# Two Sets

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## 1 Problem

Little X has $n$ distinct integers: $p_1, p_2, ..., p_n$. He wants to divide all of them into two sets $A$ and $B$. The following two conditions must be satisfied:

If number $x$ belongs to set $A$, then number $a - x$ must also belong to set $A$. If number $x$ belongs to set $B$, then number $b - x$ must also belong to set $B$. Help Little X divide the numbers into two sets or determine that it's impossible.

## 2 Solution

We present an $O(n \lg n)$ time solution.

For each number $x$ call $a - x$ "the $A$ inverse of $x$" and $b - x$ "the $B$ inverse of $x$". Let $x.set$ be our assignment for $x$, $x.set = A$ if we assign $x$ to $A$ and $x.set = B$ if we assign $x$ to $B$, if $x$ has yet been assigned let $x.set = \emptyset$. Notice $x$ and all its inverses must be assigned to the same set.

First notice that if $a = b$ then we can divide the numbers into two sets iff for each element its $A$ inverse exists, thus we can answer in $O(n \lg n)$ time. Below WLOG assume $a > b$.

Our greedy strategy is to process the input in decreasing order. After we processed an input $x$, $x$ and all its inverses will be assigned. Whenever we assign an input, we make sure this assignment is absolutely correct and will never be changed. Below are all possible cases when we process $x$ :

(i) $x$ has no inverse : then impossible to divide into two sets

(ii) $x$ has exactly 1 inverse : WLOG suppose it only has $a - x$, then we can never assign $x$ to $B$. If $a - x$ has already been assigned to $B$, then it's impossible to divide into two sets. Otherwise, assign both $x, a - x$ to $A$.

(iii) $x$ has both inverses and at least one of $x, a - x, b - x$ has been assigned : If the current assignments are consistent then assign the rest to the same set. Otherwise it's impossible to divide into two sets.

(iv) $x$ has both inverses and none of $x, a - x, b - x$ has been assigned : Notice $a - (b - x)$ (the $A$ inverse of $b - x$) cannot exist, because it is larger than $x$ so if it exists it would have been processed and $b - x$ would be assigned. This means we cannot assign $x$ to $A$. If $b - (a - x)$ (the $B$ inverse of $a - x$) exist and is not assigned to $A$, then we can assign $x, a - x, b - x, b - (a - x)$ to $B$. Otherwise it's impossible to divide into two sets.

If we successfully assign all inputs then we have successfully divided the numbers into two sets. Run time is $O(n \lg n)$ to sort and do two binary searches for each input.