What can abstract mathematics tell us about programming climate models?

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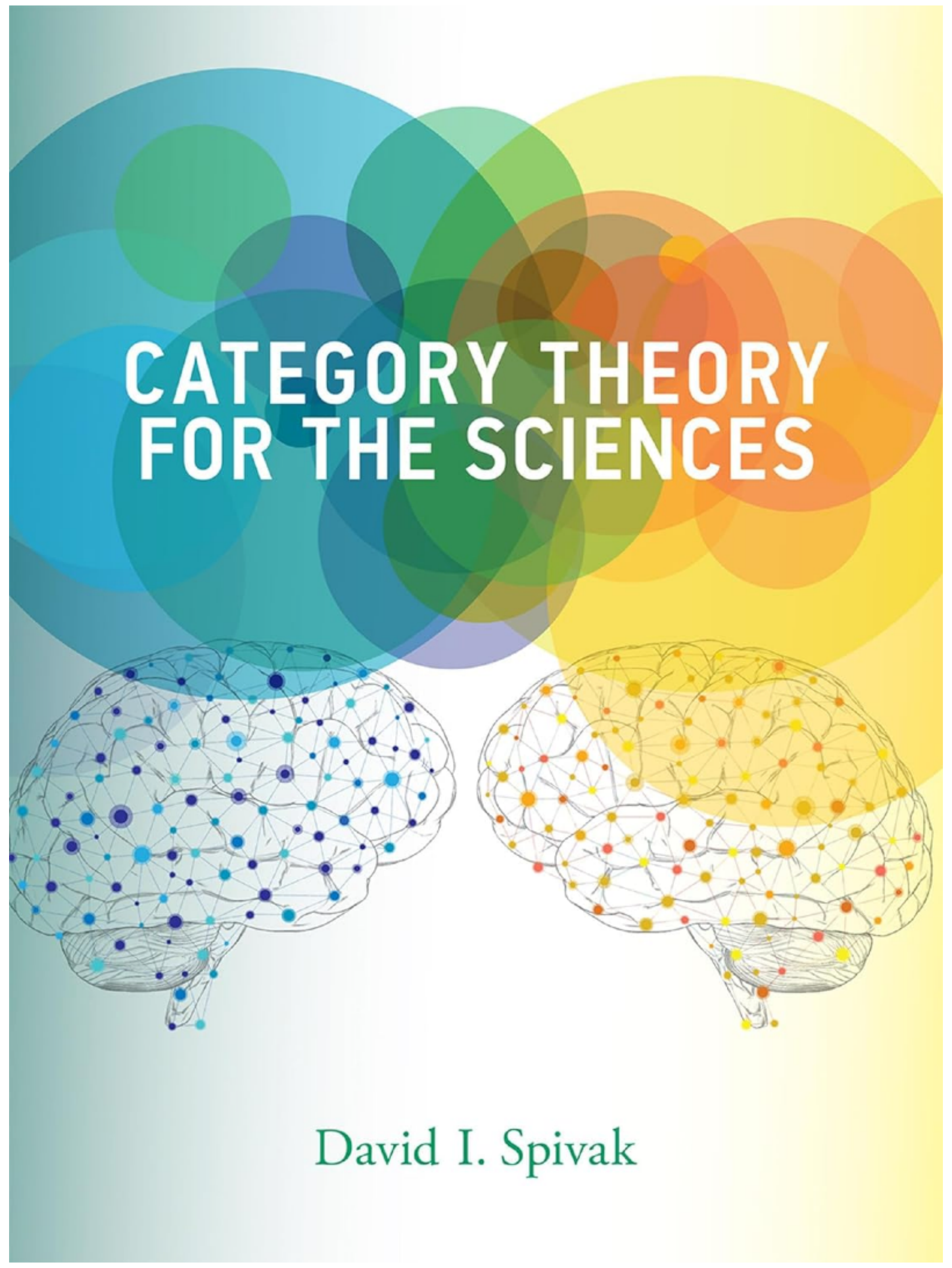
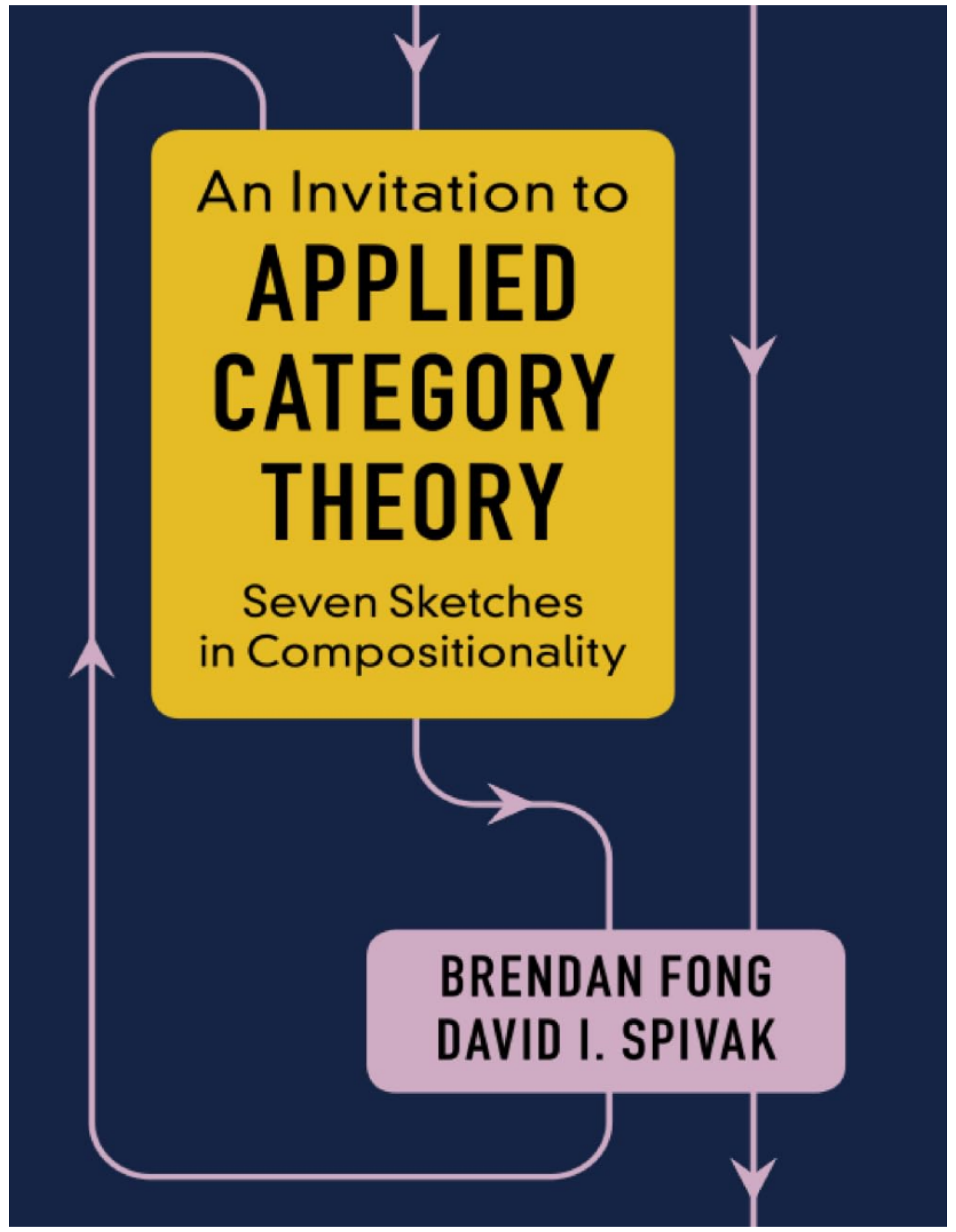
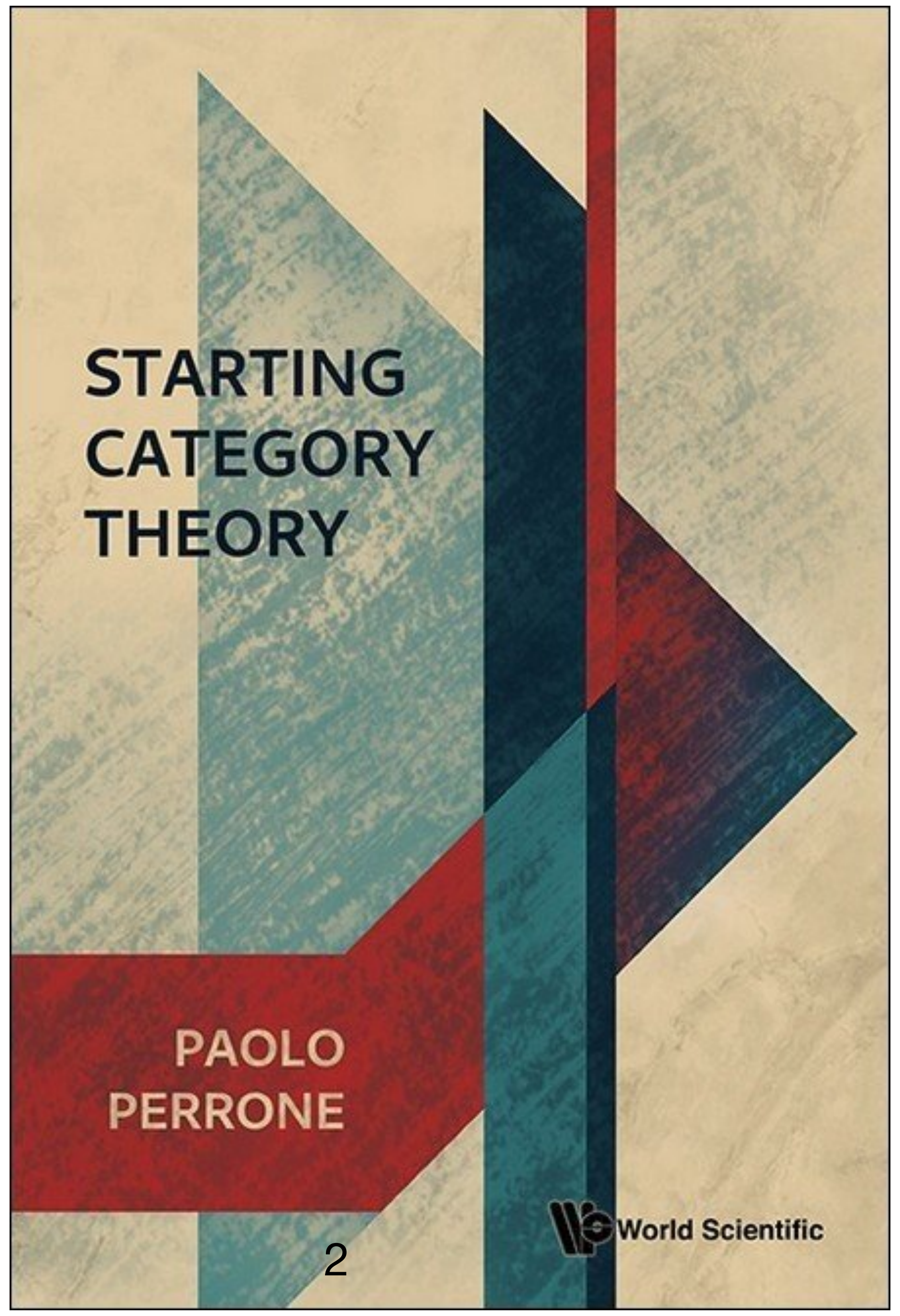
Category theory

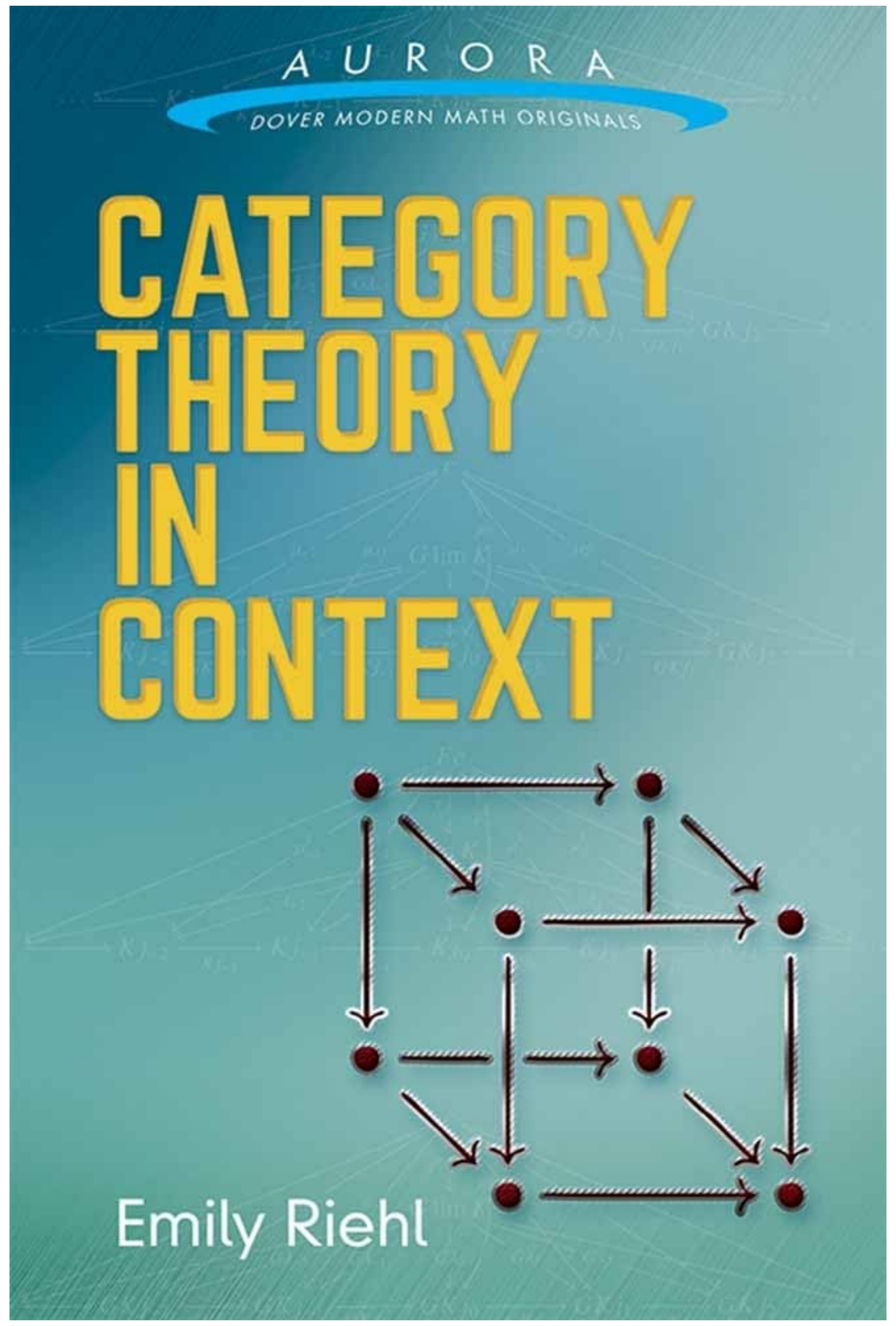
A branch of abstract mathematics

Provides organising principles exposing mathematical structure and structure within mathematics









Today's objectives

View some common ideas in numerical programming through the category theory lens

- Understand what is a category, functor, and natural transformation
- Understand their relationship to particular programming patterns
- Use these basic categorical concepts to inform **testing** and **optimisation** strategies
- Understand these basics as a launch pad

Categories

- Collection of objects obj($\mathscr{C}$)
    - often denoted $A, B, C, \ldots \in$ obj($\mathscr{C}$)
- Collection of "morphisms" (relationships) between pairs of objects morph($\mathscr{C}$)
    - often denoted $f : A \to B, g : B \to C, \ldots \in$ morph($\mathscr{C}$)
- Composition operation

$$\frac{f : A \to B \qquad g : B \to C}{f; g : A \to C}$$

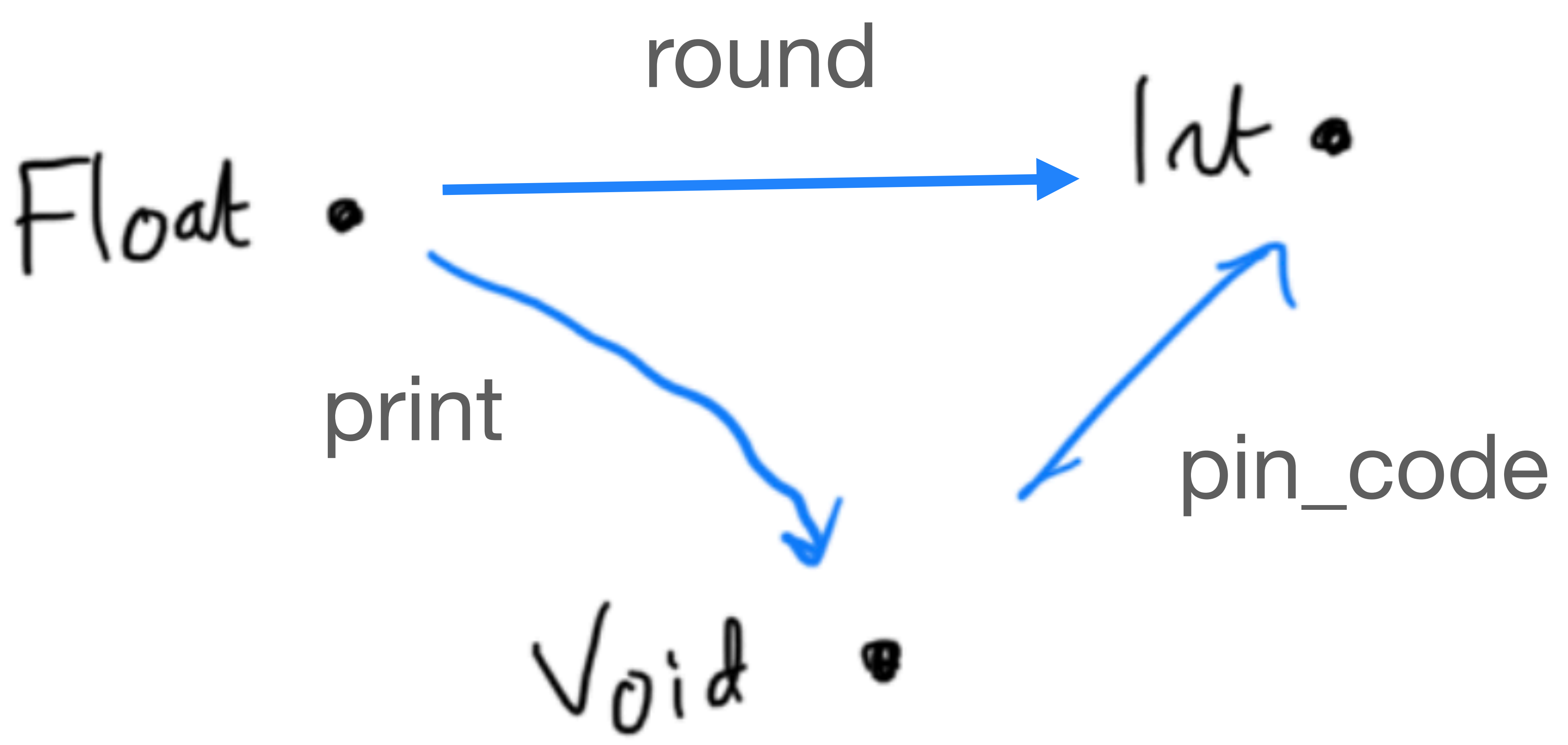
- Identity morphisms

$$\forall A \in \mathrm{obj}(\mathscr{C}) \,.\, id_A : A \to A$$

- Axioms: $f; id = id; f = f$ $\quad$ $f; (g; h) = (f; g); h$

Categories in programming

Types as objects, functions as morphisms



category.py

Functors

A morphism <u>between</u> categories

$\mathsf{F} : \mathscr{C} \to \mathscr{D}$

Has two components:

(1) mapping on objects: $\forall A \in \mathrm{obj}(\mathscr{C}) \implies \mathsf{F}(A) \in \mathrm{obj}(\mathscr{D})$

(2) mapping on morphisms:

$\forall (f : A \to B) \in \mathrm{morph}(\mathscr{C}) \implies \mathsf{F}(f) : \mathsf{F}(A) \to \mathsf{F}(B) \in \mathrm{morph}(\mathscr{D})$

Has two axioms:

$$\mathsf{F}(\mathrm{id}_A) = \mathrm{id}_{\mathsf{FA}}$$

$$\mathsf{F}(f; g) = \mathsf{F}(f); \mathsf{F}(g)$$

Functors in programming

Object mapping is a type constructor

```
list : Type -> Type
```

Morphism mapping lifts a function to apply over the type constructor

$$\frac{A \to B}{\text{list}(A) \to \text{list}(B)}$$

Natural transformations

A morphism <u>between</u> functors

$\mathsf{F} : \mathscr{C} \to \mathscr{D}$ $\qquad\qquad$ $\mathsf{G} : \mathscr{C} \to \mathscr{D}$

A natural transformation $\alpha : F \to G$ is a ***family of morphisms***

meaning $\forall A \in \mathrm{obj}(\mathscr{C})\,.\ \alpha_A : \mathsf{F}A \to \mathsf{G}A \in \mathrm{morph}(\mathscr{D})$

along with the **naturality property**

$$\begin{tikzcd}
\mathsf{F}A \arrow[r, "\alpha_A"] \arrow[d, "\mathsf{F}f"'] & \mathsf{G}A \arrow[d, "\mathsf{G}f"] \\
\mathsf{F}B \arrow[r, "\alpha_B"'] & \mathsf{G}B
\end{tikzcd}$$

Natural transformations in programming

nat_trans.py

Connecting back to climate modelling…

- Functors represent pointwise traversals (common transformation)
- Structural transformations are natural transformations
- Stencil computations are a generalisation to functors….

Comonads (*briefly and incompletely*)

An endofunctor $\mathsf{D} : \mathscr{C} \to \mathscr{C}$ **with:**

- Natural transformation $\varepsilon_A : DA \to A$
- For every morphism, a lifting $(-)^\dagger$

$$\frac{f : DA \to B}{f^\dagger : DA \to DB}$$

cf.

Functorial mapping

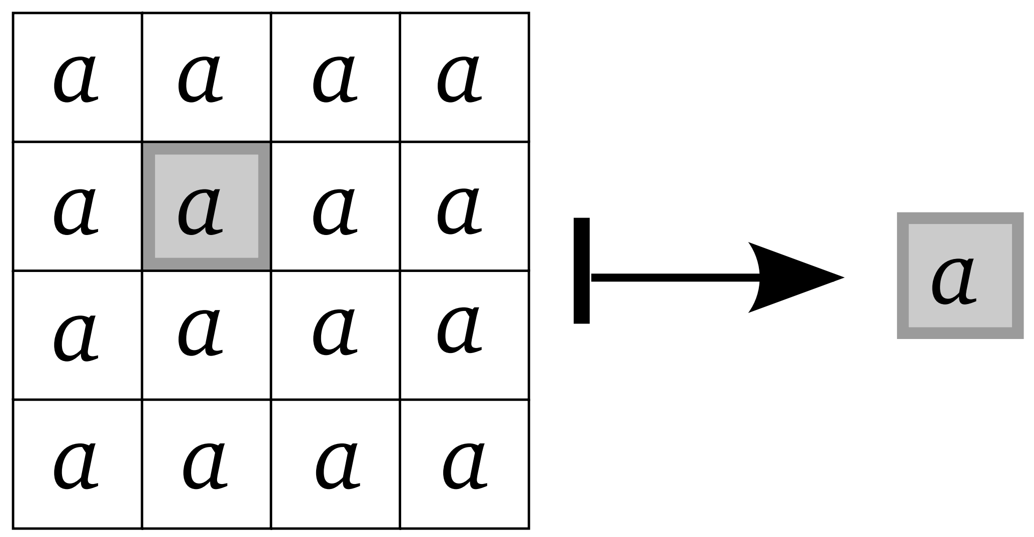
$$\frac{f : A \to B}{\mathsf{F}f : DA \to DB}$$

Comonads in programming

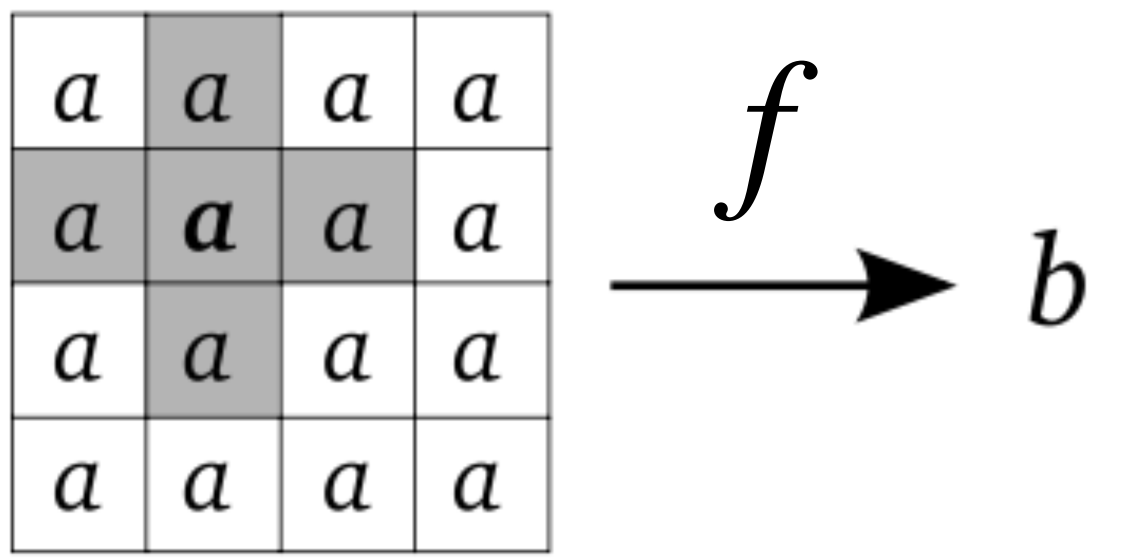
Array comonad

$$DA = \text{Array } I\ A \times I$$

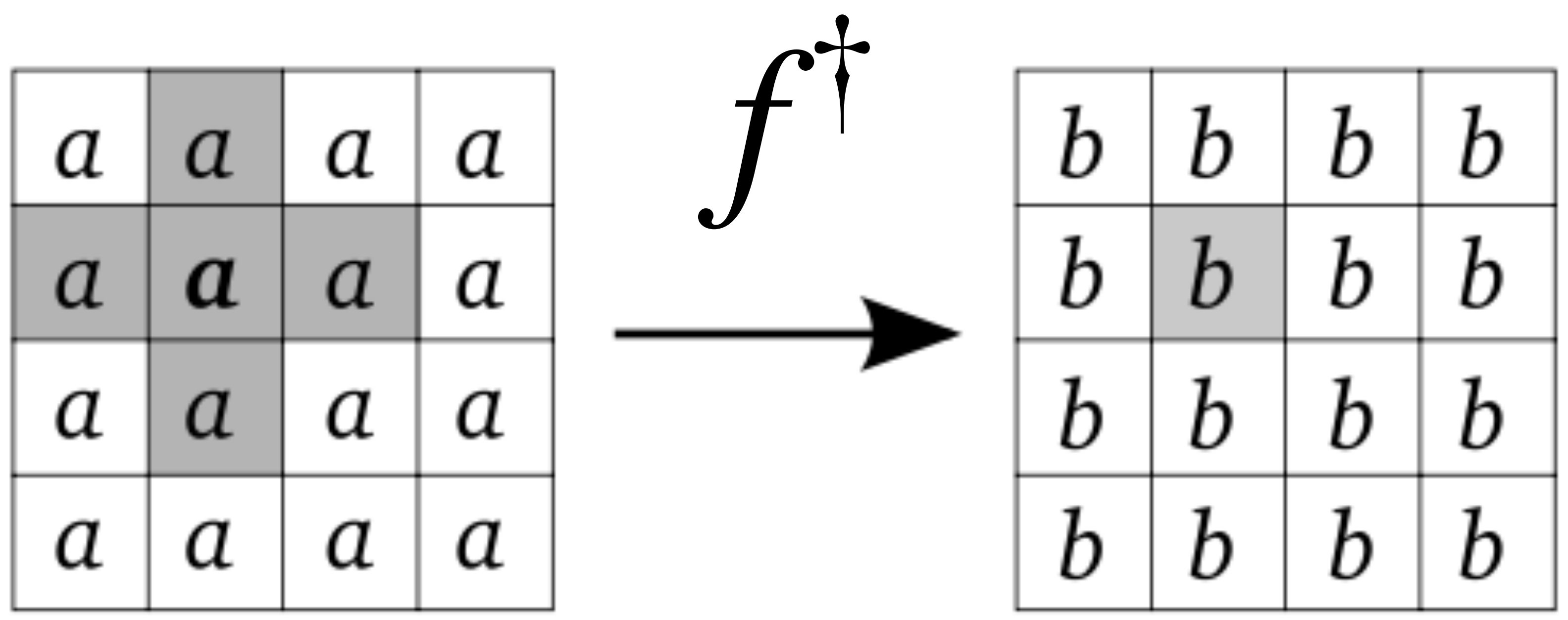
array-data × cursor



$$\varepsilon_A : DA \to A$$



Local computation (neighbourhood)



Global computation

Ypnos: declarative, parallel structured grid programming (Orchard et al. 2010)
A Categorical Outlook on Cellular Automata (Capobianco, Uustalu, 2010)

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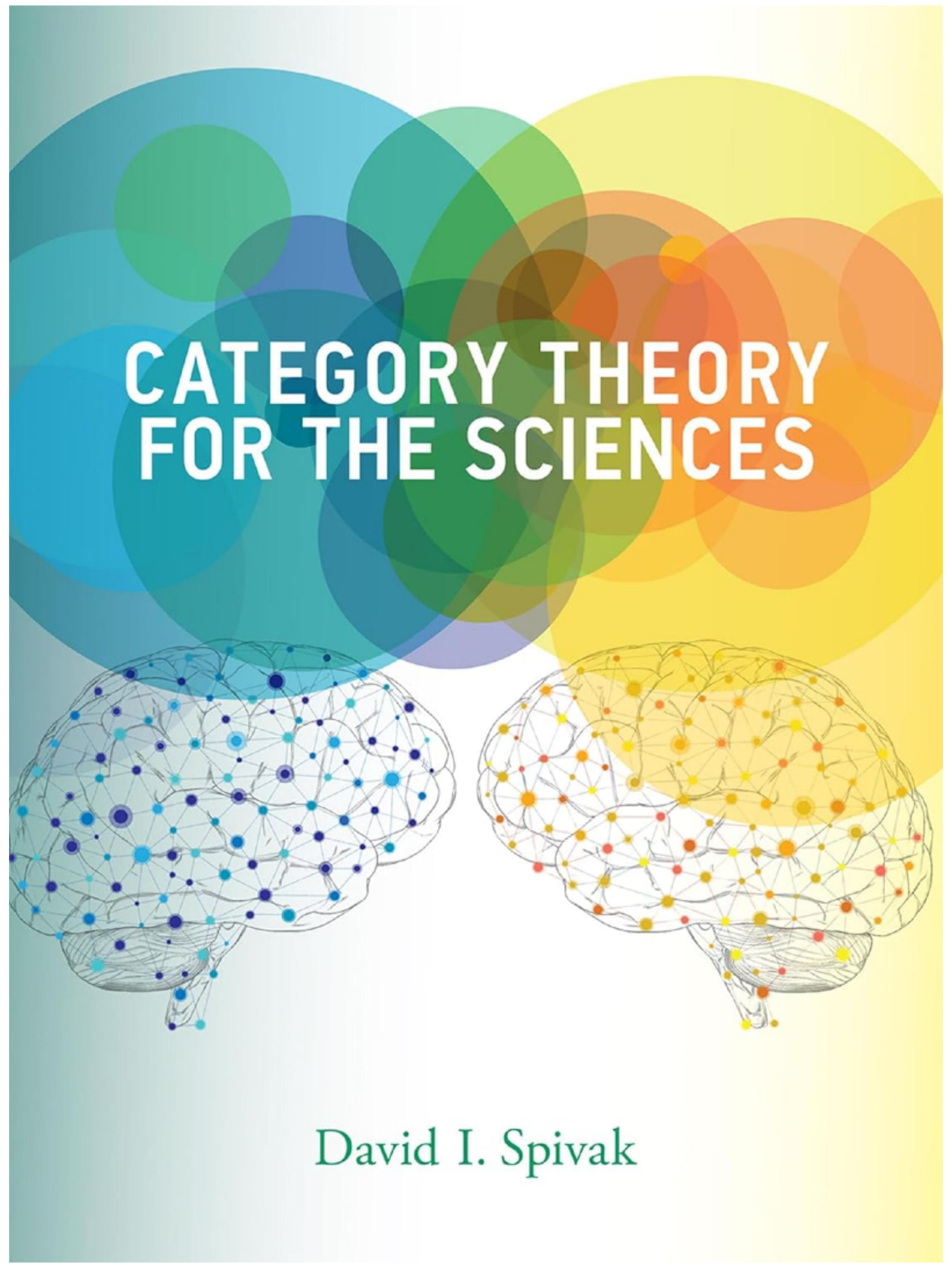
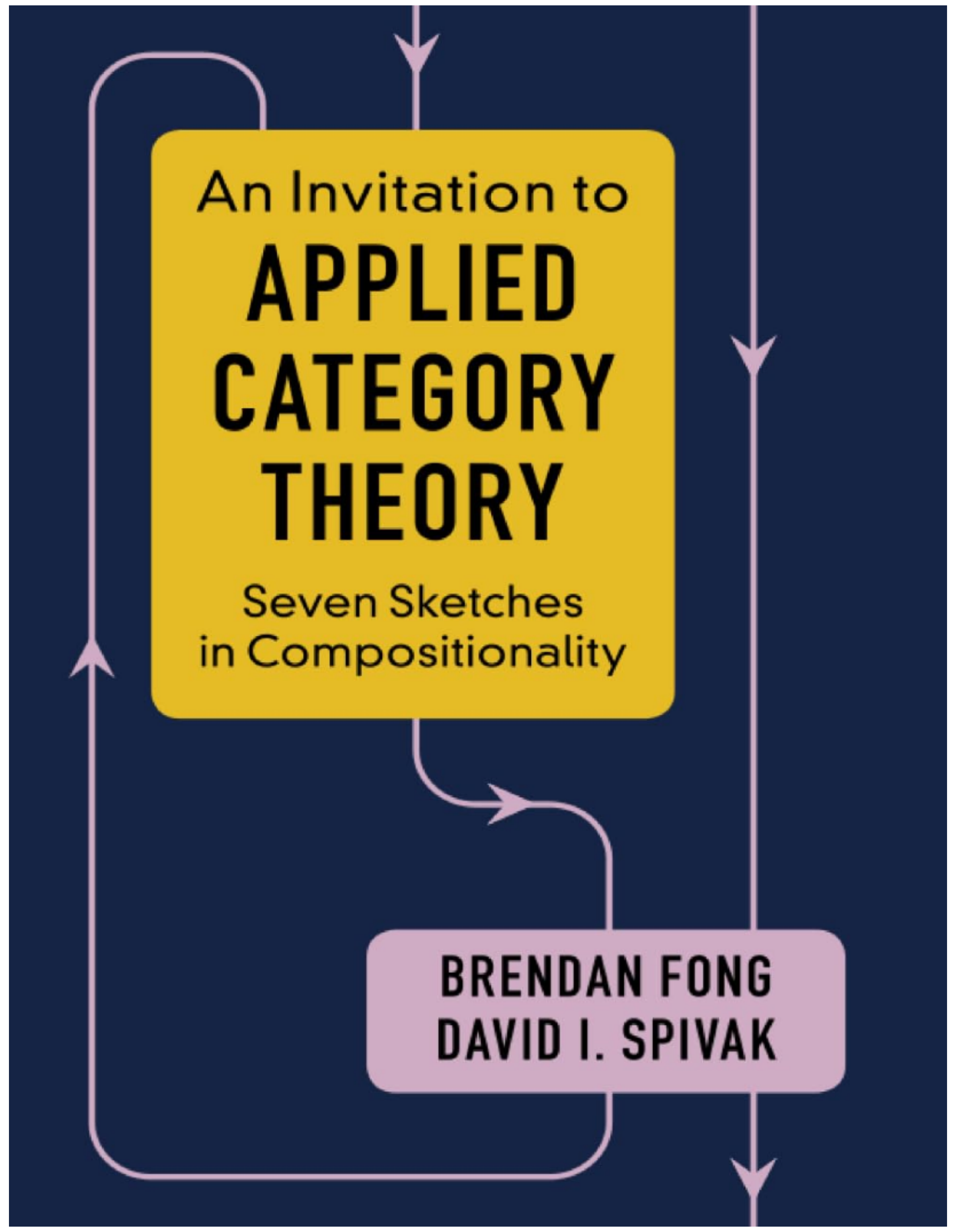
Thanks!


CATEGORY THEORY
FOR THE SCIENCES
David I. Spivak


An Invitation to
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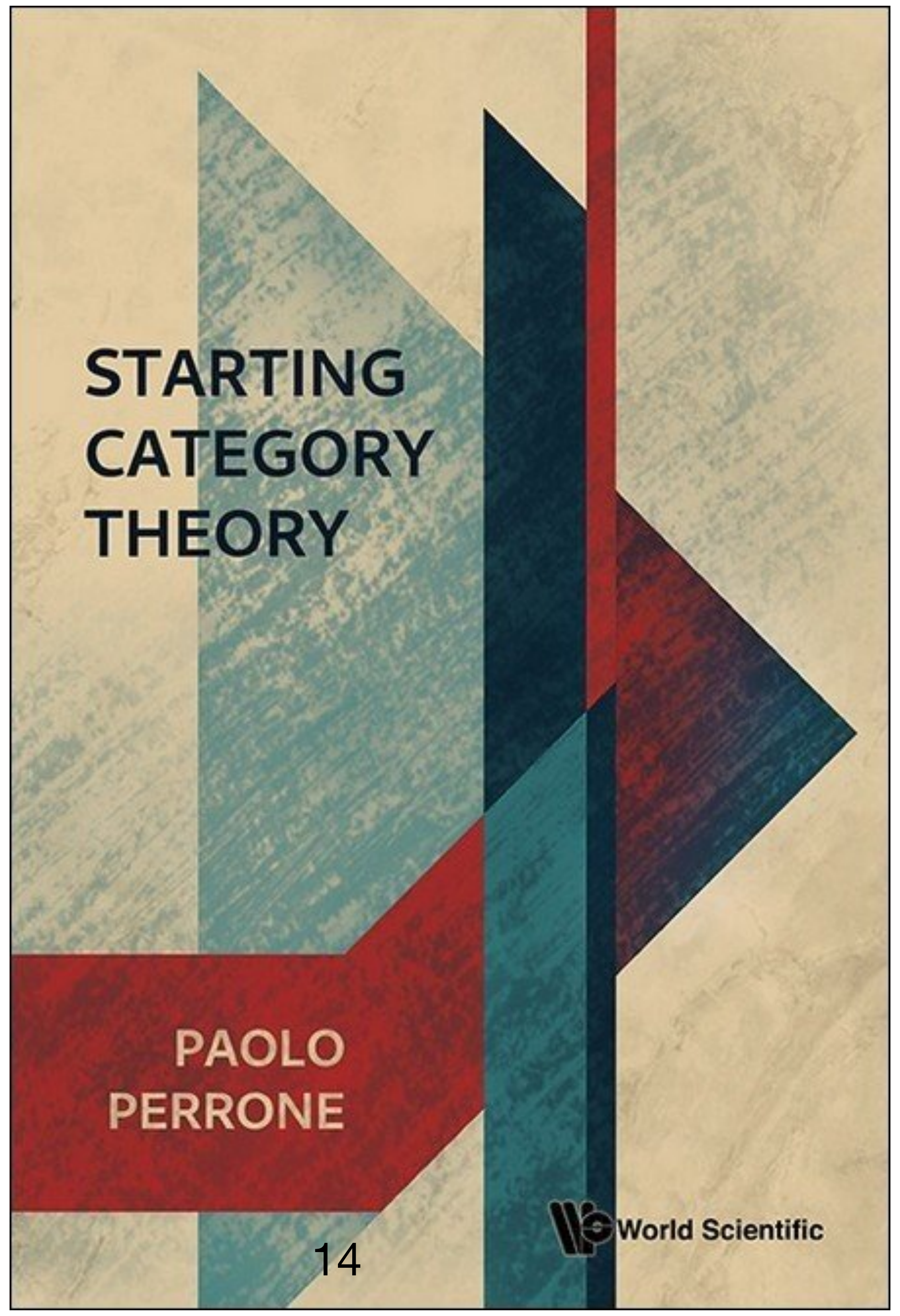
THEORY
Seven Sketches
in Compositionality
BRENDAN FONG
DAVID I. SPIVAK


STARTING
CATEGORY
THEORY
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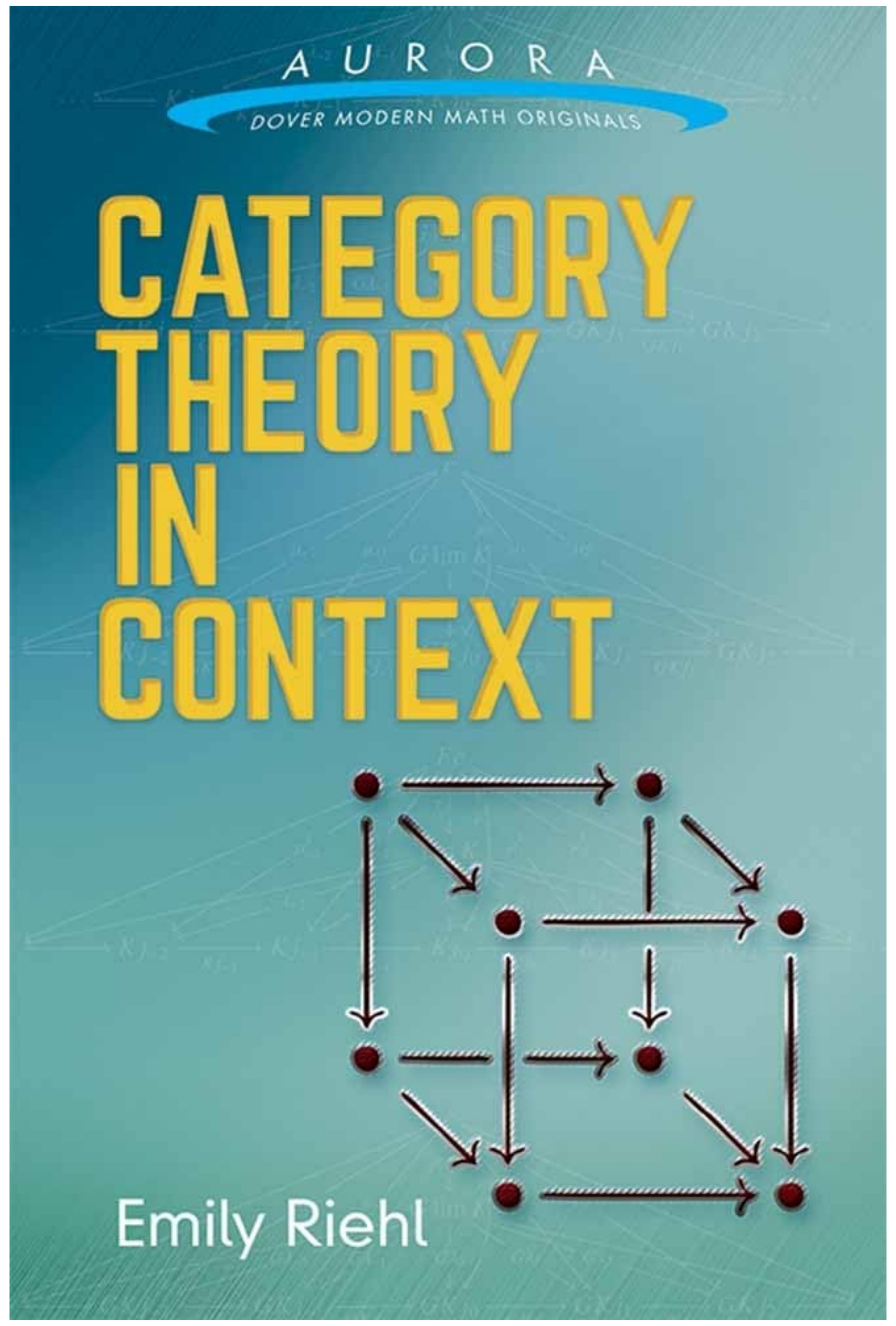
PERRONE
World Scientific


CATEGORY THEORY
FOR PROGRAMMERS
Bartosz Milewski


AURORA
DOVER MODERN MATH ORIGINALS
CATEGORY
THEORY
IN
CONTEXT
Emily Riehl