How Low (or High) Can You Go?

1. You're playing a game of "high-low," which proceeds as follows:

- First, you are presented with a random number, x_1, which is between 0 and 1.
- A new number, x_2, is about to be randomly selected between 0 and 1, independent of the first number. But before it's selected, you must guess how x_2 will compare to x_1. If you think x_2 will be greater than x_1 you guess "high." If you think x_2 will be less than x_1, you guess "low." If you guess correctly, you earn a point and advance to the next round. Otherwise, the game is over.
- If you correctly guessed how x_2 compared to x_1 then another random number, x_3, will be selected between 0 and 1. This time, you must compare x_3 to x_2, guessing whether it will be "high" or "low." If you guess correctly, you earn a point and advance to the next round. Otherwise, the game is over.

You continue playing as many rounds as you can, as long as you keep guessing correctly.

You quickly realize that the best strategy is to guess "high" whenever the previous number is less than 0.5, and "low" whenever the previous number is greater than 0.5.

With this strategy, what is the probability you will earn at least two points? That is, what are your chances of correctly comparing x_2 to x_1 and then also correctly comparing x_3 to x_2?

Solution:

We have that $X_1, X_2, X_3 \overset{\text{iid}}{\sim} \text{Unif}(0,1)$ be random variables, and x_1, x_2, x_3 be specific values of these random variables. Adopting the strategy posed in the question, we say we pass round 1 when $x_1 < \frac{1}{2} \cap x_2 > x_1$ or $x_1 > \frac{1}{2} \cap x_2 < x_1$, and similarly we pass round 2 when $x_2 < \frac{1}{2} \cap x_3 > x_2$ or $x_2 > \frac{1}{2} \cap x_3 < x_2$.

We condition on succeeding in Round 2 by the event that we succeed in Round 1, and we therefore consider four disjoint events demonstrated in the below integral:

$$\mathbb{P}(2 \text{ rounds won}) = \underbrace{\iint \mathbb{P}(X_3 > x_2)}_{\text{R1: High R2: High}} + \underbrace{\iint \mathbb{P}(X_3 < x_2)}_{\text{R1: High R2: Low}} + \underbrace{\iint \mathbb{P}(X_3 > x_2)}_{\text{R1: Low R2: High}} + \underbrace{\iint \mathbb{P}(X_3 < x_2)}_{\text{R1: Low R2: Low}}$$

Now we need to determine the bounds defining each integral:

- **R1: High R2: High :** We must have $0 < x_1 < \frac{1}{2}$ since we successfully guess high in R1, and $x_1 < x_2 < \frac{1}{2}$ since we require x_2 be greater than x_1. Since we guess high again on the second round, we also want $x_3 > x_2$, therefore x_2 must be no greater than $\frac{1}{2}$.
- **R1: High R2: Low :** We must have $0 < x_1 < \frac{1}{2}$ for the same reason as in the above case. If we successfully win with a guess of high in round 1, we must have $\frac{1}{2} < x_2 < 1$. If we guess low in the second round, we need $x_3 < x_2$.
- **R1: Low R2: High :** We require $\frac{1}{2} < x_1 < 1$ if guess low in round 1. If successful, we would have $0 < x_2 < \frac{1}{2}$, and thus for $x_3 > x_2$, we would need to guess high to win again.
- **R1: Low R2: Low :** We require $\frac{1}{2} < x_1 < 1$ for the same reason as the previous case. If successful after the first round, this would imply $\frac{1}{2} < x_2 < x_1$. For a guess of low in round 2, we would need $x_3 < x_2$.

Now we need to determine what are $\mathbb{P}(X_3 > x_2)$ and $\mathbb{P}(X_3 < x_2)$. Since X_3 is a uniform random variable:

$$\mathbb{P}(X_3 > x_2) = \int_{x_2}^{1} 1 dx_3 = 1 - x_2$$
$$\mathbb{P}(X_3 < x_2) = \int_{0}^{x_2} 1 dx_3 = x_2$$

And so finally we have:

$$\begin{aligned}
\mathbb{P}(2 \text{ rounds won}) &= \int_0^{\frac{1}{2}} \int_{x_1}^{\frac{1}{2}} (1 - x_2) dx_2 dx_1 + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{0} x_2 dx_2 dx_1 \\
&+ \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} (1 - x_2) dx_2 dx_1 + \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{x_1} x_2 dx_2 dx_1 \\
&= \frac{1}{12} + \frac{3}{16} + \frac{3}{16} + \frac{1}{12} \\
&= \boxed{\frac{13}{24}}
\end{aligned}$$

Which is roughly $\boxed{0.54}$ chance of winning.