The Morning Star

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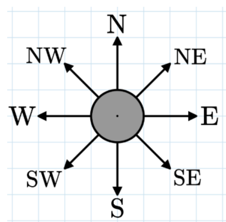
1 Problem

A compass points directly toward the morning star. It can only point in one of eight directions: the four cardinal directions (N, S, E, W) or some combination (NW, NE, SW, SE). Otherwise, it will break.



The directions the compass can point.

There are n distinct points with integer coordinates on a plane. How many ways can you put a compass at one point and the morning star at another so that the compass does not break?

2 Solution

We present an algorithm that runs in randomized $O(n)$ time.

Define 4 special lines, $l_0 : x = 0, l_1 : y = x, l_2 : y = 0$ and $l_3 : y = -x$.

Let P be the input set of points. We say a pair of points $(p, q), p \neq q \in P$ is in group $G_i, i = 0, ..., 3$ if the line passing through them is parallel to l_i.

It's easy to see that we can put a a compass at point p and the morning star at point q such that the compass does not break iff (p, q) is in one of the groups ie. $\in \cup_{i=0}^{3} G_i$. Furthermore, no point can belong to 2 different groups because no pair of lines l_i, l_j where $i \neq j$ are parallel to each other. It follows that the desired answer is $\sum_{i=0}^{3} |G_i|$.

We show how to compute $|G_1|$ correctly in randomized $O(n)$ time (the other G_i will be similar) :

First we create an empty hash table H.

Then for each $p \in P$, there's a unique line l that's parallel to l_1 passing through it, in particular $l : y = x + c$ where $c = p.y - p.x$. If the entry $H[c]$ already exists, we do $H[c] = H[c] + 1$, else we create a new entry $H[c] = 1$.

Finally, after initializing $res = 0$, for each $(key, value) \in H$ where $value > 1$, we do $res = res + 2 * \binom{value}{2}$. We return res at the end.

correctness : Clearly each entry $H[c]$ stores the number of points in P that lies on $l : y = x + c$ which is parallel to l_1. Notice any pair of points lying on l will be in G_1, there're $2 * \binom{H[c]}{2}$ such pairs. Combining with the fact that a point cannot lie on 2 different lines with slope 1 so we never double count any pair in G_1, it follows $res \leq |G_1|$.

On the other hand, for any $(p, q) \in G_1$, both p, q must lie on $y = x + c$ for some c, thus the combination (p, q) will be counted in $2 * \binom{H[c]}{2}$. Further, for 2 different pairs $(p, q) \neq (p', q') \in G_1$, if they correspond to different combinations then clearly they'll be counted twice, otherwise they correspond to same combinations but different order ie. $p' = q, q' = p$, then they'll be still counted twice as we double count each combination ie. $2 * \binom{H[c]}{2}$. It follows $res \geq |G_1|$. ■