

Problem Set 5: Branching Processes

Due on Friday March 10th, 2025 by 11:59 pm on CrowdMark

Note that to receive full credit, you must show your work (show your calculations and explain your reasoning).

1 A Royal Family

A careful study of the reproductive behavior of a royal family revealed that each member of the family has a probability of

- $\frac{1}{3}$ of producing no children,
- $\frac{2}{3}$ of producing two children,
- $\frac{1}{9}$ of producing one child,
- $\frac{2}{9}$ of producing three children.

Suppose we can model this as the branching process, and we look at the royal family that starts with the king.

- (a) Is this a subcritical, critical, or supercritical process?
- (b) Find the probability generating function (PGF) for the second generation.
- (c) What is the probability that the king has exactly 2 grandchildren?
- (d) What is the probability that the royal family goes extinct in the second generation?
- (e) What is the probability of eventual extinction of the royal family?
- (f) After n generations, what is the probability that there is the maximum number of royal blooded individuals?

2 Another Offspring

Consider the offspring distribution X defined by for $k \geq 0$:

$$\mathbb{P}(X = k) = 2^{-k-1}$$

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- (a) Find the extinction probability.
- (b) Let G be its probability generating function, and let G_n be the composition of G with itself n times. Then, for $n \geq 1$, we have:

$$G_n(s) = \frac{a_n - b_n s}{c_n - d_n s}$$

Where a_n , b_n , c_n , and d_n are constants. What are these constants?

3 Raccoon Family

A wholesome family of raccoons always have identical quadruplets (four offspring). Each of the 4 little raccoons has a $1/3$ chance of becoming a doctor, a lawyer or a scientist, independently of its 3 siblings. A doctor raccoon will reproduce further with probability $2/3$, a lawyer with probability $1/2$ and a scientist with probability $1/4$, again, independently of everything else. If it reproduces at all, a raccoon reproduces only once in its life, and then leaves the raccoon scene. (For the purposes of this problem assume that raccoons reproduce asexually). Let us call the raccoons who have offspring fertile.

- (a) What is the distribution of the number of fertile offspring? Write down its generating function.
- (b) What is the generating function for the number of great-grandchildren a raccoon will have? What is its expectation?
- (c) Let the raccoon population be modeled by a branching process, and let's suppose that it starts from exactly one individual at time 0. Is it certain that the population will go extinct sooner or later?

4 All Who Every Lived

For a branching process $(Z_n : n \geq 0)$, denote S as the total number of individuals that ever lived, or rather:

$$S = \sum_{n=0}^{\infty} Z_n = 1 + \sum_{n=1}^{\infty} Z_n$$

- (a) Assume that the offspring distribution has the generating function given by:

$$G(s) = p + qs.$$

Find the generating function G_S , in this case.

- (b) Assume instead the offspring distribution has the generating function given by:

$$G(s) = \frac{p}{1 - qs}.$$

Find G_S , in this case.

- (c) Find the general expression for the expected number of individuals who ever lived in the cases described in parts (a) and (b).