Homework 1

Problem: We will use Taylor series expansions in a lot of the work we do in Math 4610. Expand the following functions about the given center x_0. Find the radius of convergence of each series.

a. $f(x) = \sin(2x)$ and $x_0 = 0$
b. $f(x) = \ln(2x)$ and $x_0 = 1$
c. $f(x) = e^{2x}$ and $x_0 = 1$
d. $f(x) = 3x^2 - 2x + 5$ and $x_0 = 0$
e. $f(x) = 3x^2 - 2x + 5$ and $x_0 = 1$
f. $f(x) = (3x^2 - 2x + 5)^{-1}$ and $x_0 = 1$
g. $f(x) = \cosh(x - 3)$ and $x_0 = 1$
h. $f(x)$ and $x_0 = a$
i. $f(a)$ and $x_0 = x$
j. $f(a + h)$ and $x_0 = a$

Solution: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots$

a. $\sin(2x) = \sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots$
$L = \lim_{n\to\infty} |[(-1)^{n+1}\frac{(2x)^{2(n+1)+1}}{(2(n+1)+1)!}][\frac{1}{(-1)^n}\frac{(2n+1)!}{(2x)^{2n+1}}]| = \lim_{n\to\infty} |(-1)[\frac{(2x)^{2n+3}}{(2n+3)!}][\frac{(2n+1)!}{(2x)^{2n+1}}]| = \lim_{n\to\infty}$ $|\frac{(2x)^2}{(2n+3)(2n+2)}| = |4x^2| \lim_{n\to\infty} \frac{1}{4n^2+10n+6}$, where the series converges if $L < 1$. This occurs regardless of x, so the radius of convergence of the series is $-\infty < x < \infty$.

b. $\ln(2x) = \ln(2) + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n} = \ln(2) + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$
$L = \lim_{n\to\infty} |[(-1)^{(n+1)+1}\frac{(x-1)^{n+1}}{n+1}][\frac{1}{(-1)^{n+1}}\frac{n}{(x-1)^n}]| = \lim_{n\to\infty} |(-1)\frac{n(x-1)}{n+1}| = |x-1| \lim_{n\to\infty}\frac{n}{n+1}$, where the series converges if $L < 1$. This occurs when $-1 < x-1 < 1$, so the radius of convergence of the series is $0 < x \leq 2$.

c. $e^{2x} = \sum_{n=0}^{\infty} 2^n e^2 \frac{(x-1)^n}{n!} = e^2 + 2e^2(x-1) + \frac{2^2e^2}{2!}(x-1)^2 + \frac{2^3e^2}{3!}(x-1)^3 + \cdots$
$L = \lim_{n\to\infty} |[2^{n+1}e^2\frac{(x-1)^{n+1}}{(n+1)!}][\frac{1}{2^n e^2}\frac{n!}{(x-1)^n}]| = \lim_{n\to\infty} |2\frac{(x-1)}{(n+1)}| = |2x-2| \lim_{n\to\infty}\frac{1}{n+1}$, where the series converges if $L < 1$. This occurs regardless of x, so the radius of convergence of the series is $-\infty < x < \infty$.

d. $f(x) = [3(0)^2 - 2(0) + 5] + [6(0) - 2](x) + [\frac{6}{2!}](x)^2 = 5 - 2x + 3x^2$. $f(x)$ has a finite number of terms, so the radius of convergence of the series is $-\infty < x < \infty$.

e. $f(x) = [3(1)^2 - 2(1) + 5] + [6(1) - 2](x-1) + [\frac{6}{2!}](x-1)^2 = 6 + 4(x-1) + 3(x^2 - 2x + 1) = 6 + 4x - 4 + 3x^2 - 6x + 3 = 5 - 2x + 3x^2$. $f(x)$ has a finite number of terms, so the radius of convergence of the series is $-\infty < x < \infty$.

f. $f(x) = \frac{1}{3(1)^2-2(1)+5} + \frac{-6(1)+2}{[3(1)^2-2(1)+5]^2}(x-1) + \cdots = \frac{1}{6} + \frac{-4}{36}(x-1) + \cdots = \frac{1}{6} - \frac{x-1}{9} + \cdots$

g. $\cosh(x-3) = \sum_{n=0}^{\infty}\cosh(2)\frac{(x-1)^{2n}}{(2n)!} - \sum_{n=0}^{\infty}\sinh(2)\frac{(x-1)^{2n+1}}{(2n+1)!} = \cosh(2) - \sinh(2)(x-1) + \frac{\cosh(2)}{2!}(x-1)^2 - \frac{\sinh(2)}{3!}(x-1)^3 + \cdots$
$L = \lim_{n\to\infty} |[\cosh(2)\frac{(x-1)^{2(n+1)}}{(2(n+1))!}][\frac{1}{\cosh(2)}\frac{(2n)!}{(x-1)^{2n}}]| = \lim_{n\to\infty} |[\frac{(x-1)^{2n+2}}{(2n+2)!}][\frac{(2n)!}{(x-1)^{2n}}]| = \lim_{n\to\infty}$ $|\frac{(x-1)^2}{(2n+2)(2n+1)}| = |x^2 - 2x + 1| \lim_{n\to\infty}\frac{1}{4n^2+5n+2}$, where the series converges if $L < 1$. This occurs regardless of x, so the first portion of the series is convergent.
$L = \lim_{n\to\infty} |[\sinh(2)\frac{(x-1)^{2(n+1)+1}}{(2(n+1)+1)!}][\frac{1}{\sinh(2)}\frac{(2n+1)!}{(x-1)^{2n+1}}]| = \lim_{n\to\infty} |[\frac{(x-1)^{2n+3}}{(2n+3)!}][\frac{(2n+1)!}{(x-1)^{2n+1}}]| = \lim_{n\to\infty}$ $|\frac{(x-1)^2}{(2n+3)(2n+2)}| = |x^2 - 2x + 1| \lim_{n\to\infty}\frac{1}{4n^2+10n+6}$, where the series converges if $L < 1$. This occurs regardless of x, so the second portion of the series is convergence.
Since the sum of two convergent series is also convergent, then the radius of convergence of the entire series is $-\infty < x < \infty$.

h. $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$

i. $f(a) = f(x) + f'(x)(a-x) + \frac{f''(x)}{2!}(a-x)^2 + \frac{f'''(x)}{3!}(a-x)^3 + \cdots$

j. $f(a+h) = f(a) + f'(a)(a+h-a) + \frac{f''(a)}{2!}(a+h-a) + \frac{f'''(a)}{3!}(a+h-a) + \cdots = f(a) + hf'(a) + \frac{hf''(a)}{2!} + \frac{hf'''(a)}{3!} + \cdots$

Problem: Compute the following antiderivatives.

a. $\int x\sin(2x)dx$ (by parts)

b. $\int xe^{x^2}dx$ (by substitution)

c. $\int xe^{x}dx$ (by parts)

d. $\int e^{x^2}dx$ (expand integrand in a Taylor series)

e. $\int x\sqrt{1+x}dx$

f. $\int \sec(\theta)d\theta$

g. $\int \sec^2(\theta)d\theta$

h. $\int \text{ sech}^2(\theta)d\theta$

i. $\int \frac{x^2+2}{7-x^2}dx$

j. $\int \frac{1}{ap-bp^2}dp$

Solution: a. $u=x, \quad dv=\sin(2x)dx, \quad du=dx, \quad v=-\frac{\cos(2x)}{2} \quad \rightarrow \quad \int udv = uv - \int vdu$
$\int x\sin(2x)dx = -\frac{x\cos(2x)}{2} + \int \frac{\cos(2x)}{2}dx = -\frac{x\cos(2x)}{2} + \frac{\sin(2x)}{4} + C = \frac{\sin(2x)-2x\cos(2x)}{4} + C$

b. $u=e^{x^2}, \quad du = 2xe^{x^2}dx \quad \rightarrow \quad \int xe^{x^2}dx = \int \frac{1}{2}du = \frac{u}{2} + C = \frac{e^{x^2}}{2} + C$

c. $u=x, \quad dv=e^xdx, \quad du=dx, \quad v=e^x \quad \rightarrow \quad \int udv = uv - \int vdu$
$\int xe^xdx = xe^x - \int e^xdx = xe^x - e^x + C = e^x(x-1) + C$

d. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \rightarrow \quad e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$
$\int e^{x^2}dx = \int(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots)dx = x + \frac{x^3}{3} + \frac{x^5}{5(2!)} + \frac{x^7}{7(3!)} + \cdots + C$

e. $u=1+x, \quad du=dx \quad \rightarrow \quad \int x\sqrt{1+x}dx = \int(u-1)\sqrt{u}du = \int u^{3/2} - \sqrt{u}du = \frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3} + C$
$\int x\sqrt{1+x}dx = \frac{2(1+x)^{5/2}}{5} - \frac{2(1+x)^{3/2}}{3} + C = \frac{6(1+x)^{5/2}-10(1+x)^{3/2}}{15} + C = \frac{(1+x)^{3/2}(6x-4)}{15} + C$

f. $\int \sec(\theta)d\theta = \int \frac{\sec(\theta)\tan(\theta)+\sec^2(\theta)}{\tan(\theta)+\sec(\theta)}d\theta \quad \rightarrow \quad u=\tan(\theta)+\sec(\theta), \quad du = \sec(\theta)\tan(\theta)+\sec^2(\theta)d\theta$
$\int \sec(\theta)d\theta = \int \frac{1}{u}du = \ln(u) + C = \ln(\tan(\theta)+\sec(\theta)) + C = \ln|\tan(\theta)+\sec(\theta)| + C$

g. $\int \sec^2(\theta)d\theta = \tan(\theta) + C$

h. $\int \text{ sech}^2(\theta)d\theta = \tanh(\theta) + C$

i. $\int \frac{x^2+2}{7-x^2}dx = -\int \frac{x^2+2}{x^2-7}dx = -\int \frac{x^2-7}{x^2-7} + \frac{9}{x^2-7}dx = -\int \frac{9}{x^2-7} + 1dx = -\int \frac{9}{(x-\sqrt{7})(x+\sqrt{7})}dx + 1dx$
$\int \frac{x^2+2}{7-x^2}dx = -9\int \frac{A}{x-\sqrt{7}} + \frac{B}{x+\sqrt{7}}dx - \int 1dx = -9\int \frac{Ax+\sqrt{7}A+Bx-\sqrt{7}B}{(x-\sqrt{7})(x+\sqrt{7})}dx - x + C$
$\int \frac{x^2+2}{7-x^2}dx = -9[\int \frac{1}{2\sqrt{7}(x-\sqrt{7})}dx - \int \frac{1}{2\sqrt{7}(x+\sqrt{7})}dx] - x + C = -9[\frac{\ln(x-\sqrt{7})}{2\sqrt{7}} - \frac{\ln(x+\sqrt{7})}{2\sqrt{7}}] - x + C$
$\int \frac{x^2+2}{7-x^2}dx = -\frac{9(\ln|x-\sqrt{7}|-\ln|x+\sqrt{7}|)}{2\sqrt{7}} - x + C = \frac{9(\ln|x+\sqrt{7}|-\ln|x-\sqrt{7}|)}{2\sqrt{7}} - x + C$

j. $\int \frac{1}{ap-bp^2}dp = -\int \frac{1}{bp^2-ap}dp = -\int \frac{1}{(b-a/p)p^2}dp \quad \rightarrow \quad u = b - \frac{a}{p}, \quad du = \frac{a}{p^2}dp$
$\int \frac{1}{ap-bp^2}dp = -\int \frac{1}{au}du = -\frac{\ln(u)}{a} + C = -\frac{\ln(b-a/p)}{a} + C = -\frac{\ln|a/p-b|}{a} + C$

Problem: Compute the solutions for the following simple initial value problems.

a. $\frac{dx}{dt} = 3x$ and $x(0) = 1.0$

b. $\frac{dx}{dt} = 3tx$ and $x(0) = 1.0$

c. $\frac{dx}{dt} = 0.1x - 0.003x^2$ and $x(0) = 4$

d. $\frac{dx}{dt} = 0.1x - 0.003x^2$ and $x(0) = 400$

Solution: a. $\frac{dx}{dt} = 3x \quad \rightarrow \quad \frac{dx}{dt} - 3x = 0 \quad \rightarrow \quad r - 3 = 0 \quad \rightarrow \quad r = 3$
$x(t) = c_1e^{3t} \quad \rightarrow \quad x(0) = c_1 = 1 \quad \rightarrow \quad x(t) = e^{3t}$

b. $\frac{dx}{dt} = 3tx \quad \rightarrow \quad \frac{dx}{x} = 3tdt \quad \rightarrow \quad \int \frac{dx}{x} = \int 3tdt \quad \rightarrow \quad \ln(x) = \frac{3t^2}{2} + c_1$
$x(t) = c_1 e^{3t^2/2} \quad \rightarrow \quad x(0) = c_1 = 1 \quad \rightarrow \quad x(t) = e^{3t^2/2}$

c. $\frac{dx}{dt} = 0.1x - 0.003x^2 \quad \rightarrow \quad x^{-2}\frac{dx}{dt} = 0.1x^{-1} - 0.003 \quad \rightarrow \quad v = x^{-1} \quad \rightarrow \quad \frac{dv}{dt} = -x^{-2}\frac{dx}{dt}$
$-\frac{dv}{dt} = 0.1v - 0.003 \quad \rightarrow \quad \frac{dv}{dt} + 0.1v = 0.003 \quad \rightarrow \quad \mu = e^{\int 0.1dt} = e^{0.1t}$
$e^{0.1t}\frac{dv}{dt}' + 0.1e^{0.1t}v = 0.003e^{0.1t} \quad \rightarrow \quad \frac{d}{dt}(e^{0.1t}v) = 0.003e^{0.1t} \quad \rightarrow \quad e^{0.1t}v = \int 0.003e^{0.1t}dt$
$v = 0.03e^{-0.1t}e^{0.1t} + c_1e^{-0.1t} \quad \rightarrow \quad x^{-1} = 0.03 + c_1e^{-0.1t} \quad \rightarrow \quad x(t) = \frac{1}{0.03+c_1e^{-0.1t}}$
$x(0) = \frac{1}{0.03+c_1} = 4 \quad \rightarrow \quad 0.12 + 4c_1 = 1 \quad \rightarrow \quad c_1 = 0.22 \quad \rightarrow \quad x(t) = \frac{1}{0.03+0.22e^{-0.1t}}$

d. $x(0) = \frac{1}{0.03+c_1} = 400 \quad \rightarrow \quad 12 + 400c_1 = 1 \quad \rightarrow \quad c_1 = -0.0275 \quad \rightarrow \quad x(t) = \frac{1}{0.03-0.0275e^{-0.1t}}$

Problem: Graph the solutions of the last two differential equations in the Problem above. That is:

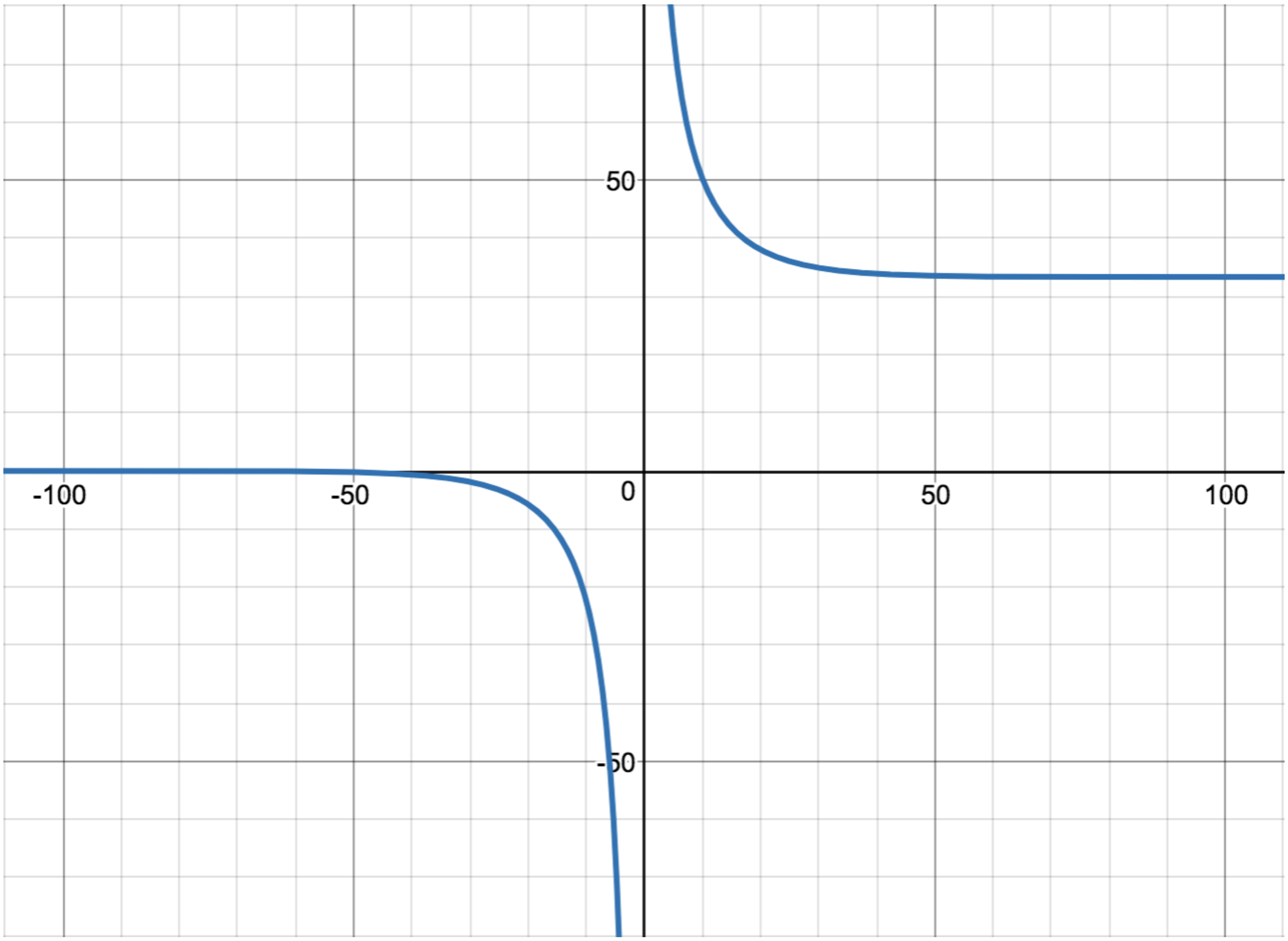
a. $\frac{dx}{dt} = 0.1x - 0.003x^2$ and $x(0) = 4$

b. $\frac{dx}{dt} = 0.1x - 0.003x^2$ and $x(0) = 400$

You do not need to recompute the solutions. Just graph them.

Solution: a.

b.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