1. A particle is trapped in an infinite square well with a width a. At $t = 0$ is is equally probable for the particle to be found anywhere on the left-side of the well and impossible for it to be found on the right-side of the well.

   (a) What is the normalized wavefunction $\Psi(x, 0)$ that represents this initial state? Note that Ψ breaks one of the fundamental rules about wavefunctions.

   The wavefunction, disregarding the cases of x outside the range $[0, a]$, is given by

$$\Psi(x,0) = \begin{cases} A & 0 \le x \le \frac{a}{2} \\ 0 & \frac{a}{2} < x \le a \end{cases}$$

   where A is the normalization constant. We see the issue with Ψ at $x = a/2$ where we have a discontinuity. The normalization constant can be found by the typical normalization process:

$$1 = \int_{-\infty}^{\infty} |\Psi|^2 \, \mathrm{d}x.$$

   We recognize $\Psi(x, t = 0)$ takes on non-zero values only on the interval $[0, \frac{a}{2}]$ and that $\Psi(x, t = 0)$ is real.

$$1 = A^2 \int_0^{a/2} \mathrm{d}x = A^2 \frac{a}{2}.$$

   Thus,

$$A = \sqrt{\frac{2}{a}}$$

   and consequently

$$\Psi(x,0) = \begin{cases} \sqrt{\frac{2}{a}} & 0 \le x \le \frac{a}{2} \\ 0 & \frac{a}{2} < x \le a \end{cases}$$

(b) What is the probability that you would measure an energy of

$$E = \frac{4\pi^2\hbar^2}{2ma^2}$$

at $t = 0$.

The energy of any infinitely-square-well-ed particle is given by

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}.$$

By inspection, our particle is in the $n = 2$ state. The probability of finding a particle in the $n = 2$ state is given by $|C_2|^2$ where C_2 is found through linearization of Ψ. The nth linear coefficient is given as

$$C_n = \int_{-\infty}^{\infty} \psi_n^* \Psi \, \mathrm{d}x = \frac{2}{a} \int_0^{a/2} \sin\left(\frac{n\pi}{a} x\right) \, \mathrm{d}x = \frac{2}{n\pi} \left(1 - \cos\left(\frac{n\pi}{2}\right)\right).$$

Thus, C_2 is given as

$$C_2 = \frac{2}{\pi},$$

hence, the probability of measuring the 2nd energy state is given as

$$P\left(E_2\right) = |C_2|^2 = \left(\frac{2}{\pi}\right)^2 \approx 0.41.$$

2. Consider the standard infinite square well with width a. The stationary state solutions are $\psi_n(x)$.

   (a) Compute $\langle x \rangle$ and $\langle x^2 \rangle$ for $\psi_n(x)$.

   The expectation value of x is given by

$$
\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} \psi_n^*[x]\psi_n \,\mathrm{d}x \\
&= \int_0^a x\psi_n^*\psi_n \,\mathrm{d}x \\
&= \frac{2}{a}\int_0^a x\sin^2\left(\frac{n\pi}{a}x\right)\,\mathrm{d}x \\
&= \frac{2}{a}\left[\frac{x^2}{4} - \frac{x\sin\left(\frac{2\pi n}{a}x\right)}{4\left(\frac{n\pi}{a}\right)} - \frac{\cos\left(\frac{2\pi n}{a}x\right)}{8\left(\frac{n\pi}{a}\right)^2}\right]_0^a \\
&= \frac{2}{a}\left(\frac{a^2}{4} - \frac{a\sin(2\pi n)}{4\left(\frac{n\pi}{a}\right)} - \frac{\cos(2\pi n)}{8\left(\frac{n\pi}{a}\right)^2} - 0 + 0 + \frac{1}{8\left(\frac{n\pi}{a}\right)^2}\right) \\
&= \frac{2}{a}\left(\frac{a^2}{2} - \frac{1}{8\left(\frac{n\pi}{a}\right)^2} + \frac{1}{8\left(\frac{n\pi}{a}\right)^2}\right) \\
&= \frac{a}{2}.
\end{aligned}
$$

   The mean square displacement $\langle x^2 \rangle$ is given by

$$
\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} \psi_n^*\left[x^2\right]\psi_n \,\mathrm{d}x \\
&= \frac{2}{a}\int_0^a x^2\sin^2\left(\frac{n\pi}{a}x\right)\,\mathrm{d}x \\
&= \frac{2}{a}\left[\frac{x^3}{6} - \left(\frac{x^2}{4\left(\frac{n\pi}{a}\right)} - \frac{1}{8\left(\frac{n\pi}{a}\right)^3}\right)\sin\left(\frac{2\pi n}{a}x\right) - \frac{x\cos\left(\frac{2\pi n}{a}x\right)}{4\left(\frac{n\pi}{a}\right)^2}\right]_0^a \\
&= \frac{2}{a}\left(\frac{a^3}{6} - \left(\frac{a^2}{4\left(\frac{n\pi}{a}\right)} - \frac{1}{8\left(\frac{n\pi}{a}\right)^3}\right)\sin(2\pi n) - \frac{a\cos(2\pi n)}{4\left(\frac{n\pi}{a}\right)^2} - 0\right) \\
&\frac{2}{a}\left(\frac{a^3}{6} - 0 - \frac{a}{4\left(\frac{n\pi}{a}\right)^2}\right) \\
&= a^2\left(\frac{1}{3} - \frac{1}{2(n\pi)^2}\right).
\end{aligned}
$$

(b) Compute $\langle p \rangle$ and $\langle p^2 \rangle$ for $\psi_n(x)$.

The mean momentum $\langle p \rangle$ is given by

$$\langle p \rangle = \frac{\mathrm{d}\langle x \rangle}{\mathrm{d}t} = 0.$$

The mean square momentum $\langle p^2 \rangle$ is given by

$$\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \psi_n^* \left[-i\hbar \frac{\partial}{\partial x}\right]^2 \psi_n \, \mathrm{d}x \\
&= \hbar^2 \int_0^a \psi_n^* \frac{\partial^2}{\partial x^2} \psi \, \mathrm{d}x \\
&= \hbar^2 \left(\frac{n\pi}{a}\right)^2 \int_0^a \psi_n^* \psi \, \mathrm{d}x \\
&= \frac{n^2 \hbar^2 \pi^2}{a^2} \\
&= 2mE_n.
\end{aligned}$$

(c) Compute σ_x and σ_p and confirm that the uncertainty principle is satisfied for all allowed n.

The standard deviation of position σ_x is given by

$$\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \left(a^2 \left(\frac{1}{3} - \frac{1}{2(n\pi)^2} \right) - \frac{a^2}{4} \right) \\
&= a\sqrt{\frac{1}{12} - \frac{1}{2(n\pi)^2}}.
\end{aligned}$$

The standard deviation of momentum σ_p is given by

$$\begin{aligned}
\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \sqrt{2mE_n - (0)^2} \\
&= \sqrt{2mE_n}.
\end{aligned}$$

To satisfy the uncertainty principle

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

for all n. The product of standard deviations is given by

$$\begin{aligned}
\sigma_x \sigma_p &= a\sqrt{\frac{1}{12} - \frac{1}{2(n\pi)^2}} \sqrt{2mE_n} \\
&= \hbar n\pi \sqrt{\frac{1}{12} - \frac{1}{2(n\pi)^2}}.
\end{aligned}$$

For all allowed n $\sigma_x \sigma_p$ is smallest for $n = 1$. Thus, if $\sigma_x \sigma_p$ satisfies the uncertainty principle at $n = 1$ it satisfies the uncertainty principle at all n. For $n = 1$

$$\sigma_x \sigma_p = \hbar\pi \sqrt{\frac{1}{12} - \frac{1}{2\pi^2}}.$$

Thus our inequality can be expressed as

$$\pi \sqrt{\frac{1}{12} - \frac{1}{2\pi^2}} \geq \frac{1}{2}.$$

The quantity on the left is approximately 0.56 which is greater than the quantity on the right 0.5. Thus, σ_x and σ_p satisfy the uncertainty principle for all allowed n.

3. A particle infinitely-square-well-ed to width a is initially observed in a quantum state described by the wavefunction

$$\Psi(x,0) = A\left(\psi_1(x) + \psi_3(x)\right),$$

where A is a real positive constant and both $\psi_1(x)$ and $\psi_3(x)$ are solutions to the time-independent Schrödinger equation for $n = 1$ and $n = 3$ respectively.

(a) Normalize $\Psi(x,0)$ assuming ψ_1 and ψ_3 are separately normalized.

We begin with typical normalization.

$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, dx = A^2 \left(\int_{-\infty}^{\infty} |\psi_1|^2 \, dx + \int_{\infty}^{\infty} |\psi_3|^2 \, dx \right) = 2A^2,$$

where we have used the assumption that ψ_1 and ψ_3 are separately normalized to evaluate the integrals. Thus,

$$A = 2^{-1/2}.$$

(b) Compute $|\Psi|^2$ and simplify as much as possible.

The time-dependent Ψ is obtained by adding the typical $\phi(t)$ to each stationary state describing Ψ.

$$\Psi = \frac{1}{\sqrt{2}} \left(\psi_1 e^{-it\frac{E_1}{\hbar}} + \psi_3 e^{-it\frac{E_3}{\hbar}} \right).$$

Then,

$$\begin{aligned} |\Psi|^2 &= \Psi^* \Psi \\ &= \frac{1}{2} \left(\psi_1 e^{it\frac{E_1}{\hbar}} + \psi_3 e^{it\frac{E_3}{\hbar}} \right) \left(\psi_1 e^{-it\frac{E_1}{\hbar}} + \psi_3 e^{-it\frac{E_3}{\hbar}} \right) \\ &= \frac{1}{2} \left(\psi_1^2 + \psi_3^2 + \psi_1 \psi_3 \exp\left(\frac{1}{\hbar} \left(E_1 - E_3 - E_1 + E_3 \right) \right) \right) \\ &= \frac{1}{2} \left(\psi_1^2 + \psi_3^2 \right). \end{aligned}$$

(c) If you measured the particle's energy, what value(s) might you possibly obtain and what is the probability of measuring them?

The factor $1/2$ can be interpreted as both $|C_1|^2$ and $|C_3|^2$. Since this is the representation of Ψ as a linear combination of stationary states we say that all other coefficients C_n are 0 for $n \neq 1, 3$. Thus, the energy states E_1 and E_3 are the only possible energy states each with an equal probability of being measured.