Oscillatory Motion and Chaos

In this chapter, the focus is on oscillatory phenomena, in particular the effects that occur in a real oscillatory system (considering friction and a
reasonable angle, such as the movement is not considered simple harmonic motion. The simplest one is a pendulum (mass connected by a string to a support,
able to swing freely in response to the force of gravity.

Simple harmonic motion (3.1)

Object of interest: Simple pendulum (particle of mass m connected by a massless string to a rigid support).

Analytic solution

θ is the angle that the strings makes with
the vertical. Two forces (parallel and perp to the string) are acting on the particle, gravity and tension of the string.
Net parallel force = 0
Net perpendicular force = -mgsin(θ) --> This force is equal to (newton 2 law) mass times acceleration of particle along the particle's trajectory
(arc = Fθ = md^2/dt). The displacement along this arc is s = lθ, where l is the lenght of the string. If we assume that θ is so small that sinθ = θ we
obtain d^2θ/ dt^2 = -(g/l)*θ --> Which is the central equation of simple harmonic motion.
According to this solution, the oscillations are sinusoidal and continue forever without decaying.

Numerical approach

Letting θi and wi be the numerical approximated angular displacement and velocity of the pendulum at time step i:
This is the pseudocode for euler_calculate:
For each time step i calculate w and θ at time step i+1

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  wi+1 = wi -(g/l)θi*Δt
  θi+1 = θi + wi*Δt
  ti+1 = ti + Δt
  Repeat for desired number of time steps
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The result of this will be an oscillatory motion with the amplitude that grows with time. This is contrary to the exact solution.

The problem is in the energy that increases with time, but since the problem has no source of energy or friction, the energy should remain constant.

Correct numerical approach

To find the correct answer we should use the Euler-Cromer method:
For each time step i calculate w and θ at time step i+1

$w_{i+1} = w_i - (g/l)\theta_i * \Delta t$

$\theta_{i+1} = \theta_i + w_{i+1} * \Delta t$

$t_{i+1} = t_i + \Delta t$

Repeat for desired numer of time steps

With this method, the previous values of w and θ are used to calulate the new value of w, but the new value of w is used to calculate the new value of θ

Chaos in the driven nonlinear pendulum (3.3)

Let's add disipation, an external driving force and nonlinearity all together in the same model.
* Not assuming the small angle approximation and not expanding sinθ.
* Including friction
* Adding to the model a sinusidal driving force Fd.

Model will be called the physical pendulum.

Let's construct a program to calculate the numerical solution to the equation of motion:
$d^2\theta/dt^2 = -(g/l)\sin\theta - q(d\theta/dt) + F_d * \text{sing}(\Omega_d * t)$

For each time tep i (start with i = 0) calculate w and θ at time step i + 1.

$w_{i+1} = w_i - [(g/l)\sin\theta_i - q w_i + F_d * \text{sing}(\Omega_d * t_i)]\Delta t$

$\theta_{i+1} = \theta_i + w_{i+1} * \Delta t$

If θ_{i+1} is out of range (-pi, +pi) add or subtract 2pi to keep it in the range

If we plot the behavior for several values of driving force, with all other parameters beign fixed:
* Fd = 0: motionis damped and the pensulum comes to rest after few oscilations. Oscillations have a frequenct close to the natural frequency of the undamped pendulum.
* Fd = 0.5: First, the oscillations are affected by the decay of an initial transient as in the case of no driving force. The initial displacement of the pendulum leads to a component of the motion that decays with time. After this transient is damped away, the pensulum settles into a steady oscillation in response to the driving force. Amplitude determined by a balance btw the energy added by the driving force and the energy dissipated by the damping.
* Fd = 1.2: Pensulum doesn't settle into a repeating steady-state behavior. Example of cahotic behavior.

Results: At low drive the motion is a simple oscillation, at high drive the motion is chaotic. Behavior is random and unpredictable --> then how was the program able to calculate it? Because the behavior of the pendulum is described by a differential equation, once the initial conditions are specified, the solution for θ is determined. --> The behaviour is both deterministic and unpredictable at the same time (contradiction btw the analytic theory and numerical calculations).
To make a certain accurate prediction concerning θ, instead of plotting θ as a function of time, plot angular velocity w as a function of θ (phase-space plot).