Party

June 9, 2025

1 Problem

A company has n employees numbered from 1 to n. Each employee either has no immediate manager or exactly one immediate manager, who is another employee with a different number. An employee A is said to be the superior of another employee B if at least one of the following is true:

- Employee A is the immediate manager of employee B

- Employee B has an immediate manager employee C such that employee A is the superior of employee C.

The company will not have a managerial cycle. That is, there will not exist an employee who is the superior of his/her own immediate manager.

Today the company is going to arrange a party. This involves dividing all n employees into several groups: every employee must belong to exactly one group. Furthermore, within any single group, there must not be two employees A and B such that A is the superior of B.

What is the minimum number of groups that must be formed ?

2 Solution

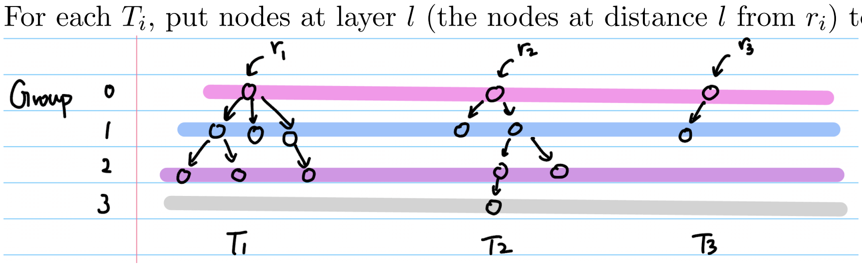
Construct directed graph $G = (V, E)$ where vertices are employees and there's an edge (u, v) iff u is immediate manager of v.

Lemma. The minimum number of groups that must be formed is the length of a longest path in G.

proof. Let P^* be a longest path and OPT be the minimum number of groups that must be formed. As no 2 employees on P^* can be in the same group, $OPT \geq |P^*|$.

We now show there's a feasible solution of size $|P^*|$. Consider the nodes with 0 in-degree $r_1, ..., r_k$, do BFS starting at each of them (and avoid visiting previously visited nodes), we'll have the corresponding breath first trees (BFT) $T_1, ..., T_k$. Notice each $v \in V$ must be in one of those trees, because there's no directed cycle v must lie on a path that starts with a node with 0 in-degree.

For each T_i, put nodes at layer l (the nodes at distance l from r_i) to group l.



Clearly there're at most $|P^*|$ groups and from above we know each $v \in V$ will be in exactly a group. As each employee either has none or exactly one immediate manager, no node can have ≥ 2 in-degree, it follows

1. In the same tree, there's no edge between 2 nodes in same layer.
2. There's no edge between 2 different trees.

Thus there's also no edge within each group. ■

It follows we can solve the problem by finding the length of a longest path in G using DFS/BFS in $O(n)$ time because $|V|, |E| \leq n$.