Can Your Team Self-Organize?

1. In a recent team building exercise at work, a group of people (including myself) was asked to quantify themselves in various ways. For example: "What outdoor temperature do you prefer?" No one reveals their answer at first. Instead, each person places a card with their name on an unmarked line, one at a time. In this example, folks who prefer higher temperatures would place their names farther right; folks who prefer lower temperatures would place their names farther left. However, the line is unmarked and doesn't have any units.

Once all the names are placed on the line, their values are revealed. For example, a group of six might have generated the following numbers, in order from left to right on the line: 60, 67, 65, 74, 70, 80.

The team's score is the length of the "longest increasing subsequence." In other words, it's the maximum number of elements in the list you can keep such that they form an increasing subsequence. In the example above, you can remove the 67 and the 74 to get the following increasing subsequence with four numbers: 60, 65, 70, 80. There are a few other ways to get an increasing subsequence of length four, but there's no way to get a sequence of length five or more, so the team's score is four.

Suppose a total of four people are participating in this team building exercise. They all write down different numbers, and then independently place their names at random positions on the line.

On average, what do you expect the team's score to be?

Solution:

To simplify, let's suppose that each of the four group members are assigned numbers $1, 2, 3,$ and 4, and they place their numbers randomly and independently on the number line. There are $4! = 24$ total ways they can do this.

Of course, there is only one way for the team to get a score of four, and that's if the sequence is 1234. Similarly, there's only one way for the team to get a score of 1, which is the sequence 4321.

There might be a clever way to determine the number of ways to achieve a score of 3, but I'm just going to list them all out:

$$1243 \to (1, 2, 4) \qquad 1324 \to (1, 2, 4) \text{ or } (1, 3, 4) \qquad 1342 \to (1, 3, 4)$$
$$1423 \to (1, 2, 3) \qquad 2134 \to (1, 3, 4) \text{ or } (2, 3, 4) \qquad 2314 \to (2, 3, 4)$$
$$2341 \to (2, 3, 4) \qquad 3124 \to (1, 2, 4) \qquad 4123 \to (1, 2, 3)$$

So there are 9 ways of obtaining a score of 3. This means that there are $24 - 1 - 1 - 9 = 13$ ways of obtaining a score of 2.

And so, our expected score is:

$$\mathbb{E}[\text{Score}] = \frac{1(1) + 2(13) + 3(9) + 4(1)}{24} = \boxed{2.42}$$