1. A decision maker is described by the utility function $u(w) = w^{1/3}$. She is given the choice between two random amounts X_1 and X_2, in exchange for her entire present wealth w_0. Suppose that

$$X_1 = \begin{cases} 8 & \text{with probability } 0.5 \\ 27 & \text{with probability } 0.5 \end{cases}$$

and

$$X_2 = \begin{cases} 1 & \text{with probability } 0.6 \\ 64 & \text{with probability } 0.4 \end{cases}$$

(a) Show that she prefers X_1 to X_2.

The expected utility of option 1 is given by

$$E(u(X_1)) = \frac{1}{2}8^{1/3} + \frac{1}{2}27^{1/3} = \frac{5}{2} = 2.5.$$

The expected utility of option 2 is given by

$$E(u(X_2)) = 0.6 \times 1^{1/3} + 0.4 \times 64^{1/3} = 2.2.$$

The expected utility of option 1 is greater than that of option 2 so option 1 is preferred.

(b) Determine for what values of w_0 she should decline the offer.

The decision make should decline the offer if the expected utility of an option, say X_1 or X_2, is less than the expected utility of doing nothing. The expected utility of doing nothing is to simply keep the wealth, that is,

$$E(u(w_0)) = w_0^{1/3}.$$

Since X_1 is preferred to X_2 we need only consider how doing nothing compares to X_1. Thus we consider when $E(u(w_0)) > E(u(X_1))$ which is given by $w_0^{1/3} > 5/2$ which is given by

$$w_0 > \left(\frac{5}{2}\right)^3 \approx 15.63.$$

Thus, the offer should be declined for a starting wealth of 1.52 units.

(c) Give an example of a utility function in which she would prefer X_2 to X_1.

The utility function
$$u(w) = \frac{w^{0.9} - 1}{0.9}$$
results in the following expected utilities

$$\begin{aligned} E(u(X_1)) &= 13.29, \\ E(u(X_2)) &= 18.32. \end{aligned}$$

Thus for this utility function X_2 is preferred to X_1.

2. Recall that the iso-elastic property says that for any $k > 0$, $u(kw) = f(k)u(w)+g(k)$ for some $f(k)$ and $g(k)$.

(a) Identify the functions $f(k)$ and $g(k)$ in the case of $u(w) = \ln(w)$.

$$u(kw) = \ln(kw) = \ln(w) + \ln(k).$$

Thus $f(k) = 1$ and $g(k) = \ln(k)$.

(b) Identify the functions $f(k)$ and $g(k)$ in the case of $u(w) = \frac{w^\lambda - 1}{\lambda}$.

$$\begin{aligned} u(kw) &= \frac{k^\lambda w^\lambda}{\lambda} \\ &= \frac{k^\lambda w^\lambda}{\lambda} + \frac{k^\lambda}{\lambda} - \frac{k^\lambda}{\lambda} \\ &= \frac{(k^\lambda w^\lambda - k^\lambda) + k^{\lambda - 1}}{\lambda} \\ &= \frac{k^\lambda (w^\lambda - 1) + k^\lambda - 1}{\lambda} \\ &= k^\lambda \frac{w^\lambda - 1}{\lambda} + \frac{k^\lambda - 1}{\lambda}. \end{aligned}$$

Thus $f(k) = k^\lambda$ and $g(k) = u(k)$.

3. Recall that the Arrow-Pratt absolute risk aversion function is given by

$$A(w) = -\frac{\frac{\mathrm{d}^2 u(w)}{\mathrm{d}w^2}}{\frac{\mathrm{d}u(w)}{\mathrm{d}w}}.$$

(a) Compute $A(w)$ in the case of $u(w) = \ln(w)$. Is $A(w)$ non-increasing?

$$\begin{aligned} A(w) &= -\frac{\frac{\mathrm{d}^2 u(w)}{\mathrm{d}w^2}}{\frac{\mathrm{d}u(w)}{\mathrm{d}w}} \\ &= -\frac{\frac{\mathrm{d}^2 \ln(w)}{\mathrm{d}w^2}}{\frac{\mathrm{d}\ln(w)}{\mathrm{d}w}} \\ &= -\frac{-w^{-2}}{w^{-1}} \\ &= w^{-1}. \end{aligned}$$

$A(w)$ is non-increasing.

(b) Compute $A(w)$ in the case of $u(w) = \frac{w^\lambda - 1}{\lambda}$. Is $A(w)$ non-increasing.

$$\begin{aligned} A(w) &= -\frac{\frac{\mathrm{d}^2 u(w)}{\mathrm{d}w^2}}{\frac{\mathrm{d}u(w)}{\mathrm{d}w}} \\ &= -\frac{\frac{\mathrm{d}^2 \frac{w^\lambda - 1}{\lambda}}{\mathrm{d}w^2}}{\frac{\mathrm{d}\frac{w^\lambda - 1}{\lambda}}{\mathrm{d}w}} \\ &= \frac{-(\lambda - 1)w^{\lambda-2}}{w^{\lambda-1}} = -(\lambda - 1)w^{-1}. \end{aligned}$$

$A(w)$ is non-increasing.