 we find

$$i\hbar \frac{1}{\psi'} \frac{\mathrm{d}\psi'}{\mathrm{d}t} - V_0 = E.$$

   Proceeding to solve this differential equation we find

$$\psi'(t) = \exp\left(\frac{-iEt}{\hbar}\right) \exp\left(\frac{-iV_0 t}{\hbar}\right).$$

   Behold! Phase factor! It is unclear to me why this should be the "correct" thing to do but keeping V_0 in the "position-side" is not. Perhaps keeping it on the "position-side" will work out in the end and produce a similar result. Nonetheless, it seems the only difference between Ψ and Ψ' is a phase-factor in the time-dependent component.

2. A particle is observed in a quantum state described by the wavefunction

$$\Psi(x,t) = A \exp\left(-a\left(\frac{mx^2}{\hbar} + it\right)\right),$$

where A and a are real positive constants.

(a) Normalize Ψ.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi|^2 \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \Psi^* \Psi \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} A \exp\left(-a\left(\frac{mx^2}{\hbar} + it\right)\right) A \exp\left(-a\left(\frac{mx^2}{\hbar} - it\right)\right) \, \mathrm{d}x \\
&= A^2 \int_{-\infty}^{\infty} \exp\left(-a\left(\frac{mx^2}{\hbar} + it + \frac{mx^2}{\hbar} - it\right)\right) \, \mathrm{d}x \\
&= A^2 \int_{-\infty}^{\infty} \exp\left(-\frac{2am}{\hbar}x^2\right) \, \mathrm{d}x \\
&= A^2 \sqrt{\frac{\pi}{\frac{2am}{\hbar}}}.
\end{aligned}$$

Solving for A we find

$$A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}.$$

(b) What is the potential $V(x)$ that this particle finds itself within? Ψ is, by definition, a solution to the Schrödinger equation. Thus,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi.$$

Arranging for the potential energy function V we get

$$V = \frac{1}{\Psi}\left(i\hbar\frac{\partial\Psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}\right).$$

To make these partial derivatives less threatening, we can begin by writing Ψ in the form $\Psi = A\psi(x)\phi(t)$. The A component is simply A, as found by normalization. The exponential term in Ψ can be separated into an x-dependent and t-dependent component;

$$\exp\left(\frac{-amx^2}{\hbar} - ait\right) = \exp\left(-\frac{amx^2}{\hbar}\right)\exp\left(-ait\right),$$

which become ψ and ϕ respectively. We know

$$\phi(t) = \exp\left(-\frac{iEt}{\hbar}\right) = \exp\left(-ait\right),$$

thus $a = E/\hbar$. We can now write V in a more approachable way with ordinary derivatives:

$$V = \frac{1}{A\psi\phi}\left(i\hbar A\psi\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\hbar^2}{2m}A\phi\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}\right).$$

The ordinary derivatives are

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(-ait)\right] = -ai\exp(-ait) = -ai\phi,$$

and

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = \frac{\mathrm{d}\psi}{\mathrm{d}x}\left[-\frac{2amx}{\hbar}\exp\left(-\frac{amx^2}{\hbar}\right)\right] = \frac{-2am}{\hbar}\left(1 - \frac{2am}{\hbar}x^2\right)\psi.$$

Thus the potential energy function V is given by

$$\begin{aligned}
V &= \frac{1}{A\psi\phi}\left(i\hbar A\psi\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\hbar^2}{2m}A\phi\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}\right)\\
&= \frac{1}{A\psi\phi}\left(-i\hbar A\psi\frac{E}{\hbar}i\phi + \frac{\hbar^2}{2m}A\phi\frac{-2Em}{\hbar^2}\left(1 - \frac{2Em}{\hbar^2}x^2\right)\psi\right)\\
&= E - E\left(1 - \frac{2Em}{\hbar^2}x^2\right)\\
&= \frac{2mE^2}{\hbar^2}x^2.
\end{aligned}$$

(c) Determine the expectation values $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$.

i. The expectation value of x, $\langle x \rangle$, is given by

$$\langle x \rangle = \int_{-\infty}^{\infty} x \left|\Psi\right|^2 \, \mathrm{d}x$$

which is an odd function evaluated over symmetric limits and therefore

$$\langle x \rangle = 0.$$

ii. The mean square position, $\langle x^2 \rangle$, is given by

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 \left|\Psi\right|^2 \, \mathrm{d}x \\
&= A^2 \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{2am}{\hbar}x^2\right) \, \mathrm{d}x \\
&= \sqrt{\frac{2am}{\pi\hbar}} \frac{\sqrt{\pi}}{2\left(\frac{2am}{\hbar}\right)^{3/2}} \\
&= \frac{\hbar^2}{4Em}.
\end{aligned}$$

iii. The expectation momentum, $\langle p \rangle$, is given by

$$\langle p \rangle = \frac{\mathrm{d}\langle x \rangle}{\mathrm{d}t} = 0.$$

iv. The mean square momentum, $\langle p^2 \rangle$, is given by

$$\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \Psi^* \left[-i\hbar\frac{\partial}{\partial x}\right]^2 \Psi \, \mathrm{d}x \\
&= -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{\partial^2}{\partial x^2} \Psi \, \mathrm{d}x \\
&= -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \left(\frac{-2am}{\hbar^2}(\hbar - 2amx^2)\right) \Psi \, \mathrm{d}x \\
&= 2am \int_{-\infty}^{\infty} |\Psi|^2(\hbar - 2amx^2) \, \mathrm{d}x \\
&= 2am \left(\hbar \int_{-\infty}^{\infty} |\Psi|^2 \, \mathrm{d}x - 2am \int_{-\infty}^{\infty} x^2|\Psi|^2 \, \mathrm{d}x\right) \\
&= 2am \left(\hbar - 2am \langle x^2 \rangle\right) \\
&= 2am \left(\hbar - 2am\frac{\hbar^2}{4Em}\right) \\
&= \frac{2Em}{\hbar} \left(\hbar - \frac{2Em}{\hbar}\frac{\hbar^2}{4Em}\right) \\
&= Em.
\end{aligned}$$

(d) Determine the standard deviations for position, σ_x, and momentum, σ_p.

The standard deviation of position is given by

$$\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\left(\frac{\hbar^2}{4Em}\right) - (0)^2} \\
&= \frac{\hbar}{2\sqrt{Em}}.
\end{aligned}$$

The standard deviation of momentum is given by

$$\begin{aligned}
\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \sqrt{Em - (0)^2} \\
&= \sqrt{Em}.
\end{aligned}$$

(e) Are your values for σ_x and σ_p consistent with the uncertainty principle?

The uncertainty principle, in terms of position and momentum, states

$$\sigma_x \sigma_p \leq \frac{\hbar}{2}.$$

Substituting known values we find

$$\sigma_x \sigma_p = \frac{\hbar}{2\sqrt{Em}} \sqrt{Em} = \frac{\hbar}{2},$$

which satisfies the uncertainty principle. Though, $\sigma_x \sigma_p$ are the limit of precision allowed by the uncertainty principle.

3. An electron is trapped in a harmonic quadratic potential. Suppose the expectation value for its position is given by $\langle x \rangle = \frac{a}{2}\sin(\omega t)$. Here, a is a real constant with units of length and ω is an angular frequency. What, if anything, can be concluded about the electron's momentum?

   The expectation value of momentum can be found in terms of the expectation value of position:

$$\begin{aligned}\langle p \rangle &= m\frac{\mathrm{d}\langle x \rangle}{\mathrm{d}t} \\ &= m\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{a}{2}\sin(\omega t)\right] \\ &= \frac{ma\omega}{2}\cos(\omega t),\end{aligned}$$

   where is is assumed that ω is not a function of time.

   We can begin to analyze the uncertainty, but ultimately without knowing the expectation of the square position or expectation of square momentum not much insight can be gained (trust me I tried).