Shortest Cycle

Calvin Fung

August 7, 2025

1 Problem

You are given integer numbers $a_1, ..., a_n$. Consider graph on n nodes, in which nodes i , j ($i \neq j$) are connected if and only if, a_i AND $a_j \neq 0$, where AND denotes the bitwise AND operation.

Find the length of the shortest cycle in this graph or determine that it doesn't have cycles at all.

2 Solution

We present an $O(n)$ time solution.

Suppose the largest integer in the input has b bits, a usual we assume b is constant. Let C be an 0-indexed array of size b such that $C[i]$ is the list of all nodes u where the ith bit of a_u is 1. Let $G = (V, E)$ be the graph, notice each entry $C[i]$ corresponds to a clique of size $|C[i]|$ in G. It follows that if there exits an entry $C[i]$ such that $|C[i]| \geq 3$, then the length of the shortest cycle in G is 3. Otherwise, $|C[i]| \leq 2, \forall i = 1, ..., b$, then we claim that

$$E = \bigcup_{i:|C[i]|=2} \{(u, v) : u \neq v \text{ and } u, v \in C[i]\}$$

Clearly given $u, v \in C[i], u \neq v$, each (u, v) is an edge in G. On the other hand, consider any edge (u, v) in G, we know applying the bitwise AND operation on a_u, a_v gives us some $z \neq 0$, so there must exist $i \in \{1, ..., b\}$ such that the ith bit of z is 1. This implies the ith bit of a_u, a_v must both be 1. Therefore, $u, v \in C[i]$ and (u, v) will appear in the union on the RHS. This leads to the following :

- Since $|\{(u, v) : u \neq v \text{ and } u, v \in C[i]\}| = 1$ for $|C[i]| = 2$ we have $|E| \leq b$ ie. only have a constant number of edges.
- One can construct G in $O(n)$ time using the union expression on the RHS

After constructing G, we can find the shortest cycle in $O(E + V(\text{size of largest connect component})) = O(V)$ time (notice each component can have at most $|E| + 1$ vertices and $|E|$ edges) using the algorithm we designed for leetcode 2608 [1].

[1] `https://github.com/Calvinfwc/leetcode/blob/main/2608Shortest_Cycle_in_a_Graph.pdf`