CS 61A
Fall 2023

# Linked Lists, Efficiency

Discussion 8: October 18, 2023

**Note:** For formal explanations of the concepts on this discussion, feel free to look at the **Appendix** section on the back of the worksheet.

# Linked Lists

A **linked list** is a recursive data structure that represents sequences. The `Link` class implements linked lists in Python. Each `Link` instance has a `first` attribute (which stores the first value of the linked list) and a `rest` attribute (which points to the rest of the linked list). An empty linked list is represented as `Link.empty`, and a non-empty linked list is represented as a `Link` instance.

### Q1: WWPD: Linked Lists

What would Python display? Try drawing the box-and-pointer diagram if you get stuck!

```
>>> link = Link(1, Link(2, Link(3)))
>>> link.first
```

1

```
>>> link.rest.first
```

2

```
>>> link.rest.rest.rest is Link.empty
```

True

```
>>> link.rest = link.rest.rest
>>> link.rest.first
```

3

```
>>> link = Link(1)
>>> link.rest = link
>>> link.rest.rest.rest.rest.first
```

1

```
>>> link = Link(2, Link(3, Link(4)))
>>> link2 = Link(1, link)
>>> link2.rest.first
```

2

```
>>> link = Link(1000, 2000)
```

Error (second argument of Link constructor must be a Link instance or Link.empty)

```
>>> link = Link(1000, Link())
```

Error (Link constructor requires at least one argument)

```
>>> link = Link(Link("Hello"), Link(2))
>>> link.first
```

Link('Hello')

```
>>> link = Link(Link("Hello"), Link(2))
>>> link.first.rest is Link.Empty
```

True

### Q2: Sum Nums

Write a function `sum_nums` that receives a linked list `s` and returns the sum of its elements. You may assume the elements of `s` are all integers. Try to implement `sum_nums` with recursion!

```
def sum_nums(s):
    """
    Returns the sum of the elements in s.

    >>> a = Link(1, Link(6, Link(7)))
    >>> sum_nums(a)
    14
    """
    if s == Link.empty:
        return 0
    return s.first + sum_nums(s.rest)
```

### Q3: Remove All

Write a function `remove_all` that takes a linked list and a `value` as input. This function mutates the linked list by removing all nodes that store `value`.

You may assume the first element of the linked list is not equal to `value`. You should mutate the input linked list; `remove_all` does not return anything.

```
def remove_all(link, value):
    """Removes all nodes in link that contain value. The first element of
    link is never equal to value.

    >>> l1 = Link(0, Link(2, Link(2, Link(3, Link(1, Link(2, Link(3)))))))
    >>> print(l1)
    <0 2 2 3 1 2 3>
    >>> remove_all(l1, 2)
    >>> print(l1)
    <0 3 1 3>
    >>> remove_all(l1, 3)
    >>> print(l1)
    <0 1>
    >>> remove_all(l1, 3)
    >>> print(l1)
    <0 1>
    """
    if link is Link.empty or link.rest is Link.empty:
        return
    if link.rest.first == value:
        link.rest = link.rest.rest
        remove_all(link, value)
    else:
        remove_all(link.rest, value)
```

```
def remove_all(link, value):

    # A different recursive solution
    if link is not Link.empty and link.rest is not Link.empty:
        remove_all(link.rest, value)
        if link.rest.first == value:
            link.rest = link.rest.rest

    # An iterative solution
    ptr = link
    while ptr is not Link.empty and ptr.rest is not Link.empty:
        if ptr.rest.first == value:
            ptr.rest = ptr.rest.rest
        else:
            ptr = ptr.rest
```

### Q4: Flip Two

Write a recursive function `flip_two` that receives a linked list `s` and flips every pair of values in `s`.

```
def flip_two(s):
    """
    Flips every pair of values in s.

    >>> one_lnk = Link(1)
    >>> flip_two(one_lnk)
    >>> one_lnk
    Link(1)
    >>> lnk = Link(1, Link(2, Link(3, Link(4, Link(5)))))
    >>> flip_two(lnk)
    >>> lnk
    Link(2, Link(1, Link(4, Link(3, Link(5)))))
    """
    # Recursive solution:
    if s is Link.empty or s.rest is Link.empty:
        return
    s.first, s.rest.first = s.rest.first, s.first
    flip_two(s.rest.rest)

# Iterative solution
def iterative_flip_two(s):
    while s is not Link.empty and s.rest is not Link.empty:
        s.first, s.rest.first = s.rest.first, s.first
        s = s.rest.rest
```

If there's only a single item (or no item) in `s`, then we're done.

Otherwise, we swap the first and second items in `s`. Then we recurse on `s.rest.rest`.

### Q5: Make Circular

Write a function `make_circular` that takes in a non-circular, non-empty linked list `s` and mutates `s` so that it becomes circular.

```
def make_circular(s):
    """Mutates linked list s into a circular linked list.

    >>> lnk = Link(1, Link(2, Link(3)))
    >>> make_circular(lnk)
    >>> lnk.rest.first
    2
    >>> lnk.rest.rest.first
    3
    >>> lnk.rest.rest.rest.first
    1
    >>> lnk.rest.rest.rest.rest.first
    2
    """
    last = s
    while last.rest is not Link.empty:
        last = last.rest
    last.rest = s
```

We find the last `Link` instance we can reach from `s` and modify its `rest` attribute to point to `s`. Iteration is easier than recursion in this problem because we need to keep track of `s`.

# Efficiency

A function's runtime complexity is a measure of how the runtime of the function changes as its input changes. A function `f(n)` has...

- constant runtime if the runtime of `f` does not depend on `n`. Its runtime is Θ(`1`).
- logarithmic runtime if the runtime of `f` is proportional to `log(n)`. Its runtime is Θ(`log(n)`).
- linear runtime if the runtime of `f` is proportional to `n`. Its runtime is Θ(`n`).
- quadratic runtime if the runtime of `f` is proportional to `n^2`. Its runtime is Θ(`n^2`).
- exponential runtime if the runtime of `f` is proportional to `b^n`, for some constant `b`. Its runtime is Θ(`b^n`).

### Q6: WWPD: Orders of Growth

What is the *worst-case* runtime of `is_prime`?

```
def is_prime(n):
    for i in range(2, n):
        if n % i == 0:
            return False
    return True
```

Choose one of:

- Constant
- Logarithmic
- Linear
- Quadratic
- Exponential
- None of these

The worst-case runtime of `is_prime` occurs when `n` is actually prime. Each iteration of the for-loop takes constant time, and we have up to `n - 2` iterations. Therefore, the worst-case runtime of `is_prime` is linear.

What is the order of growth of the runtime of `bar(n)` with respect to `n`?

```
def bar(n):
    i, sum = 1, 0
    while i <= n:
        sum += biz(n)
        i += 1
    return sum

def biz(n):
    i, sum = 1, 0
    while i <= n:
        sum += i**3
        i += 1
    return sum
```

Choose one of:

- Constant
- Logarithmic
- Linear
- Quadratic
- Exponential
- None of these

The while-loop in `bar` iterates for `n` loops, so `n` calls to `biz(n)` are made.

A single `biz(n)` call runs in linear time because the while-loop in `biz` iterates for `n` constant-time loops. Don't be confused by `i**3`: evaluating `i**3` takes constant time even though the result is cubic.

The runtime complexity of `bar` is $\Theta(n * n) = \Theta(n\hat{}2)$ and the runtime is quadratic.

What is the order of growth of the runtime of `foo` in terms of `n`, where `n` is the length of `lst`? Assume that slicing a list and evaluating `len(lst)` take constant time.

Express your answer with $\Theta$ notation.

```
def foo(lst, i):
    mid = len(lst) // 2
    if mid == 0:
        return lst
    elif i > 0:
        return foo(lst[mid:], -1)
    else:
        return foo(lst[:mid], 1)
```

A `foo` call makes a single recursive call that halves the length of the argument for `lst`. We need approximately log(n) calls to reach the base case of a `lst` with length one or less.

The nonrecursive portion of each call takes constant time, so the overall runtime of `foo` is logarithmic and the runtime complexity of `foo` is $\Theta(\log(n))$.

Note: We made this problem easier by assuming that slicing a list takes constant time; in reality, slicing a list generally takes linear time with respect to the size of the slice.

# Appendix: Explanation of Material

## Linked Lists

The `Link` class implements linked lists in Python. Each `Link` instance has a `first` attribute (which stores the first value of the linked list) and a `rest` attribute (which points to the rest of the linked list).

```
class Link:
    empty = ()

    def __init__(self, first, rest=empty):
        assert rest is Link.empty or isinstance(rest, Link)
        self.first = first
        self.rest = rest

    def __repr__(self):
        if self.rest is not Link.empty:
            rest_repr = ', ' + repr(self.rest)
        else:
            rest_repr = ''
        return 'Link(' + repr(self.first) + rest_repr + ')'

    def __str__(self):
        string = '<'
        while self.rest is not Link.empty:
            string += str(self.first) + ' '
            self = self.rest
        return string + str(self.first) + '>'
```

An empty linked list is represented as `Link.empty`, and a non-empty linked list is represented as a `Link` instance.

The `rest` attribute of a `Link` instance is always another linked list! When `Link` instances are linked via their `rest` attributes, a sequence is formed.

To check if a linked list is empty, compare it to the class attribute `Link.empty`.

# Efficiency

Throughout this class, we have mainly focused on *correctness* — whether a program produces the correct output. However, computer scientists are also interested in creating *efficient* solutions to problems. One way to quantify efficiency is to determine how a function's *runtime* changes as its input changes. In this class, we measure a function's runtime by the number of operations it performs.

A function `f(n)` has...

- constant runtime if the runtime of `f` does not depend on `n`. Its runtime is Θ(1).
- logarithmic runtime if the runtime of `f` is proportional to `log(n)`. Its runtime is Θ(`log(n)`).
- linear runtime if the runtime of `f` is proportional to `n`. Its runtime is Θ(`n`).
- quadratic runtime if the runtime of `f` is proportional to `n^2`. Its runtime is Θ(`n^2`).
- exponential runtime if the runtime of `f` is proportional to `b^n`, for some constant `b`. Its runtime is Θ(`b^n`).

**Example 1:** It takes a single multiplication operation to compute `square(1)`, and it takes a single multiplication operation to compute `square(100)`. In general, calling `square(n)` results in a *constant* number of operations that does not vary according to `n`. We say `square` has a runtime complexity of Θ(1).

| input | function call | return value | operations |
|---|---|---|---|
| 1 | `square(1)` | 1*1 | 1 |
| 2 | `square(2)` | 2*2 | 1 |
| ... | ... | ... | ... |
| 100 | `square(100)` | 100*100 | 1 |
| ... | ... | ... | ... |
| n | `square(n)` | n*n | 1 |

**Example 2:** It takes a single multiplication operation to compute `factorial(1)`, and it takes 100 multiplication operations to compute `factorial(100)`. As `n` increases, the runtime of `factorial` increases *linearly*. We say `factorial` has a runtime complexity of Θ(`n`).

| input | function call | return value | operations |
|---|---|---|---|
| 1 | `factorial(1)` | 1*1 | 1 |
| 2 | `factorial(2)` | 2*1*1 | 2 |
| ... | ... | ... | ... |
| 100 | `factorial(100)` | 100*99*...*1*1 | 100 |
| ... | ... | ... | ... |
| n | `factorial(n)` | n*(n-1)*...*1*1 | n |

**Example 3:** Consider the following function:

```
def bar(n):
    for a in range(n):
        for b in range(n):
            print(a,b)
```

Evaluating `bar(1)` results in a single `print` call, while evalulating `bar(100)` results in 10,000 `print` calls. As `n`

*Note: This worksheet is a problem bank—most TAs will not cover all the problems in discussion section.*

increases, the runtime of `bar` increases *quadratically*. We say `bar` has a runtime complexity of Θ(`n^2`).

| input | function call | operations (prints) |
|---|---|---|
| 1 | `bar(1)` | 1 |
| 2 | `bar(2)` | 4 |
| ... | ... | ... |
| 100 | `bar(100)` | 10000 |
| ... | ... | ... |
| n | `bar(n)` | n^2 |

**Example 4:** Consider the following function:

```
def rec(n):
    if n == 0:
        return 1
    else:
        return rec(n - 1) + rec(n - 1)
```

Evaluating `rec(1)` results in a single addition operation. Evaluating `rec(4)` results in `2^4 - 1` = 15 addition operations, as shown by the diagram below.

During the evaulation of `rec(4)`, there are two calls to `rec(3)`, four calls to `rec(2)`, eight calls to `rec(1)`, and 16 calls to `rec(0)`.

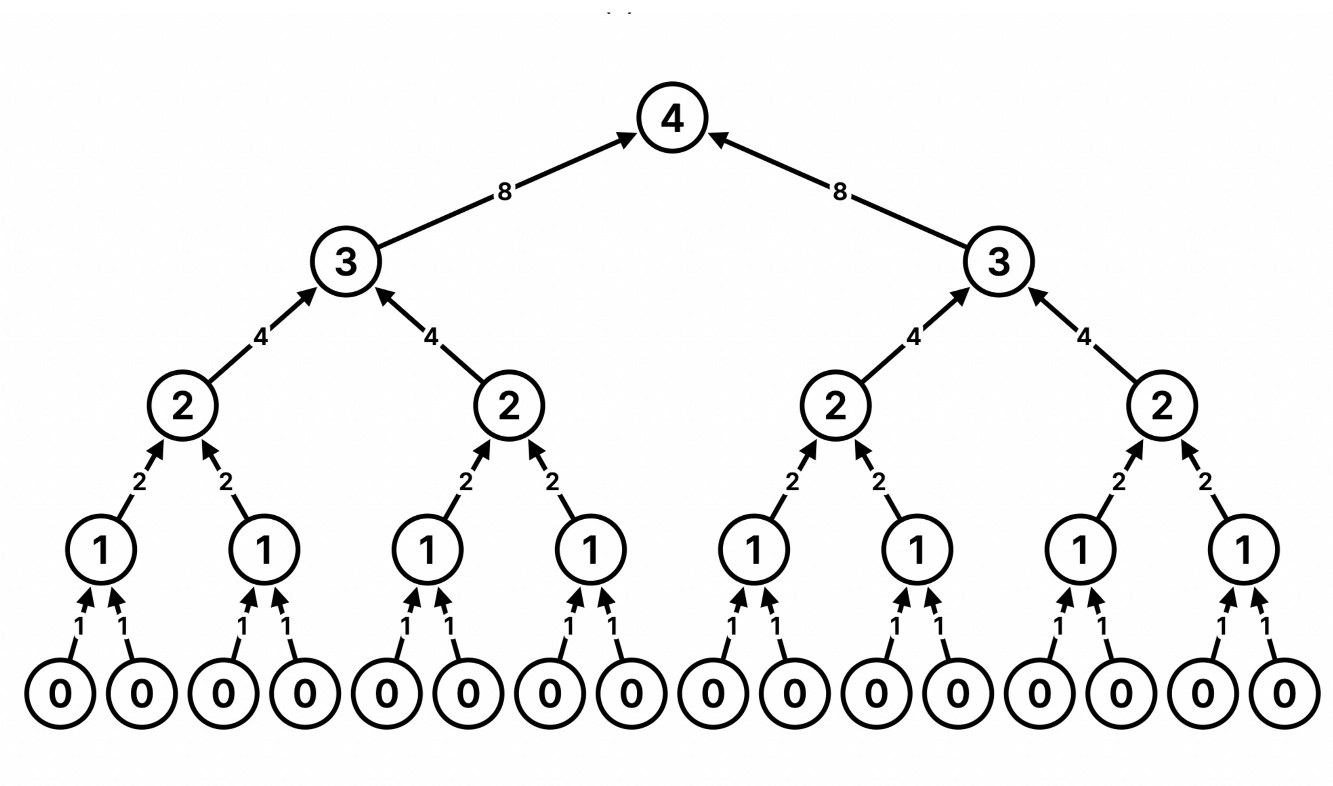
So we have eight instances of `rec(0) + rec(0)`, four instances of `rec(1) + rec(1)`, two instances of `rec(2) + rec (2)`, and a single instance of `rec(3) + rec(3)`, for a total of $1 + 2 + 4 + 8 = 15$ addition operations.

As `n` increases, the runtime of `rec` increases *exponentially*. In particular, the runtime of `rec` approximately doubles when we increase `n` by 1. We say `rec` has a runtime complexity of Θ(`2^n`).

| input | function call | return value | operations |
|---|---|---|---|
| 1 | `rec(1)` | 2 | 1 |
| 2 | `rec(2)` | 4 | 3 |
| ... | ... | ... | ... |
| 10 | `rec(10)` | 1024 | 1023 |
| ... | ... | ... | ... |
| n | `rec(n)` | 2^n | 2^n - 1 |

Tips for finding the order of growth of a function's runtime:

- If the function is recursive, determine the number of recursive calls and the runtime of each recursive call.
- If the function is iterative, determine the number of inner loops and the runtime of each loop.
- Ignore coefficients. A function that performs `n` operations and a function that performs `100 * n` operations are both linear.
- Choose the largest order of growth. If the first part of a function has a linear runtime and the second part has a quadratic runtime, the overall function has a quadratic runtime.
- In this course, we only consider constant, logarithmic, linear, quadratic, and exponential runtimes.




**Above: Call structure of rec(4).**