Experiment 81 – ELEC207 coursework

Design of a Stable Martian Segway

Report template

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1. Mathematical Modelling

A) Please define the values for *l, m* and t_s that you will use for your coursework. [1E]

Since my birthday is June 6, 2000, the following data was derived:

l is the day of the month when you were born (where l us in meters): l = 6 meters

m is the month of the year when you were born (where m is in kg): m = 6 kg

t_s is the year when you were born divided by 250 (where t_s is in seconds): t_s = 8 seconds

B) Now derive the transfer function, $H(s) = \theta(s)/T(s)$, of the Segway in terms of *l*, *m* and *g.* [1E]

First, we know that the original equation is:

$$\frac{g}{l}\theta(t) + T(t) = m\frac{d^2\theta(t)}{dt^2}$$

To obtain the transfer function, we need to first perform the Laplace transform on it:

$$\frac{g}{l}\theta(s) + T(s) = ms^2\theta(s)$$

Further conversion gives the transfer function:

$$H(s) = \frac{\theta(s)}{T(s)} = \frac{1}{ms^2 - \frac{g}{l}}$$

C) Using your values for *l* and *m* along with *g*=3.711 ms^{-2}, write the transfer function with the denominator and numerator of your transfer function in polynomial form. [1E]

Substituting the m and l we obtained before and the g given in the question into the formula gives:

$$H(s) = \frac{\theta(s)}{T(s)} = \frac{1}{ms^2 - \frac{g}{l}} = \frac{1}{6s^2 - 0.6185}$$

D) Calculate the position of the poles for your Segway and plot the poles on the complex plane. [1E]

If the location of the pole is to be found then the point at

$$\lim_{s \to pi} H(s) = \infty$$

Thus:

$$6s^2 - 0.6185 \to 0$$

The solution gives:

$$s_1 \approx -0.3211 \qquad s_2 \approx 0.3211$$

The plot on the complex plane is as follows:

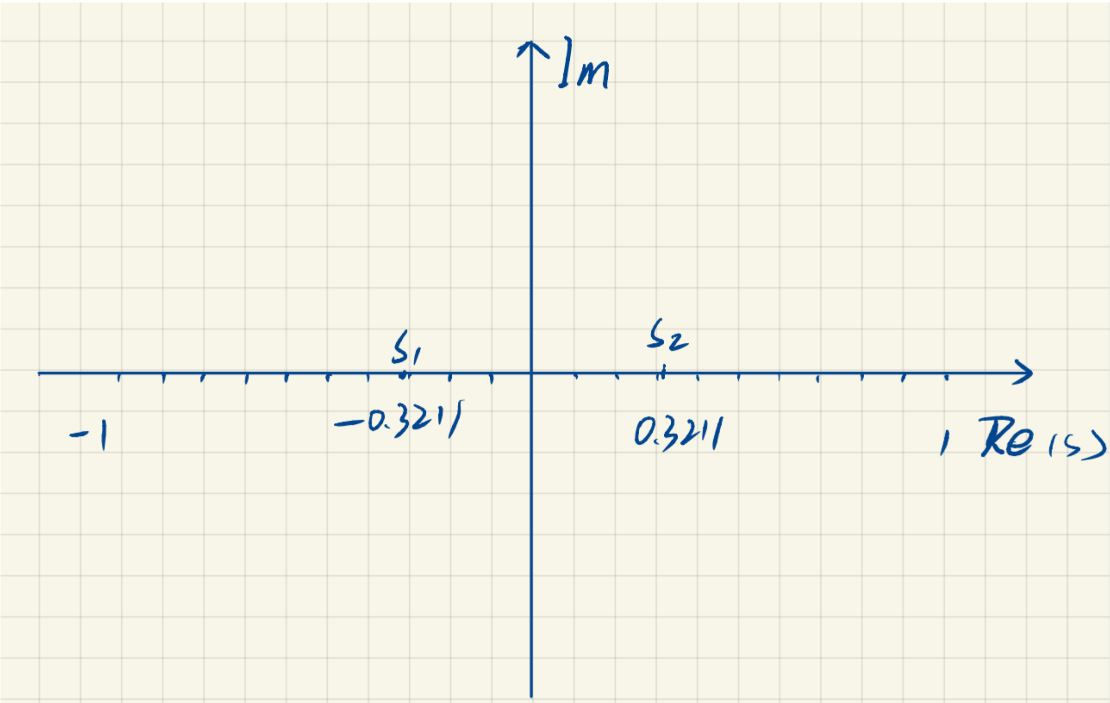

Fig.1. Plot on the complex plane

2. Validating that the Open-loop System is Unstable

E) Insert a picture of the time-response of your Segway to the unit-step. [2E]

First, the block structure diagram of the system is obtained after completing the work in the script.

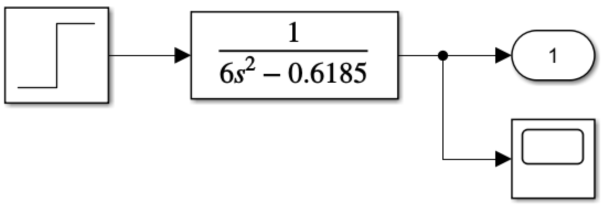

Fig.2. The Simulink blocks for system

From the block diagram, you can see that my transfer function is: $H(s) = \frac{1}{6s^2 - 0.6185}$

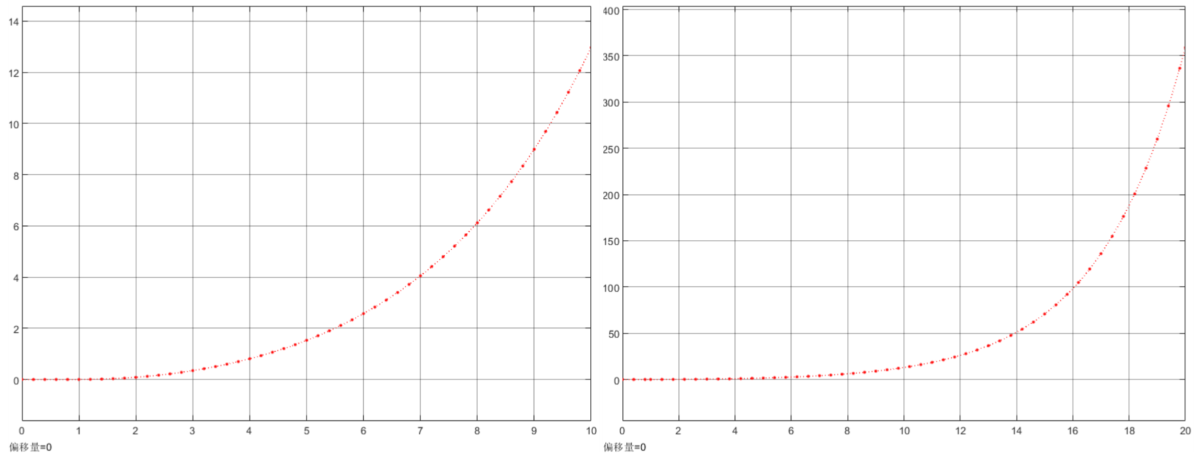

Fig.3. The time-response of my Segway to the unit-step

To observe the time-response of the Segway to a unit step more clearly, I set the stop time to 10 and 20 and ran them separately.

F) Comment on whether this time-response indicates that the open-loop system is stable. [1M]

First, for a stable system, its system output should converge to a value when the time of the input signal tends to infinity. However, the step input is the most severe operating condition for the system. If the dynamic performance of the system meets the requirements for a step input, then the system will usually still meet the dynamic performance requirements for other forms of input. The input to this system is the step signal and it can be seen from Figure 3 that the time response of this system is an exponential growth and is not convergent. Therefore, this system is unstable.

In addition, for the pole, although the left half of the pole also does not affect the system stability, its right half pole will make the system unstable. First, the pole is the point in the transfer function that will make the denominator zero, then the signal at the frequency of the point will have an infinite amplitude at the output. Therefore, the pole in the right plane, when there is noise present, will make it bigger and bigger until it reaches the supply voltage oscillation. For the left plane zero pole its frequency is negative, the reality of the frequency can only be positive cannot be negative, and the right half of the zero-pole frequency is positive, which is the reality of the frequency can be reached, so when a signal reaches the frequency at the signal will be zero or infinity. The transfer function poles are systematically stable in the left half-plane of the complex plane, critically stable on the imaginary axis, and unstable in the right half-plane. We can see in the previous Figure 1 that there are two poles in the complex plane, one of which is in the right half-plane. Therefore, it is concluded that the system is unstable.

3. Ensuring that the Closed-loop System is Stable Using PID Control

G) Write the closed-loop transfer function for your Segway in terms of K_p, K_I and K_D as a ratio of polynomials in *s*. Ensure that the highest order term in s in the denominator has a coefficient of unity. [3M]

First, we know that the transfer function of the PID controller is:

$$C(s) = K_p + \frac{K_I}{s} + K_D s = \frac{K_p s + K_I + K_D s^2}{s}$$

The closed-loop transfer function of Segway is:

$$\frac{\theta(s)}{X(s)} = \frac{C(s)H(s)}{1 + C(s)H(s)}$$

My open-loop Segway transfer function is:

$$H(s) = \frac{1}{6s^2 - 0.6185}$$

The following equation is obtained by association:

$$\frac{\theta(s)}{X(s)} = \frac{\frac{K_p s + K_I + K_D s^2}{s} \times \frac{1}{6s^2 - 0.6185}}{1 + \frac{K_p s + K_I + K_D s^2}{s} \times \frac{1}{6s^2 - 0.6185}}$$

Simplification:

$$\frac{\theta(s)}{X(s)} = \frac{\frac{K_p s + K_I + K_D s^2}{6s^3 - 0.6185s}}{\frac{6s^3 - 0.6185s + K_p s + K_I + K_D s^2}{6s^3 - 0.6185s}}$$

$$\frac{\theta(s)}{X(s)} = \frac{K_p s + K_I + K_D s^2}{6s^3 - 0.6185s + K_p s + K_I + K_D s^2}$$

Since it is necessary to ensure that the coefficient of the highest s term in the denominator is 1, we can obtain the final closed-loop transfer function:

$$\frac{\theta(s)}{X(s)} = \frac{\frac{1}{6}K_p s + \frac{1}{6}K_I + \frac{1}{6}K_D s^2}{s^3 - 0.1031s + \frac{1}{6}K_p s + \frac{1}{6}K_I + \frac{1}{6}K_D s^2}$$

H) What is the characteristic polynomial that would result in these pole positions? [1M]

First, to ensure that the system is stable, the poles are placed at the points s=-1, s=-2, s=-3.

Thus, the characteristic polynomial should be:

$$(s + 1) (s + 2) (s + 3) = 0$$

The final characteristic polynomial is obtained after simplification as:

$$s^3 + 6s^2 + 11s + 6 = 0$$

I) By equating the coefficients in the closed-loop transfer function's denominator and this characteristic function, deduce values for K_p, K_I and K_D which will ensure that the closed-loop system is stable. [3M]

To ensure that the denominator and the characteristic polynomial are equivalent, we can derive the values of K_P, K_I and K_D by the following calculation:

The denominator of the transfer function is:

$$s^3 - 0.1031s + \frac{1}{6}K_p s + \frac{1}{6}K_I + \frac{1}{6}K_D s^2$$

The cubic term of the denominator:

$$s^3 = s^3$$

The quadratic term of the denominator:

$$6s^2 = \frac{1}{6}K_D s^2$$

$$K_D = 36$$

The primary term of the denominator:

$$11s = (\frac{1}{6}K_p - 0.1031)s$$

$$K_p = 66.6186$$

The constant term of the denominator:

$$6 = \frac{1}{6}K_I$$

$$K_I = 36$$

According to our calculations, we found that $K_D = 36, \quad K_p = 66.6186, \quad K_I = 36$. we can find that all the poles are on the left half of the complex plane, and according to the previous conclusion of determining the stability of the system we can conclude that this closed-loop system is stable.

4. Validating That the Closed-loop System is Stable

J) Insert a picture of the time-response of your closed-loop system to the unit-step. [2M]

The closed-loop system structure block after adding the subtract and PID modules is shown below:

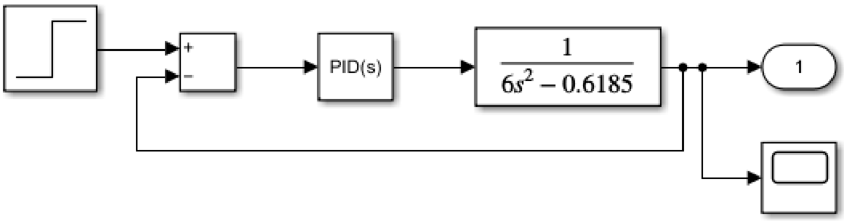

Fig.4. The block for closed-loop system

The PID parameters we have obtained are fed into the PID module and then the output of the transfer function is connected to the subtract module as negative feedback.

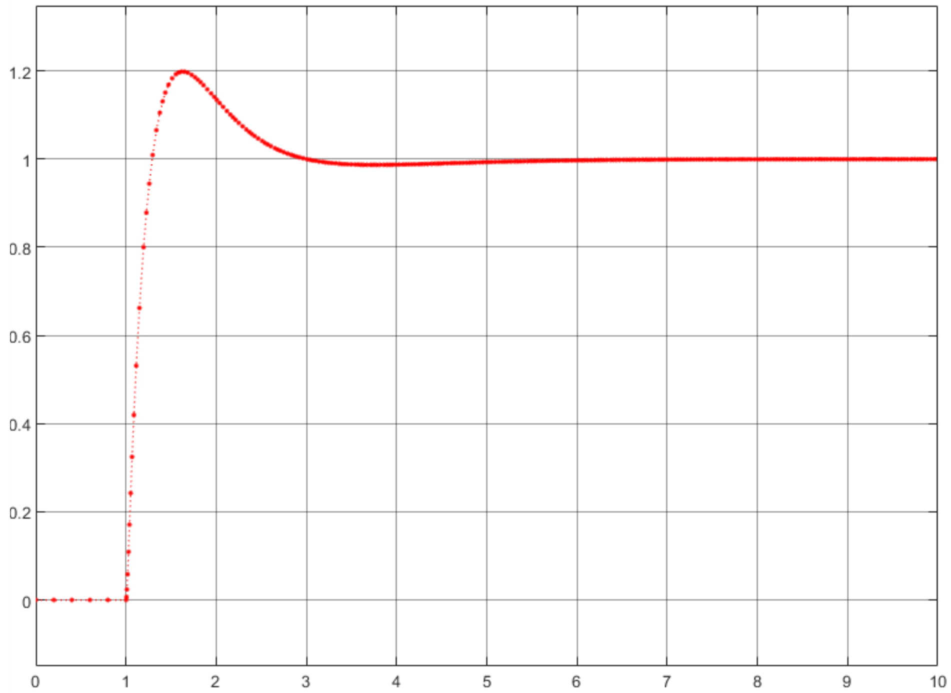

Fig.5. The time-response of my closed-loop system to the unit-step

We can see that the closed-loop system is now stable. The magnitude of the system converges to a constant 1 as time increases, so its time response is convergent for all bounded inputs.

5. Optimising the Time-Response Using Root Locus

K) Calculate the positions of the open-loop zeros (ie the zeros of $C(s)H(s)$) for the values of *l*, *m*, K_p, K_I and K_D that you have used. [1M]

First, we know $C(s) = \frac{K_p s + K_I + K_D s^2}{s}$, $H(s) = \frac{1}{6s^2 - 0.6185}$

Thus:

$$C(s)H(s) = \frac{K_p s + K_I + K_D s^2}{s} \times \frac{1}{6s^2 - 0.6185} = \frac{K_p s + K_I + K_D s^2}{6s^3 - 0.6185s}$$

Bringing in our derived PID parameters to the equation yields:

$$C(s)H(s) = \frac{66.6185s + 36 + 36s^2}{6s^3 - 0.6185s}$$

To calculate the zero point, we only need to find the point where the numerator is zero.

$$66.6185s + 36 + 36s^2 = 0$$

Solving the equation yields:

$$s_1 = -0.93 - 0.38i\,,\;\; s_2 = \; -0.93 + 0.38i$$

From this we can find the location of the open-loop zero point as follows:

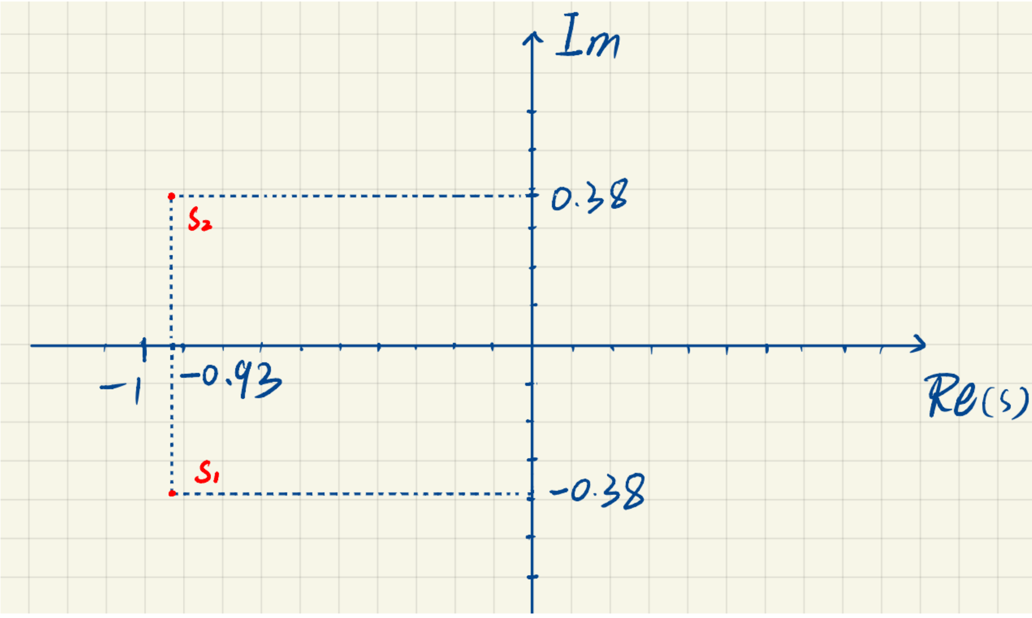

Fig.6. The position of the zeros

The positions of the open-loop zeros are therefore: s_1(-0.93, -0.38), s_2(-0.93, 0.38).

L) State the positions of the open-loop poles (ie the poles of $C(s)H(s)$) for the values of *l* and *m* that you have used. [2E]

To calculate the location of the open-loop poles we simply calculate the following equation:

$$6s^3 - 0.6185s = 0$$

Solving the equation yields:

$$s_1 \approx -0.32, \qquad s_2 = \; 0, \qquad s_3 \approx \; 0.32$$

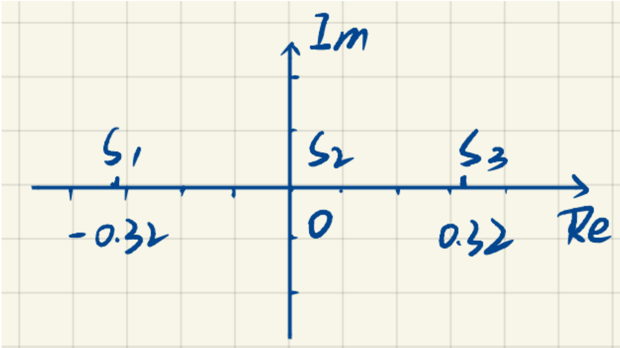

Fig.7. The position of the open-loop poles

M) Sketch the root locus for $C(s)H(s)$ and identify the points on the root locus that are such that $Re(s) = -4/t_s$. [3M]

First, we know that t_s is assumed to be 8, thus:

$$Re(s) = \frac{-4}{t_s} = \frac{-4}{8} = -0.5$$

And we know that the root branch starts at the open-loop pole and ends at the open-loop zero. And, the number of branches is equal to the greater of the number of open-loop poles and the number of open-loop zeros. For our system, the number of poles is 3, which is greater than the number of zeros 2. Therefore, the number of root branches of this system is 3.

Then, when the finite number of open-loop poles n is greater than the finite number of zeros m, there are n - m root trajectory branches converging to infinity along a set of asymptotes with intersection angle φ and intersection point σ with the following equation satisfied:

$$\varphi = \frac{(2k+1)\pi}{n-m}, k = 0,1,2, \dots n-m-1$$

$$\sigma = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{n} z_j}{n-m}$$

Substituting the values already obtained into the equation yields:

$$\sigma = \frac{-3.2 + 0 + 3.2 + 0.93 + 0.93}{3-2} = 1.86$$

$$\varphi = \frac{\pi}{3-2} = \pi$$

Since we know that the open-loop transfer function is:

$$C(s)H(s) = \frac{66.6185s + 36 + 36s^2}{6s^3 - 0.6185s} = \frac{36(1.85s + 1 + s^2)}{6s^3 - 0.6185s}$$

In MATLAB we just need to input the parameters of the numerator and denominator into the rlocus function to get the root locus.

```
num = [1 1.85 1];

den=[6 0 -0.6185 0];

sys=tf(num,den);

rlocus(sys);
```

Fig.8. The code of the root locus [1]

After running it, we can get the root locus:

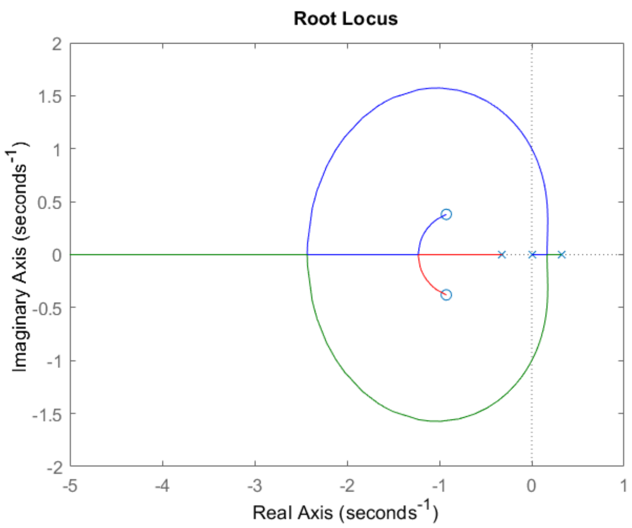

Fig.9. The root locus

We get the data for the following three points at $Re(s) = -0.5$ as requested by the script:

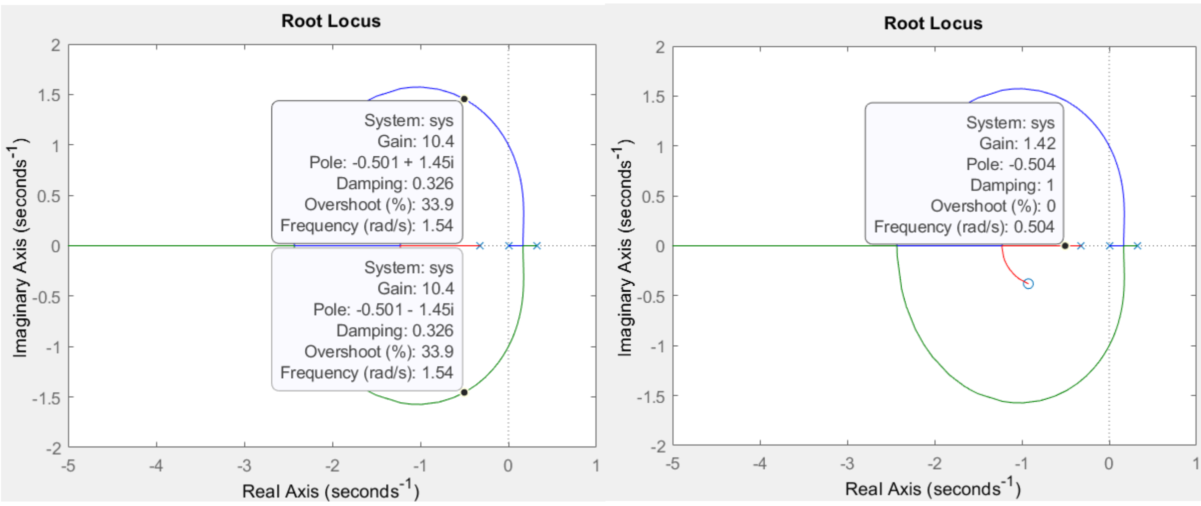

Fig.10. The data of Re(s)=-0.5 on root locus

N) Write the open-loop transfer function, $C(s)H(s)$, as a ratio of polynomials in s. [1H]

In our previous work we have obtained the open-loop transfer function:

$$C(s)H(s) = \frac{66.6185s + 36 + 36s^2}{6s^3 - 0.6185s} = \frac{36s^2 + 66.6185s + 36}{6s^3 - 0.6185s}$$

Then, we obtain a ratio polynomial in s:

$$C(s)H(s) = \frac{36s^2 + 66.6185s + 36}{6s^3 - 0.6185s}$$

O) Write $P(s) + KZ(s) = 0$ as a polynomial in s involving K. [1H]

First:

$$C(s)H(s) = \frac{Z(s)}{P(s)} = \frac{36s^2 + 66.6185s + 36}{6s^3 - 0.6185s}$$

Thus:

$$Z(s) = 36s^2 + 66.6185s + 36$$

$$P(s) = 6s^3 - 0.6185s$$

Then:

$$P(s) + KZ(s) = 6s^3 - 0.6185s + K(36s^2 + 66.6185s + 36) = 0$$

Simplify:

$$P(s) + KZ(s) = 6s^3 - 0.6185s + 36Ks^2 + 66.6185Ks + 36K = 0$$

P) Write $P(\tilde{s}) + KZ(\tilde{s}) = 0$ as a polynomial in $\tilde{s}$ involving K. [1H]

First since $\tilde{s} = s + \frac{4}{t_s}$ and $s = \tilde{s} - \frac{4}{t_s}$, we already know t_s = 8. Then we therefore obtain:

$$s = \tilde{s} - 0.5$$

$$P(\tilde{s}) + KZ(\tilde{s}) = 6(\tilde{s} - 0.5)^3 - 0.6185(\tilde{s} - 0.5) + 36K(\tilde{s} - 0.5)^2 + 66.6185K(\tilde{s} - 0.5) + 36K = 0$$

Simplify:

$$P(\tilde{s}) + KZ(\tilde{s}) = 6(\tilde{s} - 0.5)^3 - 0.6185(\tilde{s} - 0.5) + 36K(\tilde{s} - 0.5)^2 + 66.6185K(\tilde{s} - 0.5) + 36K = 0$$

Simplify:

$$P(\tilde{s}) + KZ(\tilde{s}) = 6\tilde{s}^3 + 36K\tilde{s}^2 - 9\tilde{s}^2 + 30.6185K\tilde{s} + 3.8815\tilde{s} + 11.69075K - 0.44075 = 0$$

To facilitate the computation of the following problem, we transformed it into standard form:

$$P(\tilde{s}) + KZ(\tilde{s}) = 6\tilde{s}^3 + (36K - 9)\tilde{s}^2 + (30.6185K + 3.8815)\tilde{s} + (11.69075K - 0.44075) = 0$$

Q) Complete a Routh table for $P(\tilde{s}) + KZ(\tilde{s})$. Deduce the value of K that is such that $Re(s) = -4/t_s$ [3H]

For the following equation:

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

We know the format of the Routh table as shown in the figure below:

$$\left.\begin{array}{lllllll} a_n & a_{n-2} & a_{n-4} & \cdots & a_4 & a_2 & a_0 \\ a_{n-1} & a_{n-3} & a_{n-5} & \cdots & a_3 & a_1 & a_{-1} \\ c_{n-1} & c_{n-3} & c_{n-5} & \cdots & c_3 & c_1 & \\ d_{n-1} & d_{n-3} & d_{n-5} & \cdots & d_3 & d_1 & \\ \vdots & & & & & & \\ u_{n-1} & u_{n-3} & & & & & \\ u_{n-1} & u_{n-3} & & & & & \\ v_{n-1} & & & & & & \end{array}\right\} n+1$$

Fig.11. The Routh table [2]

Where

$$c_{n-1} = \frac{-\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}}, \quad c_{n-3} = \frac{-\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}}$$

$$d_{n-1} = \frac{-\begin{vmatrix} a_{n-1} & a_{n-3} \\ c_{n-1} & c_{n-3} \end{vmatrix}}{c_{n-1}}, \quad d_{n-3} = \frac{-\begin{vmatrix} a_{n-1} & a_{n-5} \\ c_{n-1} & c_{n-5} \end{vmatrix}}{c_{n-1}}$$

Fig.12. Parameters of Routh table [2]

Then, we list the Routh table based on the resulting equation in standard form.

$$P(\tilde{s}) + KZ(\tilde{s}) = 6\tilde{s}^3 + (36K-9)\tilde{s}^2 + (30.6185K+3.8815)\tilde{s} + (11.69075K-0.44075) = 0$$

S^3	a_n	a_{n-2}
S^2	a_{n-1}	a_{n-3}
S^1	c_{n-1}	c_{n-3}
S^0	d_{n-1}	d_{n-3}

Table.1. Routh table

First, we already know:

a_n = 6, a_{n-1} = 36K-9, a_{n-2} = 30.6185K+3.8815, a_{n-3} = 11.69075K-0.44075

It is easy to see that $c_{n-3} = d_{n-3} = 0$ and d_{n-1} = 11.69075K-0.44075. Again, we calculate c_{n-1} and d_{n-1} according to the above equation.

$$c_{n-1} = -\frac{\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}}$$

Then:

$$c_{n-1} = -\frac{\begin{vmatrix} 6 & 30.6185K + 3.8815 \\ 36K - 9 & 11.69075K - 0.44075 \end{vmatrix}}{36K - 9}$$

Simplify:

$$c_{n-1} = \frac{1102.266K^2 - 205.977K - 32.289}{36K - 9}$$

Therefore, we can obtain the Routh table as follows:

S^3	**6**	**30.6185K+3.8815**
S^2	**36K-9**	**11.69075K-0.44075**
S^1	$\boldsymbol{\frac{1102.266K^2 - 205.977K - 32.289}{36K - 9}}$	**0**
S^0	**11.69075K-0.44075**	**0**

Table.2. My Routh table

We obtain by making the third row zero:

$$\frac{1102.266K^2 - 205.977K - 32.289}{36K - 9} = 0$$

The solution gives:

$$K_1 \approx 0.2919, \quad K_2 \approx -0.0723$$

Through simulation I found that -0.0723 causes instability in the closed loop system. Therefore, we take 0.2919 as the value of K.

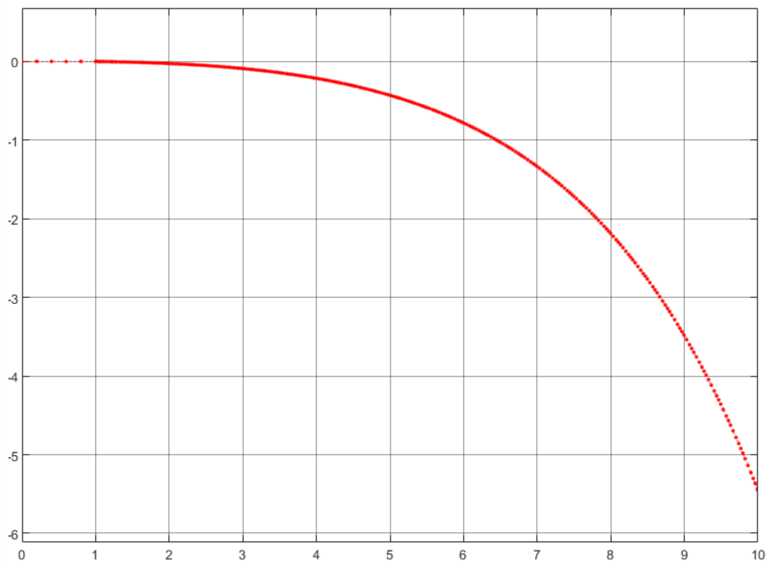

Fig.13. The time-response of the system when K = -0.0723

After substituting the value of K into the auxiliary equation, we get:

$$(36K - 9)s^2 + (11.69075K - 0.44075) = 0$$

$$1.41s^2 + 2.97 = 0$$

The solution gives:

$$s_1 \approx -1.45i, \;\; s_2 \approx 1.45i$$

This corresponds to the value we got before.

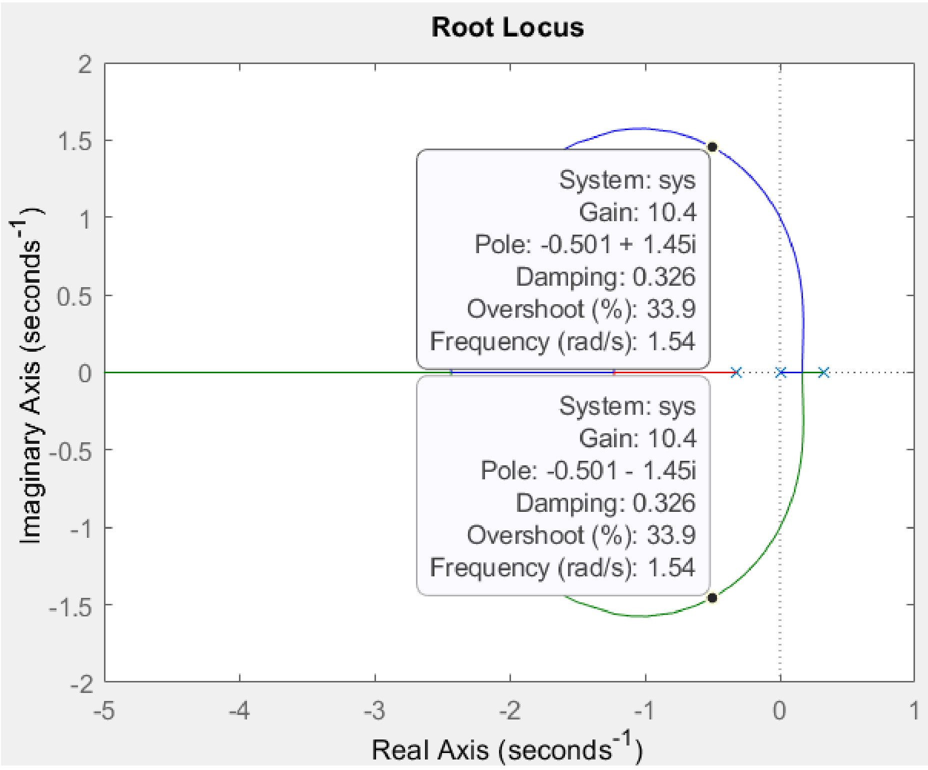

Fig.14. The root locus with Re(s)=0.5

6. Validating the Response of Optimised System

R) Insert a picture of the time-response of your improved closed-loop system to the unit-step. [2H]

After adding a gain block the resulting gain values are written to the following system block diagram:

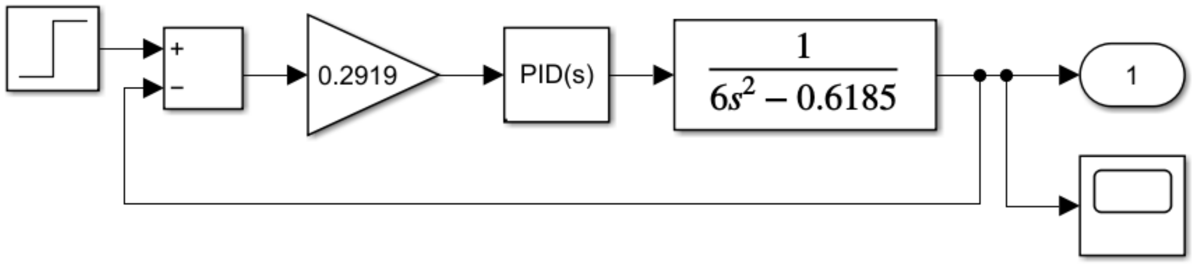

Fig.15. The improved closed-loop system block

After running the above system block, we get the following figures:

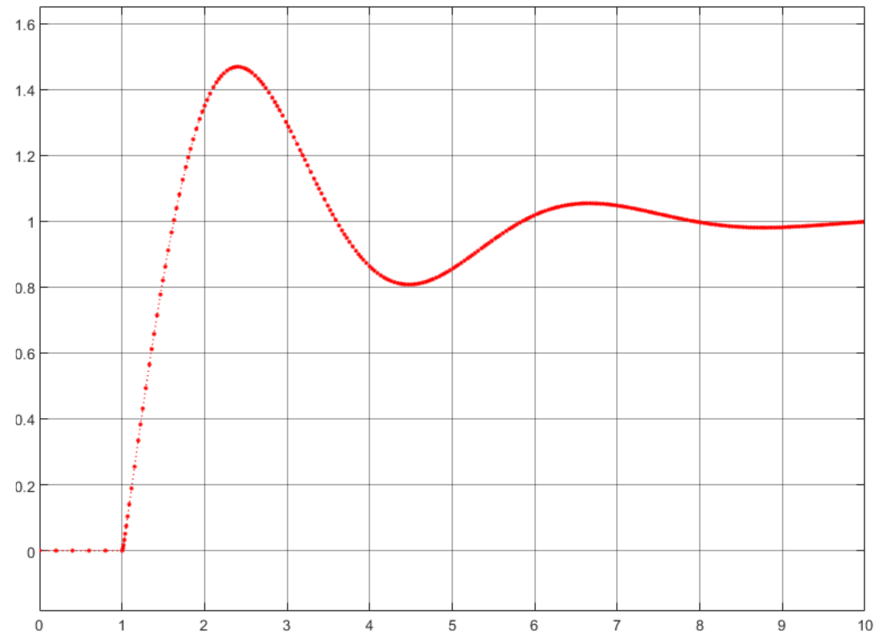

Fig.16. The time-response of my improved closed-loop system (10s)

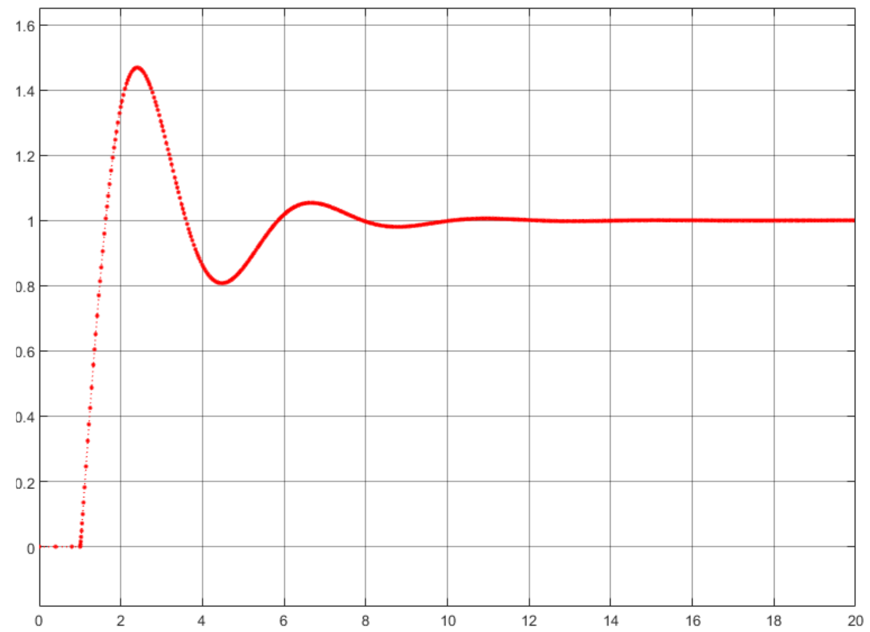

Fig.16. The time-response of my improved closed-loop system (20s)

We can find that the setup time is now about 8s, which has been achieved as an improvement. At the same time, the amplitude of the time response still converges to a constant of 1, so this improved system is still stable. Although the improved time response of my system has a few more oscillations than the original unimproved system, it does not affect the system very much.

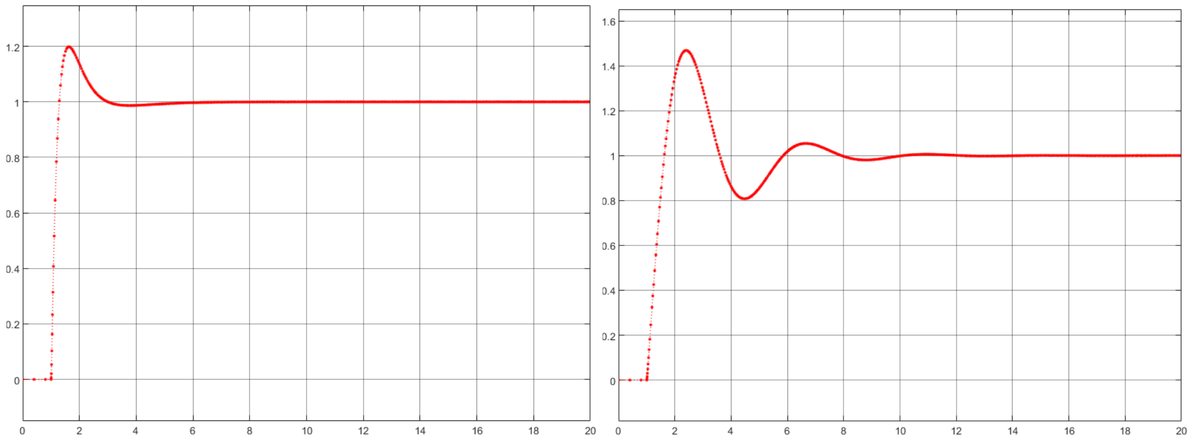

Fig.17. Time response of the improved system compared to the initial system

The above figure shows the time response of the improved system (K=0.2911) compared with the time response of the previous x system (K=1). By comparing with the improved system, we find that the set time of the time response before the improvement is about 3s and the number of oscillations is less than the set time response of the improved system. This is because when the value of K is reduced, the damping ratio of the system is also changed. When the damping ratio is reduced, the number of oscillations in the time response increases. However, the system is still a stable system.

References:

[1] D. Hua, "Root Locus Method," Lecture 9 Root Locus Method. [Online]. Available: https://www.zhihu.com/tardis/zm/art/137565062?source_id=1005. [Accessed: 13-Apr-2023].

[2] J. Sama, Routh-Hurwitz. [Online]. Available: https://zhuanlan.zhihu.com/p/105605367. [Accessed: 13-Apr-2023].