1. Determine the eigenfunctions and eigenvalues of the operator

$$\hat{Q} = \frac{\mathrm{d}^2}{\mathrm{d}\phi^2}$$

where ϕ is the angle in polar coordinates. Due to the rotational symmetry of the problem, your eigenfunctions, f, should satisfy the boundary condition

$$f(\phi) = f(\phi + 2\pi).$$

The eigenvalue equation

$$\hat{Q}f = qf,$$

where q is the eigenvalue, has the general solution

$$f = Ae^{q^b\phi}.$$

Then,

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\phi^2} = \left(q^b\right)^2 Ae^{q^b\phi} = \left(q^b\right)^2 f,$$

where $\left(q^b\right)^2$ is the eigenvalue yielding the condition

$$\left(q^b\right)^2 = q.$$

Thus, $b = 1/2$. Thusly the eigenfunction f is given by

$$f = A\exp\left(\sqrt{q}\phi\right).$$

We apply the boundary condition by letting $\phi = 0$. Thus, having neglected the amplitudes which are constant,

$$1 = \exp\left(\sqrt{q}2\pi\right).$$

This is satisfied for

$$\sqrt{q}2\pi = in2\pi.$$

Thus,

$$q = -n^2.$$

Is the spectrum of eigenvalues degenerate or non-degenerate?

The spectrum of eigenvalues, $n \in \mathbb{Z}$, is degenerate for $n \neq 0$.

2. Suppose that $f(x)$ and $g(x)$ are both eigenfunctions of an operator $\hat{Q}$. The spectrum is degenerate such that both $f(x)$ and $g(x)$ have the same eigenvalue, q.

   (a.) Prove that any linear combination of $f(x)$ and $g(x)$ is also an eigenfunction of Q. What is its eigenvalue?

   Proof. Let $f(x)$ and $g(x)$ be defined as eigenfunctions of an operator $\hat{Q}$ such that

   $$\hat{Q}f = qf, \quad \hat{Q}g = qg.$$

   Furthermore let $a, b \in \mathbb{R}$ be given. Then, let $j(x)$ be a function defined as a linear combination of f and g such that

   $$j(x) = af(x) + bg(x).$$

   Then,

   $$\begin{aligned}
   \hat{Q}j &= \hat{Q}\left[af + bg\right] \\
   &= \hat{Q}af + \hat{Q}bg \\
   &= a\hat{Q}f + b\hat{Q}g \\
   &= aqf + bqg \\
   &= q\left(af + bg\right) \\
   &= qj.
   \end{aligned}$$

   That is, a linear combination of eigenfunctions of $\hat{Q}$, each with eigenvalue q, is also an eigenfunction of $\hat{Q}$ with eigenvalue q. □

   (b.) An anti-hermitian operator obeys the following condition:

   $$\hat{Q}^{\dagger} = -\hat{Q}.$$

   Show that the expectation value of an anti-hermitian operator is imaginary.

   (c.) Show that the commutator of two hermitian operators is anti-hermitian.

   (d.) Show that the commutator of two anti-hermitian operators is also anti-hermitian.

3. We have two operators $\hat{A}$ and $\hat{B}$ each with two eigenstates. The eigenstates and corresponding eigenvalues are characterized by the equations

$$\begin{aligned}
\hat{A}\psi_1 &= a_1\psi_1, \\
\hat{A}\psi_2 &= a_2\psi_2, \\
\hat{B}\phi_1 &= b_1\phi_1, \\
\hat{B}\phi_2 &= b_2\phi_2.
\end{aligned}$$

Suppose we know that the eigenstates for each operator are related by

$$\begin{aligned}
\psi_1 &= \frac{1}{5}\left(3\phi_1 + 4\phi_2\right), \\
\psi_2 &= \frac{1}{5}\left(4\phi_1 - 3\phi_2\right).
\end{aligned}$$

(a.) If observable A is measured and we obtain a value of a_1, what is the state of the system in the instant after the measurement was made?

To measure a_1 the system must be in state ψ_1. Therefore, a measurement of a_1 puts the system in state ψ_1.

(b.) If B is now measured following the measurement in part (a.), what are the possible results and what are their associated probabilities?

Before measurement the system is in state ψ_1 which is a linear combination of the states ϕ_1 and ϕ_2. The probability of measuring each of these states, $P(\phi_n)$, is given by the square of their amplitudes. Thus,

$$\begin{aligned}
P(\phi_1) &= \left|\frac{3}{5}\right|^2 = \frac{9}{25}, \\
P(\phi_2) &= \left|\frac{4}{5}\right|^2 = \frac{16}{25}.
\end{aligned}$$

(c.) If we measure A again immediately following the measurement of B in part (b.), what is the probability of obtaining a_1? This is tricky because we do not know what value of B we obtained in part (b.).

Expressing ϕ_1 and ϕ_2 as linear combinations of ψ_1 and ψ_2 we have

$$\begin{aligned}
\phi_1 &= \frac{3}{5}\psi_1 + \frac{4}{5}\psi_2, \\
\phi_2 &= \frac{4}{5}\psi_1 - \frac{3}{5}\psi_2.
\end{aligned}$$

Futhermore, we can define a new quantum state, ξ, based on the unknown outcome of measuring the observable B. This quantum state will be a superposition of outcomes of a measurement of B with amplitudes given by the root of the probability of measuring that quantum state. That is,

$$\begin{aligned}\xi &= \frac{3}{5}\phi_1 + \frac{4}{5}\phi_2 \\ &= \frac{3}{5}\left(\frac{3}{5}\psi_1 + \frac{4}{5}\psi_2\right) + \frac{4}{5}\left(\frac{4}{5}\psi_1 - \frac{3}{5}\psi_2\right) \\ &= \psi_1.\end{aligned}$$

This, however, is wrong[1]. For as cool of a result as this would be it neglects the fact that measurement of B changes the wavefunction so we cannot construct ξ in this way. I kept this in the response because I thought it was neat. We instead must consider the probability of measuring ψ_1 given ϕ_1 or ϕ_2, $P\left(\psi_1 \mid \phi_n\right)$.

$$\begin{aligned}P\left(\psi_1 \mid \phi_1\right) &= \frac{9}{25}\left|\frac{3}{5}\right|^2 = \frac{81}{625}, \\ P\left(\psi_1 \mid \phi_2\right) &= \frac{16}{25}\left|\frac{4}{5}\right|^2 = \frac{256}{625}.\end{aligned}$$

We then sum these two probabilities to find the probability of measuring ψ_1 after making some measurement of observable B:

$$P(\psi_1) = \frac{81}{625} + \frac{256}{625} = \frac{337}{625}.$$

[1]Realization that this was wrong required a lot of “this cant be right” type thoughts followed by a probability tree from my prior stats class. I feel like there is some interpretation to be had here given that I returned ψ_1 which was the wavefunction I started with.