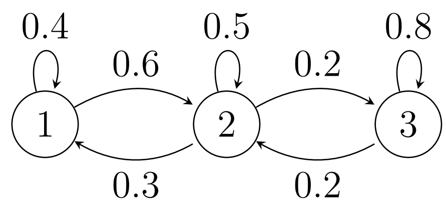
Problem Set 2: Stationary Distributions

Due on Friday January 31st, 2025 by 11:59 pm on CrowdMark

Note that to receive full credit, you must show your work (show your calculations and explain your reasoning).

1 Birth-Death Markov Chain

Suppose we have a “birth-death” Markov chain below with three states. We define the transition that results in a higher index as a birth, and a transition that results in a lower index as a death. For example, a transition from state 1 to state 2 is a birth, while the transition from state 3 to state 2 is a death. Any transition where the states do not change (for example a transition from any state to itself) is neither a birth nor a death.



Calculate each of the following quantities, assuming that when we start observing the chain, it is already in steady-state.

(a) For each state i, the probability that the current state is i.

(b) The probability that the first transition we observe is a birth.

(c) The probability that the first change of state we observe is a birth.

(d) The conditional probability that the process was in state 2 before the first transition that we observe, given that this transition was a birth.

(e) The conditional probability that the process was in state 2 before the first change of state that we observe, given that this change of state was a birth.

(f) The conditional probability that the first observed transition is a birth given that it resulted in a change of state.

(g) The conditional probability that the first observed transition leads to state 2, given that it resulted in a change of state.

Note: *A transition doesn't necessarily result in a change in state.*

2 Morning Run

Every morning I go outside for my morning run. When I leave the house for my run, I am equally likely to go out either the front door or the back door; and similarly, when I return, I am equally likely enter my house through the front door or the back door. I also own five pairs of running shoes which I take off immediately after my run and leave them at whichever door I happen to enter through. If there are no shoes at the door from which I leave to go running, I run barefooted.

(a) Draw a Markov chain of this scenario, indicating each state and the transition probabilities.

(b) Determine the proportion of time in the long run that I spend running barefooted.

3 Bank Queue

Suppose that someone arrives at a bank at time n with probability α. They wait in a queue (if any) which is served by one bank clerk in a first-come-first-serve fashion. When at the front of the queue, the person requires a service which is distributed like a random variable $\mathcal{S}$ with values in $\mathbb{N} : \mathbb{P}(\mathcal{S} = k) = p_k$ for $k = 1, 2, \ldots$ Different people require services which are independent random variables. Consider the quantity W_n which is the total waiting time at time n. In other words, if I observe the queue at time n, then W_n represents the time I have to wait in line until I finish my service. It can be shown (you don't have to prove this) that W_n obeys the recursion:

$$W_{n+1} = (W_n + \mathcal{S}_n \xi - 1)^+ ,$$

where the $\mathcal{S}_n$ are i.i.d random variables distributed like $\mathcal{S}$, independent of the ξ_n. The ξ_n are also i.i.d with $\mathbb{P}(\xi_n = 1) = \alpha$, and $\mathbb{P}(\xi_n = 0) = 1 - \alpha$. Thus, $\xi_n = 1$ indicates that there is an arrival at time n.

(a) Show that $(W_n : n \geq 0)$ is a Markov chain and compute its transition probabilities $p(k, \ell)$, for $k, \ell = 0, 1, 2 \ldots$ in terms of parameters α and p_k.

(b) Suppose that $p_1 = 1 - \beta$ and $p_2 = \beta$. Find conditions on α and β so that the stationary distribution exists.

(c) Give an interpretation of you condition in part **(b)** in context to this queuing scenario.

(d) Find the stationary distribution.

(e) Find the average waiting time in steady-state.

(f) If 4 customers arrive every 5 units of time on average, what is the maximum value of β so that a stationary distribution exists?