1. Let the wavefunction $\Psi(x, t)$ be a solution to the time-dependent Schrödinger equation when the potential energy is given by $V(x)$. What is the solution to the Shrödinger equation if we now consider a potential of $V(x) + V_0$ where V_0 is a real positive constant.

2. A particle is observed in a quantum state described by the wavefunction

$$\Psi(x,t) = A\exp\left(-a\left(\frac{mx^2}{\hbar} + it\right)\right),$$

where A and a are real positive constants.

(a) Normalize Ψ.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi|^2 \,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \Psi^*\Psi \,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} A\exp\left(-a\left(\frac{mx^2}{\hbar} + it\right)\right) A\exp\left(-a\left(\frac{mx^2}{\hbar} - it\right)\right) \,\mathrm{d}x \\
&= A^2\int_{-\infty}^{\infty} \exp\left(-a\left(\frac{mx^2}{\hbar} + it + \frac{mx^2}{\hbar} - it\right)\right) \,\mathrm{d}x \\
&= A^2\int_{-\infty}^{\infty} \exp\left(-\frac{2am}{\hbar}x^2\right) \,\mathrm{d}x \\
&= A^2\sqrt{\frac{\pi}{\frac{2am}{\hbar}}}.
\end{aligned}$$

Solving for A we find

$$A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}.$$

(b) What is the potential $V(x)$ that this particle finds itself within? Ψ is, by definition, a solution to the Schrödinger equation. Thus,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi.$$

Arranging for the potential energy function V we get

$$V = \frac{1}{\Psi}\left(i\hbar\frac{\partial\Psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}\right).$$

To make these partial derivatives less threatening, we can begin by writing Ψ in the form $\Psi = A\psi(x)\phi(t)$. The A component is simply A, as found by normalization. The exponential term in Ψ can be separated into an x-dependent and t-dependent component;

$$\exp\left(\frac{-amx^2}{\hbar} - ait\right) = \exp\left(-\frac{amx^2}{\hbar}\right)\exp\left(-ait\right),$$

which become ψ and ϕ respectively. We know

$$\phi(t) = \exp\left(-\frac{iEt}{\hbar}\right) = \exp\left(-ait\right),$$

thus $a = E/\hbar$. We can now write V in a more approachable way with ordinary derivatives:

$$V = \frac{1}{A\psi\phi}\left(i\hbar A\psi\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\hbar^2}{2m}A\phi\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}\right).$$

The ordinary derivatives are

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(-ait)\right] = -ai\exp(-ait) = -ai\phi,$$

and

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = \frac{\mathrm{d}\psi}{\mathrm{d}x}\left[-\frac{2amx}{\hbar}\exp\left(-\frac{amx^2}{\hbar}\right)\right] = \frac{-2am}{\hbar}\left(1 - \frac{2am}{\hbar}x^2\right)\psi.$$

Thus the potential energy function V is given by

$$\begin{aligned}
V &= \frac{1}{A\psi\phi}\left(i\hbar A\psi\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\hbar^2}{2m}A\phi\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}\right) \\
&= \frac{1}{A\psi\phi}\left(-i\hbar A\psi\frac{E}{\hbar}i\phi + \frac{\hbar^2}{2m}A\phi\frac{-2Em}{\hbar^2}\left(1 - \frac{2Em}{\hbar^2}x^2\right)\psi\right) \\
&= E - E\left(1 - \frac{2Em}{\hbar^2}x^2\right) \\
&= \frac{2mE^2}{\hbar^2}x^2.
\end{aligned}$$

(c) Determine the expectation values $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$.

i. The expectation value of x, $\langle x \rangle$, is given by

$$\langle x \rangle = \int_{-\infty}^{\infty} x \left|\Psi\right|^2 \, \mathrm{d}x$$

which is an odd function evaluated over symmetric limits and therefore

$$\langle x \rangle = 0.$$

ii. The mean square position, $\langle x^2 \rangle$, is given by

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 \left|\Psi\right|^2 \, \mathrm{d}x \\
&= A^2 \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{2am}{\hbar}x^2\right) \, \mathrm{d}x \\
&= \sqrt{\frac{2am}{\pi\hbar}} \frac{\sqrt{\pi}}{2\left(\frac{2am}{\hbar}\right)^{3/2}} \\
&= \frac{\hbar^2}{4Em}.
\end{aligned}$$

iii. The expectation momentum, $\langle p \rangle$, is given by

$$\langle p \rangle = \frac{\mathrm{d}\langle x \rangle}{\mathrm{d}t} = 0.$$

The mean square momentum, $\langle p^2 \rangle$, is given by

$$\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \Psi^* \left[-i\hbar\frac{\partial}{\partial x}\right]^2 \Psi \, \mathrm{d}x \\
&= -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{\partial}{\partial x}\frac{\partial}{\partial x}\Psi \, \mathrm{d}x \\
&= -\hbar^2 \int_{-\infty}^{\infty} \, \mathrm{d}x
\end{aligned}$$

(d) Determine the standard deviations for position, σ_x, and momentum, σ_p.

(e) Are your values for σ_x and σ_p consistent with the uncertainty principle?

3. An electron is trapped in a harmonic quadratic potential. Suppose the expectation value for its position is given by $\langle x \rangle = \frac{a}{2} \sin(\omega t)$. Here, a is a real constant with units of length and ω is an angular frequency. What, if anything, can be concluded about the electron's momentum?