

Card Game

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1 Problem

Alice and Bob are playing a game. They have n cards numbered from 1 to n . At the beginning of the game, some of these cards are given to Alice, and the rest are given to Bob.

Card with number i beats card with number j if and only if $i > j$, with one exception: card 1 beats card n .

The game continues as long as each player has at least one card. During each turn, the following occurs:

Alice chooses one of her cards and places it face up on the table; Bob, seeing Alice's card, chooses one of his cards and places it face up on the table; if Alice's card beats Bob's card, both cards are taken by Alice. Otherwise, both cards are taken by Bob. A player can use a card that they have taken during one of the previous turns.

The player who has no cards at the beginning of a turn loses. Determine who will win if both players play optimally.

2 Solution

The winner is set the moment the cards are distributed, it only depends on who the cards 1, $n - 1$ and n went to and whether Bob has only 1 card.

- If the cards 1, n go to the same person, that person will win by always playing n .
- If Alice has 1 and Bob has n . If Bob has no other cards, Alice can just play 1 and Bob is forced to play n and Bob will lose. Otherwise Bob has some other card $1 < x < n$, then if Alice plays some $y \neq 1$, Bob can beat it by playing n , else Alice plays 1, Bob can beat it by playing x so Bob wins.
- If Alice has n and Bob has 1. If Alice also has card $n - 1$ then she wins by always playing $n - 1$. Otherwise Bob has card $n - 1$, then if Alice plays some $y \neq n$, he can beat it using $n - 1$, else Alice plays n , then he can beat it using 1 so Bob wins.

Thus we can decide the winner in $O(n)$ time.