1. Show that the converse of p4 is not true. That is, provide a counterexample of an asymmetric binary relation R with elements x and y such that $\neg xRy$ does not imply yRx.

   The binary relation R is asymmetric if $xRy \Rightarrow yRx$ for all x, y.

   Counterexample. Let R be the binary relation "is married to". Suppose $\neg xRy$; that is, x is not married to y. It is well known that this does not imply y is married to x. Thus, $\neg xRy$ does not imply yRx.

2. Consider the set of all triples where each component is a real number; that is, $\mathbb{R}^3$. Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ and define the weak preference $x \preccurlyeq^* y$ if $x_i \leq y_i$ for at least two out of the three components, $i = 1, 2, 3$.

   (a) Show $\preccurlyeq^*$ is connected.

   The binary relation R is connected if xRy or yRx for all x, y.

   Proof. Let the binary relation $\preccurlyeq^*$ be defined for $x, y \in \mathbb{R}^3$ such that $x \preccurlyeq^* y$ if $x_i \leq y_i$ for at least two out of the three components $i = 1, 2, 3$. The binary relation $\leq$ on the real numbers is connected. Thus, $x_i \leq y_i$ or $y_i \leq x_i$ for $i = 1, 2, 3$. We now proceed with two cases:

   i. Suppose $x \preccurlyeq^* y$. Clearly, $x \preccurlyeq^* y$ or $y \preccurlyeq^* x$ is true.

   ii. Suppose $\neg(x \preccurlyeq^* y)$. That is, $x_i \leq y_i$ is not true for two out of the three components. Since $\leq$ is connected, the components for which $x_i \leq y_i$ is not true instead satisfy $y_i \leq x_i$. Thus, for two out of the three components $y_i \leq x_i$. Therefore, $y \preccurlyeq^* x$.

   The binary relation $\preccurlyeq^*$ is connected. □

   (b) Show $\preccurlyeq^*$ is not transitive by providing a counterexample.

   The binary relation R is transitive if $(xRy \wedge yRz) \Rightarrow xRz$ for all x, y, z.

   Counterexample. Let $x = (1, 2, 3)$, $y = (2, 2, 1)$, and $z = (2, 0, 2)$. It is clear that $x \preccurlyeq^* y$ and $y \preccurlyeq^* z$. Furthermore, it is clear that $\neg x \preccurlyeq^* z$. Thus, $\preccurlyeq^*$ is not transitive.

(c) Define the strict preference $x \prec^* y$ by $x \preccurlyeq^* y$ but not $y \preccurlyeq^* x$.

i. Explain why it is equivalent to say $x \prec^* y$ if $x_i < y_i$ for at least two out of the three components, $i = 1, 2, 3$.

If two of the three components satisfy $x_i \leq y_i$ then those same two may satisfy $y_i \leq x_i$ if the two components are equal. The strict preference $\prec^*$ is stronger than $\preccurlyeq^*$ in the same way that $<$ is stronger than $\leq$. Thus, if we say $x \prec^* y$ then $x \preccurlyeq^* y$ is implied.

ii. Prove or give a counterexample. $\prec^*$ is asymmetric.

The binary relation R is asymmetric if $xRy \Rightarrow \neg yRx$ for all x, y.

Proof. Let $\prec^*$ be the binary relation defined as $x \preccurlyeq^* y \wedge \neg y \preccurlyeq^* x$. We proceed with a proof by contradiction. Suppose $x \prec^* y$; that is, $x_i < y_i$ for at least two i where $i = 1, 2, 3$. Further suppose $y \prec^* x$; that is, $y_i < x_i$ for at least two i. Thus two elements of $x_i < y_i$ and two elements of $y_i < x_i$. Since there are three elements in x and y each there must be at least one element pair, x_j and y_j, that satisfies $x_j < y_j$ and $y_j < x_j$. This is, clearly, a contradiction. There are no real numbers x_j and y_j that satisfy this condition. Thus, $\neg y \prec^* x$. Therefore, $x \prec^* y \Rightarrow \neg y \prec^* x$. That is, $\prec^*$ is asymmetric. $\square$

iii. Prove or give a counterexample. $\prec^*$ is negatively transitive.

The binary relation R is negatively transitive if $\neg xRy \wedge \neg yRz \Rightarrow \neg xRz$ for all x, y.

Proof. $\neg x \prec^* y$ means $x_i \geq y_i$ for at least two i. This is equivalently $y \preccurlyeq^* x$. Similarly, $\neg y \prec^* z$ means $y_i \geq z_i$ for at least two i. This is equivalently $z \preccurlyeq^* y$. Thus negative transitivity of the strict preference can be expressed as transitivity of the weak preference. As proven in 2b, $\preccurlyeq^*$ is not transitive. Thus, $\prec^*$ is not negatively transitive. $\square$