Note that integrals have been evaluated by the provided integral table.

1. Consider the continuous Gaussian distribution, $\rho(x) = Ae^{-\lambda(x-a)^2}$, where A, a, and λ are positive, real constants. Note that this is <u>not</u> a wavefunction, but rather a distribution.

   (a) Normalize the distribution to determine A.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \rho(x)\ \mathrm{d}x \\
&= \int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2}\ \mathrm{d}x \\
&= A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}\ \mathrm{d}x \\
&= A\int_{-\infty}^{\infty} e^{-(\lambda x^2 - 2\lambda a x + \lambda a^2)}\ \mathrm{d}x \\
&= A\sqrt{\frac{\pi}{\lambda}} \exp\left(\frac{(-2\lambda a)^2 - 4\lambda^2 a^2}{4\lambda}\right) \\
&= A\sqrt{\frac{\pi}{\lambda}}.
\end{aligned}$$

   Thus, $A = \sqrt{\lambda/\pi}$.

   (b) Determine $\langle x \rangle$, $\langle x^2 \rangle$, and σ.

   The average value of x, or expectation value, is given by

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x\rho(x)\ \mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x \exp\left(-\lambda(x-a)^2\right)\ \mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} a \sqrt{\frac{\pi}{\lambda}} \\
&= a.
\end{aligned}$$

The average of the squares of x is given by

$$
\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 \rho(x) \, \mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\lambda(x-a)^2\right) \, \mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\lambda x^2 + 2\lambda a x - \lambda a^2\right) \, \mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\lambda x^2 + 2\lambda a x\right) \exp\left(-\lambda a^2\right) \, \mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \exp\left(-\lambda a^2\right) \int_{-\infty}^{\infty} x^2 \exp\left(-\lambda x^2 + 2\lambda a x\right) \, \mathrm{d}x \\
&= \sqrt{\frac{\lambda}{\pi}} \exp\left(-\lambda a^2\right) \frac{\sqrt{\pi}(2\lambda + (2\lambda a)^2)}{4\lambda^{5/2}} \exp\left(\frac{(2\lambda a)^2}{4\lambda}\right) \\
&= \frac{2\lambda + (2\lambda a)^2}{4\lambda^2} \\
&= \frac{1 + 2\lambda a^2}{2\lambda}.
\end{aligned}
$$

The standard deviation, σ, of ρ is given by

$$
\begin{aligned}
\sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{1 + 2\lambda a^2}{2\lambda} - a^2} \\
&= \frac{1}{\sqrt{2\lambda}}.
\end{aligned}
$$

2. At time $t = 0\,\mathrm{s}$, an electron is represented by the wave function,

$$\Psi(x,0) = \begin{cases} A\frac{x}{a}, & 0 \le x \le a \\ A\frac{(b-x)}{(b-a)}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

where A, a, and b are constants.

(a) Normalize Ψ.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi|^2 \,\mathrm{d}x \\
&= \int_{-\infty}^{0} (0)^2 \,\mathrm{d}x + \int_{0}^{a} \left(A\frac{x}{a}\right)^2 \,\mathrm{d}x + \int_{a}^{b} \left(A\frac{(b-x)}{(b-a)}\right)^2 \,\mathrm{d}x + \int_{b}^{\infty} (0)^2 \,\mathrm{d}x \\
&= 0 + \frac{A^2}{a^2}\int_{0}^{a} x^2 \,\mathrm{d}x + \frac{A^2}{(b-a)^2}\int_{a}^{b} (b-x)^2 \,\mathrm{d}x + 0 \\
&= \frac{A^2}{a^2}\left[\frac{1}{3}x^3\right]_0^a + \frac{A^2}{(b-a)^2}\left[-\frac{a^3}{3} + a^2 b - ab^2 + \frac{b^3}{3}\right]_a^b \\
&= \frac{A^2}{a^2}\frac{a^3}{3} + \frac{A^2}{(b-a)^2}\frac{(b-a)^3}{3} \\
&= A^2\left(\frac{a}{3} + \frac{b-a}{3}\right) \\
&= A^2\frac{b}{3}.
\end{aligned}$$

Thus, $A = \sqrt{3/b}$.

(b) Sketch $\Psi(x,0)$ as a function of x.

(c) Where is the electron most likely to be found at $t = 0\,\mathrm{s}$?

(d) What is the probability the electron will be found in the region $x \le a$? Check your result in the limiting case where $b = a$ and $b = 2a$.

(e) Determine $\langle x \rangle$.

$$
\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x \left|\Psi\right|^2 \ \mathrm{d}x \\
&= \int_0^a x \left(\sqrt{\frac{3}{b}}\frac{x}{a}\right)^2 \ \mathrm{d}x + \int_a^b x \left(\sqrt{\frac{3}{b}}\frac{(b-x)}{(b-a)}\right)^2 \ \mathrm{d}x \\
&= \frac{3}{ba^2}\int_0^a x^3 \ \mathrm{d}x + \frac{3}{b}\frac{1}{(b-a)^2}\int_a^b \left(b^2 x - 2bx^2 + x^3\right) \mathrm{d}x \\
&= \frac{3}{ba^2}\left[\frac{1}{4}x^4\right]_0^a + \frac{3}{b}\frac{1}{(b-a)^2}\left[\frac{b^2}{2}x^2 - \frac{2b}{3}x^3 + \frac{1}{4}x^4\right]_a^b \\
&= \frac{3}{4}\frac{a^2}{b} + 3\frac{1}{b(b-a)^2}\left(\frac{b^4}{2} - \frac{2b^4}{3} + \frac{b^4}{4} - \frac{a^2b^2}{2} + \frac{2ba^3}{3} - \frac{a^4}{4}\right) \\
&=
\end{aligned}
$$