Queue at the School

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1 Problem

During the break the schoolchildren, boys and girls, formed a queue of n people in the canteen. Initially the children stood in the order they entered the canteen. However, after a while the boys started feeling awkward for standing in front of the girls in the queue and they started letting the girls move forward each second.

Let's describe the process more precisely. Let's say that the positions in the queue are sequentially numbered by integers from 1 to n, at that the person in the position number 1 is served first. Then, if at time x a boy stands on the i-th position and a girl stands on the (i+1)-th position, then at time x+1 the i-th position will have a girl and the (i+1)-th position will have a boy. The time is given in seconds.

You've got the initial position of the children, at the initial moment of time. Determine the way the queue is going to look after t seconds.

2 Solution

We claim that the naive algorithm runs in time $O(n^2)$ where n is length of input queue.

```
f(Q,n,t){
        bool stop = false;
        while(!stop){
                t = t - 1;
                L = empty list; //mark all girls that can swap
                for(i=2; i<=n;++i){
                if(Q[i-1] == "B" && Q[i] == "G")        L.push_back(i);
                }
                if (L.empty() || t == 0)        stop = true;
                for(i in L){
                        swap(Q[i-1],Q[i]);
                }
        }
        return Q;
}
```

Assume our queues run from left to right (ie. the leftmost person is at position 1). Let Σ^+ be the set of all possible queues. For each $Q \in \Sigma^+$ we can number the girls from left to right by $1, 2, ...$, we denote the i-th girl by G_i, there'll be $i-1$ girls infront of her.

Lemma 1. For all $i \leq j$ where $i, j \in \mathbb{Z}_+$. For any $Q \in \Sigma^+$, if girl G_i is located at position j in Q, then by running the algorithm $\mathcal{A}$ on Q there'll be no boys infront of G_i after iteration j.

proof. We prove by induction on j. When $j = 1$ this clearly holds as no boy is infront of the first girl. By induction hypothesis this holds for $j = k$. Now consider $j = k + 1$. This clearly holds for $i = k + 1$ because there is no boy infront of G_{k+1} if she locates at position $k + 1$.

Now consider $i \in \{1, ..., k\}$ and any $Q_0 \in \Sigma^+$ where G_i is at position $k+1$ in Q_0. There're 2 cases

case 1 : There's a boy at position k ie.

Queue	B	G_i
Position	k	$k+1$

Let Q_1 be the queue after running our algorithm on Q_0 for 1 iteration. In Q_1, we must have G_i being at position k (after swapping with the boy). Thus applying induction hypothesis we know there's no boy infront of G_i after k more iterations. Combine together, there's no boy infront of G_i after $k+1$ iterations.

case 2 : There's a girl at position k, she must be G_{i-1} ie.

Queue	G_{i-1}	G_i
Position	k	$k+1$

Applying induction hypothesis we know there's no boy infront of G_{i-1} after k more iterations. Since there's no boy in between G_{i-1} and G_i in Q_0, observe that no matter how many iteration you run the algorithm on Q_0, there'll never be more than 1 boy in between G_{i-1} and G_i. Thus after running k more iterations on Q_0, there's at most 1 boy infront of G_i. It follows there'll be no boy infront of G_i after $k+1$ iterations. ∎

Lemma 2. The algorithm runs in $O(n^2)$ time where n is length of queue.

proof. For any $Q \in \Sigma^+$, suppose there're l girls and the last girl G_l locates at position j. Clearly $l \leq j \leq n$. By lemma 1, there's no boy infront of G_l after j iterations which implies there's no boy infront a girl so the algorithm terminates after $j \leq n$ iterations. As each iteration takes $O(n)$ time the algorithm runs in $O(n^2)$ time. ∎