The Solar System - Chapter 4:

Kepler's Laws (4.1):

We will consider the motion of the Earth in orbit around the Sun and the force of gravity. The goal is to calulate the position of the Earth as a funciton of time.
From Newton's second law of motion: $Fg, x \ = - \frac{GMsMex}{r^3}$. The negative sign means that the force is directed toward the Sun, located at the origin of the coordinate system.
Units used: length as 1AU (one astronomical unit), the average distance between the Sun and the Earth; time in years.

Converting the equation of motion into differential equations, we find the pseudocode for the calculate function. In this instance, we are using the Euler - Cromer method, so using the previous values of position and velocity to update the velocities, while the previous values of position and the new values of velocity are used to update the position. We don't use the Euler method because the energy of the planet would grow with time, and the earth would spiral away from the sun.

In analyzing planetary motion, it is useful to visualize the movement of the planet, which can show a running display of the elapsed time beside the orbit.

Initial conditions of the problem:
Earth mass: 6.0*10^24
x- initial: 1.0
y-inital: 0
v: $2pir/1 \ = \ 2pi * AU/yr$

The path of the earth repeats itself to within the size of the points in the figure. This repeatability is a general feature of closed orbits in a 2 body solar system.
All 3 of Kepler's laws are consequences of the fact that the gravitational force follows an inverse-square law.

The inverse square law and the stability of planetary orbits (4.2):

For a two-body system, all three of Kepler's laws are consequences of the fact that gravitational force follows an inverse-square law.

We consider a two-body system in which the interaction force depends only on the separation r. The relative motion can be studied as if it were a one-body system (as one of the bodies is at rest).
The resultant equation describes a conic section:

- It is a circle if e = 0
- An ellipse if $\leq e \leq 1$
- A parabola if e = 1
- An hyperbola if e>1 with the focus at the origin

The value of e is known as the eccentricity. Any closed orbit in a two body system with an inverse square interaction must be elliptical with the location of one of the bodies at a focus, as in Kepler's first law.
The orbital period T can be obtained by dividing the area enclosed by the elliptical orbit by the constant rate at which area is swept out according to Kepler's second law.

In addition to Newton's law of universal gravitation, an inverse-square dependence is also found for the electric force between two charges (Coulomb's law).
To study this phenomenon, we study the lines of force, also known as field lines. Gravitational field lines emanate from all objects that have a mass and that radiate outward to infinity. The number of the lines is proportional to the mass of the object, and the force felt by a second object is proportional to the number of lines that intersect it. As the separation between the two objects increases, the number of intersecting field lines drops because the lines are spreading out over a larger surface area as they move away from their source.
The inverse-square law is thus a direct consequence of the field-line picture together with the geometry of the Euclidean space.

While the inverse-square law is a theoretical result, it can also be experimentally tested:
If the force law deviated from $\frac{1}{r^2}$, the orientation of elliptical orbits would change over time.
If instead the force is: $F(r) = -\frac{GMsMp}{r^{2+\delta}}$ even a tiny deviation from $\frac{1}{r^2}$would result in precession of planetary orbits, which we do not observe in Newtonian mechanics.
However, general relativity predicts small deviations. Thus, studying orbital motion provides an experimental method to verify the inverse-square law.

Precession of the perihelion of mercury (4.3)

Kepler's laws are not exact when considering multiple planets interacting through gravity. These small deviations, known as perturbations, arise due to:

- Gravitational interactions between planets.
- Slight departures from the idealized two-body problem.

For most planets, these perturbations are minor. However, for Mercury for example, whose orbit is the most elliptical, these effects become significant.

Circular orbits are insatiable: if the gravitational force deviates slightly from an inverse-square law, the orientation of the orbit will slowly rotate over time. Orbits that start nearly circular remain more stable than highly elliptical ones, but still experience long-term effects.
So how does Mercury's orbit evolve over time?

By the early 19th century, astronomers had precisely measured Mercury's orbit. They found that the perihelion (the closest point to the Sun) was slowly rotating over time (the phenomenon is called precession).

The measured precession of Mercury's perihelion was 566 arcseconds per century (1 arcsecond = 1/3600 of a degree). This means Mercury's perihelion completes a full rotation every 230,000 years.

This precision was achieved long before computers. Newtonian mechanics could explain some of this precession. In the 19th century, celestial mechanics calculated the effects of other planets (mainly Jupiter) on Mercury's orbit. The expected precession due to these effects was 523 arcseconds per century. This value was 43 arcseconds per century too small—a discrepancy that could not be explained by Newtonian physics.
Hypotheses were proposed:

- Some astronomers suggested there might be a small, unseen planet inside Mercury's orbit.
- Another idea was that a cloud of dust near the Sun exerted extra gravitational pull on Mercury.

By the late 19th century, the 43 arcseconds of precession erroor remained unsolved.

But in 1917, Einstein's general relativity provided the solution. General relativity modifies the way gravity works, treating it not as a force but as a curvature of spacetime.

The equation of motion for Mercury under general relativity predicts a small correction to the inverse-square law. So, Einstein's theory correctly predicted the missing 43 arcseconds per century in Mercury's precession

Then why general relativity cause precession? In Newtonian mechanics, planets move in perfect ellipses. In general relativity, spacetime curvature slightly distorts the orbit, causing a slow, continuous rotation of the perihelion.
This effect is small for most planets but strongest for Mercury because it is closest to the Sun, where gravitational effects are strongest. IIt has a highly elliptical orbit.

Although the relativistic correction can be derived analytically, it is also useful to simulate it numerically.

How:

- Modify planetary motion equations to include the relativistic correction term.
- Simulate Mercury's orbit over a long period to measure the cumulative effect on perihelion precession.
- Since the actual precession rate is small, increase the parameter alpha to make it more visible in simulations.
- Once a scaling factor CCC is determined, apply it to Mercury's actual parameters to calculate the true precession rate.

Set Parameters:

- Semi-major axis: a=0.39a = 0.39a=0.39 AU
- Eccentricity: e=0.206e = 0.206e=0.206
- Initial velocity v1v_1v1:
    - Can be estimated by trial and error until the orbit's eccentricity matches Mercury's real orbit.
    - Can be derived using energy and angular momentum conservation.
- Apply conservation of energy

  This ensures that Mercury's simulated orbit matches reality.

The Three-Body Problem and the Effect of Jupiter on Earth (4.4)

This section analyzes the interaction of three celestial bodies: the Sun, Earth, and Jupiter. While the two-body problem (Sun and a single planet) can be solved using Kepler's laws, adding a third body complicates the system, making the equations of motion complex. For this reason exact solutions are rare, and approximate methods are used to study such systems.

Jupiter, being the most massive planet in the solar system, affects the Earth's orbit. The Earth's orbit is usually stable without Jupiter, but the gravitational interaction between Earth and Jupiter must be considered to understand the precise effect Jupiter has.

The gravitational force between Earth and Jupiter is computed using the inverse-square law, where the gravitational force is proportional to the product of their masses and inversely proportional to the square of the distance between them. Sun's gravitational force on Earth is also included in the calculations. To simplify the model, assume the orbits of both planets are on the same plane.

The gravitational force between Jupiter and Earth is given by:

$Fej\ =\ \frac{GMjMe}{r^2 ej}$. G is the gravitational constant, MJ is the mass of Jupiter, ME is the mass of Earth, and rej is the distance between Earth and Jupiter.

This force is added to the force from the Sun on Earth to calculate the total gravitational effect on Earth's motion. The motion of both planets is then computed by updating their positions and velocities step-by-step using numerical methods.

The Euler-Cromer method is used (updates velocities before positions) in the program

The program calculates the distances between Earth, Jupiter, and the Sun at each time step. The velocities are then updated based on the forces, and the new positions of both planets are computed.