You trap an electron in an infinite-square well with size a. At time $t = 0$, we know that the electron is in the following state:

$$\Psi(x,0) = \begin{cases} bx^2, & 0 \leq x \leq a \\ 0, & \text{otherwise} \end{cases}$$

(a) (*5 points*) Assuming it is a real constant, determine b by normalizing $\Psi(x,0)$.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \Psi^*(x,0)\Psi(x,0) \, \mathrm{d}x \\
&= \int_{-\infty}^{0} 0 \, \mathrm{d}x + \int_{0}^{a} (bx^2)(bx^2) \, \mathrm{d}x + \int_{-\infty}^{0} 0 \, \mathrm{d}x \\
&= b^2 \int_0^a x^4 \, \mathrm{d}x \\
&= b^2 \left[\frac{1}{5}x^5\right]_0^a \\
&= b^2 \frac{a^5}{5}.
\end{aligned}$$

Solving for b we find

$$b = \sqrt{\frac{5}{a^5}}.$$

(b) (*8 points*) Construct the function $\Psi(x,t)$ for this electron (*i.e.*, including time-dependence).

We will express Ψ as a linear combination of stationary states $\psi_n(x)$ with associated time-dependence $\phi_n(t)$:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x)\phi_n(t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \exp\left(-\frac{iE_n}{\hbar}t\right).$$

The coefficients c_n can be found by applying Fourier's trick:

$$\begin{aligned}
c_n &= \int_{-\infty}^{\infty} \psi_n \Psi(x,0)\,\mathrm{d}x \\
&= 0 + \int_0^a \psi_n \Psi(x,0)\,\mathrm{d}x + 0 \\
&= \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) bx^2\,\mathrm{d}x \\
&= \sqrt{\frac{5}{a^5}}\sqrt{\frac{2}{a}} \int_0^a x^2 \sin\left(\frac{n\pi}{a}x\right)\,\mathrm{d}x \\
&= \frac{\sqrt{10}}{a^3} \int_0^a x^2 \sin\left(\frac{n\pi}{a}x\right)\,\mathrm{d}x,
\end{aligned}$$

which is given by integral (16) of the supplied integral table. Then,

$$c_n = \frac{\sqrt{10}}{a^3}\left[\frac{2x}{\left(\frac{n\pi}{a}\right)^2}\sin\left(\frac{n\pi}{a}x\right) + \left(\frac{2}{\left(\frac{n\pi}{a}\right)^3} - \frac{x^2}{\left(\frac{n\pi}{a}\right)}\right)\cos\left(\frac{n\pi}{a}x\right)\right]_0^a.$$

Evaluating the limits yields

$$\begin{aligned}
c_n &= \frac{\sqrt{10}}{a^3}\left(\frac{2a^3}{(n\pi)^2}\sin(n\pi) + \left(\frac{2a^3}{(n\pi)^3} - \frac{a^3}{n\pi}\right)\cos(n\pi) - 0 - \left(\frac{2a^3}{(n\pi)^3} - 0\right)\cos(0)\right) \\
&= \frac{\sqrt{10}}{a^3}\left(\frac{2a^3}{(n\pi)^3}\cos(n\pi) - \frac{a^3}{n\pi}\cos(n\pi) - \frac{2a^3}{(n\pi)^3}\right) \\
&= \frac{\sqrt{10}}{n\pi}\left(\frac{2}{(n\pi)^2}\left(\cos(n\pi) - 1\right) - \cos(n\pi)\right).
\end{aligned}$$

Since n takes in integer values $n \geq 1$, c_n can be expressed as a piecewise function:

$$c_n = \begin{cases} -\frac{\sqrt{10}}{n\pi}, & n \text{ is even} \\ \frac{\sqrt{10}}{n\pi}\left(\frac{-4}{(n\pi)^2} + 1\right), & n \text{ is odd} \end{cases}.$$

(c) (*5 points*) If you measure the electron's energy, the probability of obtaining E_n can be denoted $P(E_n)$. Determine $P(E_n)$ for $n = \{1, 2, 3, 4, 5\}$.

The probability of measuring the energy E_n is given by the probability of finding the electron in stationary state n; this is given by

$$P(E_n) = |c_n|^2 .$$

Thus,

$$\begin{aligned}
P(E_1) = |c_1|^2 &= \left(\frac{\sqrt{10}}{\pi}\left(\frac{-4}{(\pi)^2} + 1\right)\right)^2 = \frac{10}{\pi^2}\left(\frac{16}{\pi^4} - \frac{8}{\pi^2} + 1\right) \approx 0.3584, \\
P(E_2) = |c_2|^2 &= \left(-\frac{\sqrt{10}}{2\pi}\right)^2 \approx 0.2533, \\
P(E_3) = |c_3|^2 &= \left(\frac{\sqrt{10}}{3\pi}\left(\frac{-4}{(3\pi)^2} + 1\right)\right)^2 = \frac{10}{9\pi^2}\left(\frac{16}{9\pi^4} - \frac{8}{9\pi^2} + 1\right) \approx 0.1027, \\
P(E_4) = |c_4|^2 &= \left(-\frac{\sqrt{10}}{4\pi}\right)^2 \approx 0.06333, \\
P(E_5) = |c_5|^2 &= \left(\frac{\sqrt{10}}{5\pi}\left(\frac{-4}{(5\pi)^2} + 1\right)\right)^2 = \frac{10}{25\pi^2}\left(\frac{16}{25\pi^4} - \frac{8}{25\pi^2} + 1\right) \approx 0.03923.
\end{aligned}$$

(d) (*2 points*) What is the probability of obtaining an energy larger than E_5 in an energy measurement on this electron?

The probability of obtaining energy, H, greater than E_5 can be denoted

$$P(H > E_5) = \sum_{n=6}^{\infty} P(E_n).$$

Since there is a guarantee of obtaining some energy, we can instead write

$$P(H > E_5) = 1 - P(H < 5) = 1 - \sum_{n=1}^{5} P(E_n).$$

The probability of obtaining energy $P(E_n)$ for $n = 1, 2, 3, 4, 5$ was found above. Thus,

$$\begin{aligned}
P(H > E_5) &= 1 - \sum_{n=1}^{5} P(E_n) \\
&= 1 - (P(E_1) + P(E_2) + P(E_3) + P(E_4) + P(E_5)) \\
&\approx 1 - (0.3584 + 0.2533 + 0.1027 + 0.06333 + 0.03923) \\
&\approx 0.1831.
\end{aligned}$$