1. You trap an electron in an infinite-square well with size a. At time $t = 0$, we know that the electron is in the following state:

$$\Psi(x,0) = \begin{cases} bx^2, & 0 \leq x \leq a \\ 0, & \text{otherwise} \end{cases}$$

(a) (*5 points*) Assuming it is a real constant, determine b by normalizing $\Psi(x, 0)$.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \ \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \Psi^*(x,0)\Psi(x,0) \ \mathrm{d}x \\ &= \int_{-\infty}^{0} 0 \ \mathrm{d}x + \int_{0}^{a} (bx^2)(bx^2) \ \mathrm{d}x + \int_{-\infty}^{0} 0 \ \mathrm{d}x \\ &= b^2 \int_0^a x^4 \ \mathrm{d}x \\ &= b^2 \left[\frac{1}{5}x^5\right]_0^a \\ &= b^2 \frac{a^5}{5}. \end{aligned}$$

Solving for b we find

$$b = \sqrt{\frac{5}{a^5}}.$$

(b) (*8 points*) Construct the function $\Psi(x,t)$ for this electron (*i.e.*, including time-dependence).

We will express Ψ as a linear combination of stationary states $\psi_n(x)$ with associated time-dependence $\phi_n(t)$:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x)\phi_n(t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \exp\left(-\frac{iE_n}{\hbar}t\right).$$

The coefficients c_n can be found by applying Fourier's trick:

$$\begin{aligned}
c_n &= \int_{-\infty}^{\infty} \psi_n \Psi(x,0)\,\mathrm{d}x \\
&= \int_{-\infty}^{0} \psi_n \times 0\,\mathrm{d}x + \int_{0}^{a} \psi_n \Psi(x,0)\,\mathrm{d}x + \int_{a}^{\infty} \psi_n \times 0\,\mathrm{d}x \\
&= \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)\,\mathrm{d}x \\
&= b\sqrt{\frac{2}{a}} \int_0^a x^2 \sin\left(\frac{n\pi}{a}x\right)\,\mathrm{d}x \\
&= \sqrt{\frac{10}{a^6}} \left[\frac{2x}{\left(\frac{n\pi}{a}\right)^2} \sin\left(\frac{n\pi}{a}x\right) + \left(\frac{2}{\left(\frac{n\pi}{a}\right)^3} - \frac{x^2}{\left(\frac{n\pi}{a}\right)} \right) \cos\left(\frac{n\pi}{a}x\right) \right]_0^a \\
&= \sqrt{\frac{10}{a^6}} \left[\frac{2a}{\left(\frac{n\pi}{a}\right)^2} \sin(n\pi) + \left(\frac{2}{\left(\frac{n\pi}{a}\right)^3} - \frac{a^2}{\left(\frac{n\pi}{a}\right)} \right) \cos(n\pi) \right. \\
&\qquad \left. -0 - \left(\frac{2}{\left(\frac{n\pi}{a}\right)^3} \right) \cos(0) \right] \\
&= \sqrt{\frac{10}{a^6}} \left(0 + \left(\frac{2}{\left(\frac{n\pi}{a}\right)^3} - \frac{a^2}{\left(\frac{n\pi}{a}\right)} \right) \cos(n\pi) - 0 - \left(\frac{2}{\left(\frac{n\pi}{a}\right)^3} \right) \right) \\
&= \sqrt{\frac{10}{a^6}} \left(\left(\frac{2}{\left(\frac{n\pi}{a}\right)^3} - \frac{a^2}{\left(\frac{n\pi}{a}\right)} \right) \cos(n\pi) - \frac{2}{\left(\frac{n\pi}{a}\right)^3} \right) \\
&= \begin{cases} -\sqrt{\frac{10}{a^6}} \frac{a^3}{n\pi}, & n \text{ is even} \\ -\sqrt{\frac{10}{a^6}} \left(\frac{4a^3}{(n\pi)^3} - \frac{a^3}{n\pi} \right), & n \text{ is odd} \end{cases}.
\end{aligned}$$

(c) (*5 points*) If you measure the electron's energy, the probability of obtaining E_n can be denoted $P(E_n)$. Determine $P(E_n)$ for $n = \{1, 2, 3, 4, 5\}$.

The probability of measuring the energy E_n is given by the probability of finding the electron in stationary state n; this is given by

$$P(E_n) = |c_n|^2 .$$

Thus,

$$\begin{aligned}
P(E_1) &= |c_1|^2 =, \\
P(E_3) &= |c_3|^2 =, \\
P(E_2) &= |c_2|^2 =, \\
P(E_4) &= |c_4|^2 =, \\
P(E_5) &= |c_5|^2 = .
\end{aligned}$$

(d) (*2 points*) What is the probability of obtaining an energy larger than E_5 in an energy measurement on this electron?

The probability of obtaining energy, H, greater than E_5 can be denoted

$$P(H > E_5) = \sum_{n=6}^{\infty} P(E_n).$$

Since there is a probability 1 of obtaining an energy we can instead write

$$P(H > E_5) = 1 - P(H < 5) = 1 - \sum_{n=1}^{5} P(E_n).$$

The probability of obtaining energy $P(E_n)$ for $n = 1, 2, 3, 4, 5$ was found above. Thus,

$$\begin{aligned}
P(H > E_5) &= 1 - \sum_{n=1}^{5} P(E_n) \\
&= 1 - (P(E_1) + P(E_2) + P(E_3) + P(E_4) + P(E_5)) \\
&= 1 - () \\
&= .
\end{aligned}$$