Problem Set 6: Poisson Processes

Due on Friday March 21st, 2025 by 11:59 pm on CrowdMark

Note that to receive full credit, you must show your work (show your calculations and explain your reasoning).

1 Trains

A train bridge is constructed across a wide rive. Trains arrive at the bridge according to a Poisson process with average rate of 3 trains per day.

(a) If a train arrives on day 0, find the probability that there will be no trains on days 1, 2, and 3.

(b) Find the probability that the next train to arrive after the first train on day 0, takes more than 3 days to arrive.

(c) Find the probability that no trains arrive in the first 2 days, but 4 trains arrive on the 4th day.

(d) Find the probability that it takes more than 2 days for the 5th train to arrive at the bridge.

2 Casino Bell

Suppose you enter the Poisson casino and play their most popular game: "Press the button". The game is incredibly simple:

- There is a bell that rings at instantaneous times according to a Poisson process with rate $\lambda > 0$ rings per hour.
- The game lasts exactly one hour, and you are presented with a button that you press only when you hear the bell ring. You are allowed to press the button only once during the hour-long game.
- You win the game if you press the button before the one-hour time limit is up and, you press the button on the bell's *last ring of the game*. Meaning, the bell should no longer ring for the rest of the game after you press the button.
- If before the one-hour mark the bell rings even once after you have pressed the button, you lose the game.

Suppose you develop the following strategy: You wait until time s, ($0 \leq s \leq 1$ hours), doing nothing before time s. After time s, you press the button the first time you hear the bell ring.

(a) Find the probability (in terms of s) that you win this game.

(b) Find the optimal value of s and the corresponding winning probability.

3 Go Fish

A fisherman catches fish according to a Poisson process with rate $\lambda = 0.6$ per hour. The fisherman will keep fishing for two hours. If he has caught at least one fish, he quits. Otherwise, he continues until he catches at least one fish.

(a) Find the probability that he stays for more than two hours.

(b) Find the probability that the total time he spends fishing is between two and five hours.

(c) Find the probability that he catches at least two fish.

(d) Find the expected number of fish that he catches.

(e) Find the expected total fishing time, given that he has been fishing for four hours.

Suppose the lake he is fishing from is populated by three different coloured fish: red fish, blue fish, and green fish, that he can catch with probability 1/2, 2/5, and 1/10 respectively.

(f) Given that he caught exactly two fish in the first two hours, what is the probability that both are red?

(g) Suppose he decides to stop fishing only if he catches a green fish, and gives up after two hours if not. What is the probability that he fishes for the full two hours?

(h) What is the expected time to catch both a red fish and a green fish, assuming that he fishes indefinitely until he catches both colors?

4 What to Watch

Bob and Joe are two friends hanging out and want to decide on a movie to watch together. Bob will look through Netflix for a movie, and Joe will look through Amazon Prime Video. They find potential movie candidates following a Poisson process.

Suppose Bob finds potential movies on Netflix at an average rate of 2 movies per minute, while Joe finds potential movies on Amazon Prime video with an average rate of 4 movies per minute. Bob is ready to watch a movie when he encounters the third potential movie on Netflix, while Joe is ready to watch a movie when he encounters the second potential movie on Amazon Prime Video. The two friends watch a movie when either of them is ready. Assume that process of finding potential movies on Netflix and Amazon Prime Video are independent of each other.

(a) Find the probability that Joe finds exactly one movie on Amazon Prime Video before Bob finds any movie on Netflix.

(b) Find the expected number of potential movies Joe finds on Amazon Prime Video before Bob finds any movie on Netflix.

(c) What is the probability that the friends watch a movie on Netflix?

(d) What is the expected time needed to find a movie they'll both watch?