# Problem Set 3: Hitting Times

Due on Friday February 14th, 2025 by 11:59 pm on CrowdMark

*Note that to receive full credit, you must show your work (show your calculations and explain your reasoning).*

## 1 Add-Drop Dilemma

Bob is currently a student enrolled in Math 447. However, during the add-drop period, he is manically changing his course schedule every day. On each day that he is a Math 447 student, he has a probability of 1/2 of being a Math 447 the next day. Otherwise, he has an equally likely chance of dropping Math 447 and becoming a Math 547 student, a Math 463 student, a Math 324 student, or a Math 319 student the next day. On any day that he is a Math 463 student, he has a probability of 1/4 of switching to Math 324, a probability of 3/8 of switching to Math 447, and a probability of 3/8 of switching to Math 547 the next day. On any day he is a Math 547 student, he has a probability of 1/2 of switching to Math 319, a probability of 3/8 of switching to Math 447, and a probability of 1/8 of switching to Math 463 the next day.

For each question, assume that Bob will be a student forever, and he is able to change his course everyday while he is a student. Also, for parts **(a)**-**(f)**, assume that if Bob switches to Math 324 or Math 319, he will stay there and will not change his course again.

**(a)** What is the probability that he will eventually leave any 400-level course?

**(b)** What is the probability that he will eventually be in Math 319?

**(c)** What is the expected number of days until he leaves any 400-level course?

**(d)** Every time he switches into Math 447 from Math 547 or Math 463, he buys himself an ice cream cone at Frostbite. However, he can only afford so much ice cream, so after he's eaten 2 ice cream cones, he stops buying himself ice cream. What is the expected number of ice cream cones he buys himself before he leaves any 400-level course?

**(e)** His friend Joe started out just like Bob. He is now in Math 319. You don't know how long it took him to switch. What is the expected number of days it took him to switch to Math 319?

**(f)** Bob decides that Math 319 is not in his future. Accordingly, when he is a Math 447 student, he stays in Math 447 for another day with probability 1/2, and otherwise he has an equally likely chance of switching into any of the other options. When he is a Math 547 student, his probability of entering Math 447 or Math 463 are in the same proportion as before. What is the expected number of days until he is in Math 324?

**(g)** For this part only, assume that when Bob is in Math 324, he is equally likely to stay in Math 324 or switch to Math 319. Similarly, if he is Math 319, he is equally likely to stay in Math 319, or switch to Math 324. Calculate the probability that Bob being in each course on any given day far into the future.

**(h)** Suppose that if he is in Math 324 or Math 319, he has probability of 1/8 of returning to Math 447, and otherwise he remains in his current course. What is the expected number of days until he is Math 447 again? (Note that we know today he is Math 447, so if tomorrow he is still in Math 447, then the number of days until he is Math 447 again is 1).

# 2 4-Sided Dice Game

Suppose you have four fair tetrahedral dice whose four sides are numbered from 1 through 4.

You play a game in which you roll them all and divide them into two groups: those whose values are unique, and those which are duplicates. For example, if you roll a 1, 2, 2 and 4, then the 1 and 4 will go into the "unique" group, while the 2's will go into the "duplicate" group.

Next, you re-roll all the dice in the duplicate pool and sort all the dice again. Continuing the previous example, that would mean you re-roll the 2's. If the result happens to be 1 and 3, then the "unique" group will now consist of 3 and 4, while the "duplicate" group will have two 1's.

You continue re-rolling the duplicate pool and sorting all the dice until all the dice are members of the same group. If all four dice are in the "unique" group, you win. If all four are in the "duplicate" group, you lose.

What is your probability of winning the game?

# 3 A Lazy Chain

Suppose that $P$ is a transition probability matrix of an ergodic chain with unique stationary distribution $\pi$. Let $p \in (0,1)$ and $\tilde{P}$ be the lazy chain with transition probability matrix $\tilde{P} = pP + (1-p)\mathrm{Id}$. Suppose $a$ is a recurrent state. Let $R_a$ and $\tilde{R}_a$ be the first return times of the chain to state $a$. Let $S_a$ and $\tilde{S}_a$ be the first time which is 2 or *larger* that the chain returns to state $a$.

**(a)** Compute $\mathbb{E}(R_a)$ and $\mathbb{E}(\tilde{R}_a)$ in terms of $\pi(a)$ and $p$.

**(b)** Compute $\mathbb{E}(S_a)$ and $\mathbb{E}(\tilde{S}_a)$ in terms of $\pi(a)$, $p$, and $P_{aa}$.

**(c)** If it takes, in expectation 4 steps to get from state $a$ to state $b$ in the chain $P$, how long should it take to get from $a$ to $b$ in the lazy chain $\tilde{P}$? *Explain your answer. You may prove it, but this is not required.*