Precalculus Honors **5.3 Matrix Multiplication** **December 7, 2005**
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Applications of Matrix Multiplication

Just as in the previous section we saw that matrices help us organize information, in this section we will define matrix multiplication so that it makes good sense and provides useful information in our study of real-life situations.

Example 1

The Sandwich Problem. Suppose you and your roommate go into partnership making sandwiches to sell in your dormitory. Suppose that each of you makes 3 different types of sandwiches.

Sandwich 1: made of bread, peanut butter, and jelly

Sandwich 2: made of bread, ham, and cheese

Sandwich 3: made of bread, cheese, and tomato

Each night, you (y) and your roommate (r) plan to sell a number of sandwiches of each kind shown in matrix A.

$$A = \begin{array}{c} \\ y \\ r \end{array} \begin{array}{c} \begin{array}{ccc} \mathit{1} & \mathit{2} & \mathit{3} \end{array} \\ \begin{pmatrix} 4 & 5 & 3 \\ 3 & 3 & 6 \end{pmatrix} \end{array}$$

Each sandwich is made by putting together some combination of slices of bread, peanut butter, jelly, ham, cheese, and tomato. Matrix B describes the number of pieces of bread (b), ounces of peanut butter (p), ounces of jelly (j), slices of ham (h), slices of cheese (c), and slices of tomato (t). needed for each kind of sandwich.

$$B = \begin{array}{c} \\ \mathit{1} \\ \mathit{2} \\ \mathit{3} \end{array} \begin{array}{c} \begin{array}{cccccc} b & p & j & h & c & t \end{array} \\ \begin{pmatrix} 2 & 4 & 4 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 & 2 & 0 \\ 2 & 0 & 0 & 0 & 4 & 2 \end{pmatrix} \end{array}$$

To determine the ingredients for each sandwich you need to make in matrix A, we will examine the following questions.

(a) How many slices of bread will you use to make the sandwiches?

You will make 4 Sandwich 1 *sandwiches, each of which uses 2 slices of bread; 5* Sandwich 2 *sandwiches, each of which uses 2 slices of bread; and 3* Sandwich 3 *sandwiches, each of which uses 2 slices of bread. The total number of slices of bread you will use is expressed by the sum of the products*

$$4(2) + 5(2) + 3(2) = 24,$$

so you need 24 slices of bread.

(b) How many slices of bread will your roommate use to make the sandwiches?

Your roommate will make 3 Sandwich 1 *sandwiches, each of which uses 2 slices of bread; 3* Sandwich 2 *sandwiches, each of which uses 2 slices of bread; and 3* Sandwich *3 sandwiches, each of which uses 2 slices of bread. The total number of slices of bread your roommate will use is expressed by the sum of the products*

$$3(2) + 3(2) + 6(2) = 24,$$

so your roommate also needs 24 slices of bread.

(c) How much peanut butter will you use?

The amount of peanut butter you will use is

$4(4) + 5(0) + 3(0) = 16$. *You will use 16 ounces of peanut butter.*

(d) How much jelly will you use?

$4(4) + 5(0) + 3(0) = 16$. *You will use 16 ounces of jelly.*

(e) How much ham will you use?

$4(0) + 5(3) + 3(0) = 15$ *slices of ham.*

(f) How much cheese?

$4(0) + 5(2) + 3(4) = 22$ *slices of cheese*.

(g) How much tomato?

$4(0) + 5(0) + 3(2) = 6$ *slices of tomato.*

Similarly, your roommate will use (in addition to the 24 slices of bread)

3(4) + 3(0) + 6(0) = 12 ounces of peanut butter,

3(4) + 3(0) + 6(0) = 12 ounces of jelly,

3(0) + 3(3) + 6(0) = 9 slices of ham,

3(0) + 3(2) + 6(4) = 30 slices of cheese, and

3(0) + 3(0) + 6(2) = 12 slices of tomato.

We can summarize the amount of each ingredient with the following matrix C, in which the entries are the numbers of ounces or slices of food.

$$C = \begin{array}{c} \\ y \\ r \end{array} \begin{array}{c} \begin{array}{cccccc} b & p & j & h & c & t \end{array} \\ \begin{pmatrix} 24 & 16 & 16 & 15 & 22 & 6 \\ 24 & 12 & 12 & 9 & 30 & 12 \end{pmatrix} \end{array}$$

Observe that the entry in the first row and first column of C is obtained by lining up the first row of A and the first column of B, then multiplying the corresponding entries and adding the products together. Row 1 of A and column 1 of B are

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ y & \begin{pmatrix} 4 & 5 & 3 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & b \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \end{array}$$

Multiplying pairwise term by term gives

$$4(2) + 5(2) + 3(2) = 24;$$

this sum is entry C_{11}. Likewise C_{25} is obtained by multiplying pairwise term by term the second row of A by the fifth column of B, as shown below.

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ r & \begin{pmatrix} 3 & 3 & 6 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & c \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \end{array}$$

$$= 3(0) + 3(2) + 6(4) = 30.$$

What row of A multiplied by what column of B gives entry C_{12}? *The entry in the row and second column of* C *is found by multiplying the first row of* A *by the second column of* B.

All of the entries in C can be found using the method illustrated above. This way of combining entries in two matrices to yield a third matrix is called *matrix multiplication*; matrix C is defined as the product of matrices A and B. The operation can be written in the form shown below.

$C = AB$

$$= \begin{array}{c} \\ y \\ r \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 4 & 5 & 3 \\ 3 & 3 & 6 \end{pmatrix} \end{array} \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} b & p & j & h & c & t \end{array} \\ \begin{pmatrix} 2 & 4 & 4 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 & 2 & 0 \\ 2 & 0 & 0 & 0 & 4 & 2 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \\ y \\ r \end{array} \begin{array}{c} \begin{array}{cccccc} b & p & j & h & c & t \end{array} \\ \begin{pmatrix} 24 & 16 & 16 & 15 & 22 & 6 \\ 24 & 12 & 12 & 9 & 30 & 12 \end{pmatrix} \end{array}$$

In general, the matrix multiplication $C = AB$ is defined as follows: Each entry C_{ij} is obtained by multiplying pairwise term by term the ith row of the left-hand matrix A by the jth column of the right-hand matrix B. In symbols, this definition means that

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + A_{i3}B_{3j} + \cdots + A_{in}B_{nj}.$$

Example 2

Let matrix D represent the cost in dollars per ingredient for each type of sandwich in the Sandwich Problem above.

$$D = \begin{array}{c} \\ b \\ p \\ j \\ h \\ c \\ t \end{array} \begin{array}{c} \text{cost} \\ \begin{pmatrix} 0.12 \\ 0.15 \\ 0.10 \\ 0.30 \\ 0.25 \\ 0.12 \end{pmatrix} \end{array}$$

(a) You would like to determine a selling price for the sandwiches. What is the total cost of the ingredients for one sandwich of each kind?

The cost of making one Sandwich 1 *sandwich is equal to the number of each ingredient used multiplied by the cost per ingredient, or*

$$2(0.12) + 4(0.15) + 4(0.15) + 0(0.30) + 0(0.25) + 0(0.12) = 1.44.$$

The cost of the ingredients used to make a *Sandwich 1* sandwich is \$1.44. This number was calculated by multiplying pairwise term by term the first row of B by the column in D. Using similar reasoning for *Sandwiches 2* and *3*, we see that the matrix product BD gives the required information for each type of sandwich.

$$BD = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} b & p & j & h & c & t \end{array} \\ \begin{pmatrix} 2 & 4 & 4 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 & 2 & 0 \\ 2 & 0 & 0 & 0 & 4 & 2 \end{pmatrix} \end{array} \begin{array}{c} \\ b \\ p \\ j \\ h \\ c \\ t \end{array} \begin{array}{c} cost \\ \begin{pmatrix} 0.12 \\ 0.15 \\ 0.10 \\ 0.30 \\ 0.25 \\ 0.12 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} cost \\ \begin{pmatrix} 1.24 \\ 1.64 \\ 1.48 \end{pmatrix} \end{array}$$

The cost of the ingredients to make a *Sandwich 1* sandwich is \$1.24, the cost for a *Sandwich 2* sandwich is \$1.64, and the cost of a *Sandwich 3* sandwich is \$1.48.

(b) You and your roommate would like to know how much money to budget for purchasing the ingredients to make all of the sandwiches. What is the total cost of producing all of the sandwiches listed in matrix A?

The product AB *from Example 1 gives the number of each ingredient needed by you and your roommate. Multiplying* AB *by* D *results in a matrix containing the costs for you and your roommate, as shown below.*

*(*AB*)(*D*)*

$$= \begin{array}{c} \\ y \\ r \end{array} \begin{array}{c} \begin{array}{cccccc} b & p & j & h & c & t \end{array} \\ \begin{pmatrix} 24 & 16 & 16 & 15 & 22 & 6 \\ 24 & 12 & 12 & 9 & 30 & 12 \end{pmatrix} \end{array} \begin{array}{c} \\ b \\ p \\ j \\ h \\ c \\ t \end{array} \begin{array}{c} cost \\ \begin{pmatrix} 0.12 \\ 0.15 \\ 0.10 \\ 0.30 \\ 0.25 \\ 0.12 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \\ y \\ r \end{array} \begin{array}{c} cost \\ \begin{pmatrix} 17.60 \\ 17.52 \end{pmatrix} \end{array}$$

The total cost for you is \$17.60; the total cost for your roommate is \$17.52.

The rows and columns of data in the matrices in Example 1 and 2 are described by labels. In matrix A, the row labels are names (you and your roommate) and the column labels are types of sandwiches (1, 2, and 3), so that matrix A classifies data according to name and type; we refer to matrix A as a name-and-type matrix. Consistent with this notation, matrix B is a type-by-ingredient matrix. The row and column labels of matrices are especially helpful in interpreting the results of matrix multiplication.

Observe that in Example 1 we multiplied a name-by-type matrix (A) by a type-by-ingredient matrix (B) to get a name-by-ingredient matrix (C). In Example 2, we multiplied a type-by-ingredient matrix (B) by an ingredient-by cost matrix (D) to get a type-by-cost matrix. We also found that the product of a name-by-ingredient matrix (AB) and an ingredient-by-cost matrix (D) is a name-by-cost matrix. In each example, matrix multiplication eliminated the labels of the first factor's columns and the second factor's rows, leaving a product matrix with exactly the row and column labels we desired in our answer. Matrix multiplication, which at first glance may seem very strange, is actually designed to give us the information we want in a straightforward manner.

Two matrices are multiplied by multiplying the elements of a row of the left-hand matrix by the corresponding elements of a column of the right-hand matrix and then adding the products. If matrix S is multiplied by matrix T, the number of columns of S must equal the number of rows of T. If $ST = U$, the product matrix U has the same number of rows as S and the same number of columns as T. In symbols, this is

$$S_{m \times n} T_{n \times p} = U_{m \times p}.$$

In order for the product of ST to be meaningful, the column labels of S must be the same as the row labels of T. If $ST = U$, then U has the row labels of S and the column labels of T.