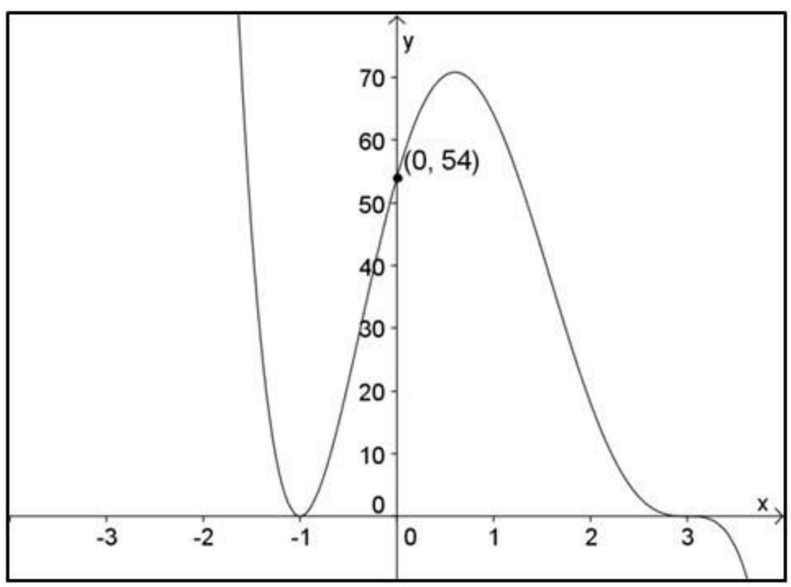
MAT 171

Pre-Calculus Algebra Lab Manual

Wake Technical Community College
Edition 6.0
Fall 2021 - Spring 2022 - Summer 2022



Contents

Interpreting Graphs (1)

Student Learning Objectives:

- From the graph of a function determine its domain and range and analyze the graph for intervals of increase/decrease, and continuity and state each using interval notation.
- Determine maximum/minimum values (local and absolute); output given input, input given output, and intervals where a graph is greater than/less than/equal to zero or another value.
- From the graph of a relation use the vertical line test to determine if it is a function, and if it is a function use the horizontal line test to determine if it is a one-to-one function.
- Use typical naming conventions i.e. linear, quadratic, polynomial, exponential, logarithmic, rational, radical, piecewise, and absolute value to associate the graph of a function with its name and symbolic form as well as describe the basic properties of these functions.

Part 1:

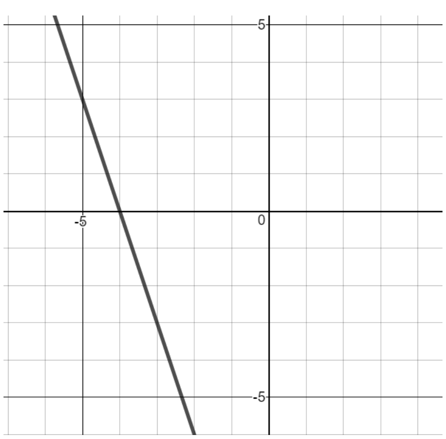
Each of the following six points are important for understanding the basics of this lesson. Match each numbered statement on the right with the appropriate letter from the column on the left. Then, study each, and write the statements in your notebook.

1. The domain of a function is:
2. The range of a function is:
3. The input values can also be called the ______
4. The output values can also be called the ______
5. The VLT - Vertical Line Test tests to see that there are no repeating ____________ values.
6. The Horizontal Line Test tests a function to determine if it is _____ ___ _____ by inspecting to insure that there are no repeating ___ ________.

A. One-to-one; y values.
B. The set of all input/independent/x-values
C. Independent or x-values or domain
D. The set of all output/dependent/y-values
E. Range/dependent values or y values
F. x/input/independent

Part 2:

Below are the graphs of six relations. Inspect each graph carefully. Below each graph, answer the questions related to the graph. If necessary, you may use approximate values.



Graph 1

Does the graph represent a function?

Domain:

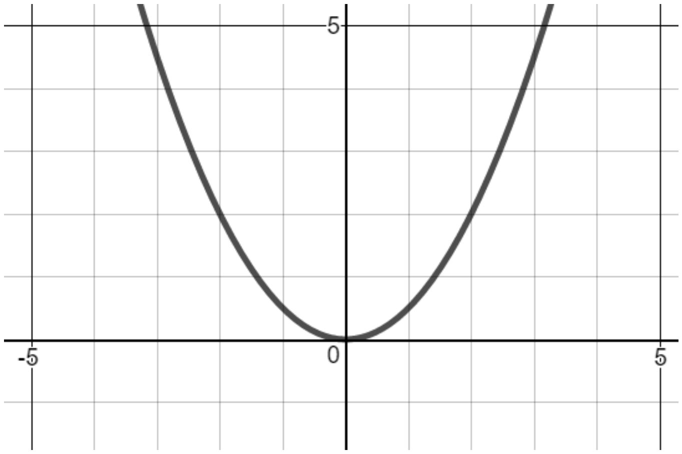
Range:

Does the graph represent a one-to-one function?

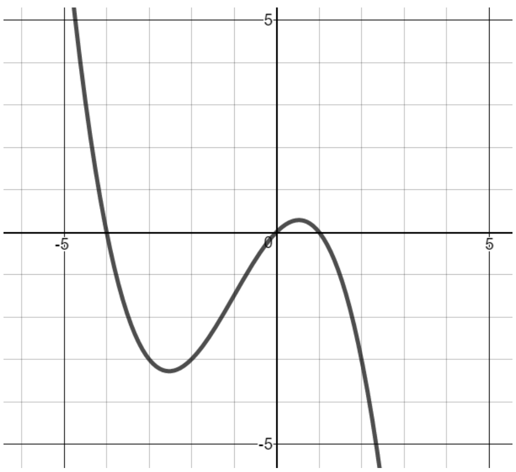
$f(-5) =$

$f(-3) =$

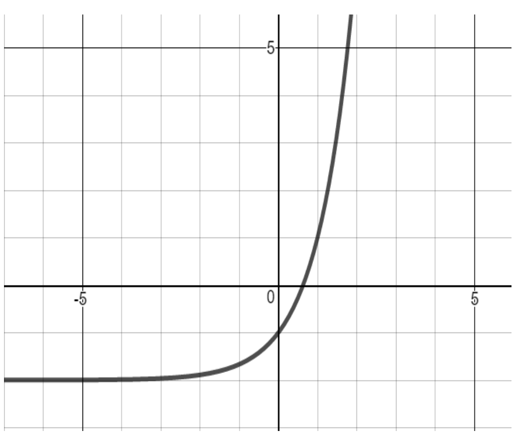
$If\ f(x) = 0$, then $x =$



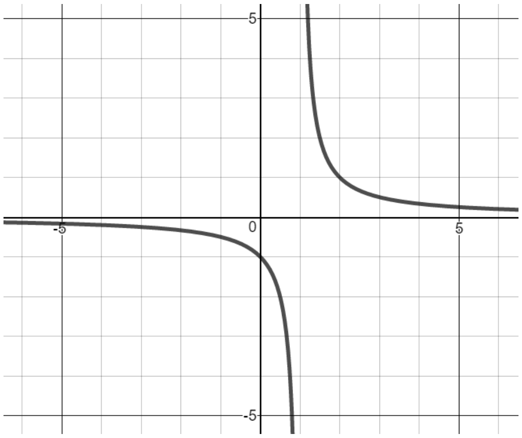
Graph 2
Does the graph represent a function?
Domain:
Range:
One-to-one?
$f(-2) =$
$f(0) =$
$f(2) =$
$If\ f(x) = 0$, then $x =$



Graph 3
Does the graph represent a function?
Domain:
Range:
Does the graph represent a one-to-one function?
$f(-3) =$
$f(0) =$
$f(2) =$
$If\ f(x) = 0$, then $x =$



Graph 4
Does the graph represent a function?
Domain:
Range:
Does the graph represent a one-to-one function?
$f(-0.5) =$
$f(0) =$
$f(1) =$
$If\ f(x) = 0$, then $x =$



Graph 5

Does the graph represent a function?

Domain:

Range:

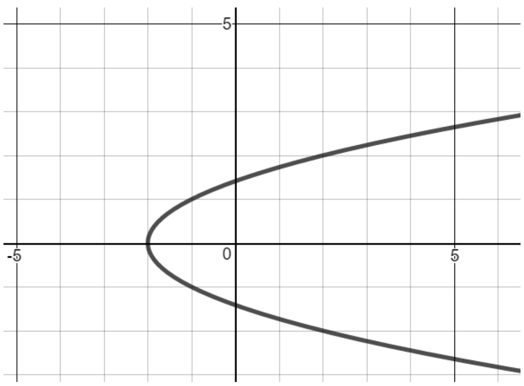
Does the graph represent a one-to-one function?

$f(-3)=$

$f(0)=$

$f(1)=$

$If\ f(x)=0$, then $x=$



Graph 6

Does the graph represent a function?

Domain:

Range:

One-to-one?

Write 6 distinct ordered pairs represented in graph 6.

What makes this graph different from the others we have studied so far?

Part 3:

Different shapes for different functions.

The graph for #1 in part 2 is a ________________________ or a _____________________ ______________________.

The graph for #2 in part 2 has a shape like a _________________ and is a _____________________ ___________________.

The graph for #3 in part 2 has a shape like a _________________ and is a _____________________ ___________________.

The graph for #4 in part 2 has a shape like a _________________ and is a _____________________ ___________________.

The graph for #5 in part 2 has two _________________ and is a _____________________ ___________________.

The graph for #6 in part 2 has a shape like a side-ways __________ and IS NOT a _____________________.

Interpreting Graphs (2)

Student Learning Objectives:

- From the graph of a function determine its domain and range and analyze the graph for intervals of increase/decrease, and continuity and state each using interval notation.
- Determine maximum/minimum values (local and absolute); output given input, input given output, and intervals where a graph is greater than/less than/equal to zero or another value.
- From the graph of a relation use the vertical line test to determine if it is a function, and if it is a function use the horizontal line test to determine if it is a one-to-one function.
- Use typical naming conventions i.e. linear, quadratic, polynomial, exponential, logarithmic, rational, radical, piecewise, and absolute value to associate the graph of a function with its name and symbolic form as well as describe the basic properties of these functions.

Part 1:

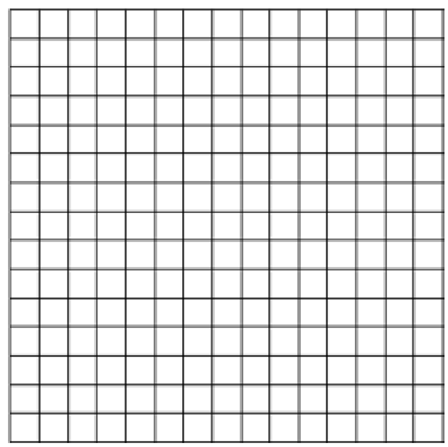
Using a graphing utility on your device (**Desmos is preferred, www.desmos.com**) enter each of the following functions, and ***sketch the graph obtained.*** All of these graphs will appear on a standard 10 by 10 coordinate grid. State the appropriate name of the curve obtained if we have reviewed it above. If we have not looked at the function yet, the name of the function is given. Once you have the graph, state the domain and range in interval notation.

A. $f(x) = -x^2 + 1$;

Type of function: ________________

Domain: ________________

Range: ________________

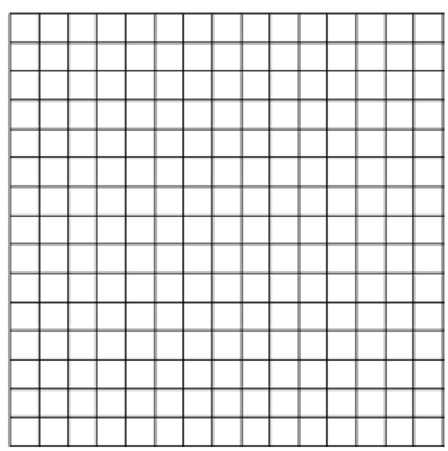


B. $f(x) = \sqrt{(x-2)}$;

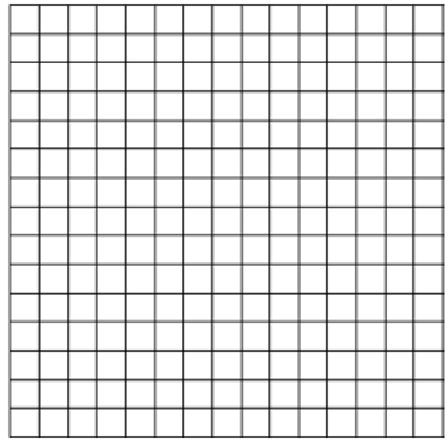
Type of function: ________________

Domain: ________________

Range: ________________



C. $f(x)=\left(\frac{1}{2}\right)^{x}$;

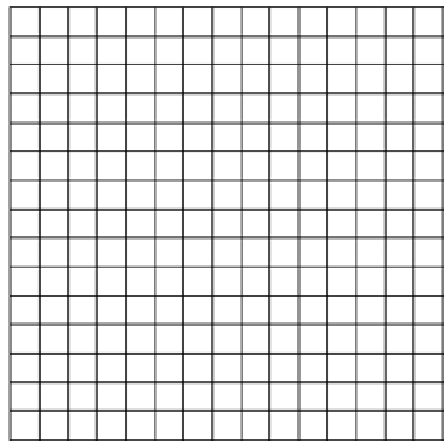


Type of function: ________________

Domain: ________________

Range: ________________

D. $f(x)=\frac{1}{(x+2)}$;



Type of function: ________________

Domain: ________________

Range: ________________

Part 2:

In Part 2 of Interpreting Graphs (1) and Part 1 of this lesson you worked with specific graphs and you have their sketches readily available.

In this part of the activity you will determine the intervals where the graphs are $>, \geq, <, \leq$ to 0.

Part 2/Interpreting Graphs (1) Graphs:

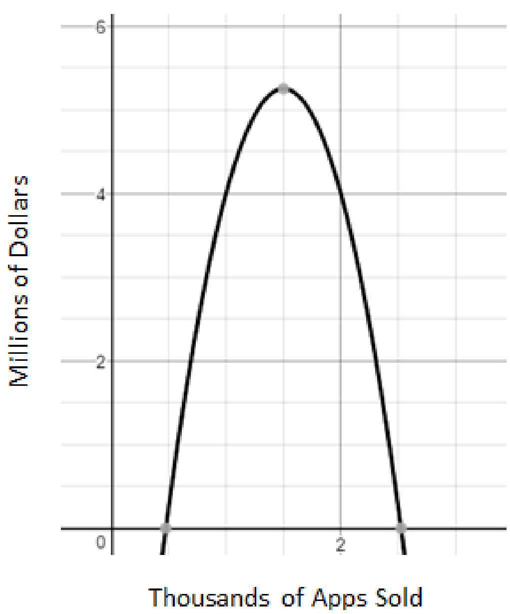
	Graph 1	Graph 2	Graph 3	Graph 4	Graph 5	Graph 6
Interval where $f(x)<0$						
Interval where $f(x)\leq 0$						
Interval where $f(x)>0$						
Interval where $f(x)\geq 0$						

Part 1/ Interpreting Graphs (2) Graphs:

	Graph A	Graph B	Graph C	Graph D
Interval where $f(x) < 0$				
Interval where $f(x) \leq 0$				
Interval where $f(x) > 0$				
Interval where $f(x) \geq 0$				

Part 3: Real World

Oftentimes it is necessary to know when values are above or below a value other than zero. The following graph depicts the profit function for a very popular Smart Phone App. The input/x-values are in thousands of apps sold and the profits are in millions of dollars.



1. What is a reasonable domain for this profit function if the company wants to "break even" or better?

2. What is the profit when one thousand apps are sold?

3. At approximately what interval(s) are the profits below 3 million dollars?

4. At approximately what interval(s) are the profits **at least** 3 million dollars?

Graphing By Point Plotting

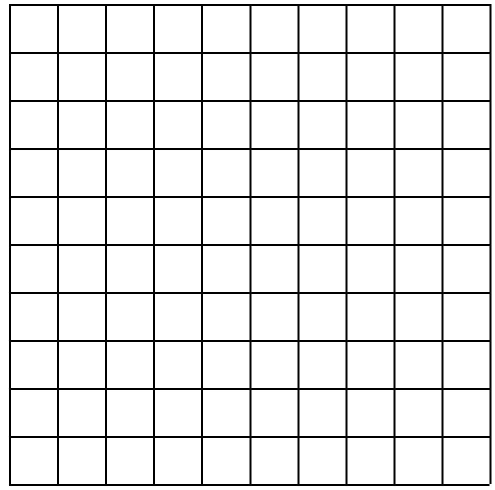
Student Learning Objective:

- From a function in symbolic form, build a table of values, form ordered pairs, and sketch the graph of the function by hand.

For each of the functions below create a table of values and graph the function by hand.

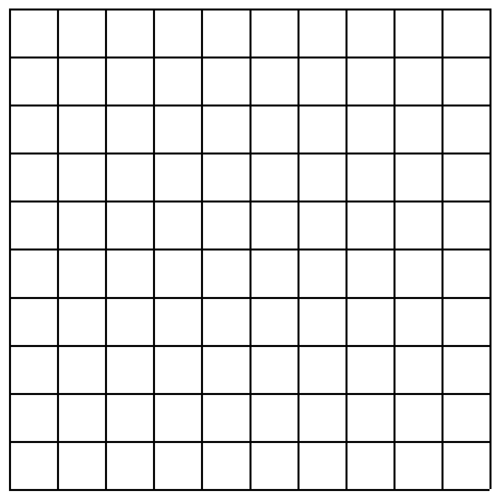
1. $f(x) = x$

x	$f(x)$	$(x, f(x))$



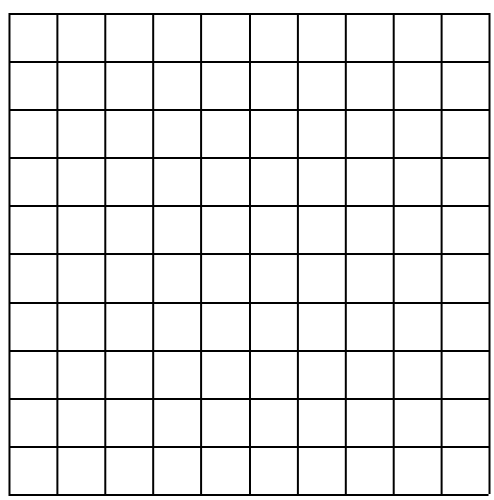
2. $f(x) = x^2$

x	$f(x)$	$(x, f(x))$



3. $f(x) = x^3$

x	$f(x)$	$(x, f(x))$



4. $f(x) = |x|$

x	$f(x)$	$(x, f(x))$

5. $f(x) = \dfrac{1}{x}$

x	$f(x)$	$(x, f(x))$

6. $f(x) = \sqrt{x}$

x	$f(x)$	$(x, f(x))$

7. $f(x) = 2^x$

x	$f(x)$	$(x, f(x))$

8. $f(x) = \left(\frac{1}{3}\right)^x$

x	$f(x)$	$(x, f(x))$

9. $f(x) = \log(x)$

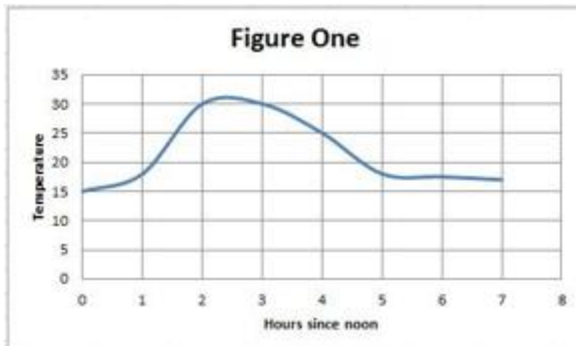
x	$f(x)$	$(x, f(x))$

Obtaining Information from a Function's Graph

Student Learning Objectives:

- From the graph of a function determine its domain and range and analyze the graph for intervals of increase/decrease, and continuity and state each using interval notation.
- Determine maximum/minimum values (local and absolute); output given input, input given output, and intervals where a graph is greater than/less than/equal to zero or another value.

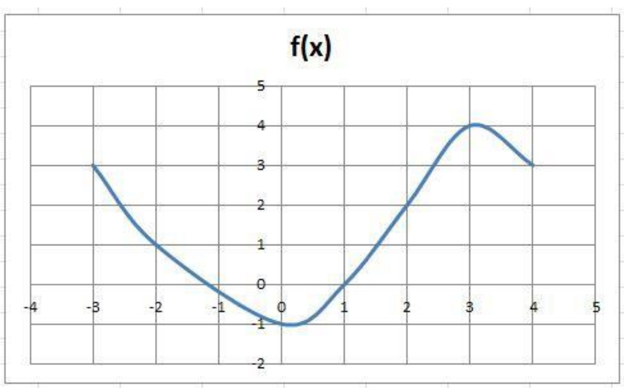
Example: The function T graphed in Figure One gives the temperature between noon and 7:00 P.M. at a certain weather station, and x represents hours since noon. Determine the values of x for which $T(x) \geq 25$.



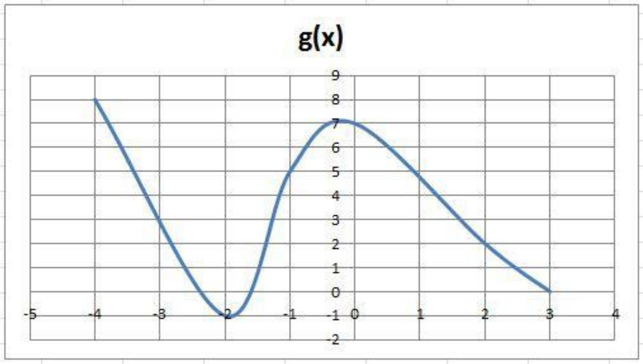
Solution: The graph is above 25 for x between 1.5 and 4. Therefore, the temperature was 25° or more between 1:30 P.M. and 4:00 P.M.

Exercises:

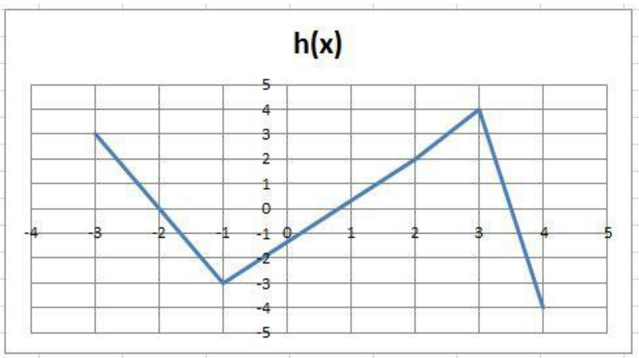
1. Determine the values of x for which $f(x) \leq 3$.



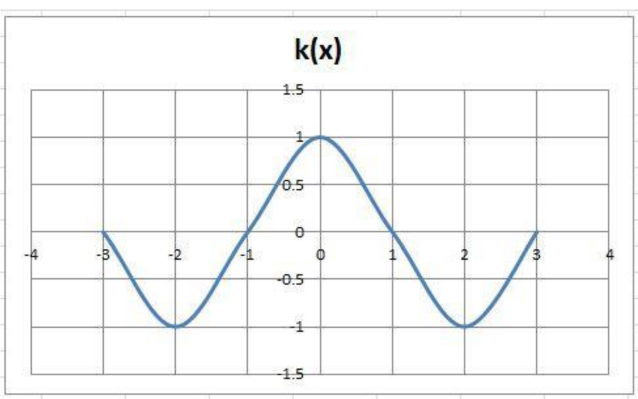
2. Determine the values of x for which $g(x) > 5$.



3. Determine the values of x for which $h(x) \geq 1$.



4. Determine the values of x for which $k(x) < 0$.



Interpreting Graphs – Increasing and Decreasing

Student Learning Objectives:

- From the graph of a function determine its domain and range and analyze the graph for intervals of increase/decrease, and continuity and state each using interval notation.
- Determine maximum/minimum values (local and absolute); output given input, input given output, and intervals where a graph is greater than/less than/equal to zero or another value.

If the function values or ____ values are getting larger from left to right, then the graph is said to be ________________. If these values are getting smaller, then the graph is said to be _____________.

We state the interval(s) where a graph is increasing or decreasing on the ____ values. We generally use ( ) to state the intervals for increase and decrease because our max/min points are said to be *turning points.*

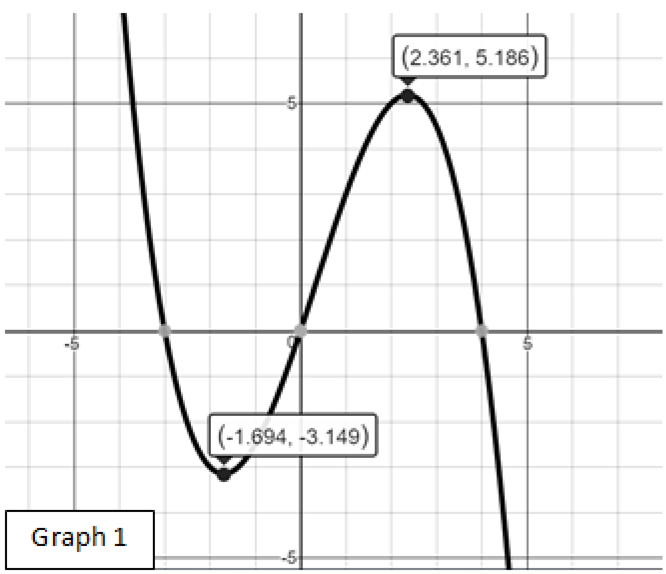
Graph 1

1. The graph below, **Graph 1,** has the following characteristics:
   a. "Reading" the graph from left to right, the first observation is that the graph is initially _________________________.

   b. The graph is decreasing on the ***intervals*** (_______, _______) and (_______, _______).

   c. The graph has turning points at which ***ordered pairs***? (_______, _______) and (_______, _______) Remember, ordered pairs and intervals might look similar, but their meanings are very different!

   d. Finally, the graph is increasing on which interval? (_______, _______)



Graph 2

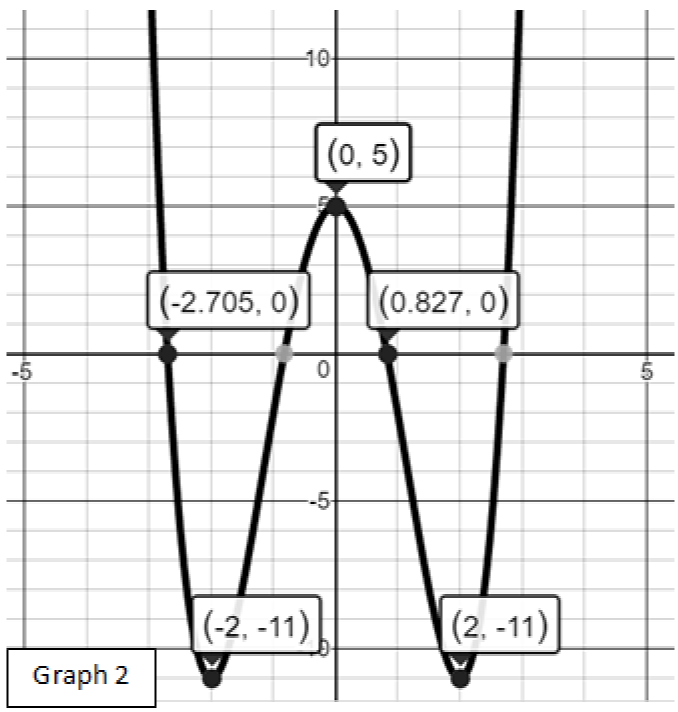
2. Look at **Graph 2** below.
   a. Using your pencil, go from left to right and trace the portion(s) of the graph where the function values are increasing.

      State those interval(s):

   b. Now, using your pencil, go from left to right and trace the portion(s) of the graph where the function values are decreasing.

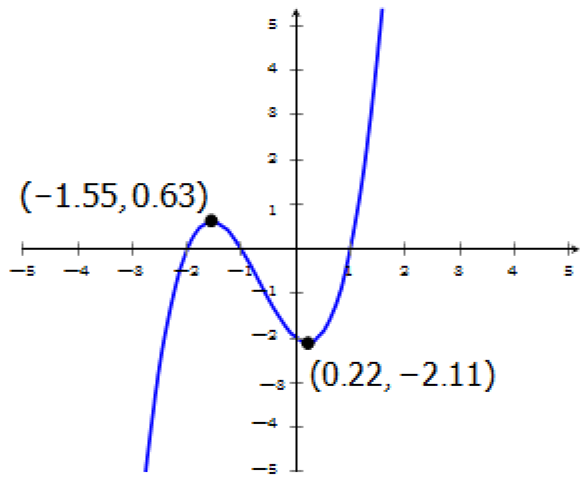
      State those interval(s):

   c. What are the coordinates of the turning points?



Graph 3

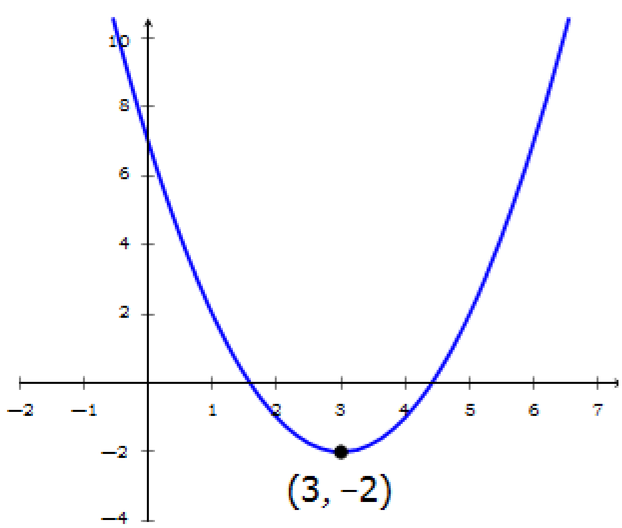
Below is the graph of the function $f(x) = x^3 + 2x^2 - x - 2$. Determine the interval(s) on which the function is a. increasing and b. decreasing.



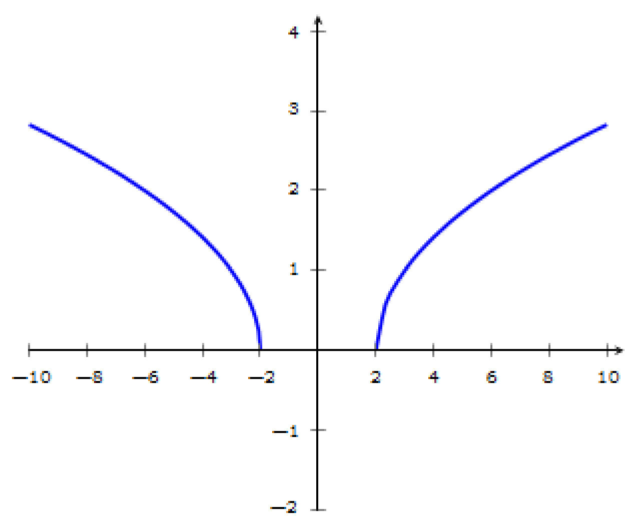
a. The function is increasing over the intervals:____________________
b. The function is decreasing over the interval:____________________

Describe the increasing and decreasing behavior of the following functions.

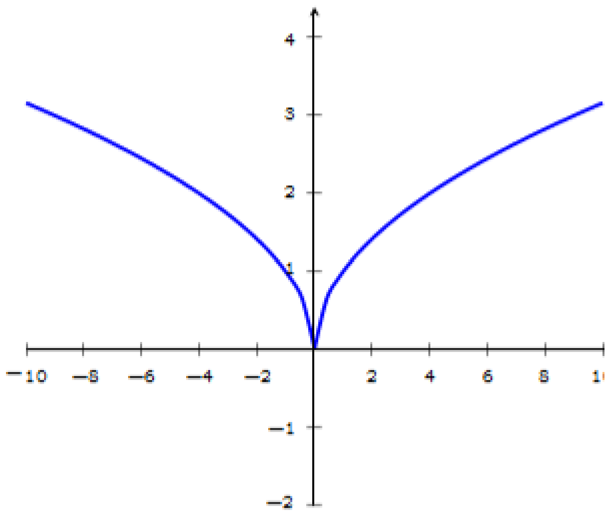
3.



4.

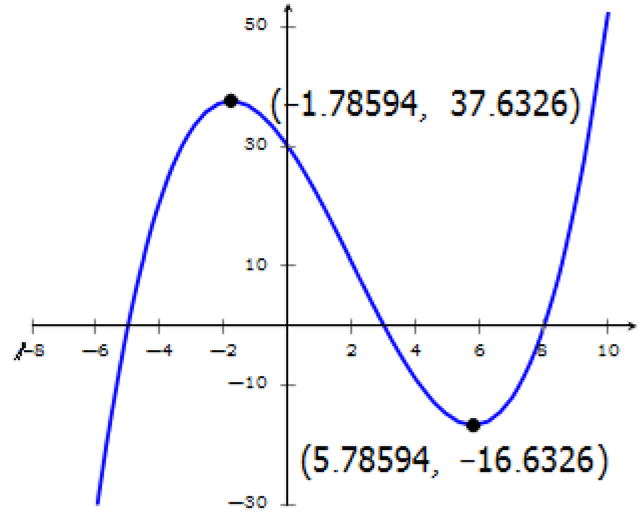


5.



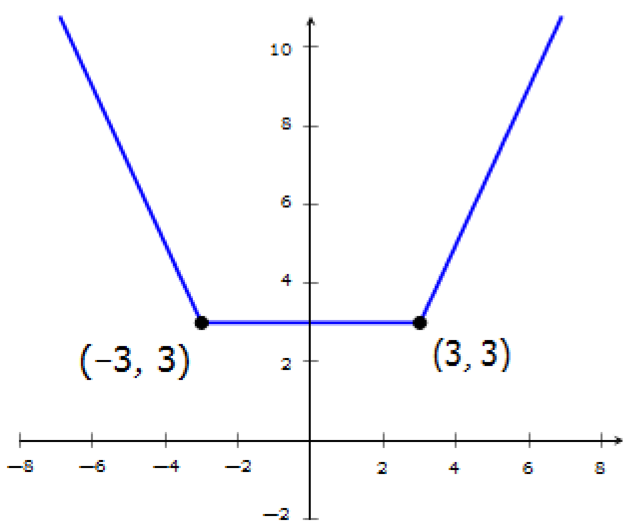
4
3
2
1
−1
−2
−10
−8
−6
−4
−2
2
4
6
8
10

7.



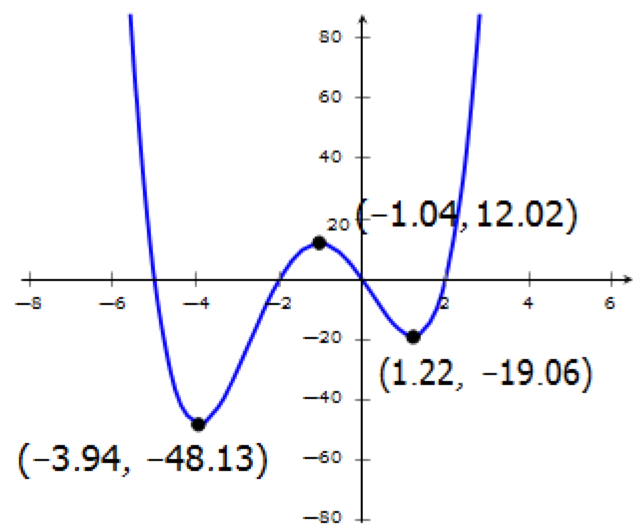
50
30
10
−10
−30
−8
−6
−4
−2
2
4
6
8
10
(−1.78594, 37.6326)
(5.78594, −16.6326)

6.



10
8
6
4
2
−2
−8
−6
−4
−2
2
4
6
8
(−3, 3)
(3, 3)

8.



80
60
40
20
−20
−40
−60
−80
−8
−6
−4
−2
2
4
6
(−1.04, 12.02)
(1.22, −19.06)
(−3.94, −48.13)

Interpreting Graphs End Behavior and Asymptotes

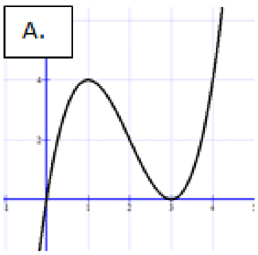
Student Learning Objectives:

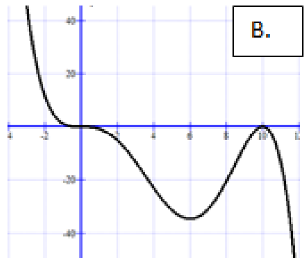
- Determine the end behavior of the graph of a function, work with asymptotes as limiting behavior, and find the asymptotes of a function given its graph.

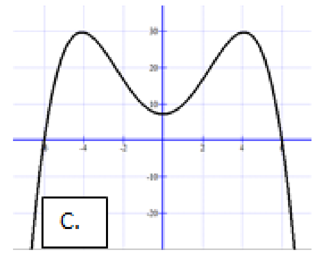
Part 1

The "end behavior" of the graph of a function describes ____________________________________.

These three graphs represent _______________________ functions.







Describe the domain and range of each graph in interval notation.

A. ______________________ B. ______________________ C. ______________________

______________________ ______________________ ______________________

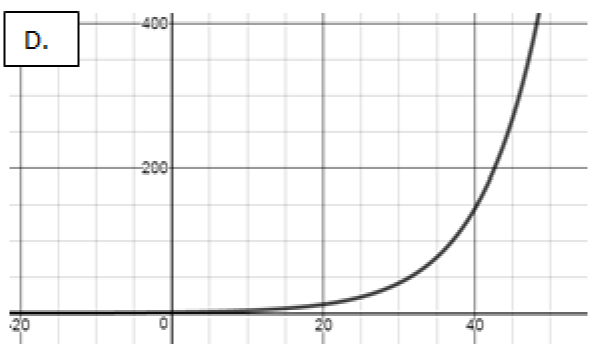
Now, let's determine the ***end behavior*** of each graph.

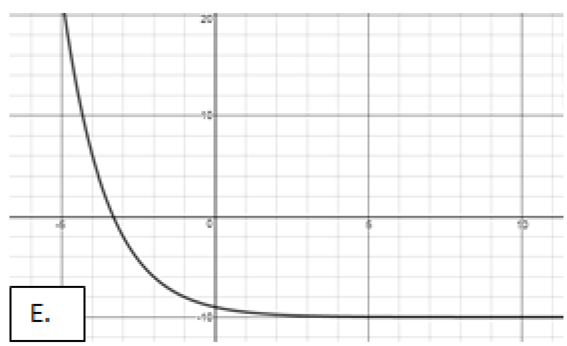
In graph A, as x approaches $-\infty$, y approaches _______________. As x approaches ∞, y approaches _______________.

In graph B, as x approaches $-\infty$, y approaches _______________. As x approaches ∞, y approaches _______________.

In graph C, as x approaches $-\infty$, y approaches _______________. As x approaches ∞, y approaches _______________.

These two graphs are __________________________ in nature.





Describe the domain and range of each graph in interval notation.

D. ______________________________ E. ______________________________

______________________________ ______________________________

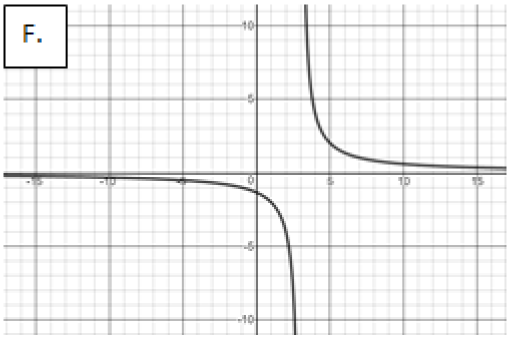
In graph D, as x approaches $-\infty$, y approaches _______________. As x approaches ∞, y approaches _______________.

In graph E, as x approaches $-\infty$, y approaches _______________. As x approaches ∞, y approaches _______________.

Graphs D and E also have asymptotes.

Asymptotes are known as the ___________________ ______________________ of the function.
These graphs have horizontal asymptotes because the y-values are limited in the _________________ of the function.
From a previous lesson it could be said that Graph D shows growth because the curve is ________________________.
Graph E shows decay because the curve is _____________________.

Graph F, below, has horizontal and vertical asymptotes.



The horizontal asymptote is _____________________.

Use your pencil and draw a vertical line where you think that there could be a vertical asymptote.

The vertical asymptote is written as the equation of a vertical line, or x = ______.

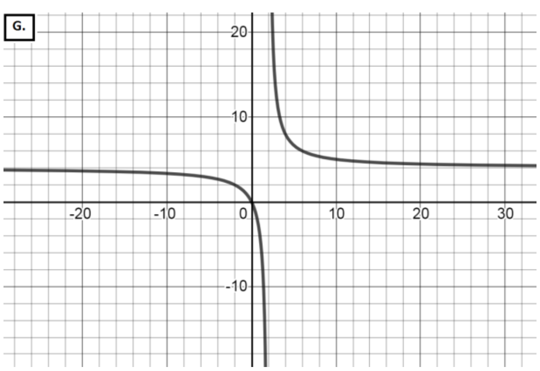
Both the horizontal and vertical asymptotes are limiting values.

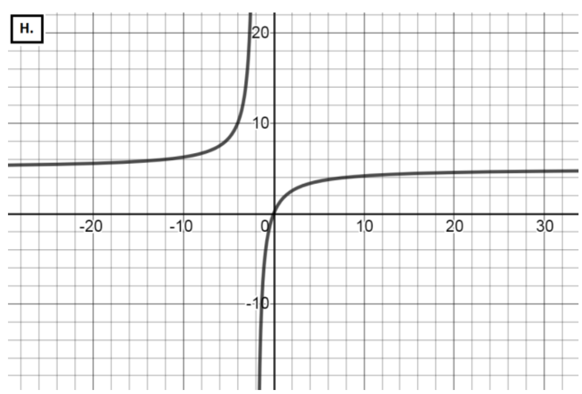
The horizontal asymptote limits the value of _________. The vertical asymptote limits the value of ______.

“Squeeze” the vertical asymptote to determine the behavior near it. As x approaches _____ from values to its left, y approaches _______. As x approaches ______ from values to its right, y approaches ________.

The function’s end behavior: As x approaches $-\infty$, y approaches ______________. As x approaches ∞, y approaches ______________.

For graphs G and H, use the ideas generated in F above to draw all of the conclusions you can about the graph shown.





G.____

____

____

____

H. ____

____

____

____

From a previous lesson, the graphs of F, G and H are all called _______________________ functions because we can see their _______________________.

Part 2:

Go to [Desmos](www.desmos.com) (www.desmos.com)

Graph each of the following functions. On this paper record the name of the function and any end behavior and/or asymptotes. You may include a sketch here if it is helpful to you ☺

1. $y = -(x-2)^3$

3. $y = \sqrt{5-x}$

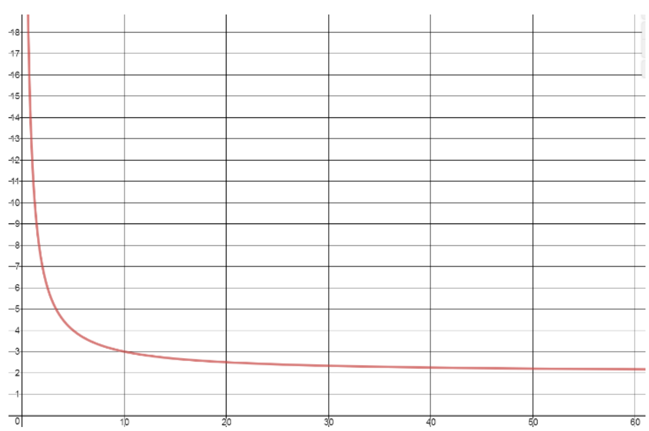
2. $f(x) = 45e^{.08x}$

4. $f(x) = \dfrac{x-3}{x-4}$

Part 3:

Real World Connection

The ***Average Cost Function*** is rational in nature. The average cost function is the cost function, $C(x)$ divided by the number of units produced, x. The average cost function graphed here is $\overline{C}(x) = \dfrac{2x+10}{x}$.



$\overline{C}(x)$ is only graphed in the ______________ quadrant because in the context of cost the domain must be $x >$ _____.

There is a horizontal asymptote at __________ and a vertical asymptote at ______________.

Part 4

Words to Graph Matching

For each of the given scenarios, match the words to the **most appropriate** graph at the right. The physical domain of each should be t ≥ 0, where t represents time.

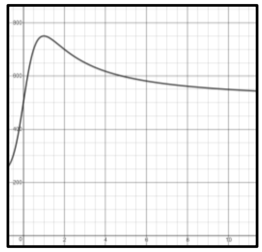
a) Cost versus time: A moving company charges a base price of $500 plus $125 per each hour that the job takes.

b) Height versus time: A pen is tossed up in the air from an initial height of 500 feet.

c) Drug in the bloodstream versus time: The initial amount of drug injected into the muscle of a person is 500 units.

d) Bacteria versus time: The initial amount of cells is 500.

e) Radioactive Substance versus time: The half-life of 500 mgs of a certain radioactive substance is 10 years.



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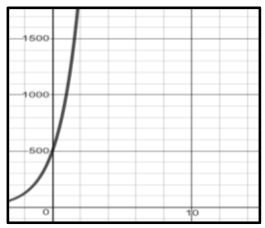

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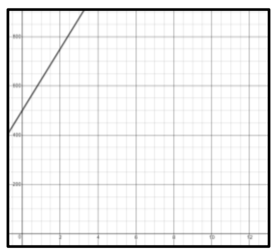

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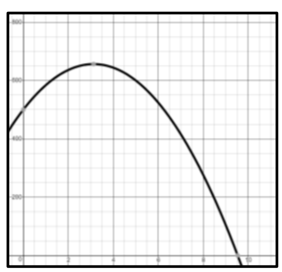

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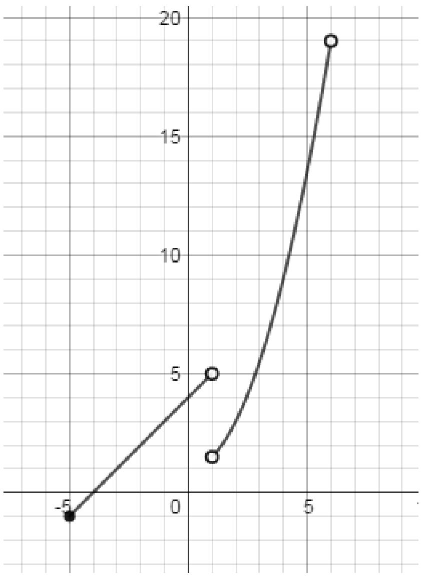
Piecewise Functions

Student Learning Objectives:

- From the graph of a function determine its domain and range and analyze the graph for intervals of increase/decrease, and continuity and state each using interval notation.
- Determine maximum/minimum values (local and absolute); output given input, input given output, and intervals where a graph is greater than/less than/equal to zero or another value.

Use the following graphs to answer the questions:



1. Write the domain in interval notation.

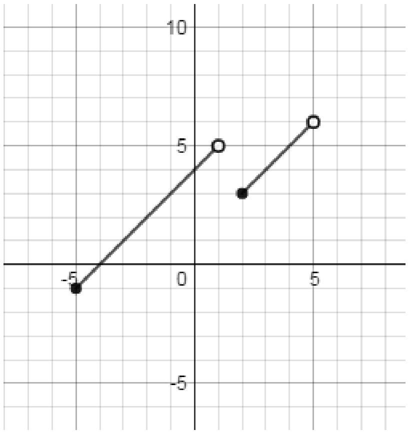
2. Write the range in interval notation.

3. $f(0) =$

4. $f(3) =$

5. If $f(x) = 0,\ x =$

6. If $f(x) = 5,\ x =$



1. Write the domain in interval notation.

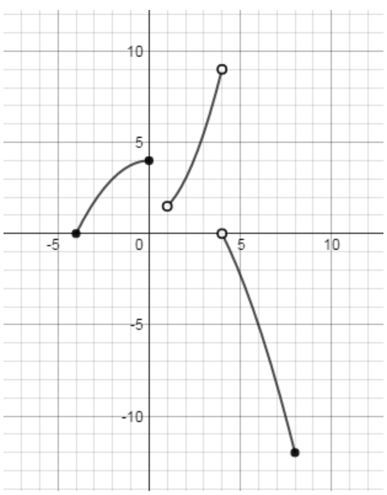
2. Write the range in interval notation.

3. $f(-2) =$

4. $f(2) =$

5. If $f(x) = 1,\ x =$

6. If $f(x) = 4,\ x =$



1. Write the domain in interval notation.

2. Write the range in interval notation.

3. $f(-2) =$

4. $f(2) =$

5. If $f(x) = 0,\ x =$

6. If $f(x) = 3,\ x =$

Transformations

Student Learning Objectives:

- Use typical naming conventions i.e. linear, quadratic, polynomial, exponential, logarithmic, rational, radical, piecewise, and absolute value to associate the graph of a function with its name and symbolic form as well as describe the basic properties of these functions.
- Transform the graph of a function given a symbolic representation and describe a transformed graph with a symbolic representation.

Part 1

Go to Desmos (www. desmos.com).

A. Type in each function from the list below to see the graph in the coordinate plane. For each function you may simply type $f(x) =$ ______

a. $f_1(\mathrm{x}) = 5$

b. $f_2(\mathrm{x}) = x$

c. $f_3(\mathrm{x}) = x^2$

d. $f_4(\mathrm{x}) = x^3$

e. $f_5(\mathrm{x}) = \sqrt{x}$

f. $f_6(\mathrm{x}) = e^x$

g. $f_7(\mathrm{x}) = \ln(x)$

h. $f_8(\mathrm{x}) = |x|$

i. $f_9(x) = \dfrac{1}{x}$

Part 2

Now, use the list provided below to identify each graph.

POLYNOMIAL FUNCTION

ABSOLUTE VALUE FUNCTION

EXPONENTIAL FUNCTION

LINEAR FUNCTION

LOGARITHMIC FUNCTION

QUADRATIC FUNCTION

RADICAL FUNCTION

RATIONAL FUNCTION

Part 3

In Desmos, for each of the nine functions, investigate each of these:

A. Add 4 to the original function and describe what happens to the graph of the function.

B. Subtract 2 from the original function and describe what happens.

C. For the function $f_3(\mathrm{x}) = x^2$, subtract 1 from the x, by typing this $f(x) = (x-2)^2$.
   Notice the 2 is subtracted from the x value, not the function value.
   Describe how this changes the original function.

D. For the function $f_5(\mathrm{x}) = \sqrt{x}$, add 5 inside the radical symbol with x.
   Describe how the graph changes.

E. For the function $f_3(\mathrm{x}) = x^2$ change the function by multiplying x^2 by 0.5.
   Describe how the graph changes.

F. Now do the same as part E, but multiply by -1.
   Describe how the graph changes.

G. Now do the same as Part E, but multiply by 3.
   Describe how the graph changes.

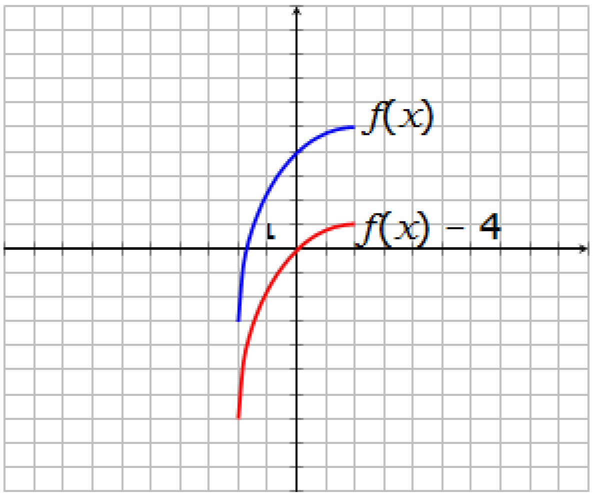
H. Pick two or three functions from the original list and do the following: add or subtract to the end of the function; add/subtract within the function; multiply the function by a value.

All of these changes to the base functions are called transformations, and they follow patterns as you have seen so far. We will study transformations in more detail in an upcoming lesson, but describe what you have learned today by writing several sentences below.

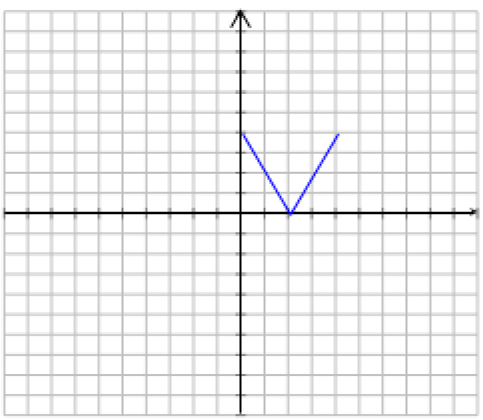
Practice

Each grid below is scaled by 1's and contains the graph of some "mystery function", $y = f(x)$. For each one, use what you have learned about shifts, reflections, and stretches to sketch the graph of the indicated function on the same coordinate plane. The first one is done as an example for you.

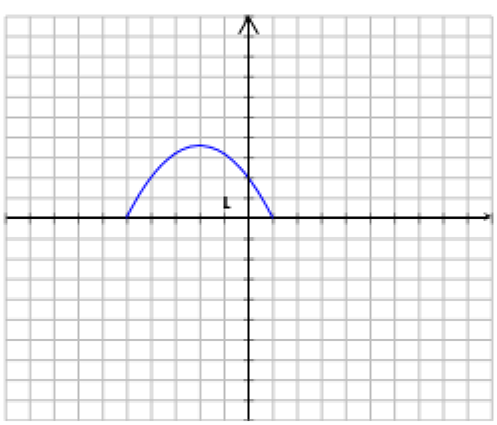
Example I. Sketch $y = f(x) - 4$



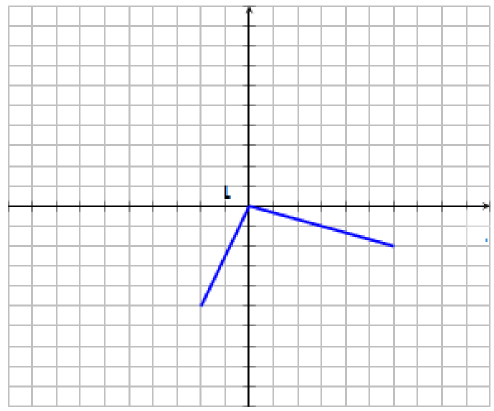
1. Sketch $y = f(x-3)$



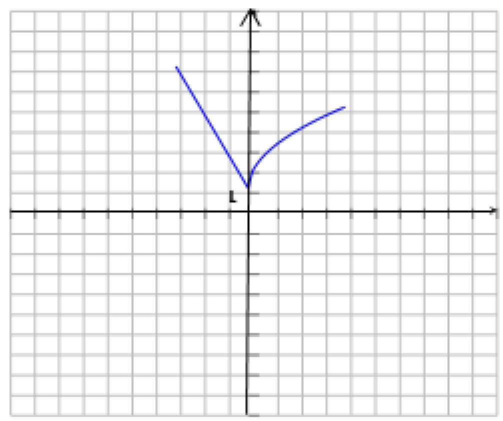
2. Sketch $y = -f(x)$



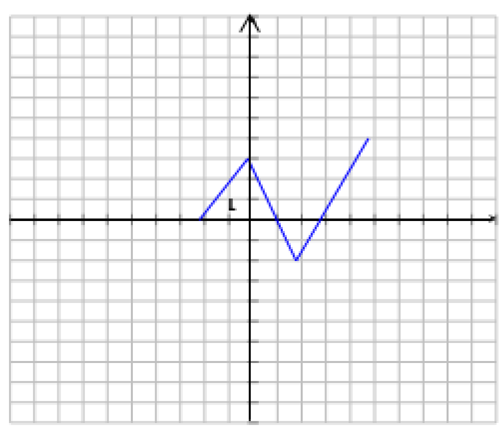
3. Sketch $y = \frac{1}{2} f(x)$



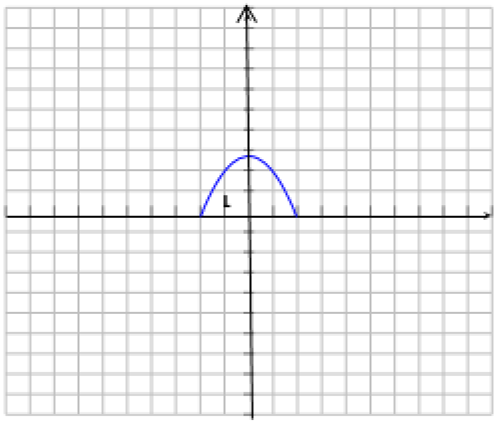
4. Sketch $y = f(x) - 2$



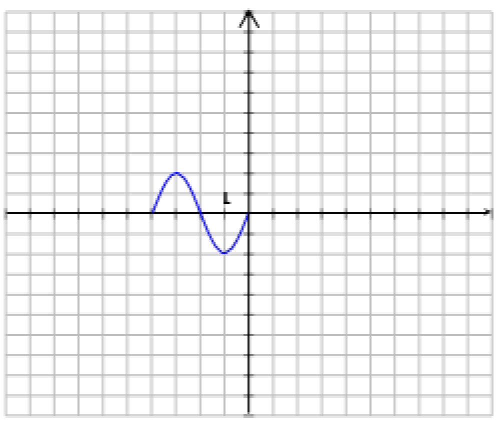
5. Sketch $y = 2f(x)$



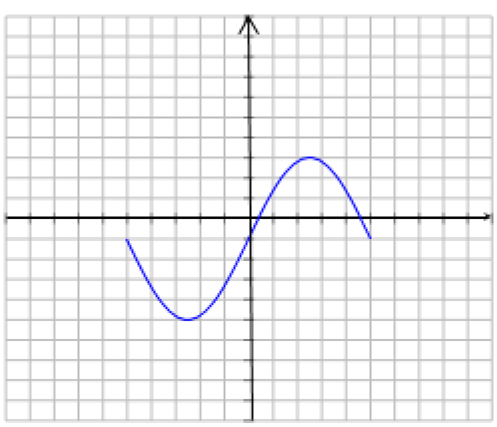
6. Sketch $y = -f(x) - 3$



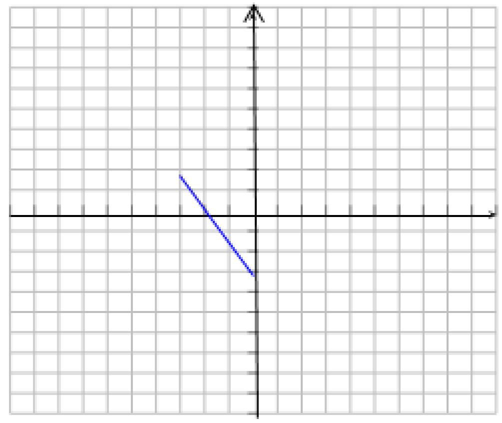
7. Sketch $y = 1 + f(x-3)$



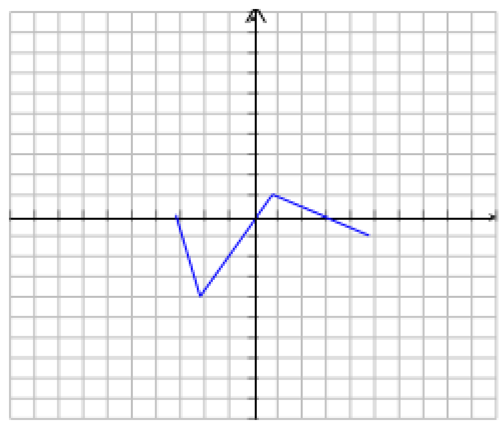
8. Sketch $y = f(x) + 4$



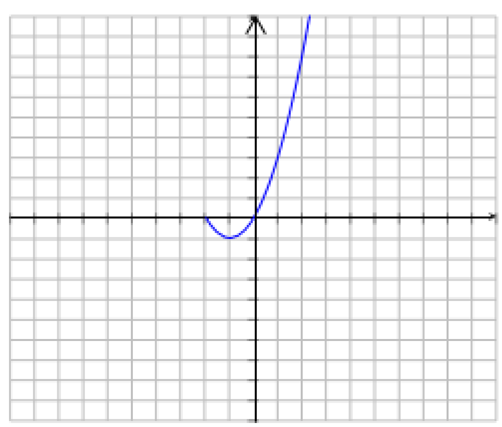
9. Sketch $y = 4 + f(x+2)$



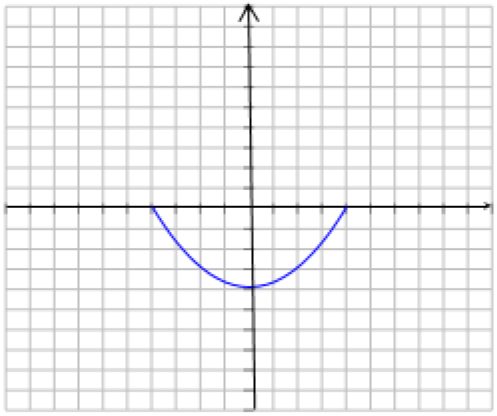
10. Sketch $y = 2 - f(x)$



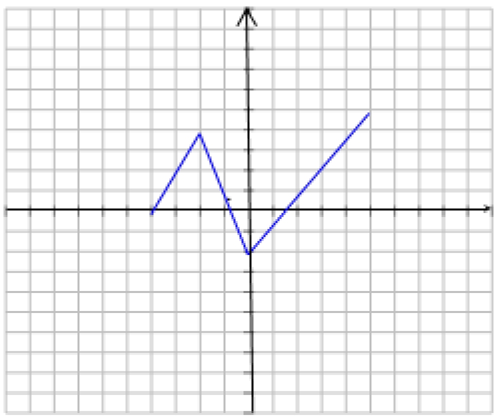
11. Sketch $y = -f(x)$



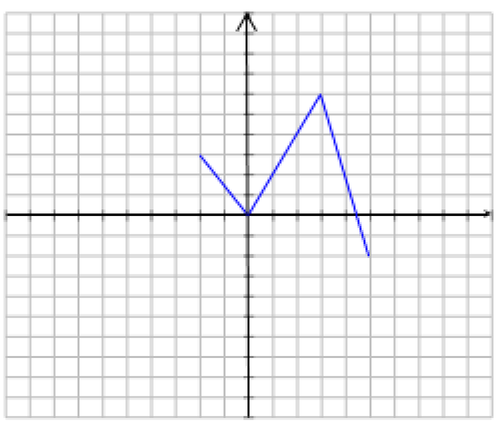
12. Sketch $y=\frac{1}{2}f\left(x\right)+1$



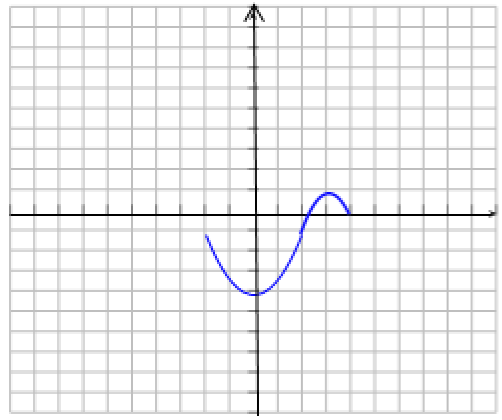
13. Sketch $y=f(x+4)$



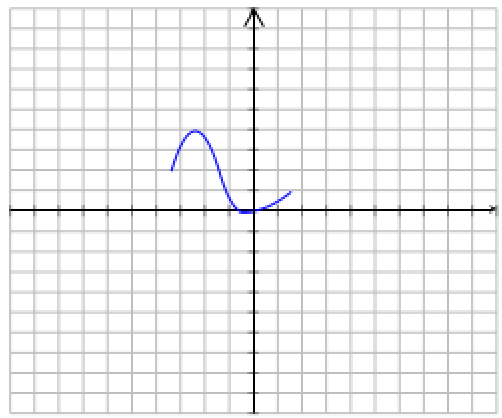
14. Sketch $y=-f\left(x-4\right)$



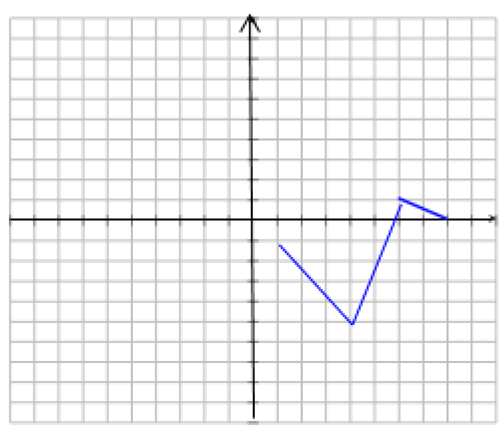
15. Sketch $y=1-2f\left(x\right)$



16. Sketch $y=-\left[f(x+3)-3\right]$



17. If this is the graph of $y=f\left(x-4\right)-5$, sketch $-f\left(x\right)$.



Transformations of basic/parent functions

Many common functions are simple transformations of the parent functions you learned while Graphing by Point Plotting. For example, the height of a rock dropped off of a cliff t seconds after it is released might be modeled by the function $h(t) = -16.1t^2 + 50$, whose graph is a transformation of the graph of $y = x^2$. We don't see the variables x and y anymore because of the real world context of height and time. There is a multiplication on the outside of the square (a vertical stretch/expansion), and a negative on the outside (a vertical flip/reflection) and finally an addition on the outside (a vertical shift up). The parabola shape results because of acceleration due to gravity; the rock will speed up as it falls, resulting in a decreasing height curve that gets steeper as time goes on.

In the following exercises, sketch a graph of each function based on its associated parent function, and state the resulting function's domain and range. Choose an appropriate scale for the grid based on the necessary transformations.

1. $f(x) = \sqrt{x+1} - 4$

2. $g(x) = 2|x-3|$

3. $h(x) = 9 - x^2$

4. $F(C) = \frac{9}{5}C + 32$

5. $J(t) = 2^{-t} + 1$

6. $H(t) = \dfrac{t^3}{4} - 2.5$

7. $y = 5 - \dfrac{1}{x^2}$

8. $y(x) = \ln(2x) - 3$

9. $G(x) = 7 - \left|\dfrac{x}{5}\right|$

10. $F(x) = \dfrac{1}{x+2} + 5.5$

Domain of a Function

Student Learning Objectives:

- From a function in symbolic form, state its domain and range in interval notation, state the output value associated with a given input value and vice versa.

As you should recall from Module 1, the domain of a function is the set of all values of the independent variable for which the function is defined. There are several things that can limit where a function is defined. Let's examine them.

Rational Functions

Functions that include a quotient, e.g., $f(x) = \dfrac{1}{x}$

For what values of x would the function be undefined? ____

The domain of this function is all real numbers except 0. We can write this as

$$\text{Domain } = \{x | x \in \Re, x \neq 0\} \text{ or } (-\infty, 0) \cup (0, \infty)$$

Can you generalize this to any rational function?

Determine the domain of the following functions. Write your answer in interval notation.

1. $g(x) = \dfrac{x-2}{x+5}$ Domain: _________________________

2. $h(x) = \dfrac{x+3}{x^2+2x-15}$ Domain: _________________________

3. $k(x) = \dfrac{x-7}{x^2+3x-4}$ Domain: _________________________

Root Functions

Functions that include a square root, a fourth root, sixth root, etc., e.g., $f(x)=\sqrt{x}$

For what values of x would this function be defined? ____

This function is defined when what is under the square root is non-negative, i.e.,

$$\text{Domain } =\{x|x\in\Re, x\geq 0\} \text{ or } [0,\infty)$$

Can you generalize this to any function involving an even indexed root?

Determine the domain of the following functions. Write your answer in interval notation.

4. $g(x)=\sqrt{2x+8}$ Domain: __________________________

5. $h(x)=\sqrt[4]{9-2x}$ Domain: __________________________

6. $k(x)=\sqrt[3]{x^3-8}$ Domain: __________________________

Real Life Limitations

Many quantities in the real world cannot take on certain values, e.g., time cannot be negative.

State the independent and dependent variables and determine the domain of the following functions. Write your answer in interval notation.

7. The cost C in dollars of producing x yards of a certain fabric is given by the function $C(x) = 1500 + 3x + 0.02x^2 + 0.0001x^3$.

8. The surface area S of a sphere is a function of its radius r given by $S(r) = 4\pi r^2$

9. The stopping distance D of a car after the brakes have been applied varies directly as the square of the speed s. The function that models this is $D = ks^2$.
   (A) What is the independent variable?

   (B) What is the domain of this function?

   (C) What is the dependent variable?

More Examples

Some functions involve a combination of these things and each must be considered in determining the domain. Determine the domain of the functions below and write your answer in interval notation.

10. $f(x) = \dfrac{(x+6)}{\sqrt{8x-7}}$ Domain: ___________________________

11. $g(x) = \dfrac{\sqrt{8-x}}{x^2 - 5x + 6}$ Domain: ____________________________

12. $h(x) = \dfrac{1+x^2}{x\sqrt{3x+4}}$ Domain: ______________________________

13. $k(x) = \dfrac{x+5}{\sqrt[3]{2x+8}}$ Domain:_______________________________

Writing the symbolic form of a function having a certain domain or range

For each exercise below you must write the symbolic form of a function (like $f(x) =$ something involving x) which has the indicated domain/range. Use the rules from the previous pages. Justify the domain and/or range using the domain rules and transformations.

1. Write the symbolic form of a function $f(x)$ having domain $(-\infty, \infty)$ and range $[2, \infty)$.

2. Write the symbolic form of a function $g(x)$ having domain $(-\infty, \infty)$ and range $(-\infty, 2]$.

3. Write the symbolic form of a function $h(x)$ having domain $(-\infty, -5) \cup (-5, \infty)$ and range $(-\infty, 3) \cup (3, \infty)$.

4. Write the symbolic form of a function $F(x)$ having domain $(4, \infty)$ and range $(-\infty, \infty)$.

5. Write the symbolic form of a function $G(t)$ having domain $(-\infty, \infty)$ and range $(-\infty, 0)$.

6. Write the symbolic form of a function $H(t)$ having domain $(-\infty, 1]$.

7. Write the symbolic form of a function $j(x)$ having domain $[0, 10) \cup (10, \infty)$.

Inverses

Student Learning Objective:

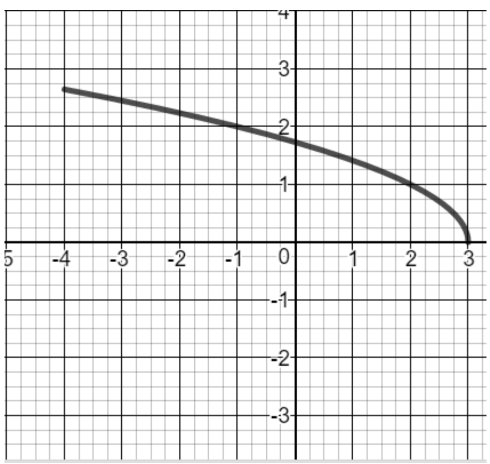
- Given the graph of a one-to-one function, sketch its inverse and compare the relationship between the domain and range of a function to its inverse.

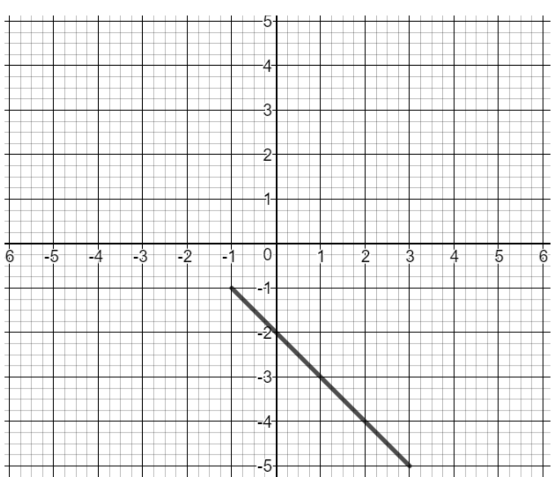
Exercises

1. State the inverse set for this relation: {(1,-3), (-2, 3), (5, 1), (6, 4)}

2. State the inverse set for this relation: {(-5, 7), (-6,-8), (1,-2),(10, 3)}

3. Use each graph to sketch the graph of its inverse. Include the identity line in your sketch.

a. 

b. 

4. Here we are given information about three sets of new inverse functions. Fill in the blanks:

	$f(x)$	$f^{-1}(x)$
Domain	$(-\infty,1)$	$(2,\infty)$
Range		
	$g(x)$	$g^{-1}(x)$
Domain	$(-\infty,\infty)$	
Range	$(-2,2)$	
	$h(x)$	$h^{-1}(x)$
Domain		
Range	$(-5,\infty)$	$(-\infty,0)$

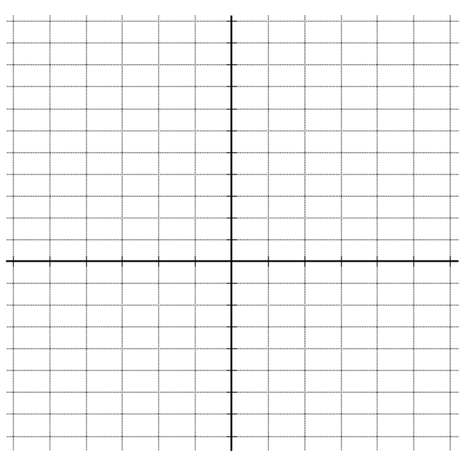
5. Use Desmos to graph the function $f(x)=\sqrt{x-3}$.

a. Is this a one-to-one function?

b. What is the domain of $f(x)$?

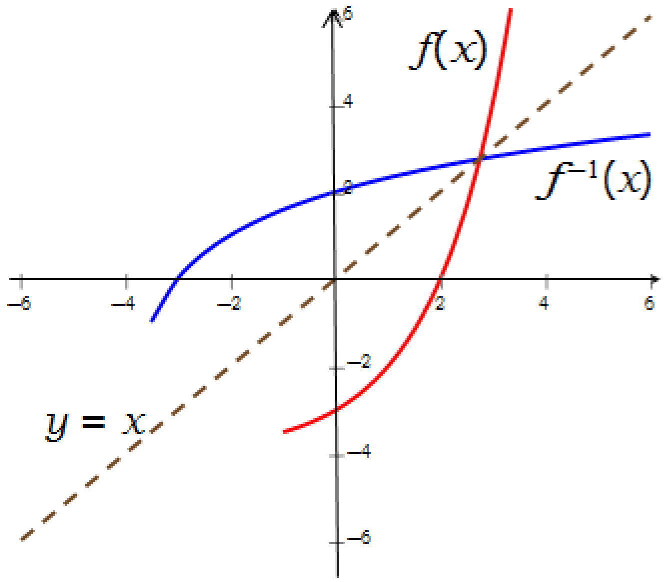
c. What is the range of $f(x)$?

d. Sketch the inverse function, $f^{-1}(x)$, below. Include the identity line, $y=x$ on your graph.



e. State the domain and range of $f^{-1}(x)$.

Recall that given a one-to-one function $f(x)$, its inverse function, $f^{-1}(x)$, has several properties that are related to the original function, $f(x)$.



First, the graph of $f^{-1}(x)$ is the reflection about the line $y = x$ of the original function $f(x)$. Second, given a table of values of the original function $f(x)$, then the values of $f^{-1}(x)$ are easily determined.

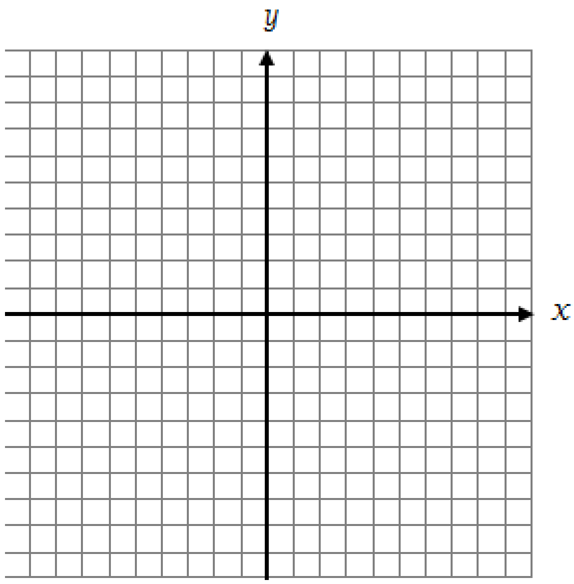
x	-2	-1	0	1	2	4	6
$f(x)$	-3	-1	1	3	5	9	13

x	-3	-1	1	3	5	9	13
$f^{-1}(x)$	-2	-1	0	1	2	4	6

Third, the domain of $f(x)$ is the range of $f^{-1}(x)$ and the range of $f(x)$ is the domain of $f^{-1}(x)$.

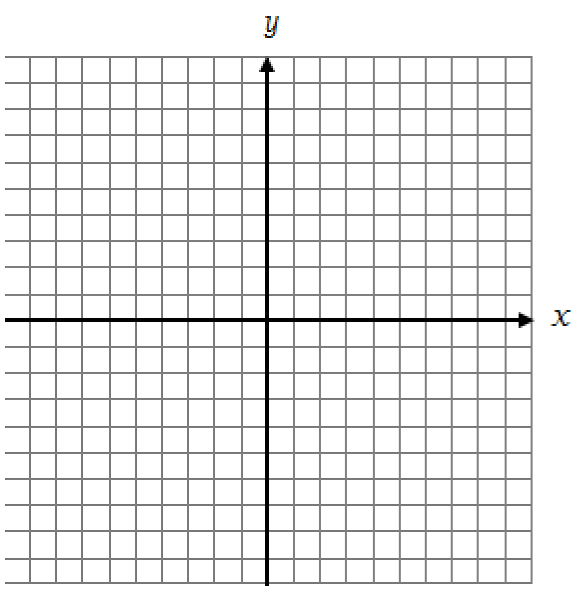
Graphical Analysis

1. Sketch $f(x)=4^x$ on the grid below. After you have done so, use the property that $f^{-1}(x)$ is the reflection about $y=x$ to sketch $f^{-1}(x)$ on the same grid.



What is the asymptote for $f(x)=4^x$? What is the asymptote for the inverse function?

2. Sketch $g(x)=\left(\frac{1}{3}\right)^x$ on the grid below. After you have done so, use the property that $g^{-1}(x)$ is the reflection about $y=x$ to sketch $g^{-1}(x)$ on the same grid.



What is the asymptote for $g(x)=\left(\frac{1}{3}\right)^x$? What is the asymptote for the inverse function?

Average Rate of Change

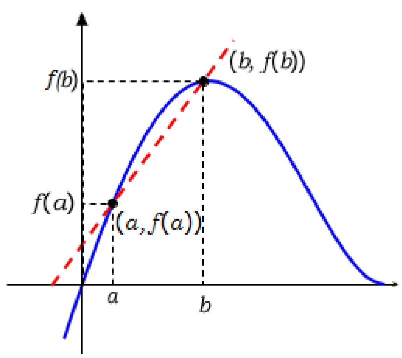
Student Learning Objectives:

- Determine the average rate of change from one point on a function to another and describe the average rate of change in the context of an application.
- From a function in symbolic form, determine the average rate of change (ARC) of the function between two given points and the difference quotient of the function.

The average rate of change of the function $y = f(x)$ between $x = a$ and $x = b$ is

$$\text{Average Rate of Change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

The average rate of change (ARC) is the slope of the secant line (a line that passes through exactly two points on a curve) between $x = a$ and $x = b$ (or, "from $x = a$ to $x = b$). It is the slope of the line that passes through the points $(a, f(a))$ and $(b, f(b))$.



Example 1. For the function $f(x) = x^3 - 4x^2$, determine the average rate of change between $x = 1$ and $x = 3$.

$f(1) = (1)^3 - 4(1)^2 = -3$

$f(3) = (3)^3 - 4(3)^2 = -9$

$$ARC = \frac{f(3) - f(1)}{3 - 1} = \frac{-9 - (-3)}{3 - 1} = -3$$

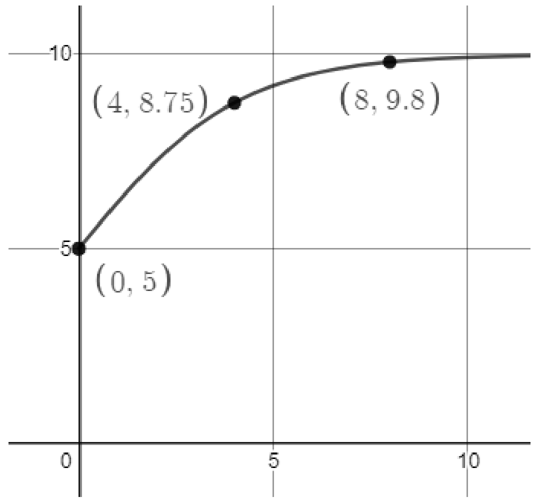
Exercises

1. Determine the average rate of change of $f(x) = 2x^3 - 7x^2 + 15$ from $x = -1$ to $x = 0$.

2. Determine the average rate of change of $f(x) = 2^x - x^2$ from $x = 3$ to $x = 5$.

3. If $f(x) = 2x - 3$, determine and simplify the average rate of change between $x = a$ and $x = a + h$. What value does this compare to in the original function?

4. Suppose a function p gives the population of wild turkey in Wake County (in thousands of turkeys) as a function of t years since enacting legislation protecting turkey from being over-hunted. The graph of $y = p(t)$ is show below.



a. Label each axis with the variable/quantity and units.

b. Use the graph to determine the average rate of change of p from $t = 0$ to $t = 4$.

c. Use the graph to determine the average rate of change of p from $t = 4$ to $t = 8$.

d. Compare these rates of change in context.

e. From this graph alone, what would you expect the turkey population in Wake County to approach as time goes on? What graphical component does this correspond to?

f. Suppose the NC Wildlife Resources Commission does a survey to recalculate the population of turkeys every 4 years. In Wake County, what would you expect the average rate of change over the 4 year intervals to approach?

5. The table below gives the outside temperature in degrees Fahrenheit on a summer afternoon.

Time (PM)	12:00	1:00	3:15	4:30	6:00
Temperature (°F)	78	82	88	93	91

a. What is the average rate of change in temperature from 12:00 to 1:00? Interpret this number in the context of the problem.

b. What is the average rate of change in temperature from 1:00 to 4:30? Interpret this number in the context of the problem.

c. What is the average rate of change in temperature from 4:30 to 6:00? Interpret this number in the context of the problem.

d. Estimate the time interval(s) when the temperature was rising the fastest.

6. Adjusted Average Salaries for Bachelor's Degree Graduates, S , are given by the following table, where $t = 0$ represents the year 1960. (Information retrieved from https://www.naceweb.org/job-market/compensation/salary-trends-through-salary-survey-a-historical-perspective-on-starting-salaries-for-new-college-graduates/)

t	0	10	20	30	40
S	$47,442	$58,650	$51,047	$48,832	$54,304

a. Find the average rate of change for the adjusted average salaries between $t = 10$ and $t = 40$ Round your answer to 2 decimal places and include appropriate units.

b. Using a complete sentence, interpret your answer from part a within the context of the problem.

Just in Time Learning…Working with Functions

Evaluate each by substituting a numerical value or variable into the function equation and simplify.

1. Evaluate $f(-1)$ if $f(x)=x^2-1$

2. Evaluate $h(3)$ if $h(x)=3x^2$

3. Evaluate $f(-7)$ if $f(w)=16+3w-w^2$

4. Evaluate $g(w)$ if $g(x)=2x^6-10x^4-x^2+5$

5. Evaluate $k(m+2)$ if $k(x)=3x+4$

6. Evaluate $h(-y)$ if $h(x)=2x^2-x+3$

Combining Functions by Addition, Subtraction, Multiplication

Student Learning Objectives:

- For a given function or set of functions in symbolic form, apply the operations of function addition, subtraction, multiplication, division, as well as composition and decomposition.

Adding, Subtracting, and Multiplying functions algebraically

Operations with Functions: Given functions f and g, their sum, difference, and product are defined using their expressions as well as their individual outputs.

Sum: $(f+g)(x)=f(x)+g(x)$ $\quad$ $Domain_{f+g} : Domain_f \cap Domain_g$

Difference: $(f-g)(x)=f(x)-g(x)$ $\quad$ $Domain_{f-g} : Domain_f \cap Domain_g$

Product: $(f\cdot g)(x)=f(x)\cdot g(x)$ $\quad$ $Domain_{f\cdot g} : Domain_f \cap Domain_g$

Functions can be combined in general, with the input variable, by performing the indicated operation with the expressions, or just at a particular value, by performing the indicated operation with values.

Example

Let $f(x)=x^3-4$ and $g(x)=5x+4$. To evaluate $(f+g)(2)$, we could construct and simplify $(f+g)(x)$ and then evaluate at $x=2$. Or, we could determine $f(2)$ and $g(2)$ separately, and then add to determine $(f+g)(2)$.

$(f+g)(x)=(x^3-4)+(5x+4)=x^3+5x$, so $(f+g)(2)=(2)^3+5(2)=8+10=18$.

Or, $f(2)=(2)^3-4=8-4=4$ and $g(2)=5(2)+4=10+4=14$, so $(f+g)(2)=f(2)+g(2)=4+14=18$.

Exercises

Given functions f and g for each problem, fill in the table below.

1. $f(x)=5x+4; g(x)=x^2-5x+4$ $\qquad$ 2. $f(x)=\sqrt{5-x}\ ; g(x)=\sqrt{x+1}$

Operation	Problem 1	Problem 2
$(f+g)(x)$		
$(f-g)(x)$		
$(f\cdot g)(x)$		
Domain of $f+g$, $f-g$, and $f\cdot g$		

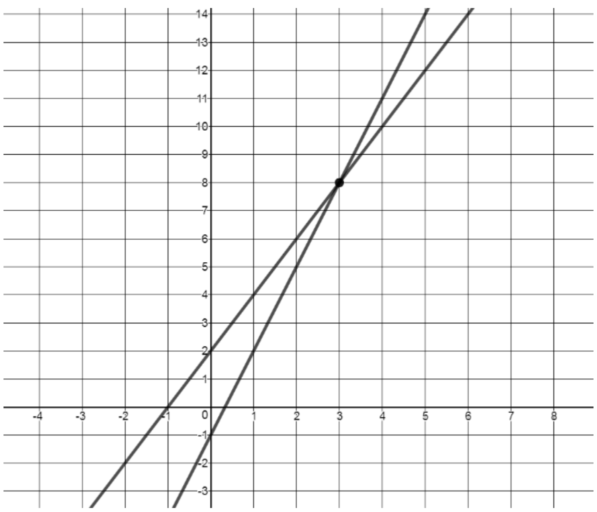
Solving a System of Equations – Graphical Method

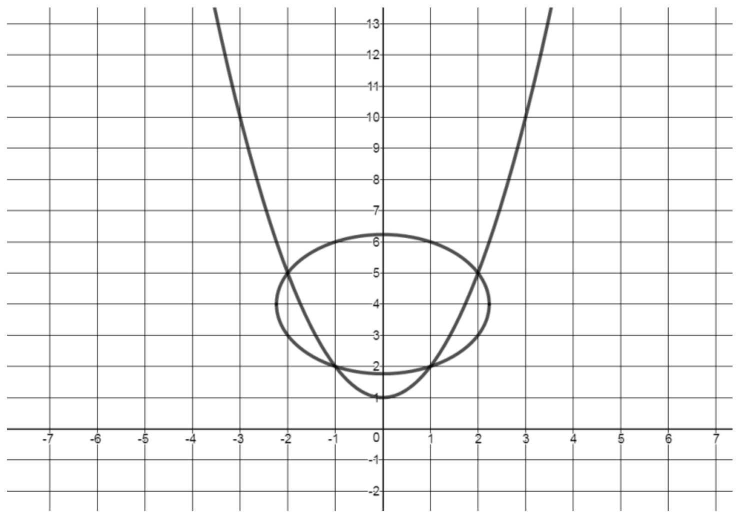
Student Learning Objective:

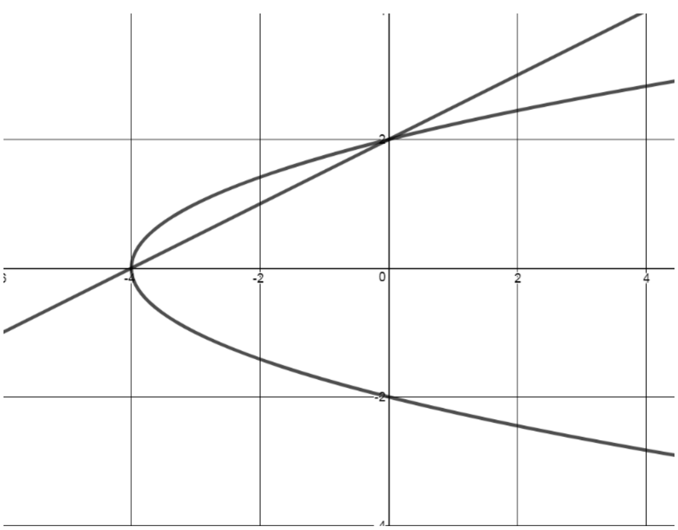
- Determine the solution to a system of equations in graphical form.

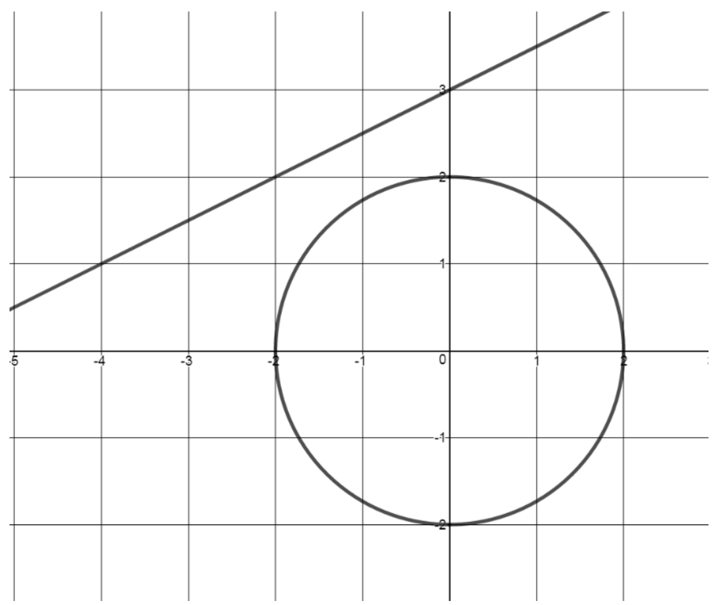
Solving Systems Graphically Practice

Determine the solutions to each of the following systems. If there is no solution write, DNE and explain why there is no solution.









Writing systems of equations satisfying certain conditions

For each exercise below you must write the symbolic form of a system of equations, such as

$$\begin{cases} y = x^2 - 1 \\ y = \frac{1}{2}x + 2 \end{cases}$$

which satisfies the indicated type of functions/graphs, as well as the number of solutions. For example, the system above consists of a line and a parabola, and has two solutions (a quick sketch of each function/equation can show you this). The individual functions/equations can be simple, as long as they satisfy the requirements. Generate a system of equations consisting of:

1. two lines with one solution
2. two lines with zero solutions
3. an absolute value function and a parabola, with two solutions
4. an absolute value function and a parabola, with one solution
5. an absolute value function and a parabola, with zero solutions
6. an exponential function and a line, with two solutions
7. a rational function and a line, with two solutions
8. a rational function and a line, with zero solutions
9. a radical function and a line, with two solutions
10. a radical function and a line, with one solution

Absolute Value Equations

Student Learning Objectives:

- Solve problems involving absolute value equations and piecewise functions.

Example: A class average on a math test was 81 and Jane scored 5 points off the class average. What did Jane score on that math test?

The difference between Jane's score, *s*, and the class average can be expressed as $s-81$. The absolute difference is five, meaning Jane score could be five more or five less than the class average.

$$|s-81|=5,$$

but the answer can be 5 less than the average (-5) or five more than the average (+5). So

$$s-81=5 \text{ or } s-81=-5.$$

Absolute value equations can help us better understand differences between quantities and solve problems in systematic ways.

Definition: The absolute value function $y=|x|$ is the number's distance from zero along the real number line, and more generally the absolute value of the difference of two real numbers is the distance between them. $y=|x|$ is also the distance between zero and -x, the additive inverse of x.

$$|x|=\begin{cases} x, & if\ x\geq 0 \\ -x, & if\ x<0 \end{cases}$$

Further, $|x-a|=d$ means that the distance between x and a is d units. So in the example above we get two cases to solve:

a. $s-81=5$ and

b. $-(s-81)=5$, which is the same as $s-81=-5$ that we had above.

Part 1

Solve the following absolute value equations.

1. $|4-5t|=11$

2. $-3|b+4|=-9$

3. $|2a-1|+7=12$

7. $\dfrac{|2-3w|}{4}=2$

4. $-3|c-5|+12=9$

8. $|x^2-5x|=6$

5. $\left|\dfrac{1}{2}x-3\right|=0$

9. $|x^2-16|=0$

6. $-8-3|x-1|=-32$

10. The equation $|x|=-2$ has no solution. Explain why.

Part 2:

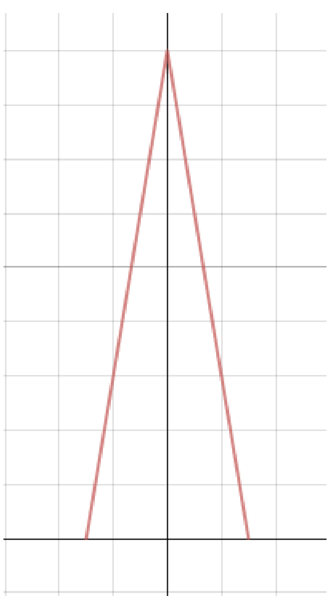
Solve the following absolute value application problems.

1. A rainstorm begins as a drizzle, builds up to a heavy rain, and then drops back to a drizzle. The rate *r* inches per hour at which it rains is given by the function $r = -0.5|t-1|+0.5$ where *t* is time in hours since the rainstorm began.

    (A) Graph the function.

    (B) For how long is the rain increasing in intensity?

2. The Transamerica pyramid is an office building in San Francisco. It stands approximately 900 feet tall and is approximately 150 feet wide at its base. Imagine that a coordinate plane is superimposed on the building's triangular structure. The horizontal and vertical axes are measured in feet and the origin of the plain is at the center of the building's base. Write an absolute value function whose graph models the V-shaped outline of the sides of the building.



Module 1 Practice

1. Match the following graphs to the function that they represent:

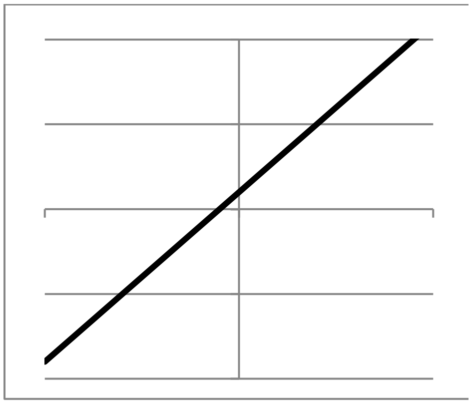


Figure 1: ______________________________

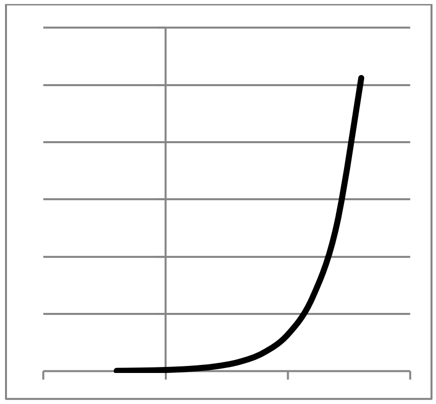


Figure 4: ______________________________

Figure 7: ______________________________

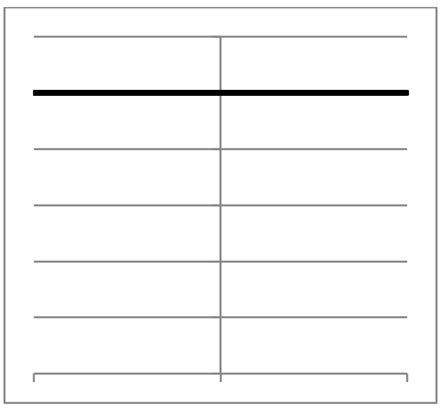


Figure 2: ______________________________

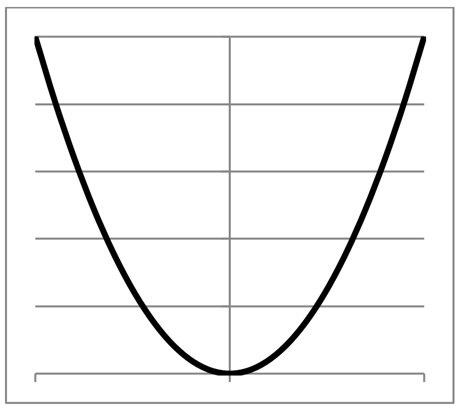


Figure 5: ______________________________

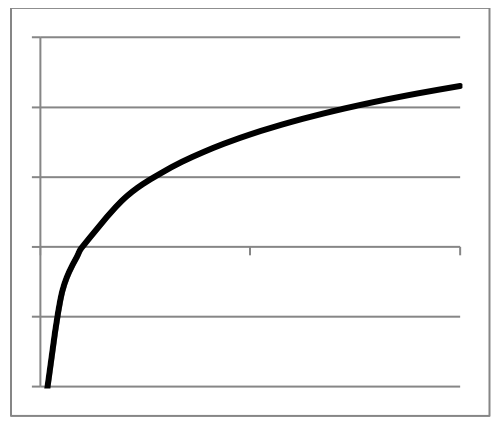


Figure 8: ______________________________

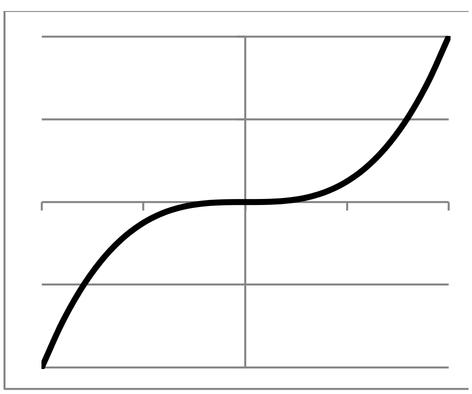


Figure 3: ______________________________

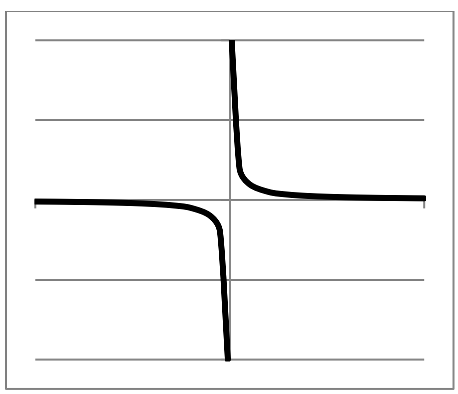


Figure 6: ______________________________

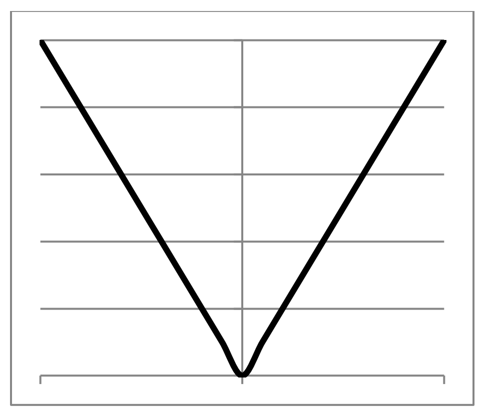
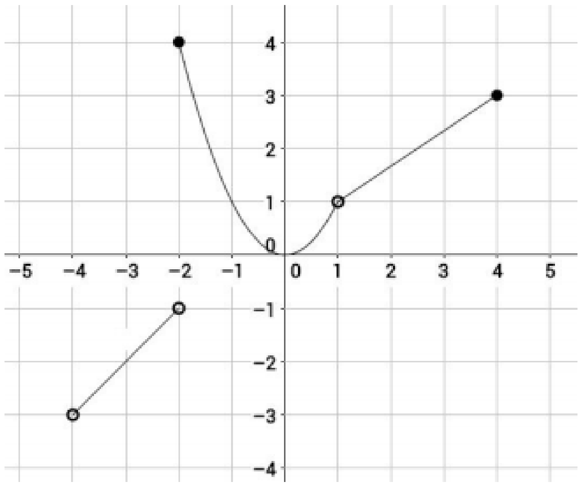


Figure 9: ______________________________

- ☐ linear function
- ☐ constant function
- ☐ rational function
- ☐ exponential function
- ☐ radical function
- ☐ quadratic function
- ☐ polynomial function
- ☐ logarithmic function
- ☐ absolute value function

2. Use the following graph of function, f to complete the following:



a. Is f one-to-one?

b. Write the domain of f using interval notation.

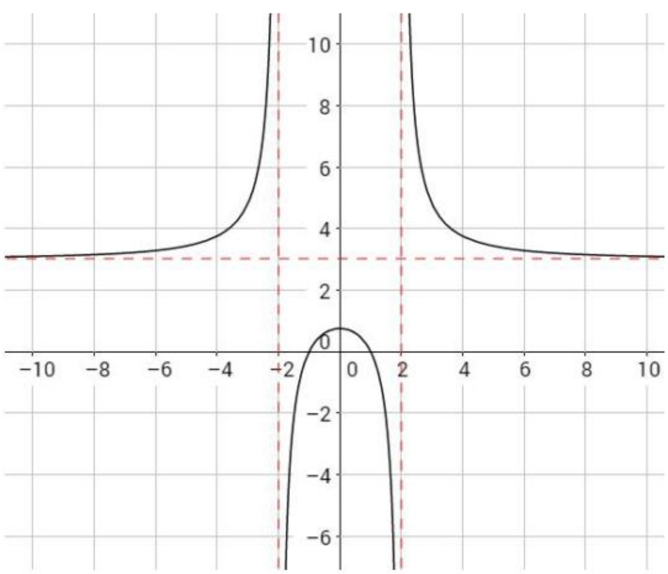
c. Write the range of f using interval notation.

d. $f(-2) =$

e. If $f(x) = -1$, then $x =$

Write the domain and range of the following in interval notation. Describe the end behavior of the following graphs and give the equations of any horizontal and vertical asymptotes:

3.



a. Domain:

b. Range:

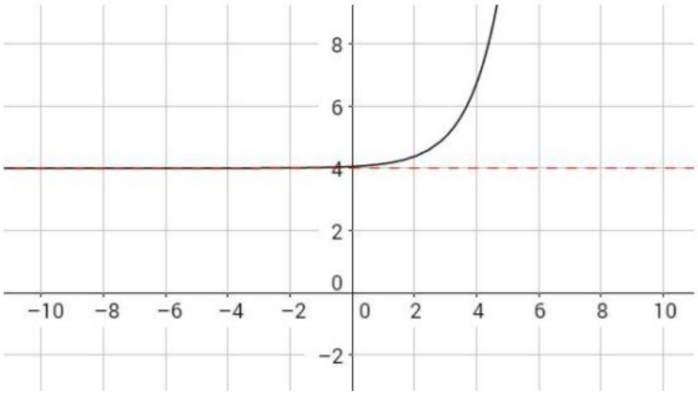
c. Vertical Asymptote(s):

d. Horizontal Asymptote:

e. As $x \to -\infty$, $f(x) \to$

f. As $x \to \infty$, $f(x) \to$

4.



a. Domain:

b. Range:

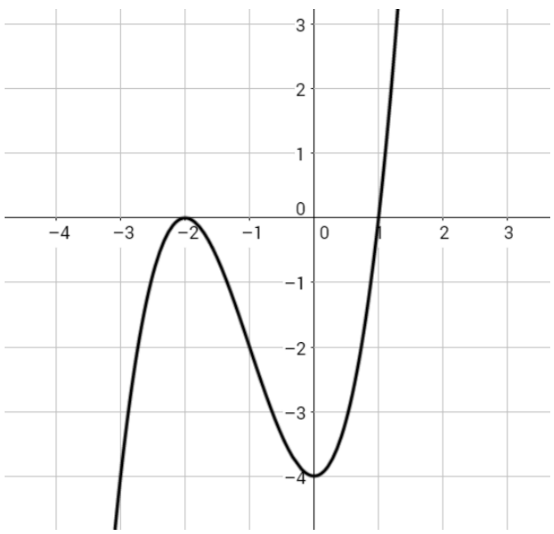
c. Vertical Asymptote:

d. Horizontal Asymptote:

e. As $x \to -\infty$, $f(x) \to$

f. As $x \to \infty$, $f(x) \to$

5.



a. Domain:

b. Range:

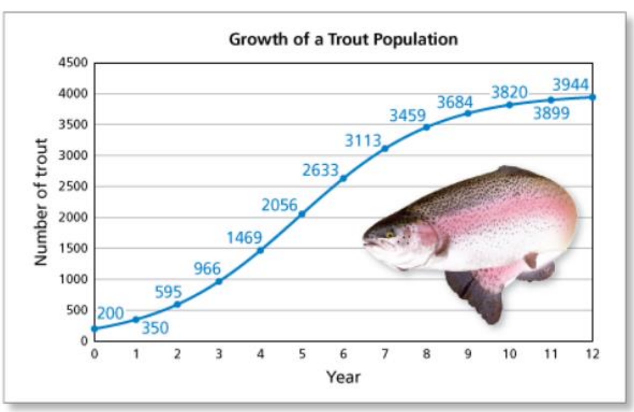
c. Vertical Asymptote:

d. Horizontal Asymptote:

e. As $x \to -\infty$, $f(x) \to$

f. As $x \to \infty$, $f(x) \to$

6. Use the graph to complete the following.



a. Determine the average rate of change from year 2 to year 4.

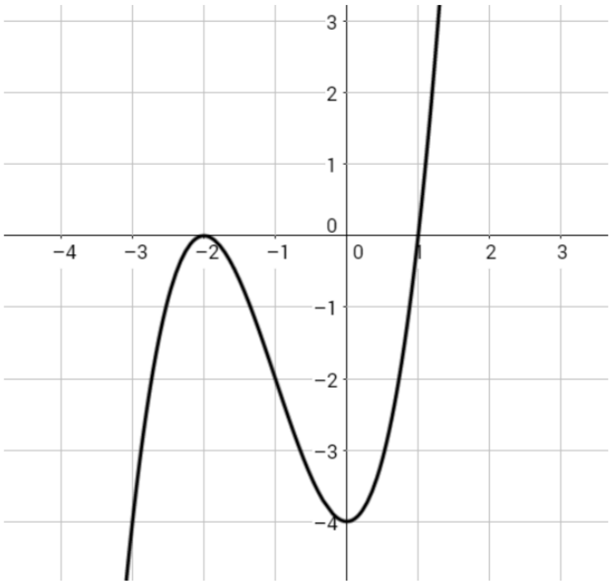
b. Interpret this rate of change in the context of the problem.

c. Determine the equation of the horizontal asymptote.

d. Write a sentence that interprets the horizontal asymptote as a limiting value.

Complete the following for the graph of f and the graph of g.

7.



$f(x)$

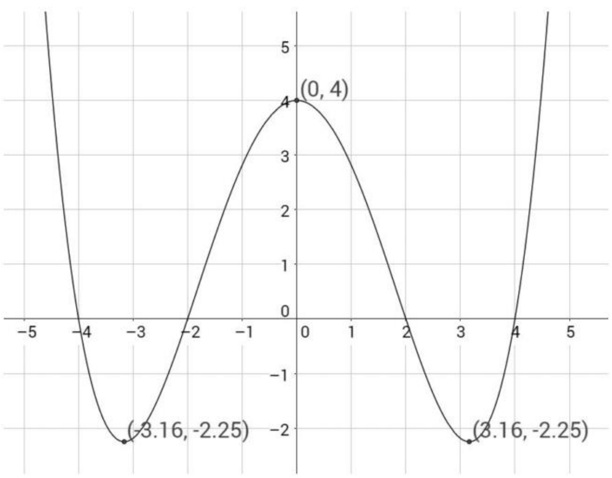
a. Determine all absolute extrema.

b. Determine intervals where the function is increasing. Write your answer in interval notation.

c. Determine intervals where the function is decreasing. Write your answer in interval notation.

d. Determine the roots (zeros) of the function.

8.



$g(x)$

a. Determine all local extrema values. Write your answer as an ordered pair.

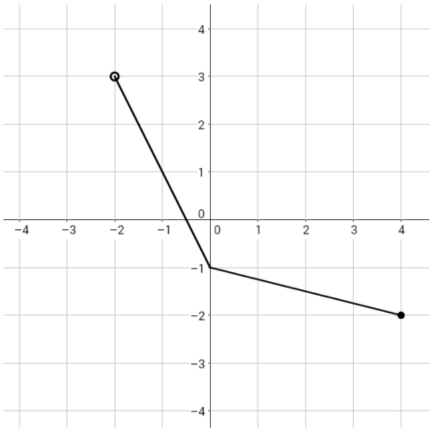
b. Determine all absolute extrema. Write your answer as an ordered pair.

c. Determine interval(s) where the function is increasing. Write your answer in interval notation.

d. Determine interval(s) where the function is decreasing. Write your answer in interval notation.

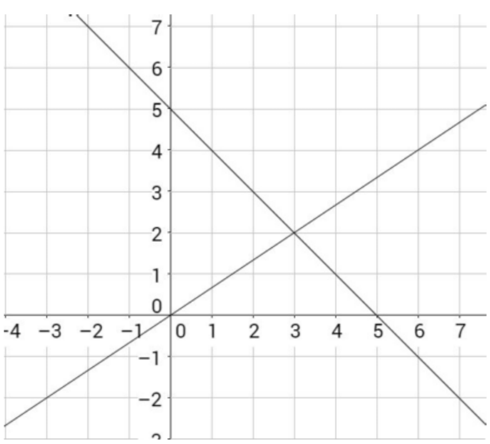
e. Determine the roots (zeros) of the function.

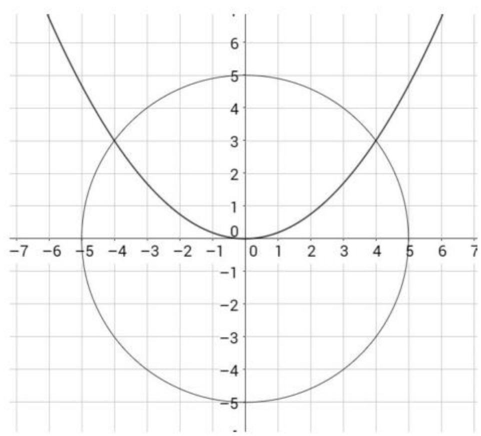
9. Use the graph of f to graph f^{-1} and fill in the table with the domain and range of each function.



Function	f	f^{-1}
Domain		
Range		

10. Determine the solutions (if they exist) to the following systems of equations.



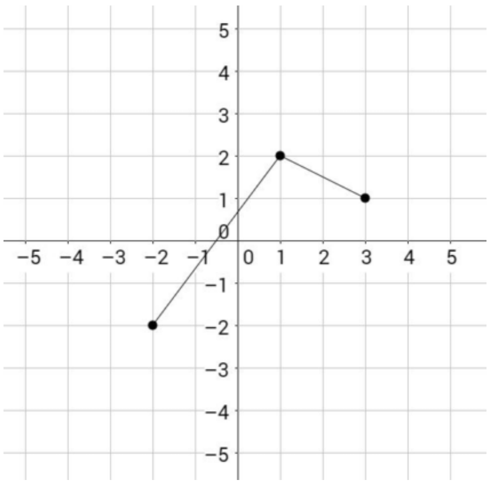


11. Generate a system of equations consisting of a quadratic function and $y = 2^x$ having

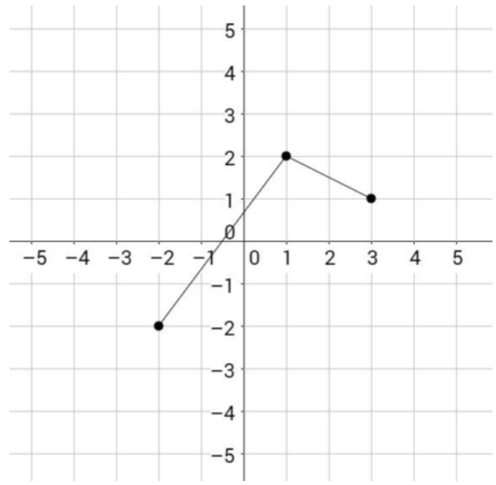
   a. Zero solutions
   b. one solution
   c. two solutions
   d. three solutions

12. Graph the following transformations of the function of f shown below.

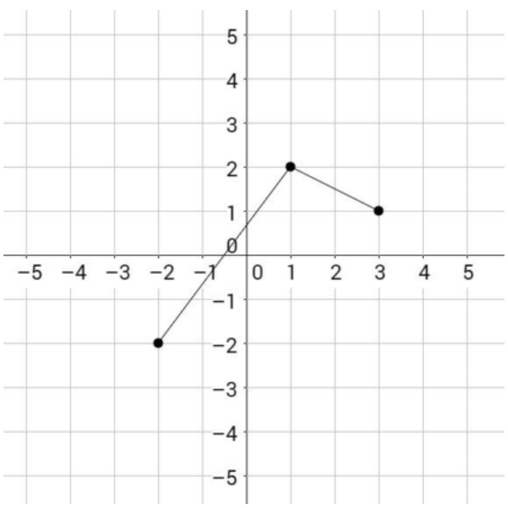
$2f(x)$



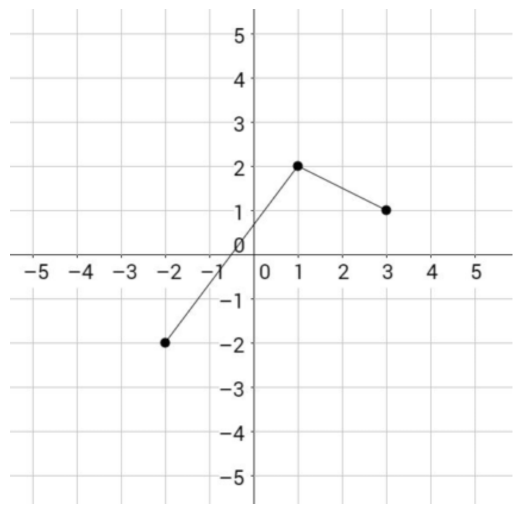
$-f(x+2)$



$3+f(x+1)$

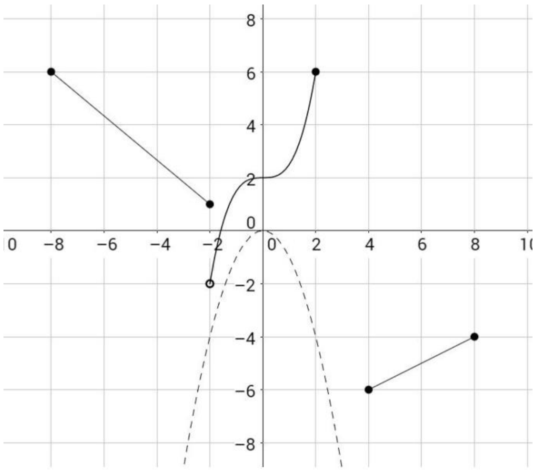


$\frac{1}{2}f(x)+2$



13. Use transformations to sketch the graphs of the functions below based off of the associated parent function. Describe each necessary transformation, in order, using terminology like "vertical, horizontal, shift, stretch, compression, and reflection."
    a. $f(x) = 10 - 4\sqrt{x}$
    b. $g(x) = e^{x+3} - 4$
    c. $h(t) = 50 - 16.1t^2$
    d. $B(X) = \frac{4}{X+3} - 1$

14. Use the graph of f, the piecewise function indicated by the solid line, and g, indicated by dashed line to complete the following:



a. Domain of f :

b. Range of f :

c. $2f(-1)+3$

d. If $g(x)=-4$ then

e. $2f(-8)+g(2)$

f. Is the inverse of function g a function? Explain.

15. Given $f(x)=5x^2-3x+2$, determine the average rate of change of f on each interval.
   b. $[-2,1]$
   c. $[0,4.5]$
   d. $[1,1.001]$

16. For each of the pairs of functions in questions 1 through 3 below, compute:
   a. The domain of f and g
   b. $(f+g)(x)$
   c. $(f-g)(x)$
   d. $(f\cdot g)(x)$
   e. The domain of $f+g$, $f-g$, and $f\cdot g$.
   f. $(f+g)(6), (f-g)(6)$, and $(f\cdot g)(6)$

i. $f(x)=3x+2;\ g(x)=-2x-5$

ii. $f(x)=\sqrt{x-5};\ g(x)=x^2+3$

iii. $f(x)=\dfrac{x}{x+1};\ g(x)=x^3$

17. Determine the domain of the following functions.

   a. $f(x)=\dfrac{2x-7}{x^2-3x-4}$

   b. $f(x)=\sqrt{4x-7}$

c. $f(x)=\dfrac{\sqrt{2-x}}{x^2+3x+2}$

d. $f(x)=\dfrac{2x^2+9x+9}{\sqrt{3x+7}}$

e. $f(x)=\sqrt[3]{x^2+4}$

f. $f(x)=\dfrac{x^2-4}{\sqrt[3]{5x-30}}$

18. Solve the following equations.

a. $-8-3|x-1|=-32$

b. $8-4|2x-5|=10$

c. $\dfrac{|2-3w|}{4}=2$

d. $2|2x-5|+3=11$

Operations with Functions and Composition of Functions (Symbolic/Algebraic)

Student Learning Objectives:

- For a given function or set of functions in symbolic form, apply the operation of function division, as well as composition and decomposition.

Part 1

Remember:

Sum: $(f+g)(x)=f(x)+g(x)$

Difference: $(f-g)(x)=f(x)-g(x)$

Product: $(f\cdot g)(x)=f(x)\cdot g(x)$

Division with Functions: given functions f and g

Quotient: $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}\, where\, g(x)\neq 0$

Given functions f and g fill in the table below.

1. $f(x)=5x+4; g(x)=x^2-1$
2. $f(x)=\sqrt{5-x}\ ; g(x)=\sqrt{x+1}$

Operation	Problem 1	Problem 2
$\frac{f(x)}{g(x)}$		
Domain of $\frac{f(x)}{g(x)}$		

Part 2

Composition of Functions: given functions f and g, **notation**: $(f \circ g)(x) = f(g(x))$.

What is a composition of functions? ____

3. Here's one that we can complete numerically: If $f(x) = 2x - 3$ and $g(x) = -x^2$ determine $f(g(2))$ and $g(f(2))$.

$f(g(2))$ =

$g(f(2))$ =

Determine $(f \circ g)(x)$ and $(g \circ f)(x)$ for each $f(x)$ and $g(x)$, and then determine the domain of $f(g(x))$:

4. $f(x) = x^2 - 1; g(x) = 3x$

5. $f(x) = \sqrt{x+1}\ ; g(x) = x - 2$

	Problem 4	Problem 5
$(f \circ g)(x)$		
$(g \circ f)(x)$		
Domain of $f(g(x))$		

6. $f(x) = x^2 - x; g(x) = x - 8$

7. $f(x) = \dfrac{1}{x^2 - 10}; g(x) = \sqrt{2 - 4x}$

	Problem 6	Problem 7
$(f \circ g)(x)$		
$(g \circ f)(x)$		
Domain of $f(g(x))$		

More Practice with Combining Functions

Problem A: Let $f(x) = \dfrac{2}{x+1}$ and $g(x) = \dfrac{x}{x+1}$. Determine the following functions and the domain of each:

1. $f + g$

3. fg

5. $\dfrac{f}{g}$

2. $f - g$

4. $\dfrac{g}{f}$

6. $g(f(1))$

Problem B: Let $a(t) = 3t - 21$ and $b(t) = \sqrt{2t-9}$. Determine the following functions and the domain of each:

1. $\left(\dfrac{b}{a}\right)$
2. $(b \cdot b)$
3. $(a \circ b)$
4. $b(a(x))$
5. $(a \circ a)$
6. $(a(b(5)))$

For each of the following functions, $h(x)$ and $g(x)$, determine a nontrivial function $f(x)$ such that $h(x) = (f \circ g)(x)$.

Example: If $h(x) = (x+3)^2$ and $g(x) = x + 3$, then $f(x) = x^2$ gives
$(f \circ g)(x) = f(g(x)) = f(x+3) = (x+3)^2 = h(x)$.

1. $h(x) = 3|x+2|$ and $g(x) = x + 2$.
2. $h(x) = -2(x-1)^2 + 8$ and $g(x) = x - 1$.
3. $h(x) = \dfrac{5}{(x+10)^3}$ and $g(x) = (x+10)^3$.
4. $h(x) = \sqrt[3]{2x^2 - 5}$ and $g(x) = 2x^2$.

For each of the following functions F, determine non-trivial functions f and g so that $F = (f \circ g)$.

1. $F(x) = (1+x^2)^3$

3. $F(x) = \dfrac{3}{2x+7}$

2. $F(x) = \sqrt{2x-9}$

4. $F(x) = |2x^2+3|$

For each of the following functions F, determine non-trivial functions $f, g,$ and h so that $F = f \circ g \circ h$.

1. $F(x) = \left|(2x+4)^2+5\right|$

2. $F(x) = \sqrt[3]{2+\sqrt{x}}$

3. $F(x) = \left(4+\sqrt{3x}\right)^5$

4. $F(x) = \dfrac{2}{\left(3+\sqrt{x}\right)^2}$

Difference Quotient of a Function

Student Learning Objectives:

- From a function in symbolic form, determine the difference quotient of the function.

The difference quotient is $\dfrac{f(x+h)-f(x)}{h}$, for $h \neq 0$.

1. Given $g(x)=3x-5$, determine and simplify the following:

   (A) $g(x+h)$ (B) $g(x+h)-g(x)$ (C) difference quotient

2. Given $f(x)=6x+2$, determine and simplify the following:

   (A) $f(x+h)$ (B) $f(x+h)-f(x)$ (C) difference quotient

3. Given $a(x)=\dfrac{1}{3}x+1$, determine and simplify the following:

   (A) $a(x+h)$ (B) $a(x+h)-a(x)$ (C) difference quotient

4. Given $r(x)=4x^2$, determine and simplify the following:

   (A) $r(x+h)$ (B) $r(x+h)-r(x)$ (C) difference quotient

5. Given $v(x)=\dfrac{1}{3x}$, determine and simplify the following:

   (A) $v(x+h)$ (B) $v(x+h)-v(x)$ (C) difference quotient

For each of the following functions, construct and simplify the difference quotient:

6. $g(x)=\frac{1}{3}x-2$

7. $s(t)=6-\frac{3}{4}t$

8. $f(a)=-\frac{1}{4a}$

9. $C(p)=p^2+1$

10. $v(t)=3t^2-2t+1$

11. $a(w)=9-\frac{1}{2}w^2$

12. $R(q)=2q^2-q$

13. $P(x)=2x^2-4x+3$

14. $W(n)=n^3+2n$

15. $T(z)=\frac{7}{z-4}$

Equations of Polynomial Functions

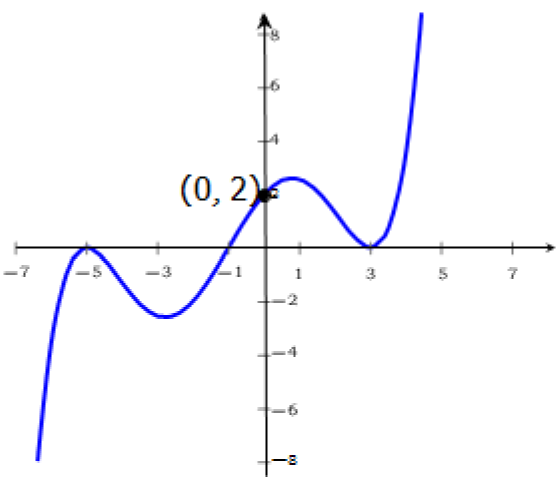
Student Learning Objectives:

- From the graph of a polynomial, rational, or exponential function, write its equation.

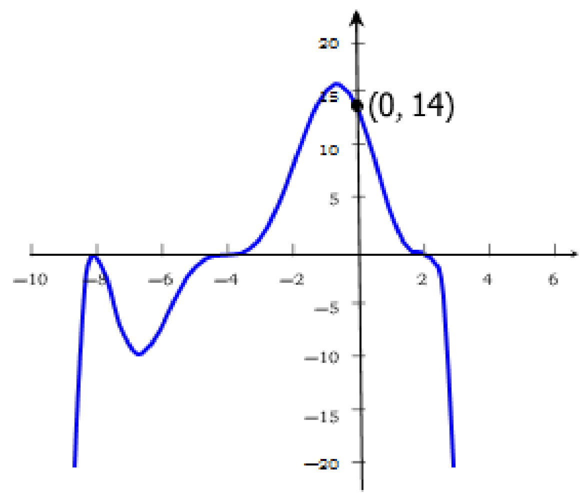
Part 1:

For each graph below, write the equation of the polynomial of lowest degree that could have the indicated graph.

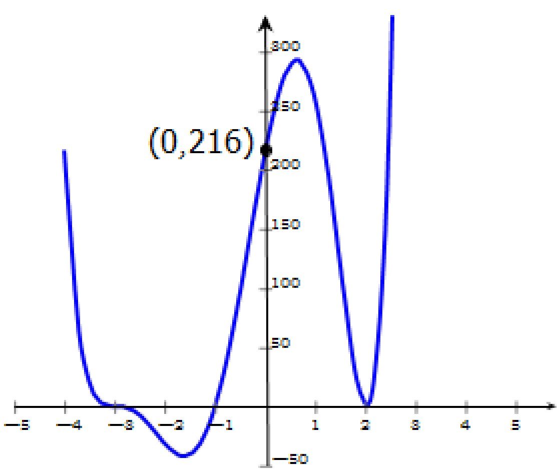
1.



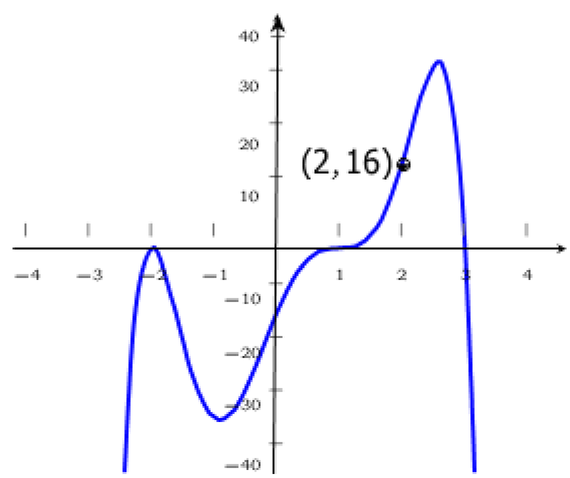
3.



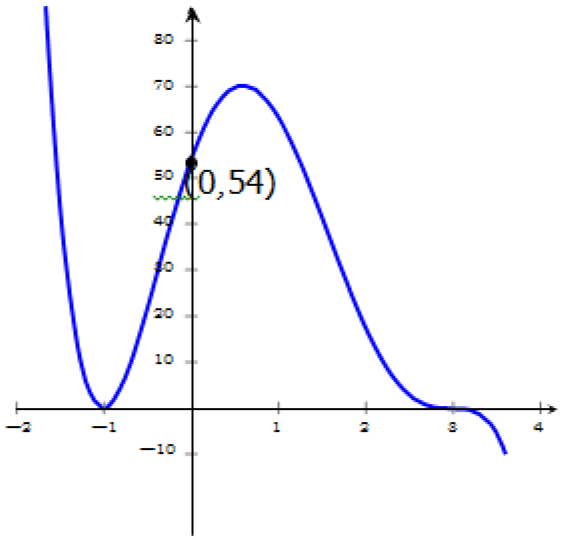
2.



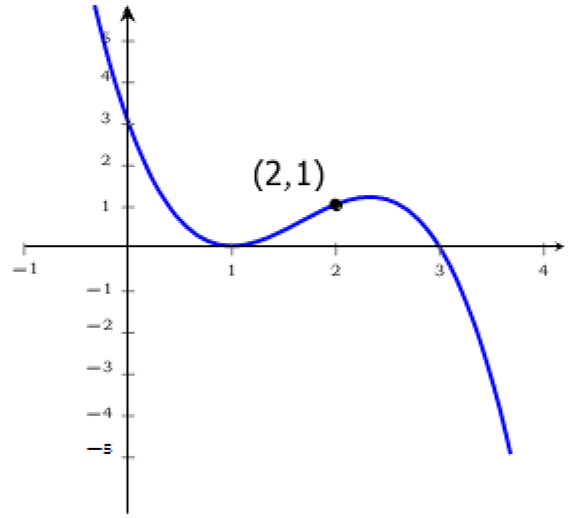
4.



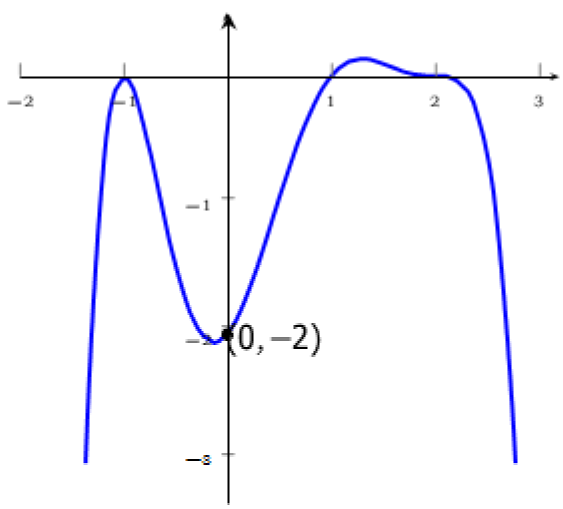
5.


80
70
60
50
(0,54)
40
30
20
10
−10
−2
−1
1
2
3
4

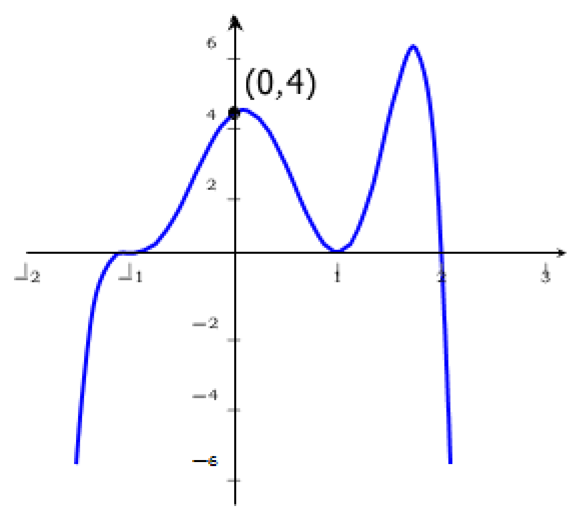
7.


5
4
3
2
1
(2,1)
−1
1
2
3
4
−1
−2
−3
−4
−5

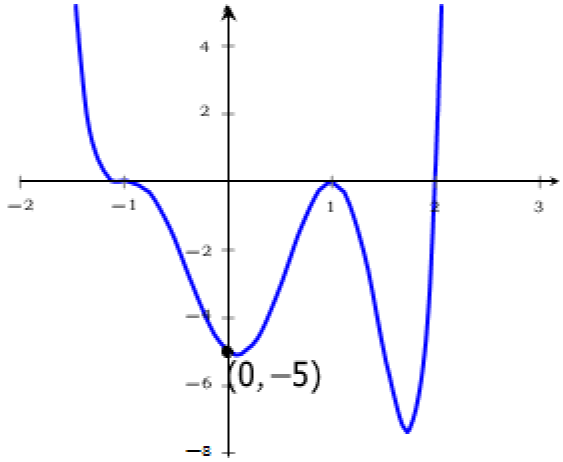
6.


−2
−1
1
2
3
−1
(0, −2)
−3

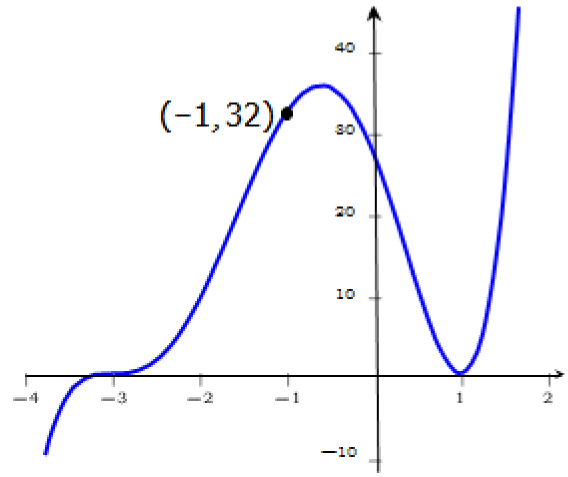
8.


6
(0,4)
4
2
−2
−1
1
2
3
−2
−4
−6

9.



10.



Part 2

Write a polynomial with real coefficients that satisfies the given conditions.

11. Q has degree 3; zeros 4,-1, and 5; and leading coefficient $\frac{1}{2}$.

12. P has degree 4; zeros $\frac{2}{3}$, 8 and -2; leading coefficient 9; and the zero -2 has multiplicity 2.

Quadratic Functions

Student Learning Objectives:

- From equation of a quadratic function in any form, determine its equation in transformation form, vertex, axis of symmetry, and x- and y-intercept(s).
  From the equation of a
- Solve quadratic equations and inequalities and related application problems using graphing, factoring, the quadratic formula, and completing the square.

<u>Part 1:</u> Given a quadratic equation in general or standard form, rewrite it in transformation form.

Steps: Given a quadratic equation of the form $f(x) = ax^2 + bx + c$,

1. The x-coordinate of the vertex, h, can be found by evaluating $h = \frac{-b}{2a}$
2. The corresponding y-coordinate of the vertex, k, can be found by evaluating $f(h)$.
3. Example: Rewrite $f(x) = 3x^2 + 4x + 5$ in transformation form.
   a. Determine h.
   $$h = \frac{-b}{2a} = \frac{-4}{2(3)} = \frac{-4}{6} = -\frac{2}{3}$$
   b. Determine k.
   $$k = f(h) = 3\left(-\frac{2}{3}\right)^2 + 4\left(-\frac{2}{3}\right) + 5$$
   $$= 3\left(\frac{4}{9}\right) + 4\left(-\frac{2}{3}\right) + 5$$
   $$= \frac{12}{9} - \frac{8}{3} + 5$$
   $$= \frac{4}{3} - \frac{8}{3} + 5$$
   $$= -\frac{4}{3} + 5$$
   $$= \frac{11}{3}$$
   c. The vertex of the function is $\left(-\frac{2}{3}, \frac{11}{3}\right)$
   d. In transformation form, the 'a' holds the same value therefore this function is

$$f(x) = 3\left(x + \frac{2}{3}\right)^2 + \frac{11}{3}$$

* The **axis of symmetry** of a quadratic function's graph is a vertical line with equation $x = h$, where h is the x-coordinate of the vertex. The graph of the quadratic function, a **parabola**, is symmetric about this line. In this previous example the axis of symmetry would be $x = -\frac{2}{3}$.

For the following functions, determine a) the vertex, b) the direction of opening (up or down), c) any max or min, d) the transformation form of the function.

1. $g(x)=2x^2-6x+9$
2. $j(x)=x^2+3x+7$
3. $n(x)=x^2+8x-3$

Part 2

Determine the x-intercepts (aka "roots" or "zeros") of a quadratic function.

Steps: Given a quadratic function of the form $f(x)=ax^2+bx+c$, the function will have:

1. **One x – intercept** when $b^2-4ac=0$
2. **Two x – intercepts** when $b^2-4ac>0$
3. **No x – intercepts** when $b^2-4ac<0$

Using Desmos, graph the following functions and then, verify the number of x-intercepts using $b^2 - 4ac$.

1. $f(x)=2x^2+9x-1$
2. $g(x)=-3x^2+4x-2$
3. $h(x)=2x^2-8x+8$

Steps: To determine the x-intercepts of a quadratic function that will not factor, use the quadratic formula.

Quadratic Formula: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Example: Determine the x-intercepts of $f(x)=2x^2-4x-9$.

$$a=2;\ b=-4;\ c=9$$

$$x=\frac{4\pm\sqrt{(-4)^2-4(2)(-9)}}{2(2)}$$

$$x=\frac{4\pm\sqrt{16+72}}{4}$$

$$x=\frac{4\pm\sqrt{88}}{4}$$

$$x=\frac{4\pm2\sqrt{22}}{4}$$

$$x=\frac{2\pm\sqrt{22}}{2}$$

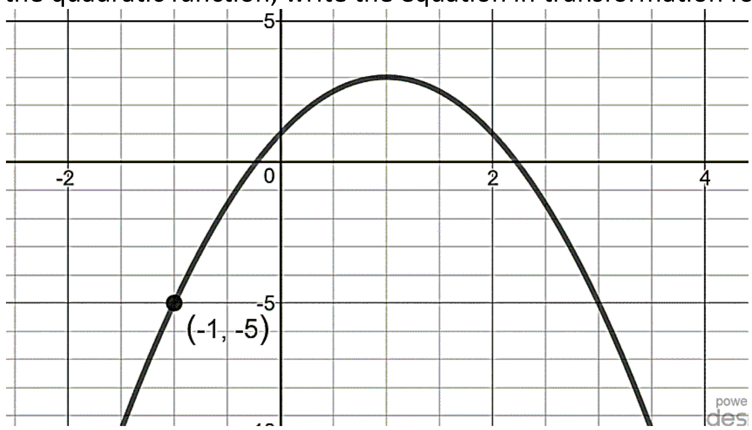
Determine the x-intercepts of the following functions algebraically and then check your answers using Desmos.

1. $f(x) = -3x^2 - 2x + 3$
2. $f(x) = x^2 + 5x - 7$

Part 3

Writing the Equation from the Graph of a Quadratic

Steps: Given the graph of the quadratic function, write the equation in transformation form:



1. Determine the ordered pair, $(h,\ k)$ of the vertex.

   $(1,\ 3)$

2. Substitute h and k into the function $f(x) = a(x - h)^2 + k$.

   $f(x) = a(x - 1)^2 + 3$

3. Now, choose another ordered pair (x, y) from the graph of the function.

   $(-1, -5)$

4. Substitute that point into the function for x and f(x) and solve for a.

   $-5 = a(-1 - 1)^2 + 3$

   $-5 = a(-2)^2 + 3$

   $-5 = 4a + 3$

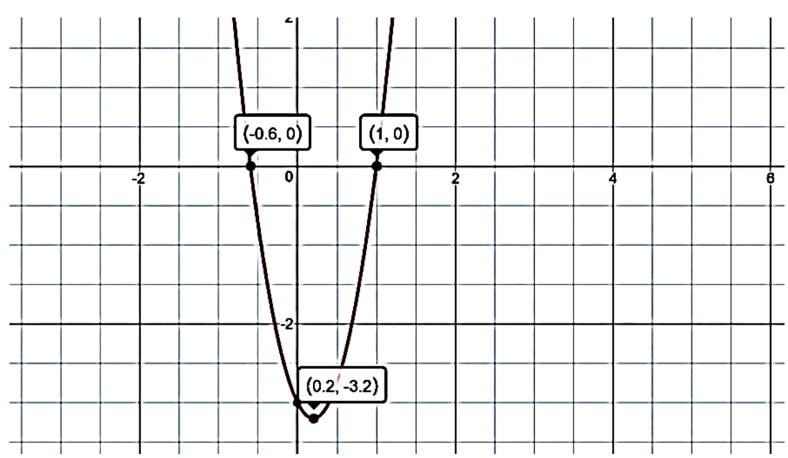
   $-8 = 4a$

   $-2 = a$

5. Write the quadratic function with h, k, and a in the correct places.

   $f(x) = -2(x - 1)^2 + 3$

<u>Practice:</u>

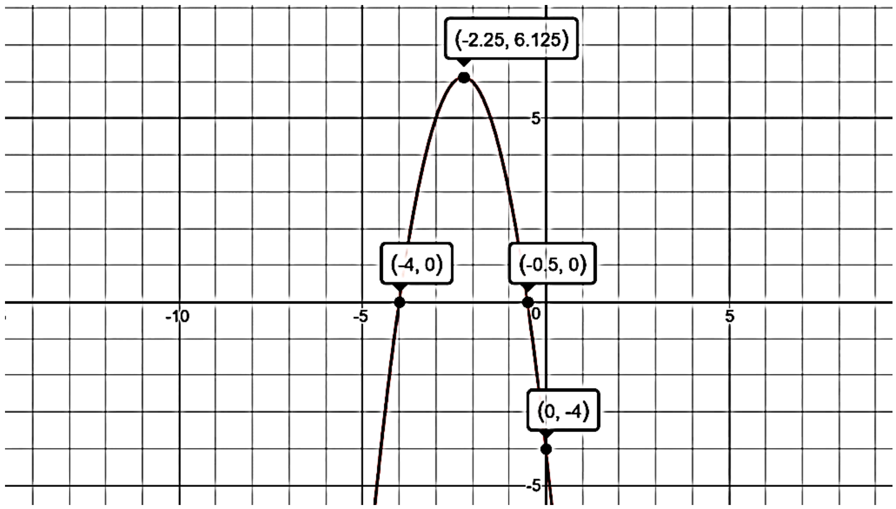


1. Given the graph to the right,
   a. Write the equation in transformation form, $f(x) = a(x - h)^2 + k$.

   b. Rewrite the equation in standard form, $f(x) = ax^2 + bx + c$.

2. Given the graph below,

   a. Write the equation in transformation form, $f(x) = a(x - h)^2 + k$.

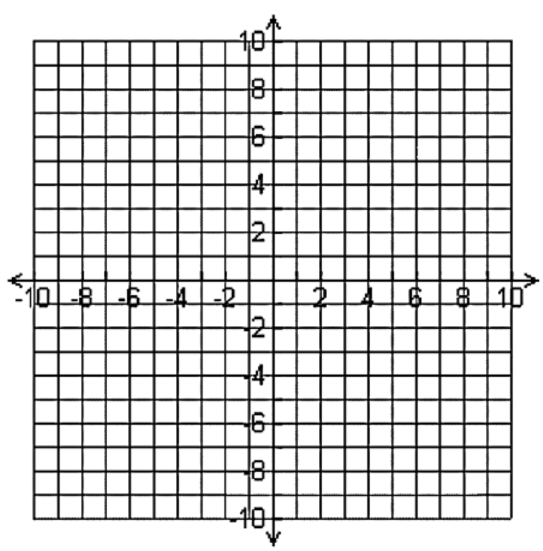


   b. Rewrite the equation in standard form, $f(x) = ax^2 + bx + c$.

3. Given the function $f(x) = -x^2 + 4x + 7$, sketch a graph of the function below.

   a. What is the vertex?
   b. What is the y-intercept?

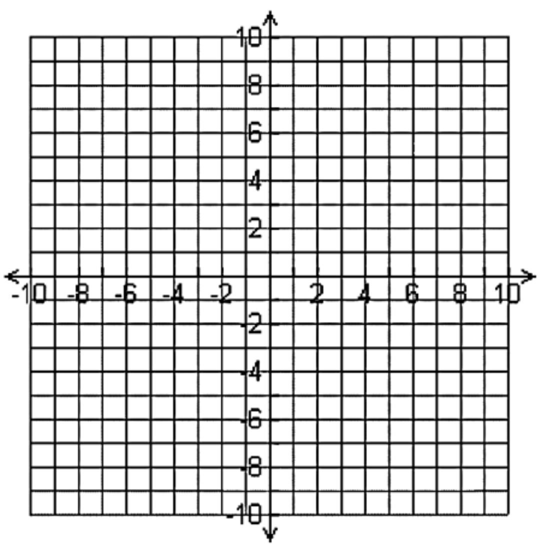
x	f(x)



4. Given the function $f(x) = 3(x-5)^2 - 8$, sketch a graph of the function below.

   a. What is the vertex?
   b. What is the y-intercept?

x	f(x)



Let’s wrap it up!
Write two quadratic functions that have the same vertex but differ in the following manner:
No other points in common; one opens upward and the other opens downward; different x-intercepts; different y intercept.

Function 1:	Function 2:
Pair of symmetric points:	Pair of symmetric points:
Opens?	Opens?
X –intercepts:	X-intercepts:
Y-intercept:	Y-intercept:

From your work on quadratic functions, write about a concept that at first was difficult but now you have mastered ☺

Solving Quadratic Inequalities

Student Learning Objectives:

- Solve quadratic equations and inequalities and related application problems using graphing, factoring, the quadratic formula, and completing the square.

Part 1 (Graphic)

Example: Solve the following inequality: $x^2 - 2x > 3$

a) Get right side of inequality to be zero, by subtracting 3 from both sides:

$$x^2 - 2x - 3 > 0$$

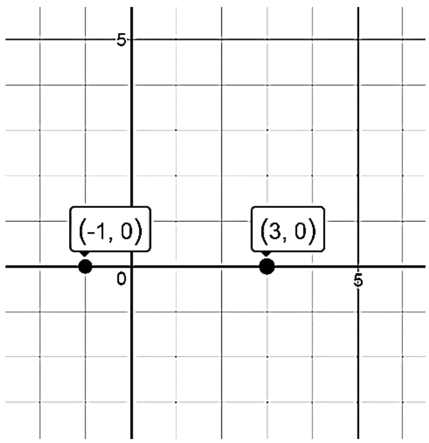
b) Factor the left side: $(x - 3)(x + 1) > 0$

c) Set each factor equal to zero and solve for x (these are the x-intercepts)

$x - 3 = 0$ $\qquad$ $x + 1 = 0$

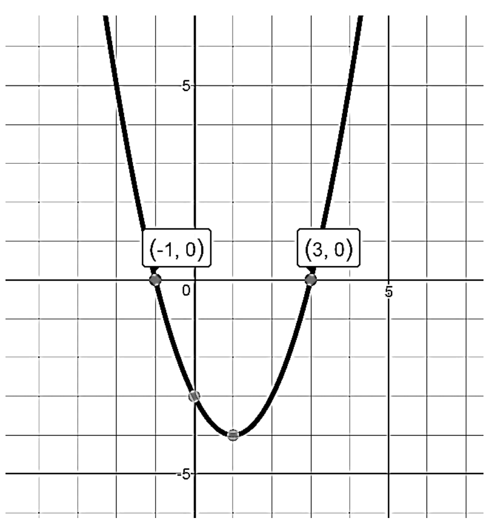
$x = 3$ $\qquad$ $x = -1$

d) On a coordinate plane, plot the x-intercepts: $(3,\ 0)$ and $(-1,\ 0)$.

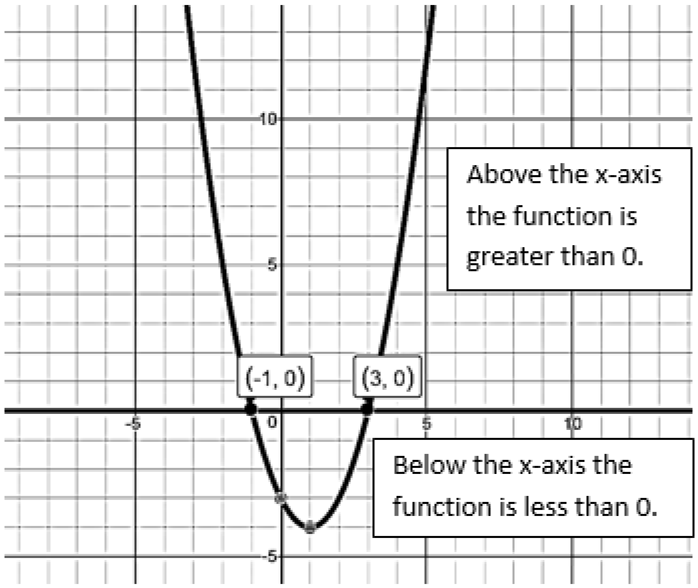


e) Using what you know about quadratic functions, sketch a graph of the quadratic function $f(x) = x^2 - 2x - 3$.

(vertex is $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$, opens up, y-intercept at $(0,-3)$, x-intercepts graphed in part d).



f) Our inequality indicates that we want to know where $x^2 - 2x - 3$ is greater than 0. Looking at our graph, this occurs where the parabola is above the x-axis. In this case, the parabola is above the x-axis for all x-values less than -1 and for all x-values greater than 3. In interval notation, this is: $(-\infty,-1)\cup(3,\infty)$.



Another way to think about this is to "test values" in the original inequalities. We want to test one value in each of the three domain regions, $x<-1$, $-1<x<3$, and $x>3$ to see which values hold true for the original inequality.

$x^2-2x-3>0$

Test $x=-2$

$$(-2)^2-2(-2)-3>0$$
$$4+4-3>0$$
$$5>0$$

TRUE, this region is included in our solution.

Test $x=0$

$$(0)^2-2(0)-3>0$$
$$0-3>0$$
$$-3>0$$

FALSE, this region is not included in our solution.

Test $x=4$

$$(4)^2-2(4)-3>0$$
$$16+8-3>0$$
$$31>0$$

TRUE, this region is included in our solution.

Practice. Solve the following quadratic inequalities using the graphic approach:

1. $x^2 + x - 12 \geq 0$

2. $4x^2 + 9 < 6x$

3. $2x^2 \leq 5x + 3$

Part 2 (Sign Chart Method)

Here is an example of the sign chart method:

Solve $2x^2 - 8x \le 0$ and state the solution using interval notation.

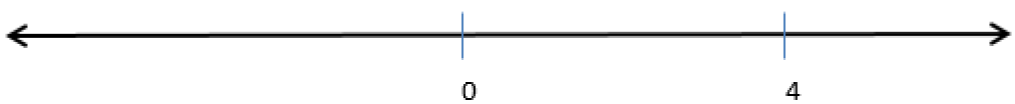
Step 1: Get the entire variable expression on one side of the inequality symbol if it is not in that form. *(This step is complete in the given example.)*

Step 2: Factor the variable expression. In this case, $2x(x-4) \le 0$.

Step 3: Set up the factors and a number line and include the values on the number line that would make each factor equal to zero.

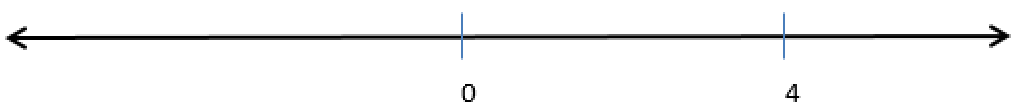
$2x$

$(x - 4)$



Step 4: Show the value of each expression before and after the zero value as shown here:

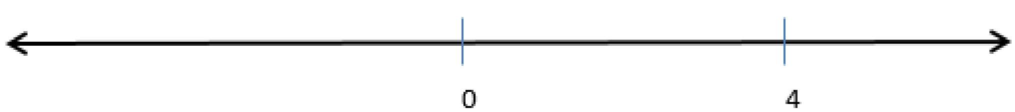
$2x$	-	-	-	-	0	+	+	+	+
$(x - 4)$	-	-	-	-	-	-	-	0	+



This step can be accomplished by simply substituting in values below and above the zero values in each expression.

Step 5: In this example the expressions are multiplied, so simply multiply the signs:

$2x$	-	-	-	-	0	+	+	+	+
$(x - 4)$	-	-	-	-	-	-	-	0	+



$2x(x-4)$	+	+	+	+	0	-	-	0	+

Step 6: In this example, the inequality is $2x(x-4) \le 0$, therefore we are looking for where there are zeros and negative symbols in the product. It appears that from 0 to 4 on our number line we see this! We have our solution.

Write the solution using interval notation, [0, 4].

Practice. Solve each using the sign chart method and write the solution using interval notation:

4. $x^2 - 6x \geq 7$

How would the solution change if the inequality symbol was $>$?

5. $2x^2 \leq 5x + 3$ (This is #3 from earlier in this activity. Did you get the same result?)

6. $49 - x^2 < 0$

How would the solution to #6 change if the inequality was $49 - x^2 > 0$?

Quadratic Applications

Student Learning Objectives:

- Solve quadratic equations and inequalities and related application problems using graphing, factoring, the quadratic formula, and completing the square.

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1. The value of Nathans's stock portfolio is given by the function $v(t) = 55 + 72t - 4t^2$ where v is the value of the portfolio in hundreds of dollars and t is the time in months.
   a. How much money did Nathan initially have in his stock portfolio?

   b. When will the value of Nathan's portfolio be at a maximum?

   c. What is the maximum value of Nathan's portfolio?

2. While playing basketball this weekend Thea shoots an air-ball. The height in feet of the ball is given by $H(t) = -16t^2 + 32t + 8$.
   a. How long will it take the ball to strike the ground?

   b. What is the domain of the given situation?

   c. What is the maximum height of the ball?

   d. When was the ball higher than 12 feet?

3. The path of a high diver is given by $H(d) = -\frac{4}{9}d^2 + \frac{24}{9}d + 10$ where h is the height in feet above the water and d is the horizontal distance from the end of the diving board in feet. Round answers to two decimal places as necessary.

   a. What is the maximum height of the diver?

   b. How far out from the end of the diving board is the diver when she reaches her maximum height?

   c. How far out is the diver when she touches the water?

4. At the beginning of 1980 to 1990 the number of mountain bike owners, M (in millions), in the US can be approximated by the model $M(t) = 0.34t^2 - 2.27t + 3.96$, where $t = 0$ represents the beginning of1980.

   a. How many US mountain bike owners were there in 1980?

   b. Determine the year between 1980 and 1990 in which there was a minimum number of US mountain bike owners.

   c. From 1980 to 1990, what was the minimum number of US mountain bike owners?

Graphing Radical Functions and Solving Radical Equations

Student Learning Objective:

- Given a radical equation, find solutions algebraically.

Skill Builder:

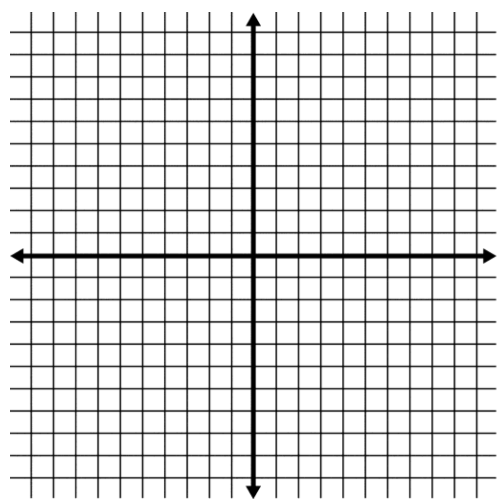
For each function given, describe the transformations that occur:

1. $g(x)=(x-1)^2+2$
2. $h(x)=-x^2-4$
3. $f(x)=3(x-3)$
4. $d(x)=\frac{1}{2}x+1$
5. $j(x)=\sqrt{(x-1)}+2$

Part 1

Graphing Radical Functions (This will help you better understand solving these equations!)

Use the coordinate plane below to graph the following function.



1. $f(x)=2\sqrt{x+3}-1$

   a. Draw the horizontal line $y=2$ through the graph. Does the line intersect the graph? If so, state the approximate x value of the intersection.

   b. Draw the horizontal line $y=1$ through the graph. Does the line intersect the graph? If so, state the approximate x value of the intersection.

   c. Draw the horizontal line $y=-2$ through the graph. Does the line intersect the graph? If so, state the approximate x value of the intersection.

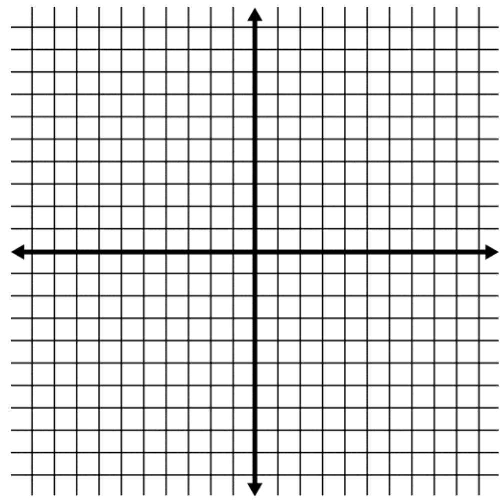
   d. Draw the horizontal line $y=5$ through the graph. Does the line intersect the graph? If so, state the approximate x value of the intersection.

e. If a y-value of b is in the _____________ (domain/range – choose one) of the function f, then there is (at least one) x-value in the _____________ (domain/range – choose one) that is a solution of $b = 2\sqrt{x+3} - 1$.

2. Graph $f(x) = 3\sqrt{x} + 5$.

   a. Determine the range of the function and state the range in interval notation.

   b. Will there be a solution to the equation $f(x) = 4$? Why or why not?



Part 2

Solving Radical Equations

Steps:

1. Isolate the radical.
2. Raise both sides to the same power as the index.
3. Solve for x.
4. If the equation has an even index, check your answers.

Example: $x = \sqrt{x+10} + 2$

1. Isolate the radical.

   $x - 2 = \sqrt{x+10}$

2. Raise both sides to the same power as the index.

   $(x-2)^2 = \left(\sqrt{x+10}\right)^2$

3. Solve for x.

$x^2 - 4x + 4 = x + 10$

$x^2 - 5x - 6 = 0$

$(x-6)(x+1) = 0$

$x - 6 = 0 \quad x + 1 = 0$

$x = 6 \qquad x = -1$

4. If the equation has an even index, check your answers.

$x = 6$ | $x = -1$

$6 = \sqrt{6+10} + 2$ | $-1 = \sqrt{-1+10} + 2$

$6 = \sqrt{16} + 2$ | $-1 = \sqrt{9} + 2$

$6 = 4 + 2$ | $-1 = 3 + 2$

$6 = \sqrt{16} + 2$ | $-1 = \sqrt{9} + 2$

☑YES! | ☒ NO!

The solution set is $\{6\}$.

Solve the following radical equations.

1. $10\sqrt{9x} = 60$

4. $\sqrt{x+6} = \sqrt{10-3x}$

2. $x + 2 = \sqrt{2x+7}$

5. $\sqrt[3]{x-5} + 1 = -1$

3. $\sqrt{3x+4} + 2 = 11$

6. $(2x+1)^{\frac{1}{2}} = 5$

Challenge: These radical equations take a couple more steps. Try them!

1. $\sqrt{3x+4}-1=\sqrt{x+5}$

2. $(3x+4)^{\frac{2}{3}}=16$

Application Problems

1. A wind powered generator is delivering P units of power. The velocity V of the wind, in miles per hour, can be determined using $V = \sqrt[3]{\frac{P}{k}}$, where k is a constant that depends on the size and efficiency of the generator.

   Given $k = 0.004$, approximately how many units of power are being delivered if the wind is blowing at 27 miles per hour?

2. The population of Pineapple Island is increasing slowly. The current population can be calculated by the function $P = 79.8\sqrt{1+0.11t}$ where P is the population in millions and t is the time in years.

   a. What is the population of Pineapple Island after 4.5 years?

   b. If the population of the island reaches 144,000,000, how many years have elapsed? Round to the nearest tenth of a year.

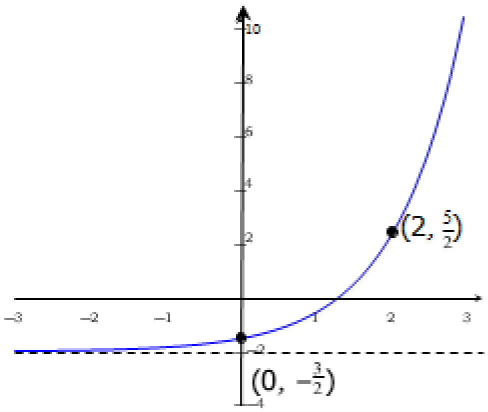
Equations of Exponential Functions

Student Learning Objectives:

- From the graph of a polynomial, rational, or exponential function, write its equation.

Write the equation of the exponential function that matches each graph below.

1.



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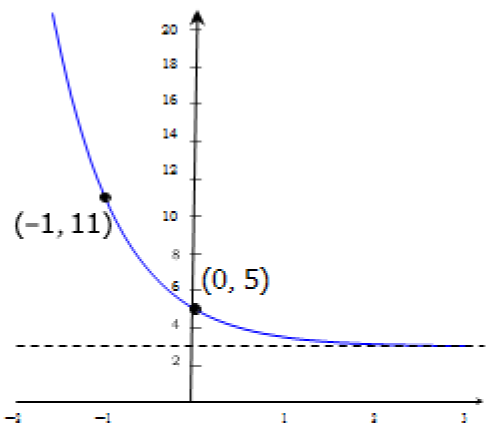

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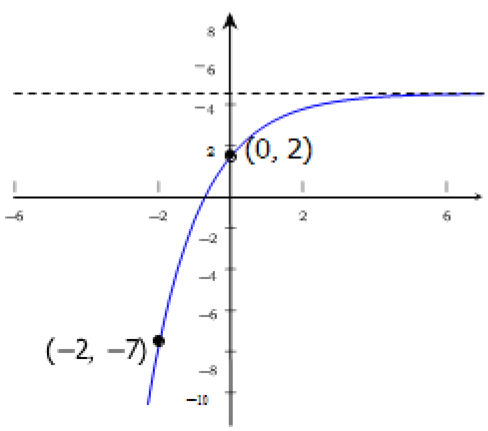

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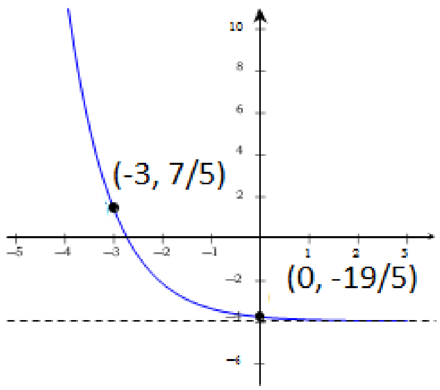

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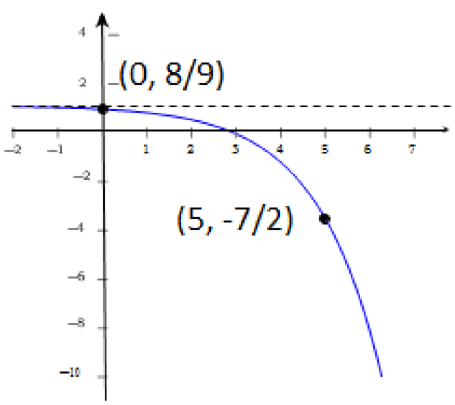

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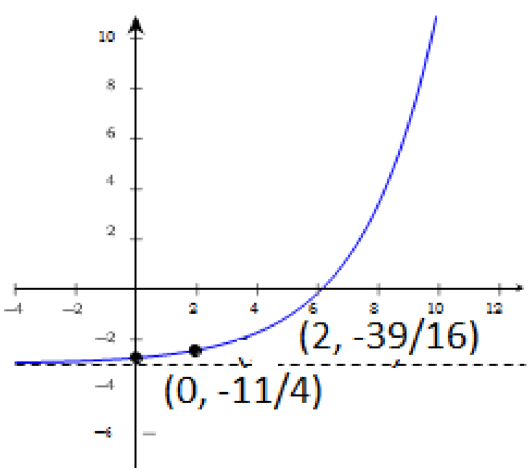

Inverses Symbolically

Student Learning Objectives:

- From a function in symbolic form, determine whether it is one-to-one, its inverse function (if one exists) and the domain and range of the inverse function.

For each function below:

A. Is the given function one-to-one?
B. If your answer to part a was yes, determine the inverse function. If you answer to part a was no, determine a restricted domain and find the inverse function with the restricted domain.
C. Determine the domain and range of the function and its inverse function.

1. $f(x) = 3x - 5$

2. $f(x) = 6x + 2$

3. $f(x) = \sqrt{x-1}$

4. $f(x) = \dfrac{1}{3x}$

5. $f(x) = 4x^2$

6. $f(x) = 9 - \frac{1}{2}x^2$

7. $f(x) = \sqrt{2x-4} + 8$

8. $f(x) = -\frac{1}{4x}$

9. $f(x) = x^2 + 1$

10. $f(x) = \frac{7}{x-4}$

Inverse Function Activity

Student Learning Objectives:

- From a function in symbolic form, determine whether it is one-to-one, its inverse function (if one exists) and the domain and range of the inverse function.

Vocabulary: The inverse function for $h(x)=b^x$ for $b>0$ is denoted $\log_b(x)$. In other words, the inverse function of $h(x)=6^x$ is $h^{-1}(x)=\log_6(x)$.

Tabular Analysis

1. Make a table of values for $f(x)=4^x$. Using this table fill in the values for the inverse function $\log_4(x)$.

x	-3	-2	-1	0	1	2	3	4
4^x								

x								
$\log_4(x)$								

You already know the domain of $f(x)=4^x$ and the relationship of domain and range for inverse functions. Use this knowledge to answer the following:

A. State the domain and range of 4^x.

B. State the domain and range of $\log_4(x)$.

C. What can you say about the relationship between the domain and range of $4x$ and $\log_4(x)$?

2. Make a table of values for $g(x)=\left(\frac{1}{3}\right)^x$. Using this table fill in the values for the inverse function $\log_{\frac{1}{3}}(x)$.

x	-3	-2	-1	0	1	2	3	4
$(1/3)^x$								

x								
$\log_{1/3}(x)$								

You already know the domain of $g(x)=\left(\frac{1}{3}\right)^x$ and the relationship of domain and range for inverse functions. Use this knowledge to answer the following:

A. State the domain and range of $\left(\frac{1}{3}\right)^x$.

B. State the domain and range of $\log_{1/3}(x)$?

C. What can you say about the relationship between the domain and range of $\left(\frac{1}{3}\right)^x$ and $\log_{1/3}(x)$?

Converting between Exponential Form and Logarithmic Form

Student Learning Objective:

- Given the symbolic form of an exponential function write it in its inverse logarithmic form, and vice versa.

$$(BASE)^{exponent} = Number \quad \& \quad \log_{BASE} Number = exponent$$

This is the basic relationship between an exponential function and its inverse log function. These basic formulas are helpful in converting between the two forms of an equation.

In the table below you are shown examples of equations written in their exponential form and their inverse log form.

Exponential Form	Log Form
$3^2 = 9$	$\log_3 9 = 2$
$\frac{1}{25} = 5^{-2}$	$\log_5 \frac{1}{25} = -2$
$e^x = y$	$\ln y = x$
$10^3 = 1000$	$\log 1000 = 3$

You should see that the **base, exponent and number** in each equation can be interchanged using the relationship shown above. Identifying the **base, exponent and number** in each form is critical in understanding this concept. ***Now, look at each example in the table and make sure that you can circle the base, put a check mark by the exponent, and create a box around the number.***

When using a base of "e", we use the symbol "ln" because "e" is the base of the natural logarithmic function "ln."

When using a base of '10', we write "log" without a base shown because '10' is the base of the common log function. Think of our number system. It is base 10. Let's show our number system as base 10:

_____ _____ _____ _____ _____ $10^0 = 1$ _____ _____ _____ _____ _____

_____ _____ _____ _____ _____ _____ _____ _____ _____ _____ _____

Using your device:

Quickly research the number "e".

What is its approximate value? ________________

Give one example of the natural log function in nature:

Part 1

Convert to the equivalent inverse form for each.

1. $\log_3(81) = 4$

2. $\ln(7.389) \approx 2$

3. $\log(1000) = 3$

4. $\log(1,000,000) = 6$

5. $5^3 = 125$

6. $4^{-2} = \frac{1}{16}$

7. $10^5 = 100,000$

8. $49^{\frac{1}{2}} = 7$

Part 2:

Solve each equation for x.

1. $x = \log_7 7$

2. $x = \log_{82} 1$

3. $x = \log_3(3^{15})$

4. $\log_2 8 = x$

5. $\log_3 x = 4$

6. $10^x = 1270$

7. $\log 100 = x$

8. $e^{2x} = 57$

Laws of Logarithms

Student Learning Objective:

- Given an expression involving logarithms and exponents, apply the properties of exponents and logarithms to condense or expand the expression as indicated.

Product Rule: $\log_b(AB) = \log_b A + \log_b B$

Quotient Rule: $\log_b\left(\dfrac{A}{B}\right) = \log_b A - \log_b B$

Power Rule: $\log_b A^p = p \cdot \log_b A$

Note: Here $A > 0$, $B > 0$, and $b > 0$ with $b \neq 1$.

Part 1

Expand the following logarithms as much as possible. Simplify your answer. Here, $a, b, c, d, f, g, h, w, x, y, z > 0$ (Why do we need these restrictions?)

1. $\log_2 \sqrt{8a^3}$

2. $\ln\left(\dfrac{b}{c^2}\right)$

3. $\log(fg)^7$

4. $\log_3\left(\frac{9x^5}{yz^4}\right)$

Part 2

Rewrite each of the following as a single logarithm. Simplify your answer. Remember, $a, b, c, d, f, g, h, w, x, y, z > 0$.

5. $\frac{1}{3}\log a + \log b$

6. $\log_4 c - 2\log_4 d$

7. $1 + 3\ln f + \ln g - 2\ln h$

8. $2\ln w + \ln x - \ln y - \frac{1}{2}\ln z$

Logarithm Practice

Student Learning Objective:

- Given an exponential or logarithmic statement, evaluate whether the properties of logarithms have been properly applied and make corrections as needed.

Determine whether the following statements are true or false. For reach true statement, justify your answer. For each false statement, change the right side of the statement to make it true, if possible. Otherwise, provide a counterexample. See the examples below.

Example 1: $\log_2(2x+3y)=\log_2(2x)+\log_2(3y)$

False: $\log_2(2x+3y)=\log_2(2x+3y)$ (or cannot be simplified further)

Furthermore we have a counterexample. By letting $x=5$ and $y=2$, the left side would become $\log_2(2x+3y)=\log_2(2(5)+3(2))=\log_2(16)=4$ and the right side would become $\log_2(2x)+\log_2(3y)=\log_2(10)+\log_2(6)\approx 3.332+2.585=5.907$ And $4\neq 5.907$

Example 2: $\log_4\left(\frac{5}{7}\right)=\log_4(5)-\log_4(7)$

True. $\log_b\left(\frac{A}{B}\right)=\log_b(A)-\log_b(B)$ as long as $A>0,\ B>0$, and $b>0$ with $b\neq 1$. Here $A=5>0,\ B=7>0,\ b=4>0$, and $b=4\neq 1$

1. $\log_a(3a^2)=2+\log_a 3$

2. $\log_3(4x^2)=2\log_3(4x)$

3. $\log_4\sqrt{3x}=\frac{1}{2}\log_4 3+\log_4 x$

4. $\log_2\left(3x^4\right) = 4\log_2 x + \log_2 3$

5. $\log\left(8x^{1/2}\right) = \sqrt{\log\left(8x\right)}$

6. $\log_2\left(\dfrac{9}{xy}\right) = \log_2 9 - \log_2 x - \log_2 y$

7. $\log\left(x-y\right) = \dfrac{\log x}{\log y}$

8. $\log_4\left(4x^4\right) = 1 + 4\log_4 x$

9. $2\ln\left(\dfrac{x}{y}\right) = \ln\left(x^2\right) + \ln y$

Prerequisite Skills: Practice with Linear Function Applications

1. In Maryland, sales tax is 6% for clothing purchases. Let x represent the amount spent on clothing pre-tax, in dollars.
    a. If you purchase a blouse for \$16, how much is the total with tax?

    b. Find a function $f(x)$, that calculates the total cost with tax.

2. A car rental company charges \$84.99 per day for a midsize SUV. There is a one-time \$15 convenience fee.
    a. Find a function, $C(d)$, to model the total cost, C, for renting a midsize SUV for d days.

    b. How much does it cost to rent a midsize SUV for 3 days?

3. Find a linear function that models the distance, d, that a person travels in t hours at an average speed of 60 miles per hour.

4. Carla has decided to start selling makeup. She will purchase a starter kit and some other supplies to have on hand. The starter kit costs \$130, and she pays half-price for any additional makeup items. Let x represent the cost of the items before the 50% discount.
    a. Find a function that models her start-up cost, $C(x)$.

    b. Carla budgeted \$250 to start. How much will her additional makeup items be worth before the discount?

Piecewise Functions

Student Learning Objectives:

- Solve problems involving absolute value equations and piecewise functions.

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1. A skydiver jumps from an aircraft flying at a height of 3000 meters. After 35 seconds, he deploys his parachute for the remainder of the fall to earth. Let H represent the height above the ground in meters and t represent the number of seconds after jumping, then

$$H(t)=\begin{cases}-51.5t+3000, & 0\le t\le 35\\ -8.9t+1509, & t>35\end{cases}$$

(A) Calculate and interpret $H(7.5)$ in this situation.

(B) Use functional notation to represent the skydiver's height after 38 seconds and calculate that value.

(C) When does the skydiver hit the ground?

(D) What is the domain of $H(t)$? Write your answer in interval notation.

2. The Olympic winning time in the Men's 400 Meter Individual Medley (400 IM) can be approximated by the function given below, where $R(t)$ represents the number of seconds required to swim the 400 IM and t is the time in years since 1984.

$$R(t)=\begin{cases}0.052t^2-0.826t+257.36, & 0\le t\le 12\\ -0.99t+267.75, & t>12\end{cases}$$

(A) Calculate $R(4)$.

(B) Interpret $R(4)$ in the context of the problem.

(C) Use functional notation to predict the winning time for the 400 IM in 2012.

(D) In what year(s) was the winning 400 IM time 4 minutes and 8 seconds?

3. According to the Bureau of Labor Statistics, the North Carolina unemployment rate for 2008 can be approximated by the function given below, where $U(t)$ represents the unemployment rate in percent and t is the month in 2008 (for example, January 2008 = 1).

$$U(t) = \begin{cases} 0.28t + 4.44, & 1 \le t \le 9 \\ 0.8t - 0.93, & 9 < t \le 12 \end{cases}$$

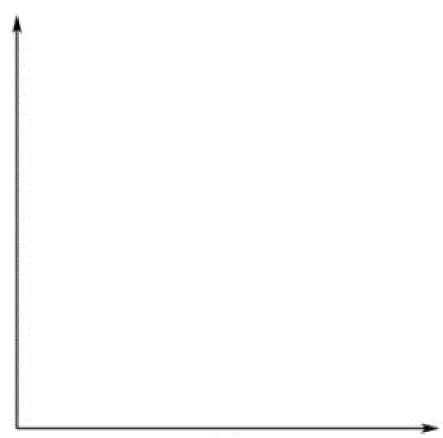
(A) Calculate $U(4)$ and interpret this value in context of the problem.

(B) Use functional notation to approximate the North Carolina unemployment rate for December 2008 and calculate the value.

(C) When in 2008 was the unemployment rate 6.12%?

4. A sales company advertises a job opening. The annual salary is to be \$10,000 plus commission. The commission is paid at the rate of 8% on all sales up through \$100,000. For sales that exceed \$100,000, the commission rate is 12%.

(A) Express salary, S, as a function of sales, x. Sketch and label the graph of $S(x)$.

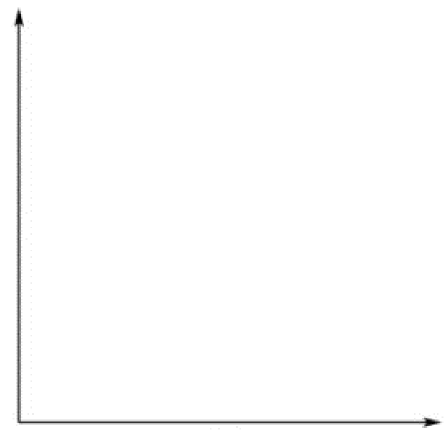


(B) Determine the annual salary if sales are \$120,000.

(C) Determine the amount of sales necessary to generate an annual salary of \$40,000.

5. Marcos will be moving and needs some extra help. In Cary, he found 919 Moving Inc. which charges a blanket cost of \$200.91 for 3 hours. The moving company charges an additional \$54.99 for each additional hour.

   (A) Express Marcos's costs, C , as a function of the number of hours needed, h . Sketch and label the graph of $C(h)$



   (B) Determine Marcos's costs if he needs 1.5 hours of help.

   (C) Determine how many hours of help he could get if he budgeted at most \$500 for the move.

6. America's dependency on foreign oil has always been a "hot" political topic, with the amount of imported oil fluctuating over the years due to political climate, public awareness, the economy, and other factors. The amount of crude oil imported can be approximated by the function given below, where $A(t)$ represents the number of barrels in billions and *t* is the time in years since 1980.

$$A(t) = \begin{cases} 0.047t^2 - 0.38t + 1.9, \; 0 \le t < 8 \\ 0.075t^2 + 1.495t - 5.265, \; 8 \le t \le 11 \\ 0.133t + 0.685, \; 11 < t \end{cases}$$

   (A) Calculate $A(9)$ and interpret $A(9)$ in the context of the problem.

   (B) Use functional notation to represent the number of barrels imported in 1995 and calculate the value.

(C) In what year(s) did America import 1.18 billion barrels of crude oil?

7. The Town of Cary uses a piecewise function to calculate a citizen's cost for water usage. Let *C* represent a citizen's cost for water usage before taxes and *g* represent the water consumption in 1000's of gallons, then

$$C(g)=\begin{cases} 3.28g+2.76, & 0\le g\le 5 \\ 3.75(g-5)+19.16, & 5<g\le 8 \\ 5.33(g-8)+30.14, & g>8 \end{cases}$$

(A) Calculate and interpret $C(7.5)$ in this situation.

(B) Use functional notation to represent a citizen's cost for water usage before taxes if he used 8200 gallons of water and calculate that value.

(C) How much water did a citizen use if her cost for water usage before taxes is $17.52?

8. You have a summer job that pays $8 every hour and time and a half for overtime. That is, if you work more than 40 hours per week, your hourly wage for the extra hours is 1.5 times your normal hourly wage of $8.

(A) Write a piecewise function that gives your weekly pay P in terms of the number h of hours you work.

(B) How much will you get paid if you work 55 hours?

9. According to https://www.irs.gov the amount of Medicare tax you pay depends on your annual income. Suppose you pay 1.45% of your income if it is less than $50,000. If your income is at least $50,000, but not more than $100,000, you pay 2.5%. If your income exceeds $100,000 you pay a fixed amount of $3000.00.

(A) Write a piecewise function that gives the Medicare tax, T in terms of annual income, a

(B) How much Medicare tax do you pay if you make $45,000 per year?

Module 2 Practice

For each of the pairs of functions in questions 1 through 5, compute:

a) $\left(\dfrac{f}{g}\right)(x)$

b) Domain of $\left(\dfrac{f}{g}\right)(x)$

c) $\left(\dfrac{g}{f}\right)(x)$

d) Domain of $\left(\dfrac{g}{f}\right)(x)$

e) $(f \circ g)(x)$

f) $(g \circ f)(x)$

g) $\left(\dfrac{f}{g}\right)(6)$, $(f \circ g)(6)$, and $(g \circ f)(6)$

1. $f(x) = 3x + 2;\ g(x) = -2x - 5$

2. $f(x)=\sqrt{x-5}; g(x)=x^2+3$

3. $f(x)=\frac{1}{x}\ ;\ g(x)=x^2$

4. $f(x)=\dfrac{x}{x+1}$; $g(x)=x^3$

5. $f(x)=x-6$; $g(x)=x^2-36$

6. Calculate the difference quotient $\frac{f(x+h)-f(x)}{h}$, $h \neq 0$ for each of the following functions.

a. $f(x) = x^2 + 2x - 1$

b. $f(x) = -2x^2 + 7$

c. $f(x) = \frac{x}{x+1}$

7. (A) Determine the inverse of given one to one function. (B) Then determine the domain and range of the given function and its inverse function.

a. $f(x) = \sqrt{2-6x}$

b. $f(x) = \frac{1}{4-x}$

c. $f(x) = 5 - 2x$

d. $f(x) = \frac{3x+2}{1-x}$

8. Convert to the equivalent inverse form:

a. $e^2 = x$

b. $2^5 = 32$

c. $4^{-2} = \frac{1}{16}$

d. $\log_3 9 = 2$

e. $\log(0.0001) = -4$

f. $\ln x = 5$

9. Fully expand each logarithm. Assume $x, y, z > 0$

a. $\log\left(\frac{x^2}{10z}\right)$

b. $\log_2 \sqrt{8x^3y^4z}$

10. Condense each logarithm. Assume $u, x, y, z > 0$

a. $2\ln(x) + \frac{1}{2}\ln(y) - \ln(z) - 3\ln(u)$

b. $\log_5(x-1) + \log_5(x-1) - 3\log_5(x)$

11. Compute exact solutions of the quadratic equations. If the solution(s) is/are real, compute them exactly. Otherwise, show why the solutions are not real.

a. $x^2 = 3x + 10$

b. $x^2 + x = 2x + 3$

c. $-2x^2 + 3x - 7 = 0$

12. Write the quadratic equations in transformation form. For each function, state the range, the vertex, and whether a max or min occurs at the vertex.

a. $f(x) = x^2 + 2x + 3$ b. $g(x) = -2x^2 + 20x - 7$ c. $h(x) = 3x^2 + 18x + 2$

13. Solve the following quadratic inequalities using either graphical approach or sign chart method.

a. $3 - 2x \geq x^2$ b. $32 - x^2 < -4$

c. $(4 - x)^2 < 2$ d. $-2x^2 - 3 \geq 7x$

14. An object is launched at 16.9 meters per second (m/s) from a 58.8-meter tall platform. The equation for the object's height s at time t seconds after launch is $s(t) = -4.9t^2 + 16.9t + 58.8$ where s is in meters.

a. When does the object reach its maximum height?

b. What is the maximum height?

c. Locate two symmetric points when $s(t) = 70$. Explain the meaning of these points in the context of the problem.

d. When does the object strike the ground?

15. The revenue, R, in dollars, is the amount of money received by a company for selling an item. Revenue is equal to the number of items sold, x, times the unit price, p. Then the equation for revenue is given by $R = xp$. A company finds a demand equation, $x = 10,000 - 50p$, that expresses the number of items in terms of the unit price.
    a. Find an equation that expresses the revenue, $R(p)$, in terms of the unit price, p.

    b. Find the unit price the company should charge in order to maximize revenue.

    c. What is the maximum revenue?

16. Dolores wants to build a rectangular fence to plant a garden. She has 60 feet of fencing.
    a. Draw a sketch and find a formula to express the area of the garden in terms of the width, w.

    b. Find the dimensions that maximize the area of the garden.

    c. What is the maximum area she can enclose?

17. Solve the following equations:

a. $\sqrt{x+8}-2=x$

b. $2\sqrt{x+5}-10=-4$

18. Determine functions f and g such that $(f \circ g)(x)=h(x)$

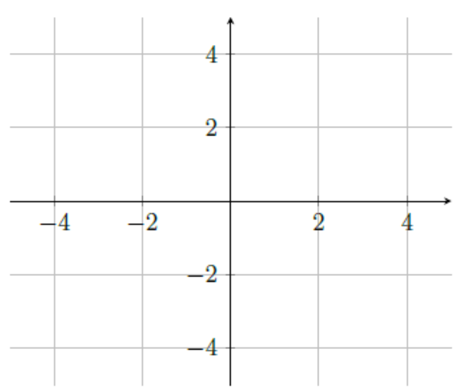
a. $h(x)=\sqrt{3x+4}-2$

b. $h(x)=\dfrac{x^3}{x^3-5}$

c. $h(x)=\sqrt{(x+1)^2+5}$

19. Graph the following function.

$$f(x)=\begin{cases} |x+1| & if \quad -4 \le x < 0 \\ -x^2 & if \quad 0 \le x \le 2 \\ 2x-2 & if \quad 2 < x < 4 \end{cases}$$



20. Write a piecewise defined function for each of the following.
    a. Your cell phone plan costs \$75 per month and gives you unlimited talk, 500 text messages per month, and no data plan. After 500 text messages, it costs \$0.10 per text you send over the 500 limit. Write a piecewise function that represents your monthly cell phone cost.

    b. In Missouri, income tax is 3.5% on the first \$9,000 of income or less, and 6% percent on any income in excess of \$9,000. Let the tax $T(x)$ be a function of the income X. Write a piecewise defined function to describe the income tax paid in Missouri.

    c. A sales representative's compensation is paid according to the following plan. Base pay is \$20,000. Commissions are paid at a rate of 5% for the first \$50,000 of sales, 10% for sales exceeding \$50,000 but no more than \$100,000. For all sales exceeding \$100,000, the commission rate is 15%. Write a piecewise function to describe the compensation plan.

21. When a diabetic takes long-acting insulin, the insulin reaches its peak effect on the blood sugar level in about three hours. This effect remains fairly constant for 5 hours, then declines, and is very low until the next injection. In a typical patient, the level of insulin might be modeled by the following function.

$$f(t)=\begin{cases} 40t+100 & if \quad 0\le t\le 3 \\ 220 & if \quad 3<t\le 8 \\ -80t+860 & if \quad 8<t\le 10 \\ 60 & if \quad 10<t\le 24 \end{cases}$$

Here, $f(t)$ represents the blood sugar level at time t hours after the time of the injection. If a patient takes insulin at 6 am, determine the blood sugar level at each of the following times.

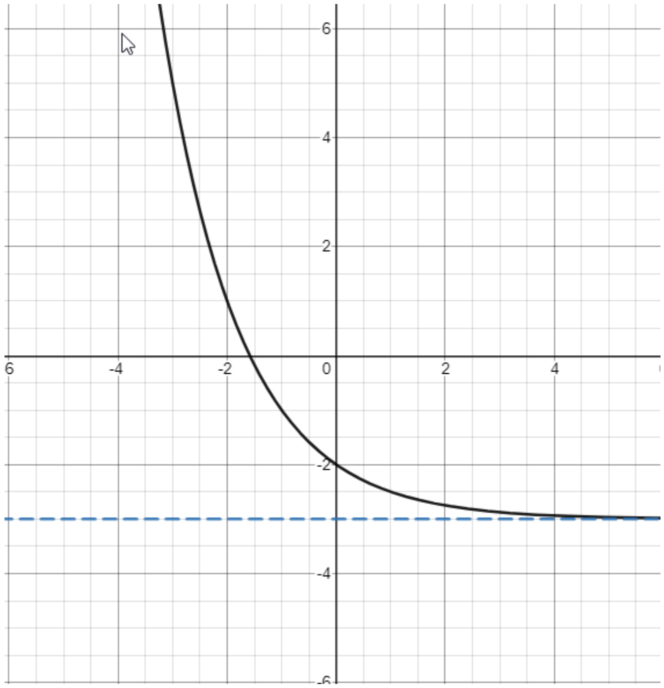
    a. 7 am

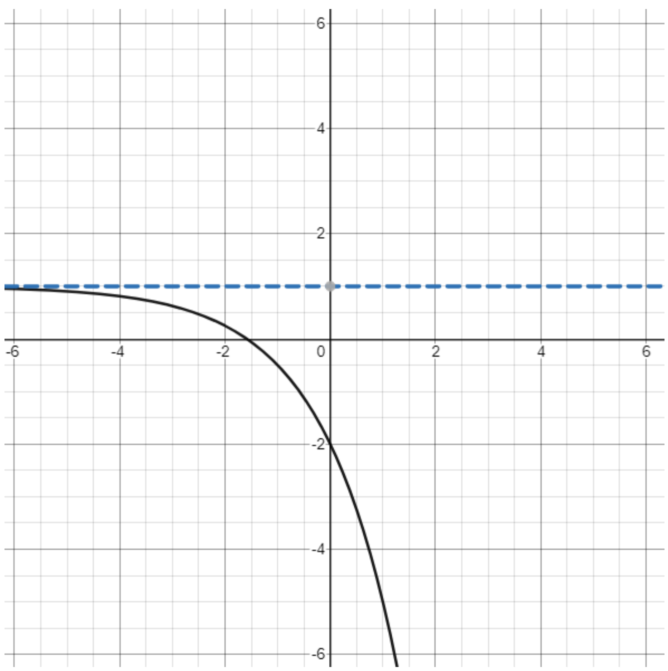
    b. 11 am

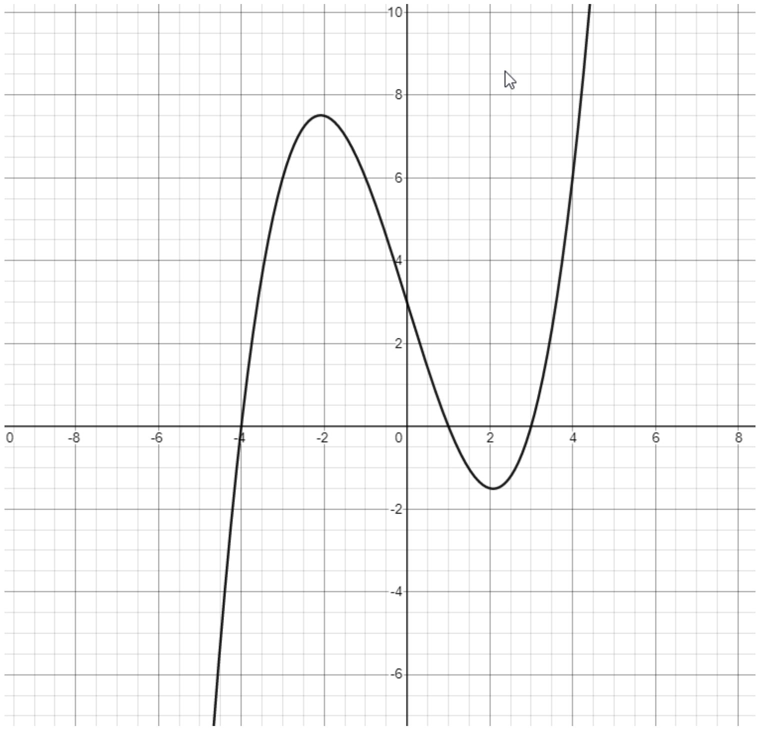
    c. 3 pm

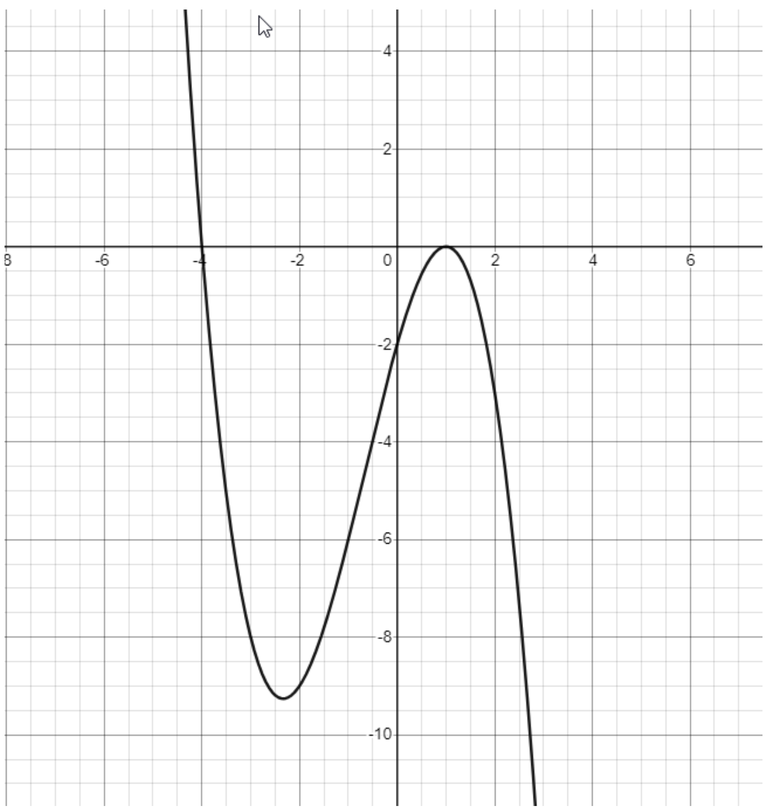
    d. 5 pm

22. Write equations of the following graphs:

a. 

b. 

c. 

d. 

Introduction to Complex Numbers

Student Learning Objective:

- Identify real and imaginary parts of complex numbers and perform addition, subtraction, and multiplication with complex numbers to evaluate complex expressions.

FACT:

We saw in Module 2 that linear factors with real coefficients correspond to x-intercepts or **zeros** of a quadratic function's graph.

Using Desmos, graph $f(x) = 3x^2 + 15x + 18$.

- Notice that the function crosses the x-axis in two places: $x = -2$ and $x = -3$.
- This means that $x = -2$ and $x = -3$ are **zeros** of $3x^2 + 15x + 18$.
- This also means that $(x+2)$ and $(x+3)$ are **factors** of $3x^2 + 15x + 18$.

EXPLORE: Consider $g(x) = x^2 + x + 2.$

a. Graph g using Desmos.

b. How many zeros do you see on the graph?

c. How many zeros would you expect this 2nd degree polynomial to have?

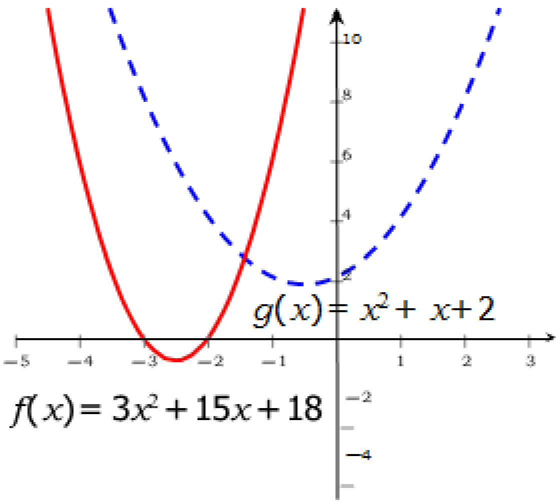
FACT: *These zeros are not real numbers. That is why we don't see them on the graph.*

We need a new number system to represent these zeros. This is the **complex number system.** Complex numbers are made up of real numbers (our "usual" numbers consisting of integers, rational numbers like $\frac{2}{3}$ and irrational numbers like π) and a new type of number:

Define $i = \sqrt{-1}$. i is called an **imaginary number**.

A **complex number** is of the form $a + bi,$ where a and b are real numbers.

Consider g , the dashed line in the graph below.



Notice how g does not cross the x-axis, so it has no x-intercepts (or real zeros). However, since it is a polynomial of degree 2, it has 2 zeros, but they are non-real complex numbers.

You can use the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to determine these zeros. You should expect the radicand, $b^2 - 4ac$, to be negative, giving a negative under the square root, because you know the zeros are not real.

By using the quadratic formula on $g(x) = x^2 + x + 2$, we get $x = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)} = \dfrac{-1 \pm \sqrt{-7}}{2} = \dfrac{-1 \pm \sqrt{7}i}{2}$.

Note: A common error that students make when writing the solutions from the quadratic formula is to forget that it gives you a common denominator of $2a$. This means you cannot just cancel the denominator with one of the terms and forget about it in the second term. Example:

$$\frac{-1 \pm \sqrt{7}i}{2} = \frac{-1}{2} \pm \frac{\sqrt{7}}{2} i \quad \text{is NOT the same as} \quad \frac{-1}{2} \pm \sqrt{7}i$$

Note: Another common error that students make when writing the solutions from the quadratic formula is to put the "*i*" under the radical. Example:

$$\frac{-1 \pm \sqrt{7}i}{2} \quad \text{is NOT the same as} \quad \frac{-1 \pm \sqrt{7i}}{2}$$

You can write $\frac{-1\pm\sqrt{7}i}{2}$ as $\frac{-1\pm i\sqrt{7}}{2}$ or as $-\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$.

These are non-real complex numbers. The real part of each zero is $-\frac{1}{2}$. The imaginary parts are $\frac{\sqrt{7}}{2}$ and $-\frac{\sqrt{7}}{2}$, respectively.

So, the zeros of $g(x)$ are: $x=-\frac{1}{2}+\frac{\sqrt{7}}{2}i$ and $x=-\frac{1}{2}-\frac{\sqrt{7}}{2}i$.

The imaginary part of each zero is the opposite of the other. This is always the case with non-real complex zeros from the quadratic formula. The roots/zeros $-\frac{1}{2}+\frac{\sqrt{7}}{2}i$ and $-\frac{1}{2}-\frac{\sqrt{7}}{2}i$ are called **complex conjugates.**

When we multiply complex numbers, we can use the "FOIL" method. Look what happens when we "FOIL" a complex number and its complex conjugate:

$$\left(-\frac{1}{2}+\frac{\sqrt{7}}{2}i\right)\left(-\frac{1}{2}-\frac{\sqrt{7}}{2}i\right)$$
$$=\frac{1}{4}+\frac{\sqrt{7}}{4}i-\frac{\sqrt{7}}{4}i+\left(\frac{\sqrt{7}}{2}i\right)^2=\frac{1}{4}-\left(\frac{\sqrt{7}}{2}\right)^2 i^2$$
$$=\frac{1}{4}-\frac{1}{7}\left(\sqrt{-1}\right)^2=\frac{1}{4}-\frac{7}{4}(-1)=\frac{1}{4}+\frac{7}{4}=2$$

We get a real number!

Since $g(x)=x^2+x+2$ has only non-real complex roots/zeros, we can call $g(x)$ an **irreducible quadratic** function.

Exercises

1. Simplify each:
   a. $3i^2$
   b. $(4i)(5i)$
   c. $(i+2)(i-2)$
   d. $(2i-3)(i-5)$
   e. $\left(\sqrt{3}+i\right)^2$

2. Simplify and leave your answers in $a+bi$ form. (Hint: First write any $\sqrt{\text{negative number}}$ as $i\sqrt{|\text{number}|}$.)

Example: $\sqrt{-6}+5\sqrt{-6}=i\sqrt{6}+5i\sqrt{6}=6i\sqrt{6}$

a. $\left(2+\sqrt{-6}\right)\left(5-\sqrt{-6}\right)$

b. $\left(5+\sqrt{-7}\right)\left(5-\sqrt{-7}\right)$

c. $\left(2\sqrt{-3}+4\right)\left(5\sqrt{-3}-1\right)$

d. $4i\left(3+\sqrt{-7}\right)$

3. Multiply and express your answer as an irreducible quadratic expression.

a. $(x+2i)(x-2i)$

b. $(x-(3+2i))(x-(3-2i))$

Quadratic Functions and Their Graphs

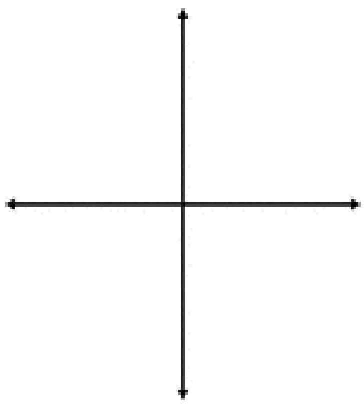
Student Learning Objectives:

- Spiraling back to Module 2: *Solve quadratic equations* and inequalities and related application problems *using* graphing, factoring, *the quadratic formula*, and completing the square.
- Given a quadratic equation in any form, graph the equation by finding intercept(s), vertex and axis of symmetry.
- Determine properties of quadratic functions that give real roots or imaginary roots by analyzing the quadratic formula.

We have already learned many things about how to graph quadratic functions from their equations: how to determine he vertex, the direction of opening, the y-intercept, the x-intercepts, the maximum or minimum value of the function, and intervals of increase and decrease. Now let's review the axis of symmetry.

The **axis of symmetry** of a quadratic function's graph is a vertical line with equation $\boldsymbol{x = h}$, where h is the x-coordinate of the vertex. The graph of the quadratic function, a **parabola**, is symmetric about this line. For example, in the graph of $f(x) = 2(x-3)^2 - 10$, the vertex is $(3, -10)$ and the y-intercept is $(0, 8)$. The equation of the axis of symmetry is $x = 3$.

1. Discuss how the vertex and y-intercept are found, and plot those two points on the axes below. Draw the axis of symmetry on the graph as a dotted line.



2. Because the axis of symmetry is the line $x = 3$ and $(0,8)$ is three units to the left of the line $x = 3$, symmetry tells us there is another point on the graph that is the reflection of $(0,8)$ located three units to the right of the axis of symmetry. Plot that point on the axes above and give its coordinates.____________

3. Evaluate $f(5)$ and write the corresponding ordered pair. ______________

4. State the coordinates of the point that is its reflection about the axis of symmetry.______________

5. For the function $f\left(x\right)=-2\left(x+1\right)^2-3$:

a. Determine the vertex.

b. Does the parabola open upward or downward?

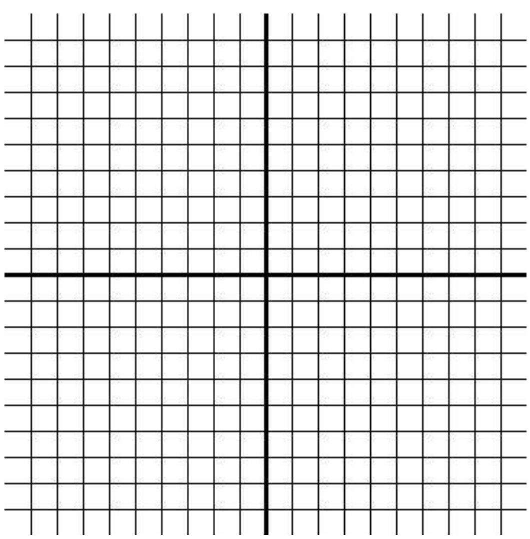
c. Write the equation of the axis of symmetry.

d. Determine the y-intercept.

e. Convert $f(x)$ to the form $f(x)=ax^2 + bx + c$.

f. Determine all real and non-real roots/zeros of the function by using the quadratic formula. If the zeros/roots are real, determine the x-intercept(s).

g. Create a graph labeling the coordinates of the vertex, labeling **one pair** of symmetric points, and showing the axis of symmetry as a dotted line.



h. Determine the maximum or minimum value of the function and state whether it is a maximum or minimum.

i. State the intervals where the function increases and decreases.

6. For the function $f(x) = 3(x-2)^2 - 5$:

   a. Determine the vertex.

   b. Does the parabola open upward or downward?

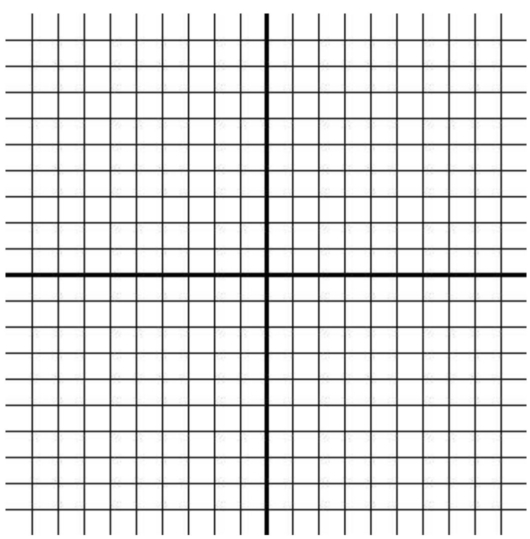
   c. Write the equation of the axis of symmetry.

   d. Determine the y-intercept.

   e. Convert $f(x)$ to the form $f(x) = ax^2 + bx + c$.

   f. Determine all real and non-real roots/zeros of the function by using the quadratic formula. If the zeros/roots are real, determine the x-intercept(s).

   g. Create a graph labeling the coordinates of the vertex, labeling **one pair** of symmetric points, and showing the axis of symmetry as a dotted line.



   h. Determine the maximum or minimum value of the function and state whether it is a maximum or minimum.

   i. State the intervals where the function increases and decreases.

7. For the function $f\left(x\right)=2x^2-12x-3$:

   a. Determine the vertex.

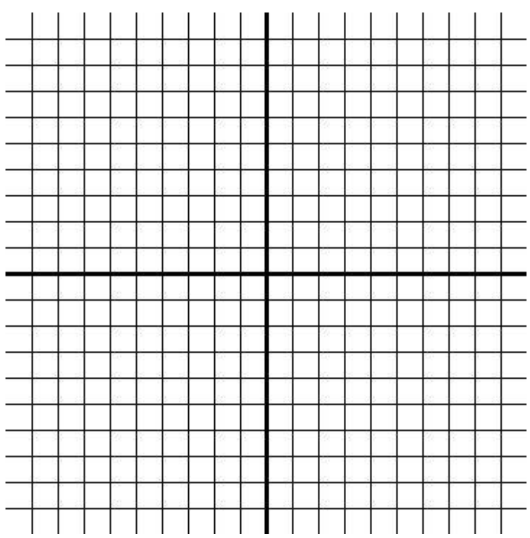
   b. Determine the direction of opening.

   c. Write the equation of the axis of symmetry.

   d. Determine the y-intercept.

   e. Without solving, determine if the roots/zeros are real or non-real complex numbers. Justify your answer.

   f. Create a graph labeling the coordinates of the vertex, labeling **one pair** of symmetric points, and showing the axis of symmetry as a dotted line.



   g. Determine the maximum or minimum value of the function and state whether it is a maximum or minimum.

   h. State the intervals where the function increases and decreases.

8. For the function $f(x) = x^2 + 8x + 20$:

a. Determine the vertex.

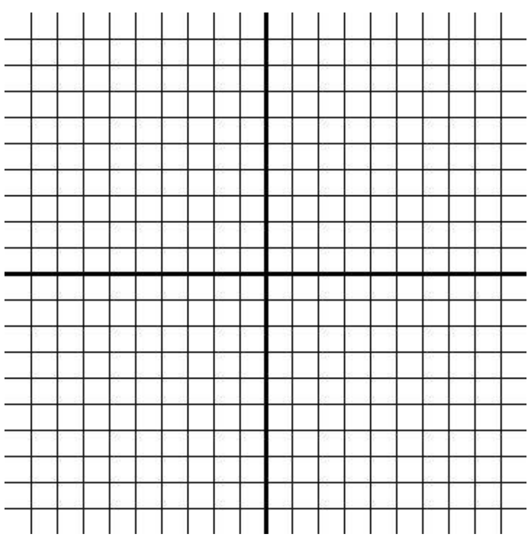
b. Determine the direction of opening.

c. Write the equation of the axis of symmetry.

d. Determine the y-intercept.

e. Without solving, determine if the roots/zeros are real or non-real complex numbers. Justify your answer.

f. Create a graph labeling the coordinates of the vertex, labeling **one pair** of symmetric points, and showing the axis of symmetry as a dotted line.



g. Determine the maximum or minimum value of the function and state whether it is a maximum or minimum.

h. State the intervals where the function increases and decreases.

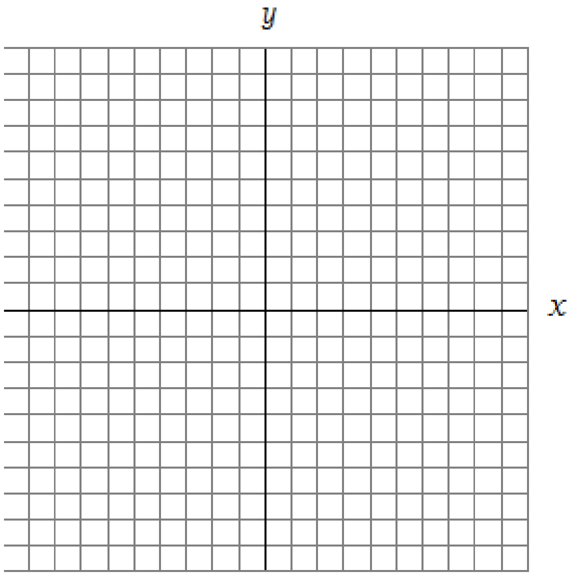
Graphing Polynomials

Student Learning Objective:

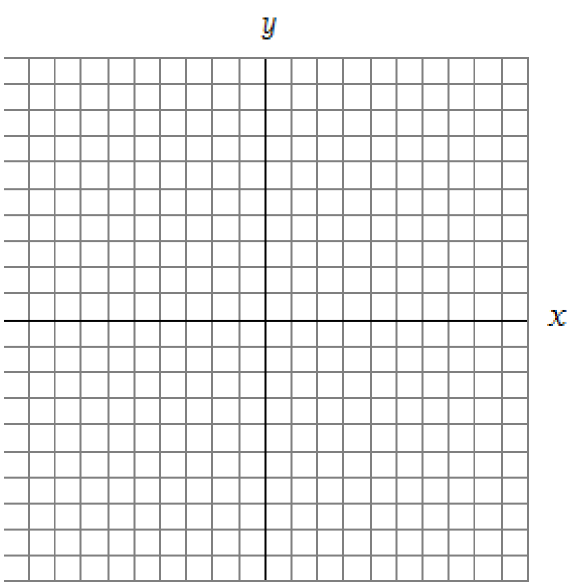
- Given a polynomial function in symbolic form, graph the function.

Sketch a graph of each polynomial function given.

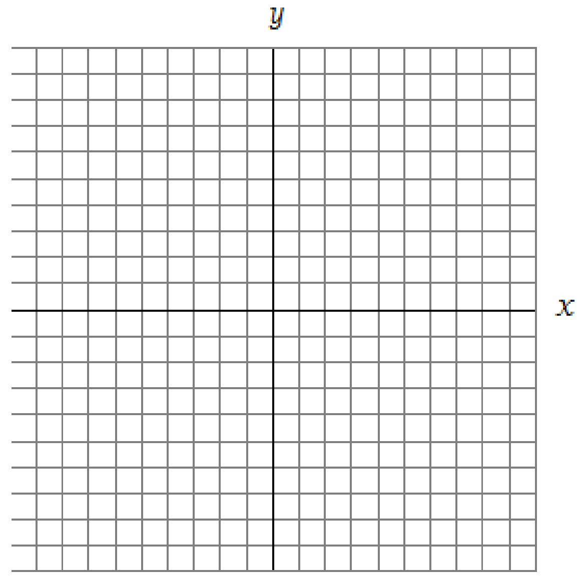
1. $g(x) = x^2 + 4x + 4$



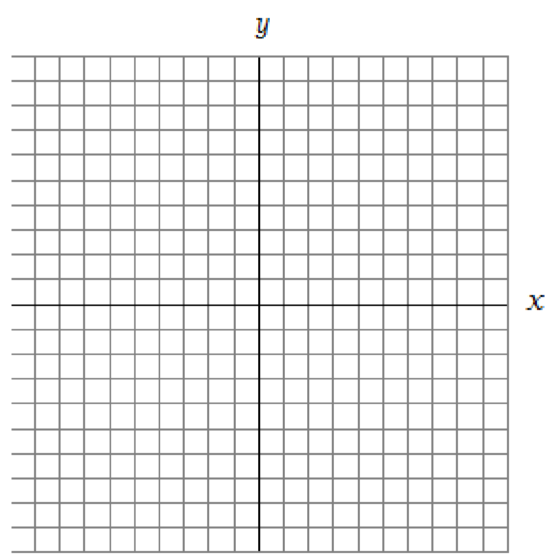
2. $h(x) = -(x-5)(x-4)(x+1)$



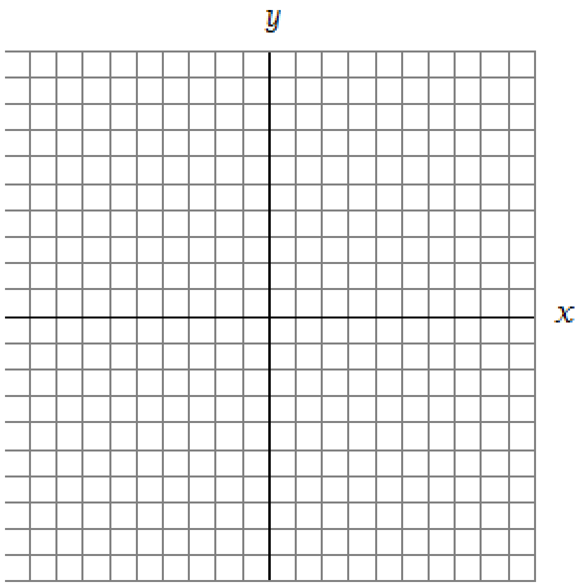
3. $f(x) = 3x^2(x+4)^3$



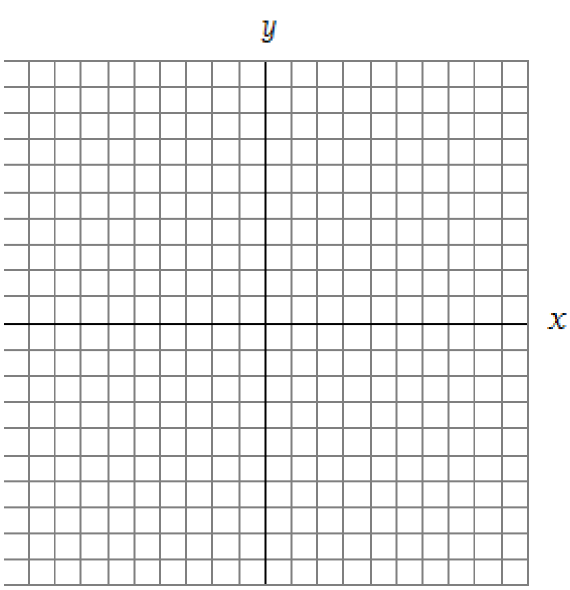
4. $p(x) = -2(x+2)(x-4)^2$



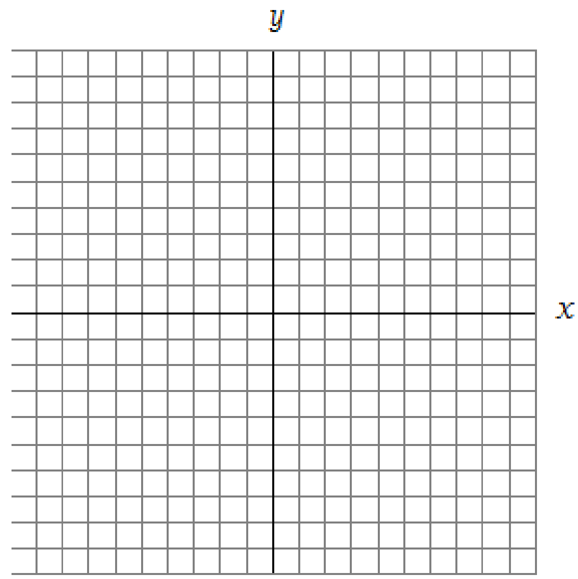
5. $q(x)=(3-x)(x-4)^3$



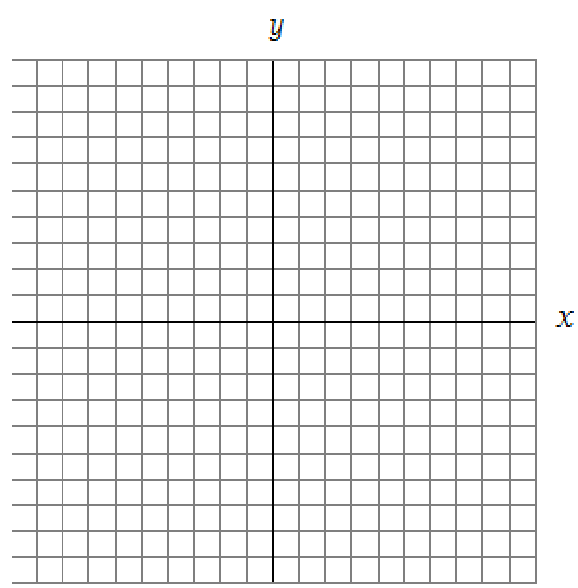
6. $c(x)=-3x(x-2)^3(x+3)^2$



7. $d(x)=2(4-x)(x+1)^2$



8. $m(x)=(x-3)^3(x-1)^2$



Dividing Polynomials

Student Learning Objective:

- Perform Long Division on polynomials and identify the quotient and remainder.

Divide the polynomials, write them as $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$ and explicitly state the quotient and remainder. Check your answer by seeing if $P(x) = D(x) \cdot Q(x) + R(x)$. For the example, $\frac{x^3 + x - 1}{x^2 + x - 2} = x - 1 + \frac{4x - 3}{x^2 + x - 2}$, the divisor is $D(x) = x^2 + x - 2$, the quotient is $Q(x) = x - 1$, and the remainder is $R(x) = 4x - 3$.

1. $\frac{3x^3 - x^2 + x - 2}{x + 2}$ Quotient: _______________ Remainder:_______________

2. $\frac{4x^4 + 3x^3 + 2x + 1}{x^2 + x + 2}$ Quotient: _______________ Remainder:_______________

3. $\dfrac{5x^4 - x^2 + x - 2}{x^2 + 2}$ Quotient: _______________ Remainder: _______________

4. $\dfrac{2x^4 + 3x^3 - 12}{x + 4}$ Quotient: _______________ Remainder: _______________

The Remainder Theorem

Student Learning Objectives:

- Given a polynomial function, use the remainder theorem to determine if a given x-value is a zero of the polynomial.
- Factor polynomials: Into linear and irreducible quadratic factors with real coefficients & completely into linear factors with complex coefficients.

Remainder Theorem: If a polynomial $P(x)$ is divided by $x-c$, then the remainder is the value $P(c)$.

Factor Theorem: If a polynomial $P(x)$ is divided by $x-c$, and the remainder is 0, then c is a zero of the polynomial and $x-c$ is a factor of the polynomial.

Example: Given the polynomial $P(x)=2x^3+x^2-13x+6$, evaluate $P(-1)$ and $P(2)$. If c is a zero, then factor the polynomial completely using the Remainder Theorem and Factor Theorems along with long division.

Solution:

$P(-1)=18\neq 0$. Therefore, -1 is not a zero of P. $P(2)=0$, therefore $c=2$ is a zero of P, and $x-2$ is a factor of P

Now using long division we see

$$\begin{array}{r|l}
 & 2x^2-x-12 \\
\hline
x+1 & 2x^3+x^2-13x+6 \\
 & \underline{-2x^3-2x^2} \\
 & \quad -x^2-13x \\
 & \quad \underline{x^2+x} \\
 & \qquad -12x+6 \\
 & \qquad \underline{12x+12} \\
 & \qquad\qquad 18
\end{array}$$

The remainder is 18, so $P(-1)=18$.

$$\begin{array}{r|l}
 & 2x^2+5x-3 \\
\hline
x-2 & 2x^3+x^2-13x+6 \\
 & \underline{-2x^3+4x^2} \\
 & \quad 5x^2-13x \\
 & \quad \underline{-5x^2+10x} \\
 & \qquad -3x+6 \\
 & \qquad \underline{3x-6} \\
 & \qquad\qquad 0
\end{array}$$

The remainder is 0, so $P(2)=0$. In other words, 2 is a zero of $P(x)$. Factoring $2x^2+5x-3$ gives us $P(x)=(x-2)(2x-1)(x+3)$

Exercises: Use division and the Remainder Theorem to evaluate $P(c)$. If c is a zero, then factor the polynomial completely.

1. $P(x)=4x^2+12x+5,\ c=-1$

2. $P(x) = x^3 - 3x^2 + 3x - 1,\ c = 1$

4. $P(x) = x^3 + 3x^2 - 7x + 6,\ c = 2$

3. $P(x) = x^4 + 3x^3 - 16x^2 - 27x + 63,\ c = 1$

5. $P(x) = x^3 + 2x^2 - 3x - 10,\ c = 1$

Zeros and Factors of Polynomials

Student Learning Objectives:

- Find all real and non-real zeros of a polynomial, and write polynomials with specified coefficients and zeros.
- Factor polynomials: Into linear and irreducible quadratic factors with real coefficients & completely into linear factors with complex coefficients.

Remember to use only exact values when factoring. Answer each of the following for the polynomials in #1 – 4:

a) Factor the polynomial as a product of linear and irreducible quadratic factors over the real numbers with real coefficients.
b) Determine all real and non-real complex zeros of the polynomial.
c) Factor the polynomial as a product of linear factors with complex coefficients.

1. $f(z) = z^3 + 6z^2 - z$

2. $g(a) = a^3 - 4a^2 + 8a - 32$

3. $h(x) = x^4 - 2x^2 - 24$

4. $v(t) = t^5 - 64t^2$

5. Let $f\left(x\right)=x^4-19x^2-30x$

a) Using division, show that $x+3$ is a factor of f. What is the quotient?

b) Determine all real and non-real complex zeros of f.

c) Write f as a product of linear factors with complex coefficients.

6. Let $g\left(x\right)=x^4+2x^3+5x^2-2x-6$

a) Using division, show that x^2-1 is a factor of g. What is the quotient?

b) Determine all real and non-real complex zeros of g.

c) Write g as a product of linear factors with complex coefficients.

Polynomial Inequalities

Student Learning Objective:

- Given a polynomial inequality, find the solutions algebraically and graphically; represent solutions using interval notation.

In Module 2 we found solutions to Quadratic Inequalities using a graphic method and a sign chart method.

We now have polynomials other than quadratics. Refer to your work from Module 2 – Solving Quadratic Inequalities.

The process is essentially the same! Rewrite the inequality by collecting all the terms on one side of the inequality sign with zero as the only term on the other side. Then, factor the polynomial using what you have learned in this unit about division and zeros. If the polynomial is already factored, then simply proceed with the solving process using the sign chart or graphical method.

Solve each inequality. State your solutions in interval notation using exact values.

1. $(x+1)(2x-3)(x-4)>0$

2. $x^3-x^2\geq 0$

3. $(x^2+6x+9)(x^2-5x-6)\leq 0$

4. $2x^3>8x$

5. $x^4-17x^2+20>4$

Polynomial Models

Student Learning Objective:

- Given a real world application, use a given quadratic or polynomial model or write an appropriate model and evaluate or solve to answer the question in context.

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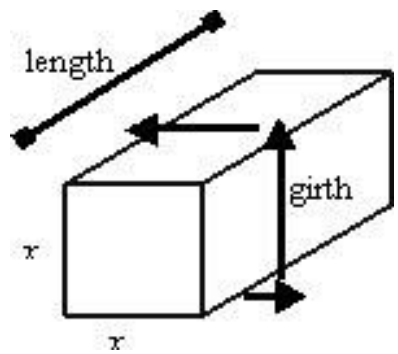
1. A parcel delivery service will only deliver packages with length plus girth (distance around) not exceeding 108 inches. A rectangular shipping box with square ends is to be used. Assume all 108 inches are used.



   a. Write an equation which relates the length and x, and solve this equation for the length.

   b. Use graphing technology to determine the length of the box if its volume is 2200 cubic inches. (Hint: you need a model for the volume first).

   c. Use Desmos to graph the volume function and determine the maximum volume of the box.

2. A wire 10 cm long is cut into two pieces, one of length x and the other of length $10 - x$. One piece is bent into the shape of a square and the other into a circle.
   a. Write a function that models the total area enclosed by the two figures.

   b. What is the domain of this function?

   c. Determine the value of x that minimizes the total area of the two figures.

3. A silo is being built in the shape of a cone where the height of the cone is 1.5 times the radius. Determine the dimensions of a silo with a volume of 1400 cubic feet.

4. A Norman Window has the shape of a rectangle beneath a semicircle. Your neighbor is designing a new house and would like a Norman Window with a perimeter of 20 ft that will let in the most light (that is, have the largest possible area). What should the height of the rectangle and the radius of the circle be to allow this?

5. A woman is going to create a rectangular garden at the back of her house using 100 feet of fencing for 3 sides and her house as the fourth.
   a. Write a function that models the area of the garden.

   b. What is the domain of this function?

   c. Determine the dimensions of the garden that maximize the area of the garden.

6. A theater owner charges \$5 per ticket and sells 250 tickets. By checking the prices and number of tickets at other nearby theaters, the owner decides that for every \$1 she raises the ticket price, she will lose 10 customers.
   a. Write a function R that models her revenue.

   b. What should she charge to maximize the revenue?

   c. What is her maximum possible revenue?

   d. How many tickets will she sell if her revenue is \$2000?

7. A box with a square base and no top is to be built with all of a \$12 budget. The base costs \$2/square foot while the walls cost \$1.50/square foot. Let the side length of the box's base be x and the height of the box be h, both in feet.
   a. Write h in terms of x.

   b. Write the box's volume V as a polynomial function of x.

   c. State the domain of V and justify it mathematically.

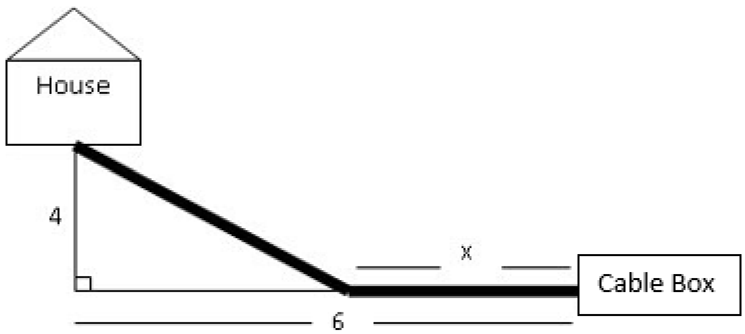
   d. Use graphing technology to determine the maximum volume of such a \$12 box.

Modeling Functions (not necessarily polynomials)

8. One leg of a right triangle is twice as long as the other. Write a function that models its perimeter, P, in terms of x, the length of the shorter leg.

9. Murray Farms packages canned apples in metal cylinders. Their medium-size can should hold 120 in^3. Use graphing technology to determine the least amount of metal needed for the medium-size can.

10. Carolina Cable Company has been asked to install cable service to a house that is located 4 miles off of the road. The nearest connection box for the cable is located 6 miles down the road (see diagram). The installation cost is \$ 10 per mile along the road and \$ 25 per mile off the road.



a. Write a function, $C(x)$, that models the total installation cost as a function of x, the length of cable (in miles) along the road.

b. What is the domain of this function?

c. Use graphing technology to determine the minimum cost to run cable to the house.

d. Repeat parts a through c but this time let $\boldsymbol{x}$ be the base of the right triangle in the diagram. Which choice of $\boldsymbol{x}$ do you prefer?

11. An apple orchard produces an annual revenue of \$ 50 per tree when planted with 1000 trees. Because of overcrowding, the annual revenue per tree is reduced by 2 cents for each additional tree planted or increased by 2 cents for each tree removed. The cost of maintaining each tree is \$ 10 per year.
    a. Write a function P that models the yearly profit of the orchard *(Profit=Revenue-Cost).*

    b. What is the domain of P ?

    c. How many total trees should be planted to maximize profit?

12. A rectangular corral of 32 square yards is to be fenced off and then divided into two sections by a stone fence that is perpendicular to two sides of the outer fence. The outer fence costs \$ 10 per yard, while the stone fence costs \$ 20 per yard.

   a. Determine the cost of fencing this corral, C, as a function of the length of the stone fence.

   b. What is the domain of C?

   c. Determine $C(8)$ and interpret what this means in the context of the problem.

Rational Functions

Student Learning Objectives:

- From the graph of a polynomial, rational, or exponential function, write its equation.
- From a function in symbolic form, determine its x- and y-intercept(s), vertical and horizontal asymptotes, and hole(s), if they exist.
- Given a rational function in symbolic form, graph the function.

Part 1: For each of the following functions, determine each if applicable:

a. Domain
b. Vertical asymptotes
c. Horizontal asymptotes
d. Zeros/x-intercepts/horizontal intercepts
e. Coordinates of hole.
f. Y-intercepts

i. $f(x)=\dfrac{2x-1}{x+4}$

a. ______________________________

b. ______________________________

c. ______________________________

d. ______________________________

e. ______________________________

f. ______________________________

ii. $f(x)=\dfrac{x(x-3)}{(x-3)(x+1)}$

a. ______________________________

b. ______________________________

c. ______________________________

d. ______________________________

e. ______________________________

f. ______________________________

iii. $f(x)=\dfrac{6(x-5)(x-3)}{(x+2)(x-1)}$

a. ______________________________

b. ______________________________

c. ______________________________

d. ______________________________

e. ______________________________

f. ______________________________

iv. $f(x)=\dfrac{x(x+3)}{(x-4)(x+3)}$

a. ______________________________

b. ______________________________

c. ______________________________

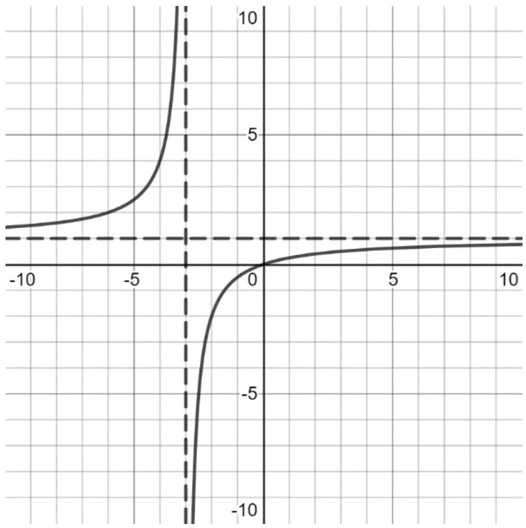
d. ______________________________

e. ______________________________

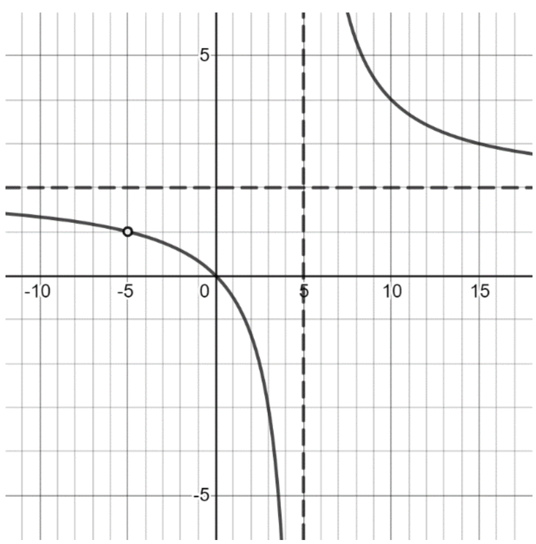
f. ______________________________

Part 2: For each of the following graphs, determine the rational function $f(x)$.

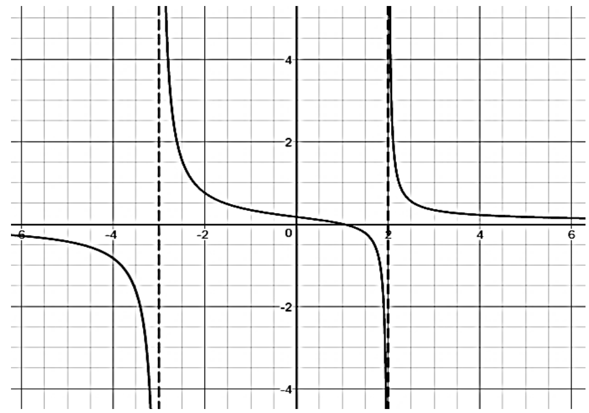
A.



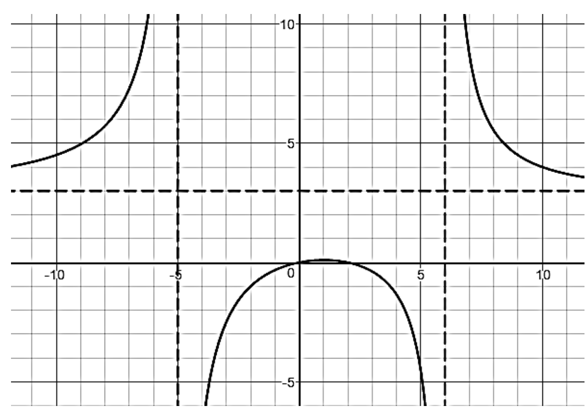
B.



C.



D.



Rational Functions Exploration

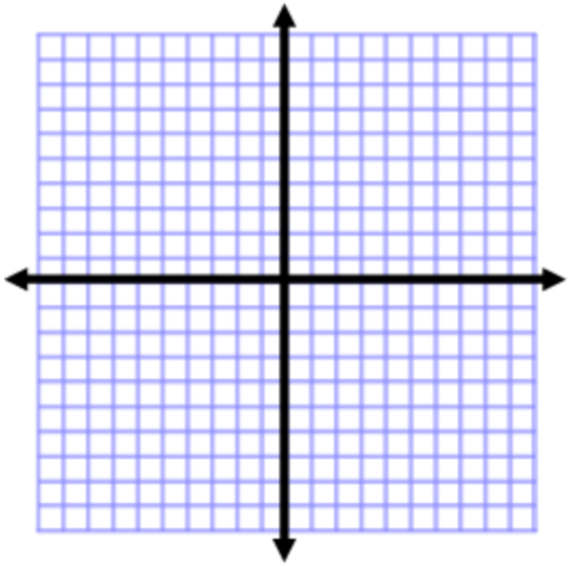
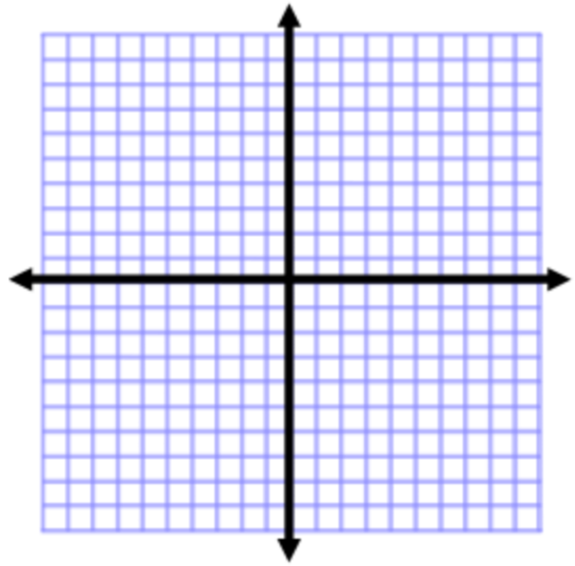
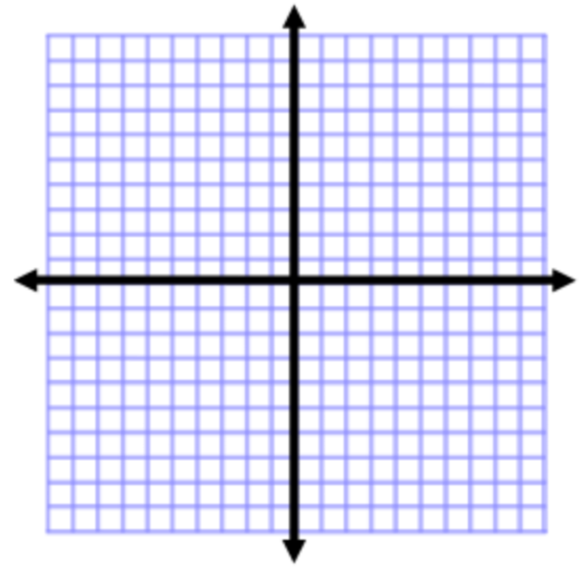
Student Learning Objective: From a function in symbolic form, determine its x- and y-intercept(s), vertical and horizontal asymptotes, and hole(s), if they exist.

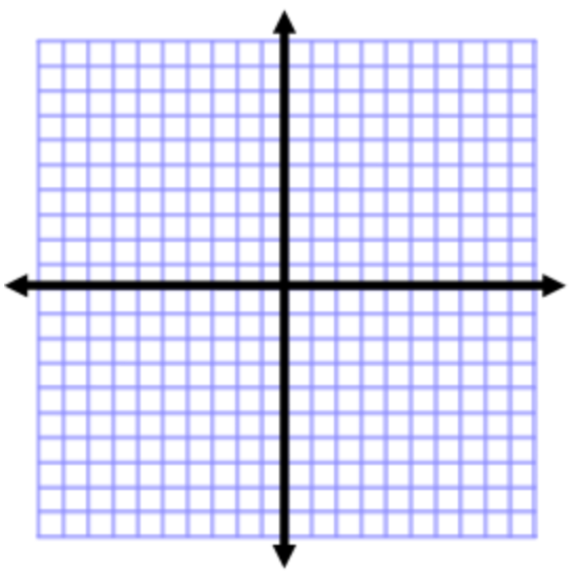
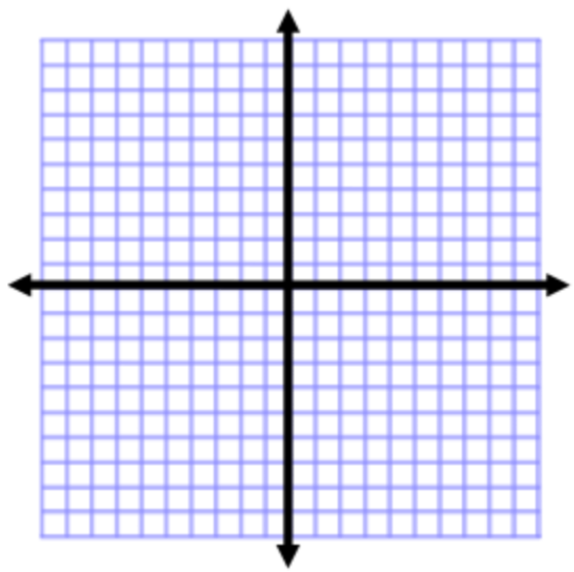
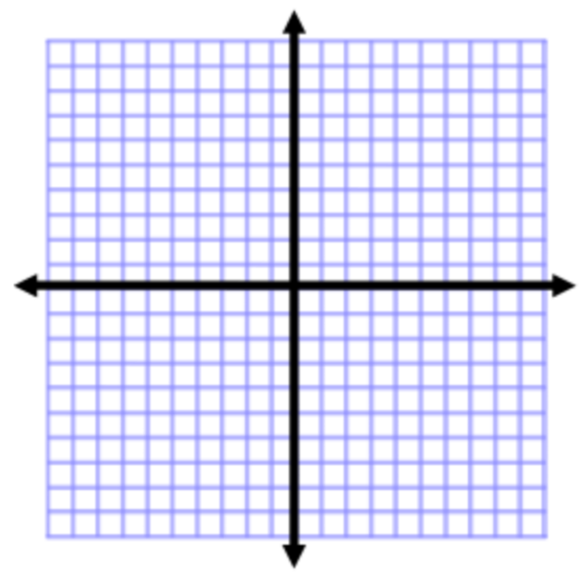
	Equation	Domain	Hole(s)	Vertical Asymptote(s) $x = b$	Horizontal Intercept(s) $(x,0)$	Vertical Intercept $(0, y)$	Horizontal Asymptote $y = a$
1.	$y=\frac{5x+3}{x-2}$						
2.	$y=\frac{7(x-3)(x+2)}{3(x-1)(x+4)}$						
3.	$y=\frac{5x^2+3}{x^2-4}$						
4.	$y=\frac{15(x-3)(x+5)}{x(x+5)}$						
5.	$y=\frac{x^2-9}{x^3+1}$						
6.	$y=\frac{5(x+2)}{(x+2)(x-4)}$						
7.	$y=\frac{x+2}{x^2+5}$						

Student Learning Objectives:

- Given a rational function, without a slant asymptote, in symbolic form, graph the function.

Graph the first six rational functions from the previous page.

Rational Functions Exploration (2)

Student Learning Objectives:

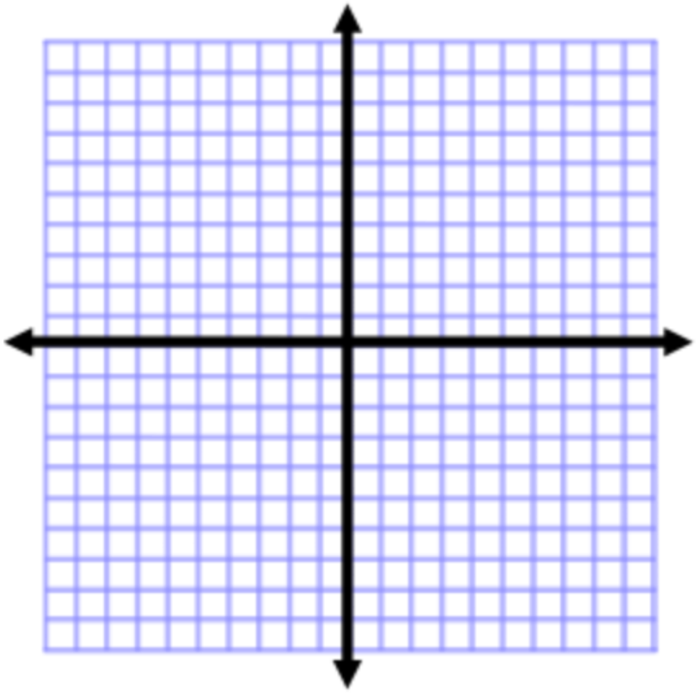
- From a function in symbolic form, determine its x- and y-intercept(s), vertical and horizontal asymptotes, and hole(s), if they exist.
- Determine if a rational function has a slant asymptote, and if one exists, identify/find it.
- Given a rational function, with a slant asymptote, in symbolic form, graph the function.

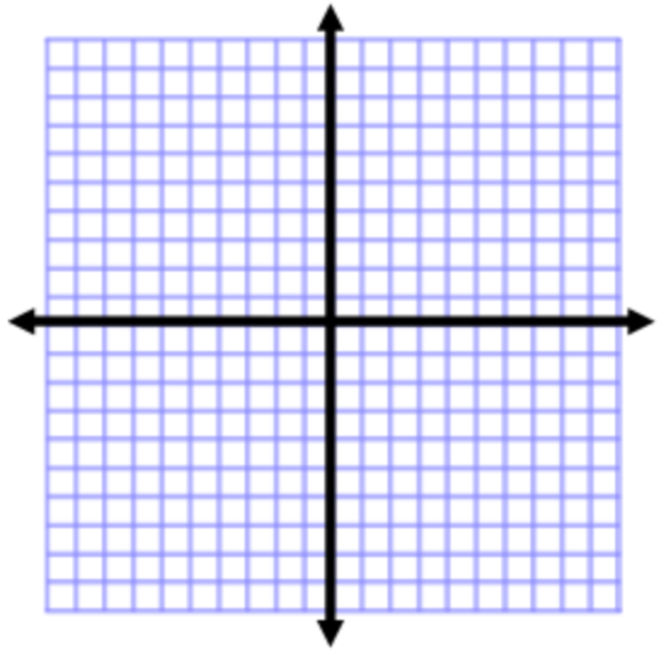
Fill in the chart below **without using a calculator**.

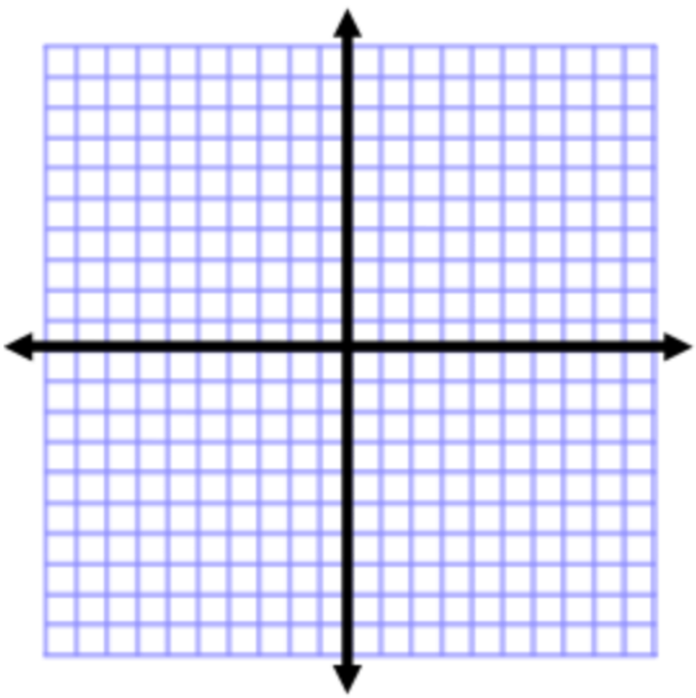
	Equation	Domain	Hole(s)	Vertical Asymptote(s) $x=b$	Horizontal Intercept(s) $(x,0)$	Vertical Intercept $(0,y)$	Horizontal Asymptote $y=a$	Write the slant asymptote a $y=mx+b$ or state that the rational function has an oblique asymptote
1.	$f(x)=\dfrac{3(x-1)(x-5)(x+7)}{x-2}$							
2.	$g(x)=\dfrac{3(x+5)(x-4)}{x-2}$							
3.	$h(x)=\dfrac{2x^2-18}{x+3}$							

Follow-up: Describe how you determine if a rational function has a horizontal asymptote, slant asymptote or an oblique asymptote. If the rational function has a slant asymptote, explain how you determine its equation.

Example: Using the information found in the chart on the previous page, graph each rational function.







Graphing Rational Functions

Student Learning Objectives:

- From a function in symbolic form, determine its x- and y-intercept(s), vertical and horizontal asymptotes, and hole(s), if they exist.
- Given a rational function, without a slant asymptote, in symbolic form, graph the function.
- Given a rational function, with a slant asymptote, in symbolic form, graph the function.

Skill Builder: Given the rational function, $f(x)=\dfrac{3(x-1)(x+3)}{(x-4)(x-1)}$, determine:

1. The ordered pair of a hole (if one exists)

2. The equation(s) of the vertical asymptotes

3. The equation of the horizontal asymptote (if one exists)

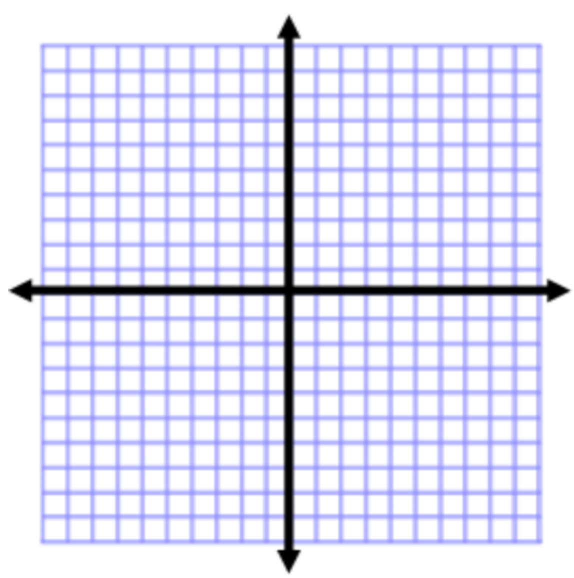
4. The equation of the slant asymptote (if one exists)

5. The x-intercept(s)

6. The y-intercept

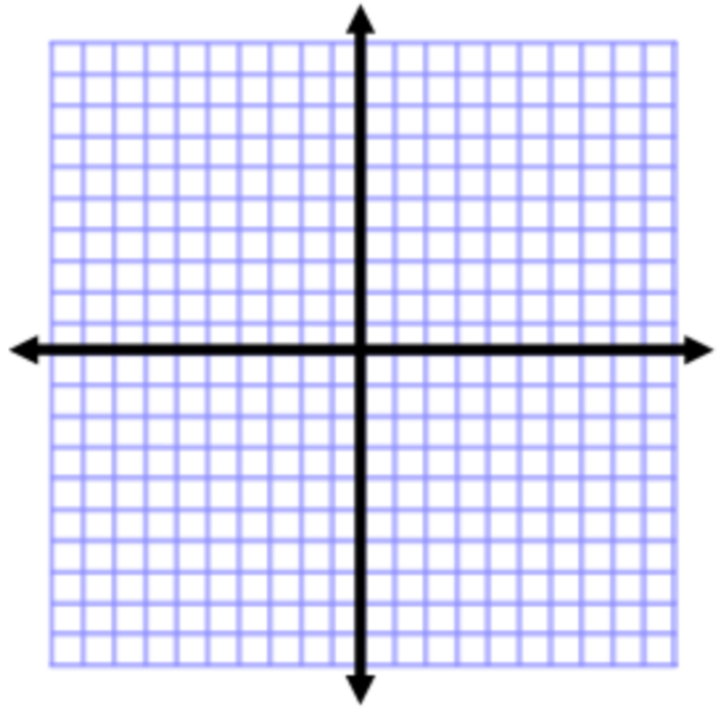
Example 1: Graph the rational function from the skill builder.

a) Graph vertical asymptotes
b) Graph horizontal or slant asymptote
c) Plot the hole (if one exists)
d) Plot the x-intercept(s)
e) Plot the y-intercept
f) Graph an additional point or two to get the basic shape.

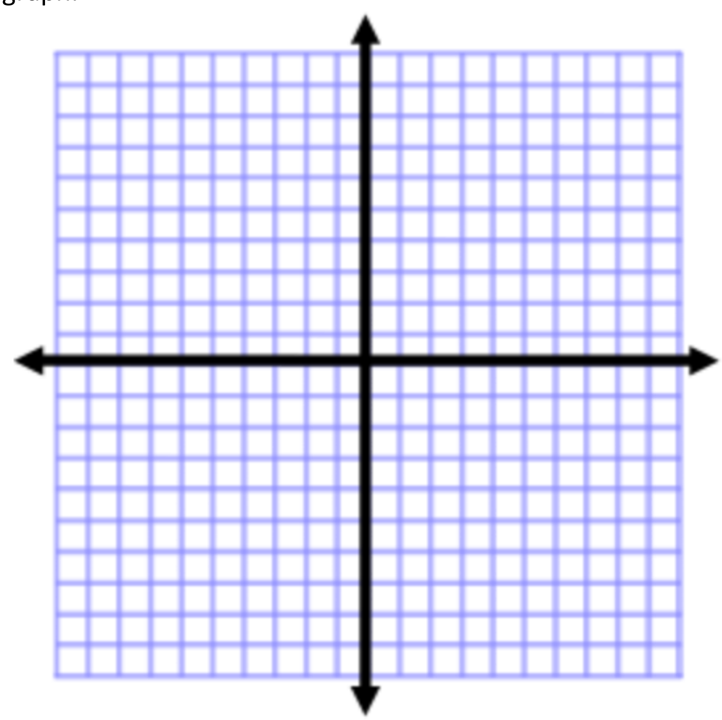


You try!

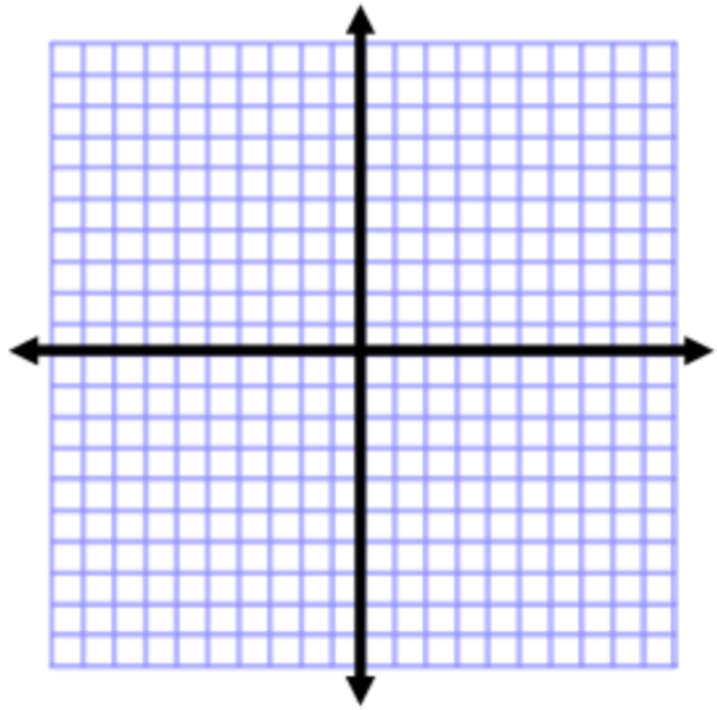
1. Given the function, $f\left(x\right)=\dfrac{\left(x+3\right)}{\left(x-1\right)\left(x+2\right)}$, state the relevant information and sketch a graph of the function. Be sure to include all relevant information with labels on your graph.



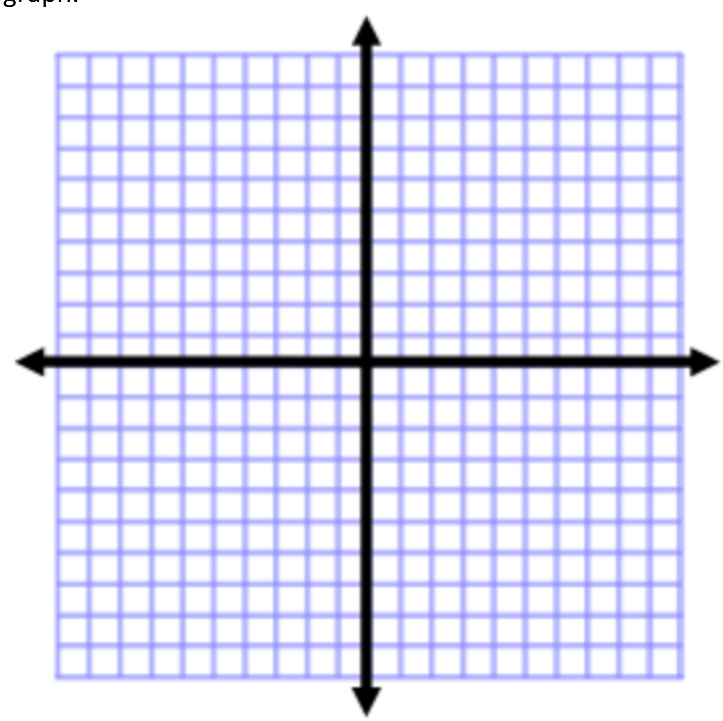
2. Given the function, $f\left(x\right)=\dfrac{3\left(x+3\right)\left(x+1\right)}{x+2}$, state the relevant information and sketch a graph of the function. Be sure to include all relevant information with labels on your graph.



3. Given the function, $f\left(x\right)=\dfrac{\left(x-1\right)}{\left(x-1\right)\left(x+2\right)}$, state the relevant information and sketch a graph of the function. Be sure to include all relevant information with labels on your graph.



4. Given the function, $f\left(x\right)=\dfrac{-3x^{2}+6x+24}{x^{2}+3x-4}$, state the relevant information and sketch a graph of the function. Be sure to include all relevant information with labels on your graph.



Module 3 Practice

1. Perform the polynomial division. Identify the quotient and remainder.

   a. $\dfrac{x^3+6x+3}{x^2-2x+2}$

   b. $\dfrac{3x^6-2x^5+2x^4+x^2-2x+1}{x^2+1}$

2. Solve the inequalities and write your solution in interval notation.

   a. $3x^2-3x \geq 2x^2+4$

   b. $(x-1)^2(x+3)(x-3)<0$

3. A polynomial, P, has the following characteristics:
   - Real coefficients
   - Degree of 5
   - Zeros at 0, 1, and $2-i$
   - The multiplicity of the zero at 1 is 2
   - $P(2)=2$

   a. Write P as a product of linear factors with complex coefficients.

   b. Write P as a product of linear and irreducible quadratic factors with real coefficients.

4. Use division to show that the given number is a zero of the given polynomial. Determine all zeros of the polynomial.

   a. $r=-2$ and $P(x)=x^3+4x^2+2x-4$

   b. $r=-5$ and $P(x)=x^4+7x^3+11x^2+5x$

5. Use $P(x)=2x^2-20x+3$ to complete the following:

   a. Write P in transformation form.

   b. What are the coordinates of the vertex?

   c. On what interval is P increasing?

   d. Determine two symmetric points that are two units on either side of the vertex.

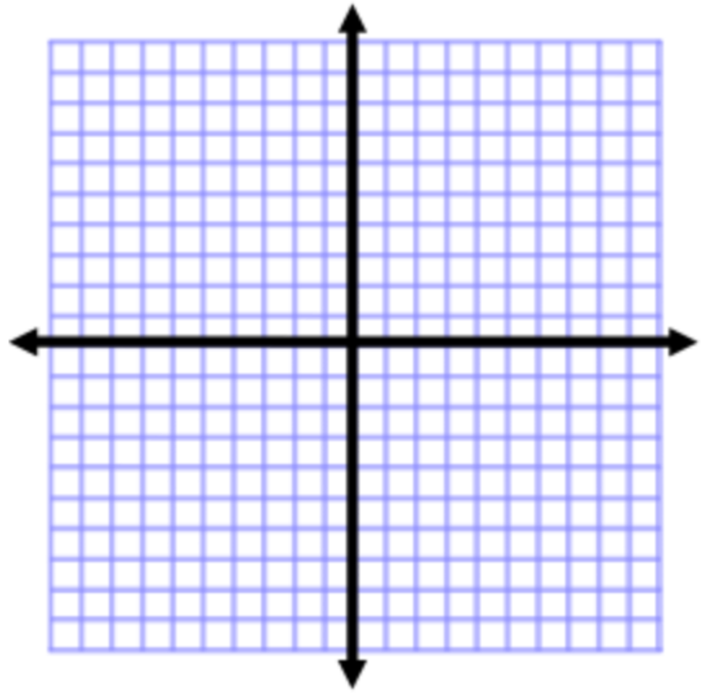
6. Perform the operation and simplify

   a. $3i(\sqrt{-24}+2i)$

   b. $((x-4)-5i)((x-4)+5i)$

7. Use the rational function $f(x)=\dfrac{3(x-2)}{x^2+3x-10}$ to determine each of the following. If one of the following does not exist, write DNE.

a. Domain in interval notation.



b. Equation of vertical asymptote(s).

c. Equation of horizontal asymptote.

d. Equation of slant asymptote.

e. y-intercept written as an ordered pair.

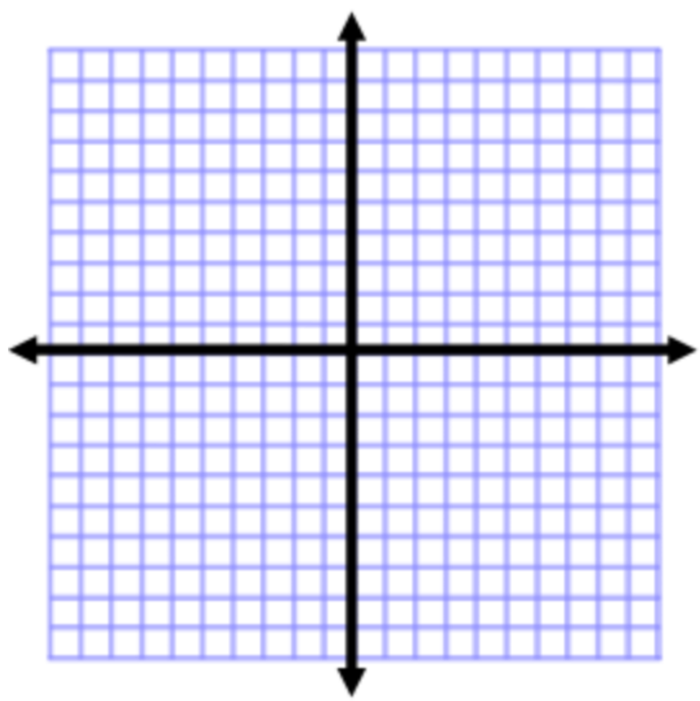
f. Location of hole written as an ordered pair.

g. x-intercept(s) written as an ordered pair.

h. Graph f.

8. Use the rational function $f(x)=\dfrac{x^3-2x^2+5x-10}{x^2+3x-10}$ to determine each of the following. If one of the following does not exist, write DNE

a. Domain in interval notation.



b. Equation of vertical asymptote(s).

c. Equation of horizontal asymptote.

d. Equation of slant asymptote.

e. y-intercept written as an ordered pair.

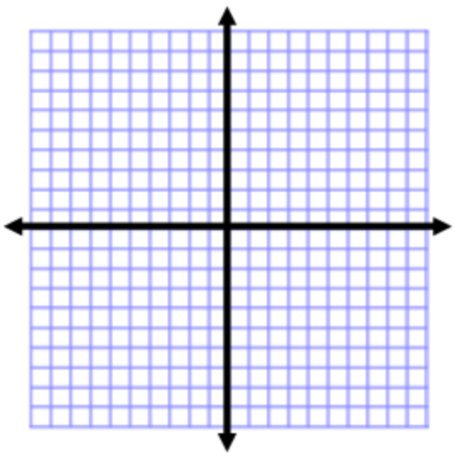
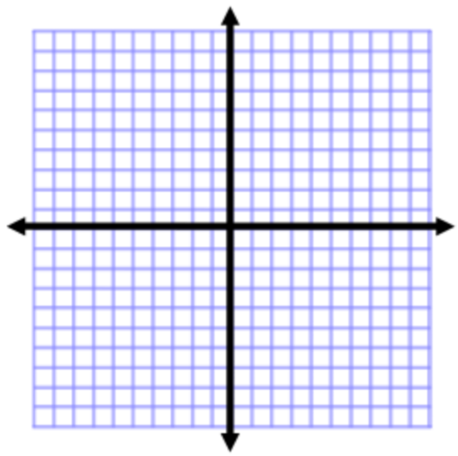
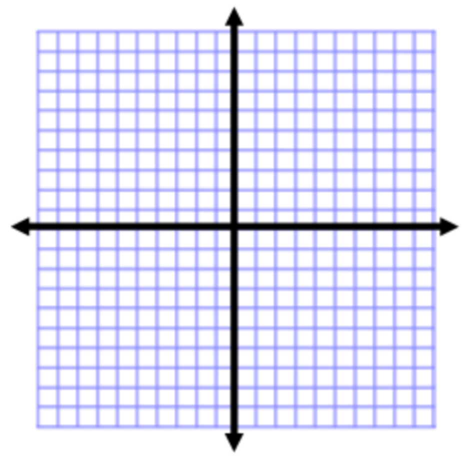
f. Location of hole written as an ordered pair.

g. x-intercept(s) written as an ordered pair.

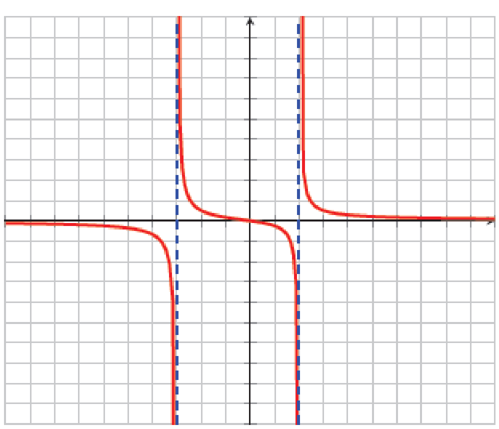
h. Graph f.

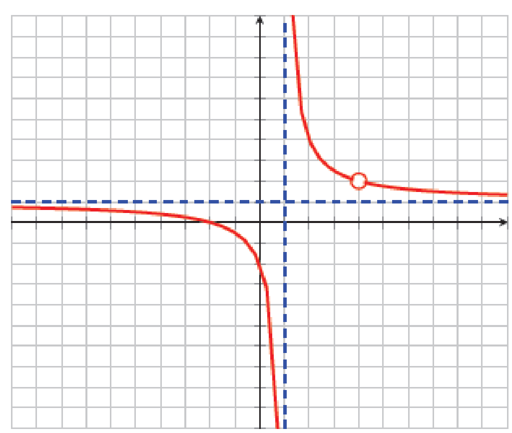
9. Determine the following for each rational function:
    a. x-intercept(s) and y-intercept.
    b. Equation of horizontal asymptote.
    c. Equation(s) of vertical asymptote(s).
    d. Coordinates of hole(s).
    e. Using the information from parts a-d and additional points, graph each rational function.

$R(x)=\dfrac{x}{x-5}$ $\qquad$ $Q(x)=\dfrac{x-8}{x^2-6x-16}$ $\qquad$ $Z(x)=\dfrac{2x^2+3x-5}{x^2-4}$

10. Write equations of the following graphs:

a.


b.


11. The length of a rectangle is twice the width. If the length and width are both decreased by 4 inches the resulting area is 44 square inches less than the original rectangle.
    a. Sketch each rectangle and label the rectangles with appropriate variables.

    b. Write an equation that shows the relationship between the areas of the original and the new rectangles.

    c. Determine the dimensions of the original rectangle.

12. You have a 1200-foot roll of fencing and a large field. You want to build a rectangular enclosure in the field that has two equal-sized rectangular parts. What are the dimensions of the largest enclosure?

13. Sketch a graph of $P(x) = -\frac{1}{4}\left(x^4 - 4x^2\right)(x+2)^2$ by first fully factoring P.

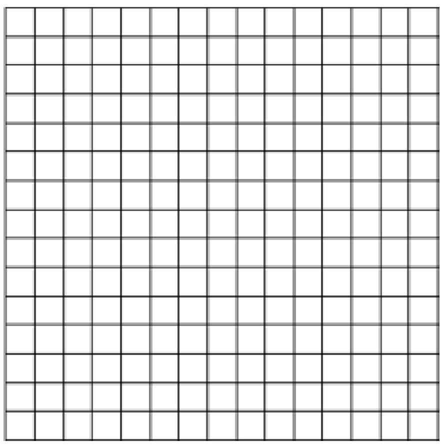
Spiral Review for Module 4

Addresses SLO's from Modules 1, 2, 3.

$$f(x)=\frac{2x}{x-4}$$

a. Determine $f(0)$, $f(3)$, and $f(x+h)$.

b. Sketch the graph of f by plotting the points $(0, f(0))$, $(3, f(3))$ and using the x-and y- intercept, vertical, horizontal, and/or slant asymptotes, and holes, if any.



c. Using the points $(0, f(0))$ and $(3, f(3))$ along with the graph, determine the average rate of change of f on the interval $[0,3]$

d. Use the difference quotient (or the slope formula) to determine the average rate of change of f on the interval $[0,3]$.

e. Using $f(x+h)$ found in part a., determine and simplify $f(x+h)-f(x)$. (Hint: this will require finding a common denominator. Leave that denominator in factored form.)

f. Use the result from part e. to state the difference quotient $\frac{f(x+h)-f(x)}{h}$, $h \neq 0$, simplifying if possible but leaving the denominator in factored form.

g. Use $h=3,\ x=0$, and the result of part f to evaluate the difference quotient.

h. Write a complete sentence comparing the answers obtained in parts c, d, and g.

i. State the interval(s) on which f is increasing.

j. State the interval(s) on which f is decreasing.

k. Write a sentence to explain how your answers to part i or j are related to the results from parts c, d and g.

l. What is the end behavior of f ?

m. Use your calculator to determine the ARC for f on the interval $[100,104]$.

n. How are the results to part l and part m related?

Solving Rational Inequalities – Algebraic Approach

Student Learning Objective:

- Solve a rational inequality and state its solution in interval notation.

---

1. Set one side of the inequality to 0.
2. Combine all terms into one rational expression.
3. Factor the numerator and denominator.
4. List any restricted values.
    a. Start your sign chart with open circles on all restricted values on your number line.
5. Determine zeros of the numerator.
    a. Add these to the number line using open circles for strict inequality and closed circles if the zero is included in the solution set.
6. Create your sign chart with a row for each factor.
7. Use a test point from each interval.
8. Write the solution using interval notation.

Try these:

1. $\frac{2x-1}{x+2} > 1$

2. Go to desmos.com and graph $y = \frac{2x-1}{x+2}$ and $y = 1$. Look at where the graph of $y = \frac{2x-1}{x+2}$ is greater than 1. Do the interval(s) of x-values match with the intervals you wrote for your answer in #1?

3. $\frac{x^2-3x-10}{x^2+2x-8} \le 0$

4. Go to desmos.com and graph $y = \frac{x^2-3x-10}{x^2+2x-8}$. Do the interval(s) above zero, match with the intervals you wrote for your answer in #3?

5. $\frac{4x+1}{x-2} \le -2$

6. $\frac{21}{x+2} > \frac{3}{x-2}$

7. $\frac{x^2(x-2)^3}{x+4} \ge 0$

8. In an earlier lesson in this course we explored graphical and algebraic sign chart methods. What topic were we studying?

Solving Rational Equations

Student Learning Objective:

- Given a rational equation, find solutions algebraically.

One way to solve rational equations is provided below.

1. Factor the denominators of each rational expression in the problem.
2. List any restrictions.
   - Restrictions are any values that would cause the denominator to be equal to zero.
3. Determine the least common denominator (LCD).
4. *Multiply each term in the equation by the LCD to* eliminate fractions.
5. Solve the resulting equation. Factor or use the quadratic formula if the resulting equation is quadratic. (Remember to set equation = 0.)
6. If any answer is one of the restricted values, do not include it in your final answer. These are called ______________________________ _____________________ !

Solve the following equations.

1. $\dfrac{2}{t+3}-\dfrac{4}{t}=4$

2. $\dfrac{3x-1}{4}+\dfrac{4}{x+1}=\dfrac{5}{2}$

3. $y+2-\dfrac{3}{2y-1}=0$

4. $\dfrac{a^2}{a-1}=\dfrac{3-2a}{a-1}$

5. $\dfrac{3z}{z+1}+\dfrac{2}{z}+5=\dfrac{3}{z^2+z}$

6. $\dfrac{1}{h}=\dfrac{4}{h-1}+1$

7. $\dfrac{m+1}{3}-\dfrac{m+1}{m+2}=0$

8. $\dfrac{4}{n+1}-\dfrac{3}{n+2}=1$

Solving Exponential and Logarithmic Equations

Student Learning Objective:

- Given an exponential or logarithmic equation of any base, use the properties of exponents and logarithms to solve the equation algebraically.

Spiral activity:

Solve each using what you have learned to date about exponents and logs.

i. $3^x = \frac{1}{27}$

ii. $\log_2 \frac{1}{8} = x$

iii. $\log_{10} 100 + \ln e^{-5} = x$

iv. $e^{x-3} = e^{2x}$

- Many equations involving exponents and logs require several steps to solve. To be successful, remember to use simple techniques of solving equations from early algebra along with concepts from Module 2 (Expanding and Condensing Logarithm Expressions). You will always show your steps to finding a solution.

- When solving equations we will see some solutions that can easily be written as an integer, fraction or exact decimal. Other times, we see answers that are written with e, ln or log in the solution.

- For straight forward, "Solve the equation…" exercises, please do not approximate answers. Simplify if you can. If you cannot, leave the answer with e, ln or log in the solution.

- With *application and word problems*, **show work first**, THEN you may approximate a solution at the end of the solving process. This will help you write and understand your answer in the context of the problem.

To begin our new processes for solving exponential and log equations, Parts 1 & 2, focus on this:

Recall from Module 1:

$\boldsymbol{b^x = y}$ is the exponential form of an equation and its inverse log form is $\log_{\boldsymbol{b}} \boldsymbol{y} = \boldsymbol{x}$.

Where $b > 0, b \neq 1$ and b, x, and y represent the same value in both equations.

Part 1:

Solving a simple equation when only one conversion is required. (You see either one exponential expression OR one log expression. Variable appears one time in equation.)

1. $8^t = 6$ This equation is in ________________________________ form.

Change form to: ________________________. Once the equation is in this form we have the EXACT value. You may leave your answer as is.

2. $\log_2 x = -3$ This equation is in ____ form.

Change form to: _______________________________. Solve for x. Because this can easily be simplified to determine the EXACT value, do it!

3. $\ln(x-2) = 5$ This equation is in _____________________________________ form.

Change form to: ____________________. Solve for x. Determine if the EXACT value stands or if it can be easily simplified.

4. $e^{-2x} = 5$ This equation is in ____ form.

Change form to: ________________________. Solve for x. Determine if the EXACT value stands or if it can be easily simplified.

5. $-1 = \log(x+3)$ Solve this equation based upon your work from above.

6. $8=\left(\frac{1}{2}\right)^{x}$ Solve this equation based upon your work from above.

7. $\log_{16}\left(x-3\right)=\frac{1}{2}$ Solve this equation based upon your work from above.

Part 2:

Exponential equations that require some transformation before the form can be converted to log form. (You see only one exponential expression along with other numeric terms. Variable appears **one time** in equation.)

8. $-2=-12+\frac{2}{3}\cdot 2^{(x-4)}$ This equation is not in either form yet. We will put it in exponential form first by using simple techniques of solving equations.

Its exponential form is ____________________________. Its logarithmic form is ____________________________.

Now, solve for x.

9. Solve for the variable in each equation.
   a. $-29+10^{t+12}=71$
   b. $-4=-3-\frac{1}{2}e^{x+4}$

Part 3:

Exponential equations with different bases. These equations require taking the log/ln of both sides of the equation - a new process. These equations can be simple or quite extensive. (You see an exponential expression on both sides of the equal sign, and a variable in more than one exponent.)

10. $7^x = 2^{x+1}$ Take the log/ln of both sides and solve.

11. $4^{2x+3} = 3^{x-1}$

Part 4:

Log equations require using the properties of logarithms ***first***. Use the properties of logs to write the equation in log form. Then solve the equation. (You see more than one log expression in the original equation.)

12. $1 = \log_4 2 + \log_4 (x+3)$

13. $\log(t+3) + \log t = 1$

14. $\log_2 (t+2) + \log_2 (t-2) = 5$

15. $-2 = \log_3 2 - \log_3 (3 + x)$

16. $\ln 4t^2 - \ln 3t = 2$

17. $\log_3 (t+1) - \log_3 (t-1) = -2$

18. $\log_3 4x = 1 - \log_3 3x$

19. $\log_2 (-x) = 3 - \log_2 (2 - x)$

Spiral Review: Inverse Functions

For each function below:

A) Is the given function one-to-one?

B) If your answer to part a was yes, determine the inverse function. If your answer to part a was no, determine a restricted domain and find the inverse function with the restricted domain, if applicable.

C) Determine the domain and range of the function and its inverse function.

1. $f(x) = \dfrac{3x}{x+1}$

2. $h(x) = \dfrac{-(x+1)}{x}$

3. $f(x) = 3\log_4(x-1)$

4. $p(x) = 3^{x+1} - 3$

5. $Q(x) = \left(\dfrac{1}{3}\right)e^{2x}$

Exponential Applications

Student Learning Objective:

- Given an initial value and either a doubling time, half-time or relative rate of change, construct an exponential function that models the exponential growth or decay and use the model to answer questions in the context of the problem.

Exponential Growth (Doubling Time)

If the initial size of a population is n_0 and the doubling time is a, then the size of the population at time *t* is

$$n(t) = n_0 2^{\frac{t}{a}}$$

where a and t are measured in the same time units (minutes, hours, days, years, etc.).

Example: A certain bacteria population doubles every 12 hours. Initially, there are 250 bacteria in the colony.

a. Write a model for the bacteria population after t hours.

b. How many bacteria are in the colony after 2 weeks?

c. When will the bacteria count reach 100,000?

Solution:

a. The population at time t is modeled by $n(t) = 250 \cdot 2^{\frac{t}{12}}$, where t is measured in hours.

b. After two weeks (336 hours), the number of bacteria is $n(336) = 250 \cdot 2^{\frac{336}{12}} = 67{,}108{,}864{,}000$ bacteria.

c. We set $n(t) = 100{,}000$ in the model and solve the resulting exponential equation for t.

$$\begin{aligned}
100{,}000 &= 250 \cdot 2^{\frac{t}{12}} \\
400 &= 2^{\frac{t}{12}} \\
\log 400 &= \log 2^{\frac{t}{12}} \\
\log 400 &= \frac{t}{2} \log 2 \\
t &= \frac{12 \log 400}{\log 2} \approx 103.726 \text{ hours}
\end{aligned}$$

Radioactive Decay Model

If m_0 is the initial mass of a radioactive substance with half-life, h, then the mass remaining at time, t, is modeled by the function

$$m(t)=m_0\left(\frac{1}{2}\right)^{\frac{t}{h}} \text{ or } m(t)=m_0 2^{\frac{-t}{h}}$$

where h and t are measured in the same time units (minutes, hours, days, years, etc.).

Example: Lithium-211 has a half-life of 2.14 minutes. A research team successfully synthesizes 1500 atoms of this element.

a. Write a function that models the number of atoms remaining after t minutes.

b. Approximate the number of atoms remaining after 5 minutes.

c. According to your model, how long will it take for there to be only one atom left?

Solution:

a. We have m_0 = 1500 and $h=2.14$ so the amount remaining after t minutes is:

$$m(t)=1500\left(\frac{1}{2}\right)^{\frac{t}{2.14}} \text{ or } m(t)=1500\cdot 2^{\frac{-t}{2.14}}$$

b. We use the function above with $t=5$ to determine $m(5)\approx 296.996$ atoms.

c. We use the function above with $m(t)=1$:

$$1500\left(\frac{1}{2}\right)^{\frac{t}{2.14}}=1$$

$$\left(\frac{1}{2}\right)^{\frac{t}{2.14}}=\frac{1}{1500}$$

$$\ln\left(\left(\frac{1}{2}\right)^{\frac{t}{2.14}}\right)=\ln\frac{1}{1500}$$

$$\frac{t}{2.14}\cdot\ln\frac{1}{2}=\ln\frac{1}{1500}$$

$$t=\frac{2.14\cdot\ln\frac{1}{1500}}{\ln\frac{1}{2}}\approx 22.6 \text{ minutes}$$

OR

$$1500\cdot 2^{\frac{-t}{2.14}}=1$$

$$2^{\frac{-t}{2.14}}=\frac{1}{1500}$$

$$\ln\left(2^{\frac{-t}{2.14}}\right)=\ln\frac{1}{1500}$$

$$\frac{-t}{2.14}\cdot\ln 2=\ln\frac{1}{1500}$$

$$t=\frac{-2.14\cdot\ln\frac{1}{1500}}{\ln 2}\approx 22.6 \text{ minutes}$$

Exponential Growth and Decay

Exponential Growth:

If an initial amount n_0 is increasing at a rate of $r\%$ per time period, then the amount after t time periods can be modeled by:

$$n(t) = n_0(1+r)^t$$

Exponential Decay:

If an initial amount n_0 is decreasing at a rate of $r\%$ per time period, then the amount after t time periods can be modeled by:

$$n(t) = n_0(1-r)^t$$

Continuous Growth:

If an initial amount n_0 is increasing at a rate proportional to its size, where r is the relative growth rate, then the amount after t time periods can be modeled by:

$$n(t) = n_0 e^{rt}$$

Continuous Decay:

If an initial amount n_0 is decreasing at a rate proportional to its size, where r is the relative decay rate, then the amount after t time periods can be modeled by:

$$n(t) = n_0 e^{-rt}$$

Exponential Applications Practice

1. A certain culture of the bacterium initially has 10 bacteria and is observed to double every 1.3 hours.

    a. Determine a model $n(t)$ for the number of bacteria in the culture after t hours.

    b. Estimate the number of bacteria present after 35 hours.

    c. When will the bacteria count reach 10,000?

2. A certain species of rabbit was introduced to France 40 years ago. Biologists observed that the population doubles every 8 years, and now the population is 9,600.

    a. What was the initial size of the rabbit population?

    b. Estimate the size of the rabbit population 5 years from now.

3. A grey squirrel population was introduced onto a small island about 4 months ago. The current grey squirrel population on the island is estimated to be about 200 and doubling every 4 months.

   a. What was the initial size of the squirrel population?

   b. Estimate the squirrel population 1 year after they were introduced onto the island.

4. The half-life of Cs-137 is 30.2 years. Suppose we have a sample with a mass of 1 kg.

   a. Write a function $m(t)$ that models the mass remaining after t years.

   b. How much Cs-137 will remain after 151 years?

5. The half-life of radium-226 is 1600 years. Suppose we have a 25 mg sample.

   a. Write a function $m(t)$ that models the mass remaining after t years.

   b. How much of the sample will remain after 3000 years?

   c. After how long will only 15 mg of the sample remain?

6. The half-life of strontium-90 is 28 years. How long will it take a 30 mg sample to decay to 20 mg?

7. A 64 g sample of Germanium-66 is left undisturbed for 12.5 hours. At the end of that period, only 2.0 g remain. What is the half-life of this material?

8. Radon has a half-life of 3.8 days. After 7.6 days, 6.5 g remain. What was the mass of the original sample?

9. A 0.5 g sample of radioactive iodine-131 has a half-life of 8.0 days. After 40 days, how much is left?

10. What is the half-life of Po-214 if after 820 seconds, a 1.0 g sample decays to 0.03125 g?

11. The number of customers, at a particular car lot, who purchase hybrid vehicles have been increasing at a rate of 3.5% per month. If at the beginning of the first month, the car lot had 1,256 recorded customers, how many recorded customers would there be at the end of 12 months who purchased a hybrid vehicle?

12. A hardware store recorded 3,256 customers visited the store in 2019. It was also recorded that the hardware owner saw a decrease of customers at a rate of 1.75% per year for the past 18 months. How many customers would have visited the hardware store in the past 18 months?

13. The frog population in a small lake grows exponentially. Currently there are 65 frogs in the lake, and the population's relative growth rate is 14% per year.
    a. Write a function $P(t)$ that models the population after t years.
    b. Determine the projected population after 3 years.
    c. Determine the number of years required for the frog population to reach 500.

14. The number of permit requests in the town of Centerville grew by an annual rate of 1.36% for the past four years. The numbers of permits requested in Centerville for the fourth year was 5,678. What were the number of permits requested in the first year?

15. Suppose the population of a city was 12,624 in 2003, and the population's relative growth rate is 0.75% per year. What is the projected population in the year 2025?

Compound Interest

Student Learning Objective:

- Given compound interest information, construct and solve a corresponding exponential equation.

Compound Interest

If P is the principal amount invested, r the interest rate per year, t the number of years for the investment, and n the number of times the interest is compounded per year, then the amount of the investment after t years is modeled by the function

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Continuously Compounded Interest

If P is the principal amount invested, r the interest rate per year, and t the number of years for the investment, then the amount of the investment after t years is modeled by the function

$$A(t) = Pe^{rt}$$

Practice

1. Eagle Bank offers a savings account that earns 2.9% compounded quarterly. You have $1000 to invest.
   a. Write a function that models the money in the account after t years.

   b. Calculate how much money you will have after 10 years.

   c. How many years will it take you to have $1700 in the account?

2. Euler Money offers a certificate of deposit (CD) that earns 5.1% compounded continuously. Talon has $10,000 to invest.
   a. Write a function, $A(t)$ that models how much the CD is worth after t years.

   b. Calculate how much money the CD will be worth after 3 years.

   c. How many years will it take for the CD to be worth $20,000?

3. Double T Finance is offering an account that earns 3.25% compounded monthly. How long will it take money in this account to double?

4. Cauchy Savings offers a savings account that earns 2.2% compounded continuously. How long will it take money in this account to triple?

Modeling with Exponential and Logarithmic Functions

Student Learning Objectives:

- Given a contextual problem and an associated exponential or logarithmic model for the problem, interpret asymptotes and intercepts of the model as well as evaluate the model when contextual values are given.
- Given an initial value and either a doubling time, half-time or relative rate of change, construct an exponential function that models the exponential growth or decay and use the model to answer questions in the context of the problem.

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2. The amount of a certain radioactive substance remaining after t years is given by a function $Q(t) = Q_0 e^{-0.025t}$.
    a. What is the relative rate of decay?

    b. Determine the half-life of the radioactive substance.

3. By measuring the amount of carbon-14 present in a fossil or piece of charcoal, scientists can estimate how old the object is. The radioactive substance C_{14} has a half-life of 5730 years and a piece of charcoal is found to contain 35% of the C_{14} that is originally had. When did the tree from which the charcoal came from die?

4. Newton's Law of Cooling is used in homicide investigations to determine the time of death. The normal body temperature is 98.6°F . Immediately following death, the body begins to cool. It has been determined experimentally that the constant in Newton's law of cooling is approximately $k = 0.197$, assuming time is measured in hours. Suppose that the temperature T_s of the surroundings is 65°F.
    a. Write a function $T(t)$ that models the temperature t hours after death.

    b. If the temperature of the body is now 73°F, how long ago was the time of death?

5. A team of archaeologists thinks they may have discovered Fred Flintstone's fossilized bowling ball. But they want to determine whether the fossil is authentic before they report their discovery. Fortunately, one of the scientists is a graduate of MAT 171, so she calls upon her experience as follows: the radioactive substance C_{14} has a half-life of 5730 years. Analysis of the "Flintstone Bowling Ball" determines that 21% of the radioactive substance has already decayed. What is the relative decay rate of C_{14} and how old is the fossil?

6. Dr. Tricia MacMillan has a problem. Every day she leaves her apartment in London at the crack of dawn and heads for Milliway's, where she purchases a delicious cup of coffee. She drinks this coffee while walking to her office. The problem is that sometimes she burns her tongue badly with the first sip, while other times she waits too long and her coffee gets cold. The latter case is worse, because besides doing a pretty bad job of keeping you warm, in her opinion cold coffee tastes terrible. Being a mathematician, Dr. MacMillan doesn't just get mad, she gets more coffee and does an experiment. She wants to figure out exactly when she can take her first sip without burning herself, and from that point how much time she has before the coffee turns bad. Every one of her mornings for the next week is spent in Milliway's with an oven thermometer and a cup of fresh coffee. After much painful experimentation, Dr. MacMillan determines that if the temperature of the coffee is above 140°F, it burns her tongue. If the temperature drops below 105°F, the coffee becomes too cold and thus undrinkable.

   Just like every other substance in the universe, coffee obeys Newton's Law of Cooling. Its temperature as a function of time is given by $T(t) = T_s + D_0 e^{-kt}$. For a typical Styrofoam cup, $k \approx 0.05$, if t is measured in minutes, but Dr. MacMillan scoffs at Styrofoam. She is the proud owner of a Sirius Cybernetics Corporation thermos (only 35% asbestos)! For this thermos the constant is $k \approx 0.03$.

   a. Why are these constants (k) positive? Why does a smaller number mean that it holds heat better?

   The next day, Dr. MacMillan leaves Milliway's with a thermos full of coffee at 160°F. It is 8:30 A.M., and the outside temperature is 50°F.

   b. Determine T_s and D_0, and rewrite $T(t) = T_s + D_0 e^{-kt}$ with the appropriate constants for this situation.

   c. How long must Dr. MacMillan wait before she is able to drink the coffee? At what time will the coffee become undrinkable? How much time does this give her to drink the coffee?

   d. How much time would she have if she were teaching at the Texas Tech University, where the outside morning temperature is usually 68°F?

   e. How much time would she have if she were teaching at the University of Southern Mercury, where the outside temperature is 440°F in winter? What do you think causes this?

Systems of Equations

Student Learning Objectives:

- Determine if a system of linear equations has a unique solution, is inconsistent, or is dependent; if the system is dependent write the complete solution.
- Solve a system of equations in three variables using algebraic methods of substitution and elimination and use these methods to solve system application problems.

There are many methods for solving systems. We looked at finding solutions graphically in Module 1. In this Module, we are going to look at algebraic methods for solving systems of equations. We will quickly review solving linear systems of two equations, we will solve non-linear systems and we will solve systems of three equations.

Review: Solving systems of two linear equations.

Systems of two linear equations will be one of the following;

Independent System – Has exactly one solution.

Inconsistent System – Has no solution

Dependent System – Has an infinite number of solutions. We will write a complete solution for dependent systems.

We have two algebraic methods we will use to solve systems of two equations, Substitution and Elimination methods.

Substitution Method:

- Solve one equation for one variable.
- Substitute the expression into the other equation for the variable you solved for to create an equation with just one variable.
- Solve the resulting equation to determine the value of one of the variables
- Substitute the found value back in an equation to determine the value of the second variable.

Solve the system using the substitution method.

1. $\begin{cases} x - y = 8 \\ 4x - 2y = 20 \end{cases}$

Elimination Method:

- Organize the system with all variables on one side and the constant on the other side. Make sure that the variables in both equations are in the same positions in the equations.
- Use a multiplier or multipliers to create equal coefficients with opposite signs for one of the variables in the system.
- Add the equations together to eliminate one of the variables.
- Solve for the value of the remaining variable.
- Substitute the found value back in an equation to determine the value of the second variable.

Solve the system using the elimination method

2. $\begin{cases} 3x+4y=5 \\ 2x-y=-9.5 \end{cases}$

Solve the following systems using the method of your choice.

3. $\begin{cases} 2x-6y=9 \\ 4x-12y=27 \end{cases}$

The answer to this system is ____________________________.

This is known as a(n) ________________________ system.

4. $\begin{cases} 3x=4y+12 \\ x-\dfrac{4}{3}y=4 \end{cases}$

Final answer can be written in the following way: $\{(x,y) \mid x=ay+b, y\, any\, real\, number\}$ (In other words, x can be an expressed in terms of y)

The answer to this system is__________________________________, this is a ______________________ solution.

This is known as a(n)________________________________ system.

Systems of three equations and three unknowns

Systems of three linear equations will be one of the following;

Independent System – Has exactly one solution.

Inconsistent System – Has no solution

Dependent System – Has an infinite number of solutions. We will write a complete solution for dependent systems.

The most famous method for solving systems is Gaussian Elimination, which uses elimination to get a system of three variables in **triangular form**, like this:

1. $$\begin{cases} x+2y+z=7 \\ -y+3z=9 \\ 2z=6 \end{cases}$$

Solve for z, we can then use **back substitution** to solve for y, and then x.

2. $$\begin{cases} 2x-y+6z=5 \\ y+4z=0 \\ -2z=1 \end{cases}$$

Sometimes getting a system in the form needed for Gaussian Elimination can be a little tedious, so we will instead use the **elimination method** to reduce the system of three equations to a system of two equations. We will then solve the system of two equations and then back substitute to determine the solution for the third variable.

3. $$\begin{cases} 2x+y-z=-8 \\ -x+y+z=3 \\ -2x+4z=18 \end{cases}$$

4. $\begin{cases} 2x+4y-z=2 \\ x+2y-3z=-4 \\ 3x-y+z=1 \end{cases}$

5. Infinitely many solutions example: $\begin{cases} y-2z=0 \\ 2x+3y=2 \\ -x-2y+z=-1 \end{cases}$

Final answer can be written in the following way: $\{(x,y,z)|\, x=az+b, y=cz+d, z\, any\, real\, number\}$. (In other words, x and y can be expressed in terms of z .)

Try These:

6. $\begin{cases} x-2y+z=1 \\ y+2z=5 \\ x+y+3z=8 \end{cases}$

7. $\begin{cases} x+2y-2z=1 \\ 2x+2y-z=6 \\ 3x+4y-3z=5 \end{cases}$

8. $\begin{cases} x+3y-2z=0 \\ 2x+4z=4 \\ 4x+6y=4 \end{cases}$

Word Problems

Define your variables, write a system of equations, use your calculator to solve the system of equations and answer the problem.

1. A boat on a river travels downstream between 2 points 20 miles apart in 1 hour. It makes its return trip against the current and it takes 2 ½ hours. What are the speed of the boat and the speed of the current?

2. The admission fee into a park is \$1.50 per child and \$4 per adult. If 2200 people entered the park and the total fees were \$5050, how many children and how many adults entered the park?

3. A man invests his savings in two accounts, one paying 6% and the other paying 10% simple interest per year. He puts twice as much in the lower-yielding account because it is less risky. His annual interest is \$3520. How much did he invest at each rate?

4. The sum of the angles of a triangle is 180°. The largest angle is equal to the sum of the other two angles. Twice the smallest angle is 10° less than the largest angle. Determine the measure for each angle in the triangle.

5. The treasurer of a club invested \$5000 of their savings into three different accounts, at annual rates of 8%, 9% and 10% simple interest per year. The total interest earned for the year was \$460. The amount earned by the 10% account was \$20 more than that earned by the 9% account. How much was invested at each rate?

6. A grocer sells peanuts at \$2.80 per pound, pecans at \$4.50 per pound, and Brazil nuts at \$5.40 per pound. He wants to make a mixture of 50 pounds of mixed nuts that to sell at \$4.44 per pound. The mixture is to contain as many pounds of Brazil nuts as peanuts and pecans combined. How many pounds of each type must he use in this mixture?

7. Write a polynomial of the form $f(x) = ax^2 + bx + c$, which passes through the points (1, -6), (-1, 2) and (-2, -3).

Systems of Non-Linear Equations

Student Learning Objective:

- Solve a system of non-linear equations by algebraic methods.

Non-Linear Systems: A system with at least one non-linear equation. Non-linear systems can have more than one solution. The same methods used for linear systems of two equations can be used to solve non-linear systems of equations.

1. Exploration:

Graph the following non-linear system at desmos.com.

$$\begin{aligned} y &= x^2 - 4 \\ y &= 2x - 1 \end{aligned}$$

How many points of intersection occur? ___________

How would you state the solution(s)? __________________________

State the solution(s): ______________________

Graphing is the method we used in Module 1 to solve systems of equations. You may use graphing to check your answers for the ones we now solve by an algebraic method.

First, let's solve the same system above using an algebraic method:

$$\begin{aligned} y &= x^2 - 4 \\ y &= 2x - 1 \end{aligned}$$

Let's continue solving by algebraic methods with these:

2. $$\begin{aligned} x^2 + y^2 &= 9 \\ x^2 - y^2 &= 1 \end{aligned}$$

3. $$\begin{aligned} x^2 + y^2 &= 1 \\ y + 2x &= -3 \end{aligned}$$

Now, you try. Solve each system of non-linear equations. Check your results at desmos.com.

1. $$\begin{aligned} x^2 + y^2 &= 8 \\ y - x &= 4 \end{aligned}$$

2. $$\begin{aligned} x^2 + 4y^2 &= 8 \\ 3x^2 - y^2 &= 11 \end{aligned}$$

3. $$\begin{cases} x^2 + y^2 = 25 \\ y = \dfrac{3}{4}x \end{cases}$$

4. $$\begin{cases} y = x^2 + 4x - 2 \\ y = 6x - 3 \end{cases}$$

5. $$\begin{cases} x^2 + y^2 = 6 \\ x^2 + y^2 = 26 \end{cases}$$

Partial Fraction Decomposition

Student Learning Objective:

- Write the partial fraction decomposition of a rational expression.

Factors:

- Linear distinct (examples: x, $x+2$, $x-2$, $x+5$, $2-x$, etc.)

Partial Fraction Decomposition for distinct linear factors: $\frac{A}{x-z_1}+\frac{B}{x-z_2}+\frac{C}{x-z_3}+\ldots$ where $x-z_i$ is the distinct factor

Part 1

Write the form of the partial fraction decomposition of the function. Do not determine the numerical values of the coefficients.

1. $\frac{1}{(x-1)(x+9)}$

2. $\frac{x}{x^2+2x-63}$

3. $\frac{x^2-4x+3}{(x-7)(x+9)}$

Part 2

Write the form of the partial fraction decomposition of the function. Determine the numerical values of the coefficients and then rewrite each as a partial fraction.

1. $\frac{x-7}{x(x+1)}$

2. $\frac{5x-6}{(x+3)(x-4)}$

3. $\dfrac{x}{(x+1)(x-1)}$

6. $\dfrac{x+1}{x^2+2x-3}$

4. $\dfrac{3x-2}{(x+2)(x-3)}$

7. $\dfrac{1}{x^2-4}$

5. $\dfrac{2x+4}{(x+4)(x-1)}$

8. $\dfrac{2x-1}{x^2-3x-10}$

Absolute Value Inequalities

Student Learning Objectives:

- Given an absolute value inequality, find solutions algebraically and graphically, and represent solutions in interval notation.

To solve absolute value inequalities, apply the definition of the absolute value function and note that the absolute value of a quantity, X, is always greater than or equal to 0.

For $a > 0$ and an algebraic expression X:

1. $|X| < a$ is equivalent to $-a < X < a$. This also works for $|X| \le a$
2. $|X| > a$ is equivalent to $X < -a$ or $X > a$. This also works for $|X| \ge a$

Solve the following absolute value inequalities and write your answer in interval notation:

1. $|3t+5| < 7$

2. $|5-2b| \ge 9$

3. $-4|4-x| \ge -12$

4. $|2c+6| - 8 > -4$

5. $4+5|2y-7| < 0$

6. $8-|3v-1| \le 4$

7. Every real number is a solution to the inequality $|x| > -0.5$. Explain why.

Solve the following absolute value application problems:

1. ”Normal” human body temperature is 98.6°F. If a temperature, x , that differs from normal by at least 1.5°F is considered unhealthy, write the condition for an unhealthy temperature x as an inequality involving an absolute value and solve for x .

2. In the United States, normal household voltage is 115 volts. However, it is not uncommon for actual voltage to differ from normal voltage by at most 5 volts. Express this situation as an inequality involving an absolute value. Use x as the actual voltage and solve for x .

3. Maria wants to purchase pavers for her garden. She determines that each paver is within a quarter inch of the stated length of 16 inches. She will line the pavers end to end with no space between them to create a straight path in her garden.

   a. Write an absolute value inequality that can be used to determine the possible lengths of a paver.

   b. Determine the possible lengths of the path if she purchases 80 pavers.

Module 4 Practice

1. Solve the following equations.

   a. $\frac{r-4}{5r}=\frac{1}{5r}+1$

   b. $\frac{2x}{x-1}+\frac{x-5}{x^2-1}=1$

   c. $\frac{4}{n+1}-\frac{1}{n^2+7n+6}=\frac{3}{n^2+7n+6}$

2. Solve the following inequalities.

   a. $2|2x-5|+3<11$

   b. $\left|\frac{1}{2}x+\frac{3}{4}\right|>5$

   c. $\frac{2x-1}{x+5}<1$

   d. $\frac{(x+3)(x-1)^2}{x+2}\geq 0$

3. Write the partial fraction decomposition for the following:

a. $\dfrac{x-25}{x^2-25}$

c. $\dfrac{x+1}{x^2-2x}$

b. $\dfrac{x-1}{x(x+4)}$

d. $\dfrac{3x-4}{x(x-1)}$

4. Write the system of equations that models the following problems. Clearly define what each of your variables represents. Then solve the system to answer the posed question.

   a. A small company borrowed \$775,000 to expand its operations. Some of the money was borrowed at 8%, some at 9%, and some at 10%. How much was borrowed at each rate if \$67,500 in interest was paid at the end of the first year and the amount borrowed at 8% was four times the amount borrowed at 10%?

   b. A passenger jet took three hours to fly 1800 miles in the direction of the jet stream. The return trip against the jet stream took four hours. What were the jet's speed in still air and the jet stream's speed?

   c. Flo's Flower Shop ordered 200 flowers for Mother's Day. They ordered sunflowers at \$1.50 each, roses at \$5.75 each, and daisies at \$2.60 each. They ordered mostly sunflowers, and 20 fewer roses than daisies. The total order came to \$589.50. How many of each type of flower was ordered?

5. Solve the system of equations by the method of your choice. You must show all of your supporting work whether you are using matrices or an algebraic method. Express your answers in (x, y, z) form. If the system is dependent, determine the complete solution. If needed, write "No Solution".

a. $\begin{cases} x-6y+4z=-12 \\ x+y-4z=12 \\ 2x+2y+5z=-15 \end{cases}$

b. $\begin{cases} x=-5y+4z+1 \\ x-2y+3z=1 \\ 2x+3y-z=2 \end{cases}$

c. $\begin{cases} 5r+4s-6t=-24 \\ -2s+2t=0 \\ s-t=2 \end{cases}$

d. $\begin{cases} x-3y+z=4 \\ 2x-7y+6z=9 \\ 2x-5y-2z=7 \end{cases}$

e. $\begin{cases} x+y-2z=2 \\ 3x-y-5z=8 \\ 2x+y+2z=-7 \end{cases}$

f. $\begin{cases} x-2y=3 \\ x+y^2=2 \end{cases}$

g. $\begin{cases} x^2+y^2=8 \\ x^2-y^2=6 \end{cases}$

6. Solve the following equations. Write your solution in exact form.

a. $\log_9(x-6)+\log_9(x+2)=1$

b. $4^{x-3}=9^{2x}$

c. $\ln(2x-5)+\ln(x)=\ln(3x)$

d. $2e^{7x-5}-10=-17$

7. You invest $15,000 in an account that earns interest at a rate of 3.9% per year, compounded quarterly. What will be the account balance after 15 years? How long will it take for the account balance to double?

8. A colony of bacteria triples every 25 hours. If the colony initially has 200 bacteria, write an equation that models the number of bacteria, n, after t hours.

9. Sarah is out to lunch with her friends and plans to order soup. The temperature of the soup t minutes after it is put into the bowl can be modeled by the function $T(t) = 72 + 128e^{-0.046t}$ where T is the temperature of the soup in degrees Fahrenheit and t is the time in minutes since the soup was placed in the bowl.
   a. What was the temperature of soup when it was put in the bowl?

   b. What is the temperature of the dining room in the restaurant?

   c. If Sarah wants to eat her soup when it is at a temperature of 160°F, how long will she need to wait to eat her soup?

10. A certain chemical compound has a half-life of 28.5 days. Consider a 100 gram amount of the compound.
   a. Write a function that models the amount A of the compound as a function of days, t.

   b. According to this model, how much of the compound remains after 15 days?

   c. How long until 35 grams of the compound remain?

MAT 171 Course Practice

1. Given $f(x) = \sqrt{x+3}$ and $g(x) = x^2 - 5,\ x > 0$, Determine:

   a. $(f \circ g)(x)$ and its domain. b. $g^{-1}(x)$ and its domain c. $(f-g)(1)$

2. Given $f(x) = \begin{cases} x-7 & x<0 \\ 7 & 0 \le x \le 3 \\ 2-x^2 & x>3 \end{cases}$, evaluate a. $f(0)$ b. $f(-3)$ c. $f(5)$

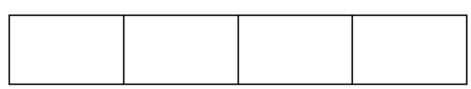
3. Given $f(x) = -x^2 + 2x + 10$
   a) Compute the average rate of change in f as x increases from 2 to 4.
   b) Determine the vertex of f and the intervals for which f is increasing and decreasing.

4. For the polynomial $P(x) = x^3 - 7x^2 + 4x - 28$ answer the following.

   a) Either 7 or -7 is a zero of this polynomial. Use division to show which is the zero.
   b) Write P as a product of linear and irreducible quadratic factors.
   c) Determine all zeros of the function.
   d) Write the function as a product of all linear factors with complex coefficients.

5. Solve $x^2 - 3x - 4 \ge 0$. Write your answer using interval notation.

6. A rectangular pen with partitions is to be built from 800 feet of fencing, as indicated in the sketch below. Express the area of the pen as a function of the width. Determine the dimensions of the pen of maximum area that can be built under these conditions.

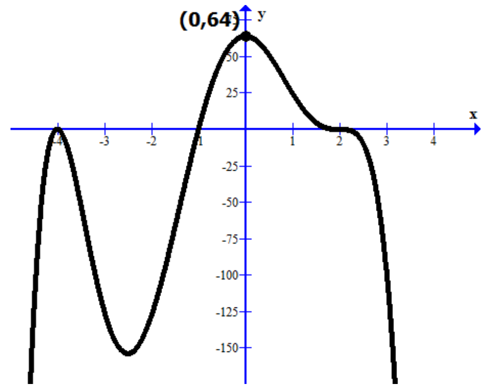


7. Solve for x: $3^{x-1} = 5^x$ without a calculator – Give an exact answer.

8. Express $(\sqrt{-25} + 3)(2 - 7i)$ in the form $a + bi$.

9. Solve the following system: $\begin{cases} x - y + 4z = 0 \\ 2x + y - z = 0 \\ -x - y + 2z = 0 \end{cases}$

10. Write the equation (in factored form) for the polynomial.



11. Use long division to determine if $(x+3)$ is a factor of $P(x)=-4x^3+5x^2+8$. State the quotient, and write P in factored form if $(x+3)$ is a factor.

12. Given the rational function $f(x)=\dfrac{-x^2+4}{x^2+5x+6}$,

a. State the domain using interval notation.
b. Determine the equations of the horizontal asymptotes, if any.
c. Determine the equations of the vertical asymptotes, if any.
d. Determine the x- and y-intercepts
e. Determine the location of any holes, or state why there are none.

13. Solve each equation. Show all work. Don't forget to check answers!

a. $\log_2(x-5)=4$

b. $\log_2(x^2+1)-\log_2(x+2)=1$

14. Factor the polynomial $P(x)=x^3+x^2+25x+25$ completely into linear factors with complex coefficients.

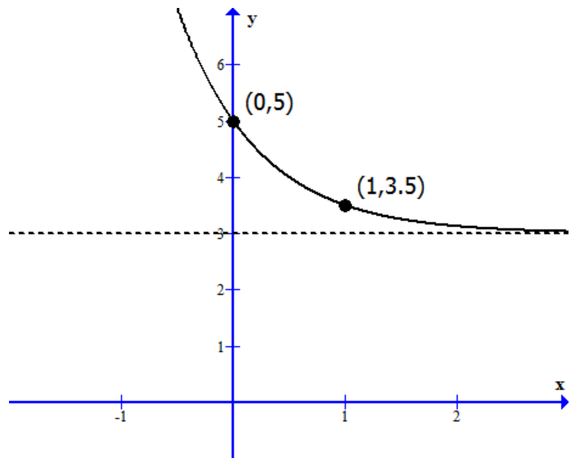
15. A function f is given, and the indicated transformations are applied to its graph (in the given order). Write the equation for the final transformed graph.

$f(x)=|x|$ shift to the right 3 units, shrink vertically by a factor of 0.8, and shift downward 2 units

16. Determine the difference quotient (and simplify) for $f(x)=x^2-2x$.

17. A substance has a half-life of 10 days. If 50 g of the substance were present initially, write a formula for the amount of substance after t days and use it to predict when 10 g of the substance will remain.

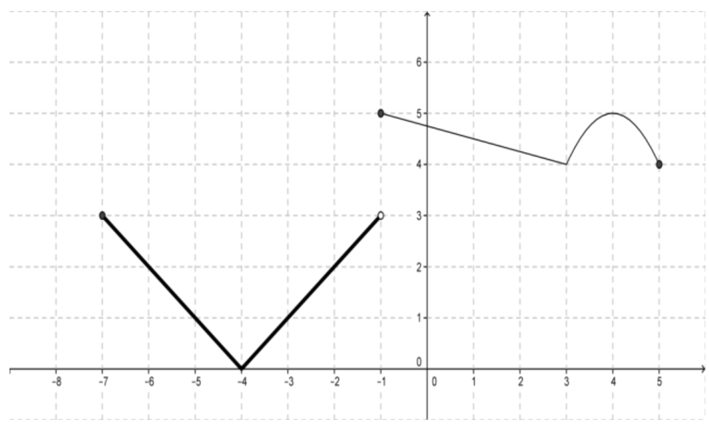
18. Write the equation for the exponential function in the graph below.



19. Determine the partial fraction decomposition for the rational expression $\dfrac{17x+5}{x^2-25}$.

20. Given the function $f(x)=-3(x+2)^2+6$ a) State the vertex b) State what its maximum value is, c) state the interval over which it is increasing.

21. Use the graph of f below to answer the following questions.



a. $f(-6)$
b. $f(-1)$
c. $f(4)$
d. domain of f
e. range of f
f. solution(s) to the equation $f(x)=4$
g. solution(s) to the equation $f(x)=3$
h. solution(s) to the equation $f(x)=1$
i. Explain why the graph of f represents a function.

22. Given $f(x)=\sqrt{x+3}$ and $r(x)=x^2-11x+28$,
a) Determine the domain of f in interval notation.
b) State the domain of $\dfrac{1}{r}$ in interval notation.
c) Determine $f^{-1}(x)$
d) Determine $(f\circ r)(x)$

23. Use the graph of the function, f , below to answer the following:

a. What is the domain of f ?

b. What is the range of f ?

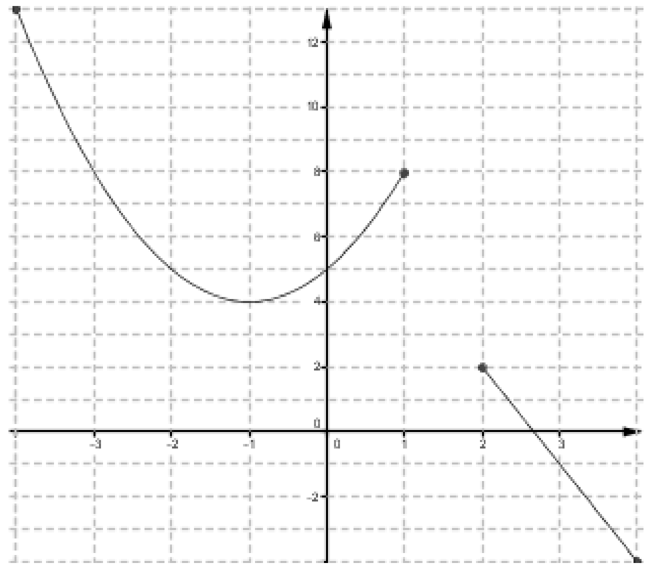
c. What is the value of $f(-3)$?

d. What is/are the solution(s) to the equation $f(x)=5$?

e. On what interval(s) is f decreasing?

f. What is the average rate of change of f as x changes from -4 to-1?

g. On the graph, sketch $g(x)=f(x+1)-2$



24. Write the factored form of the polynomial function P as a product of linear factors complex coefficients which has least degree and zeroes at -1, 0, 3 and 1+5i, leading coefficient -4, and for which the zero at 3 has multiplicity 2.

25. Determine all intercepts, asymptotes, and holes for $R(x)=\dfrac{x^3-x^2+x-1}{x^2-5x-6}$

26. Determine all solutions of the system: $\begin{cases} x-3y=10 \\ x^2+y^2=100 \end{cases}$

27. Solve.

a. $\dfrac{6}{x-1}+\dfrac{3}{x+1}=\dfrac{28}{x^2-1}$ b. $\dfrac{2x}{x+1}=\dfrac{3}{x-2}+\dfrac{1}{x^2-x-2}$

28. Solve:

a. $\sqrt{8x-4}=6$ b. $\sqrt{z-1}-z=-7$

29. Solve. Put your answer in interval notation.

a. $\dfrac{2x+1}{x-5}\leq 3$ b. $\dfrac{4x}{2x+3}>2$

30. Solve.

a. $|2t+5|>7$ b. $|x-3|\leq 5$