1. Determine the eigenfunctions and eigenvalues of the operator

$$\hat{Q} = \frac{\mathrm{d}^2}{\mathrm{d}\phi^2}$$

where ϕ is the angle in polar coordinates. Due to the rotational symmetry of the problem, your eigenfunctions, f, should satisfy the boundary condition

$$f(\phi) = f(\phi + 2\pi).$$

Is the spectrum of eigenvalues degenerate or non-degenerate?

2. Suppose that $f(x)$ and $g(x)$ are both eigenfunctions of an operator $\hat{Q}$. The spectrum is degenerate such that both $f(x)$ and $g(x)$ have the same eigenvalue, q.

   (a.) Prove that any linear combination of $f(x)$ and $g(x)$ is also an eigenfunction of Q. What is its eigenvalue?

   (b.) An anti-hermitian operator obeys the following condition:

   $$\hat{Q}^\dagger = -\hat{Q}.$$

   Show that the expectation value of an anti-hermitian operator is imaginary.

   (c.) Show that the commutator of two hermitian operators is anti-hermitian.

   (d.) Show that the commutator of two anti-hermitian operators is also anti-hermitian.

3. We have two operators $\hat{A}$ and $\hat{B}$ each with two eigenstates. The eigenstates and corresponding eigenvalues are characterized by the equations

$$\begin{aligned}
\hat{A}\psi_1 &= a_1\psi_1, \\
\hat{A}\psi_2 &= a_2\psi_2, \\
\hat{B}\phi_1 &= b_1\phi_1, \\
\hat{B}\phi_2 &= b_2\phi_2.
\end{aligned}$$

Suppose we know that the eigenstates for each operator are related by

$$\begin{aligned}
\psi_1 &= \frac{1}{5}\left(3\phi_1 + 4\phi_2\right), \\
\psi_2 &= \frac{1}{5}\left(4\phi_1 - 3\phi_2\right).
\end{aligned}$$

(a.) If observable A is measured and we obtain a value of a_1, what is the state of the system in the instant after the measurement was made?

(b.) If B is now measured following the measurement in part (a.), what are the possible results and what are their associated probabilities?

(c.) If we measure A again immediately following the measurement of B in part (b.), what is the probability of obtaining a_1? This is tricky because we do not know what value of B we obtained in part (b.).