Preferences orders and utility functions

Dr. Sooie-Hoe Loke

What is utility?

Utility theory concerns individuals' preferences or values over some set of goods (objects, services, activities, wealth).

It has been used in many decision-making applications including

- Economics ("Economics is the father of utility theory")
- Psychology
- Finance
- Many more

"Most utility theories, when stripped of all nonmathematical interpretation, amount to abstract mathematical theories of binary relations."

– P. Fishburn (1968)

We will kick off this quarter with the binary relation, based on Fishburn's book: Utility Theory for Decision Making (1970).

Notation

Let X denote a set whose elements are to be evaluated in terms of preference in a particular decision situation (alternatives, cash flows, food items, etc.).

For now, assume that X is a countable set (i.e. finite or countably infinite) with elements denoted by lowercase letters $(x, y, \ldots)$.

Define strict preference $\prec$ (read $x \prec y$ as x is less preferred than y, or y is preferred to x) as the basic binary relation on X , and indifference $\sim$ will later be defined as the absence of strict preference.

The main result here is that, under some conditions, numbers $u(x), u(y), \ldots$ can be assigned to elements $x, y, \ldots$ in X in such a way that

$$x \prec y \Leftrightarrow u(x) < u(y).$$

Binary relations

Definition

A binary relation on a set Y is a set of ordered pairs (x, y) with $x \in Y$ and $y \in Y$.

Definition

The universal binary relation on Y is the set $\{(x, y) : x, y \in Y\}$ of all ordered pairs from Y.

- If R is a binary relation on Y, then R is a subset of the universal binary relation.
- We write xRy to mean that $(x, y) \in R$. Similarly, not xRy (it is false that x stands in the relation R to y) means that $(x, y) \notin R$.
- If R is a binary relation on Y, then for each (x, y) in the universal relation either xRy or not xRy, and not both.
- (x, y) is not the same as (y, x) unless $x = y$.

If R is a binary relation on Y and if $x, y \in Y$, then exactly one of the following four cases holds:

(1) (xRy, yRx),
(2) $(xRy, \text{not } yRx)$,
(3) $(\text{not } xRy, yRx)$,
(4) $(\text{not } xRy, \text{not } yRx)$.

Example

Let Y be the set of all living people. Define R_1 as "is shorter than," so that xR_1y means that x is shorter than y.
Case (1) is impossible. Case (2) holds when x is shorter than y.
When does case (4) hold?

Example

Let R_2 be "is the brother of " (by having at least one parent in common).
Fishburn claimed that "Here cases (2) and (3) are impossible."
What do you think?

Some Relation Properties

A binary relation R on a set Y is

p1. reflexive if xRx for every $x \in Y$,

p2. irreflexive if not xRx for every $x \in Y$,

p3. symmetric if $xRy \Rightarrow yRx$, for every $x, y \in Y$,

p4. asymmetric if $xRy \Rightarrow \text{not}\, yRx$, for every $x, y \in Y$,

p5. antisymmetric if $(xRy, yRx) \Rightarrow x = y$, for every $x, y \in Y$,

p6. transitive if $(xRy, yRz) \Rightarrow xRz$, for every $x, y, z \in Y$,

p7. negatively transitive if (not xRy, not yRz) $\Rightarrow$ not xRz, for every $x, y, z \in Y$,

p8. connected or complete if xRy or yRx (possibly both) for every $x, y \in Y$,

p9. weakly connected if $x \neq y \Rightarrow$ (xRy or yRx) throughout Y.

Example

The relation R_1 (shorter than) is irreflexive, asymmetric, transitive, and negatively transitive. If no two people are of same height, R_1 is weakly connected.

Some Relation Properties

A binary relation R on a set Y is

p1. reflexive if xRx for every $x \in Y$,

p2. irreflexive if not xRx for every $x \in Y$,

p3. symmetric if $xRy \Rightarrow yRx$, for every $x, y \in Y$,

p4. asymmetric if $xRy \Rightarrow \text{not}\, yRx$, for every $x, y \in Y$,

p5. antisymmetric if $(xRy, yRx) \Rightarrow x = y$, for every $x, y \in Y$,

p6. transitive if $(xRy, yRz) \Rightarrow xRz$, for every $x, y, z \in Y$,

p7. negatively transitive if (not xRy, not yRz) $\Rightarrow$ not xRz, for every $x, y, z \in Y$,

p8. connected or complete if xRy or yRx (possibly both) for every $x, y \in Y$,

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Example

What are some properties satisfied by R_2 (brother of)? Note that Fishburn wrote that "R_2 is symmetric."

Different types of relations

Definition

A binary relation R on a set Y is

a. a weak order $\Leftrightarrow$ R on Y is asymmetric and negatively transitive;

b. a strict order $\Leftrightarrow$ R on Y is a weakly connected weak order;

c. an equivalence $\Leftrightarrow$ R on Y is reflexive, symmetric, and transitive.

Example

The relation $<$ on the real numbers is a weak order and also a strict order since $x < y$ or $y < x$ whenever $x \neq y$.

Example

The relation $=$ on the real numbers is an equivalence, since $x = x$, $x = y \Rightarrow y = x$, and $(x = y, y = z) \Rightarrow x = z$.

Equivalence relation

An equivalence on a set defines a natural partition of the set into a class of disjoint, nonempty subsets, such that two elements of the original set are in the same class if and only if they are equivalent.

If R is an equivalence, then the set $R(x) = \{y : y \in Y \text{ and } yRx\}$ is the equivalence class generated by x. In this case, $R(x) = R(y)$ if and only if xRy. When R on Y is an equivalence, we denote the set of equivalence classes as Y/R.

Example

Consider the set of all integers $(\mathbb{Z})$. For $i = 0, 1, 2$, $x \in R[i]$ provided that x is congruent to i modulo 3, that is, 3 divides $x - i$. Write down $R[0], R[1]$, and $R[2]$.

Indifferent preference

We can define indifference $\sim$ as the absence of strict preference:

$$x \sim y \Leftrightarrow (\text{not}\, x \prec y, \text{not}\, y \prec x).$$

Indifference might arise in several ways.

- An individual might feel that there is no real difference between x & y.
- They are uncertain as to their preference between x and y.
- They consider x and y incomparable on a preference basis.

Define preference-indifference $\preccurlyeq$ as the union of $\prec$ and $\sim$ via

$$x \preccurlyeq y \Leftrightarrow x \prec y \text{ or } x \sim y.$$

Theorem

Suppose $\prec$ on X is a weak order (i.e. asymmetric and negatively transitive). Then

a. *exactly one of $x \prec y, y \prec x, x \sim y$ holds for each $x, y \in X$;*

b. *$\prec$ is transitive;*

c. *$\sim$ is an equivalence (i.e. reflexive, symmetric, transitive);*

d. *$(x \prec y, y \sim z) \Rightarrow x \prec z$, and $(x \sim y, y \prec z) \Rightarrow x \prec z$;*

e. *$\preccurlyeq$ is transitive and connected;*

f. *with $\prec'$ on $X/\sim$ (the set of equivalence classes of X under $\sim$) defined by*

$$a \prec' b \Leftrightarrow x \prec y \text{ for some } x \in a \text{ and } y \in b,$$

$\prec'$ on $X/\sim$ is a strict order.

Proof.

See p. 13

An Order-Preserving Utility Function

Theorem

If $\prec$ on X is a weak order and $X/\sim$ is countable then there is a real-valued function u on X such that

$$x \prec y \Leftrightarrow u(x) < u(y), \qquad \text{for all } x, y \in X.$$

Proof.

See p. 14 and 15. $\square$

Remarks:

- Consequently, for all $x, y \in X$, $x \sim y \Leftrightarrow u(x) = u(y)$, and $x \preccurlyeq y \Leftrightarrow u(x) \leq u(y)$.
- The utility function u is said to be order-preserving since the numbers $u(x), u(y), \ldots$ as ordered by $<$ reflect the order of $x, y, \ldots$ under $\prec$.

An Order-Preserving Utility Function

Theorem

If $\prec$ on X is a weak order and $X/\sim$ is countable then there is a real-valued function u on X such that

$$x \prec y \Leftrightarrow u(x) < u(y), \qquad \textit{for all } x, y \in X. \qquad (🥔)$$

Remarks:

- If (🥔) holds, then
  $$x \prec y \Leftrightarrow v(x) < v(y), \qquad \text{for all } x, y \in X$$
  for a real-valued function v on X if and only if $[v(x) < v(y) \Leftrightarrow u(x) < u(y)]$ holds throughout X.
- Another theorem can be obtained by assuming strict partial order, in which case the $\Leftrightarrow$ in (🥔) is replaced by $\Rightarrow$.
- There are utility functions with properties beyond that of order preservation.

References

1. Fishburn, P. C. (1979). *Utility theory for decision making*. NY: Krieger.
2. Fishburn, P. C. (1968). *Utility theory*. Management science, 14(5), 335-378.