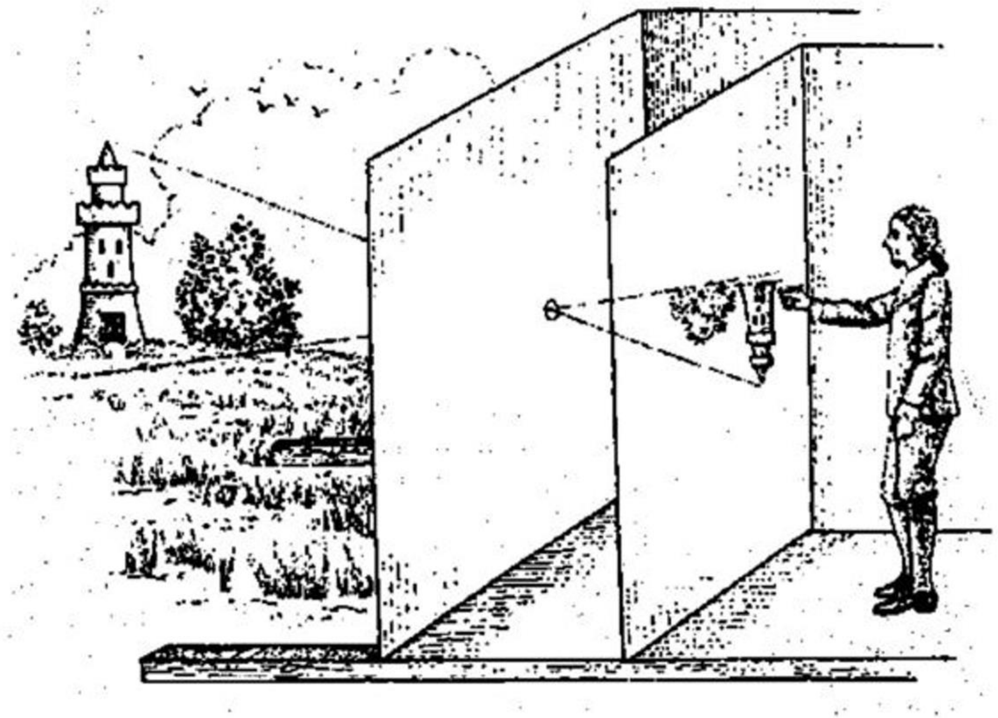
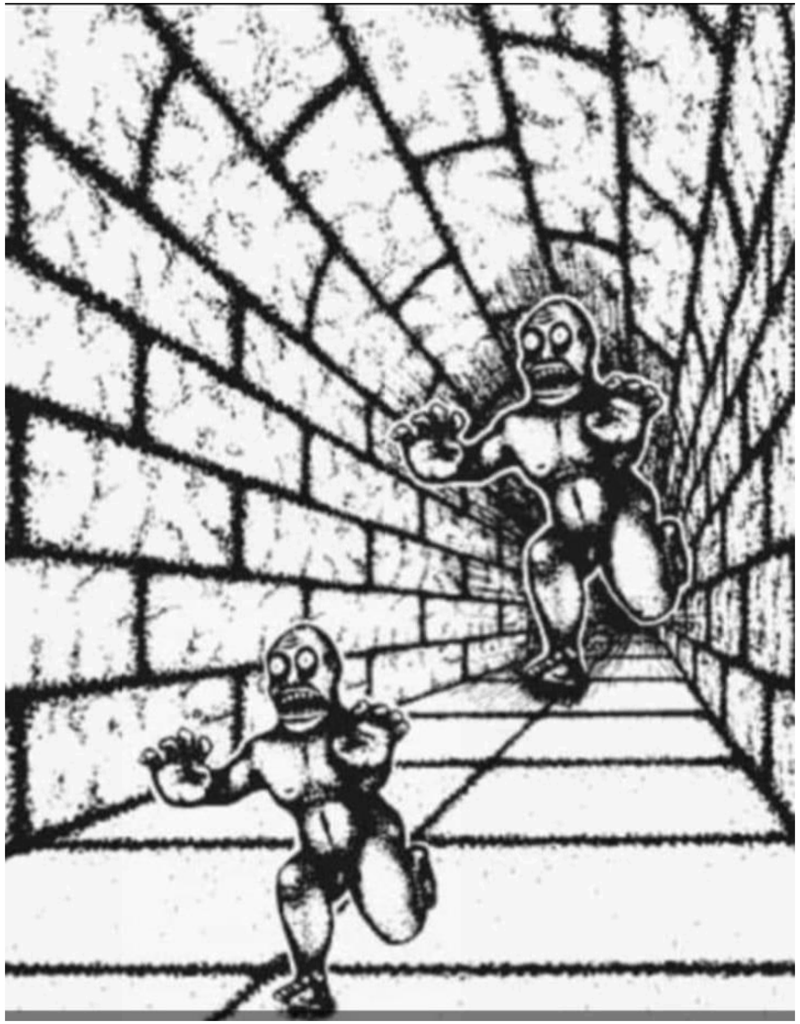
It's a 3D World, After All



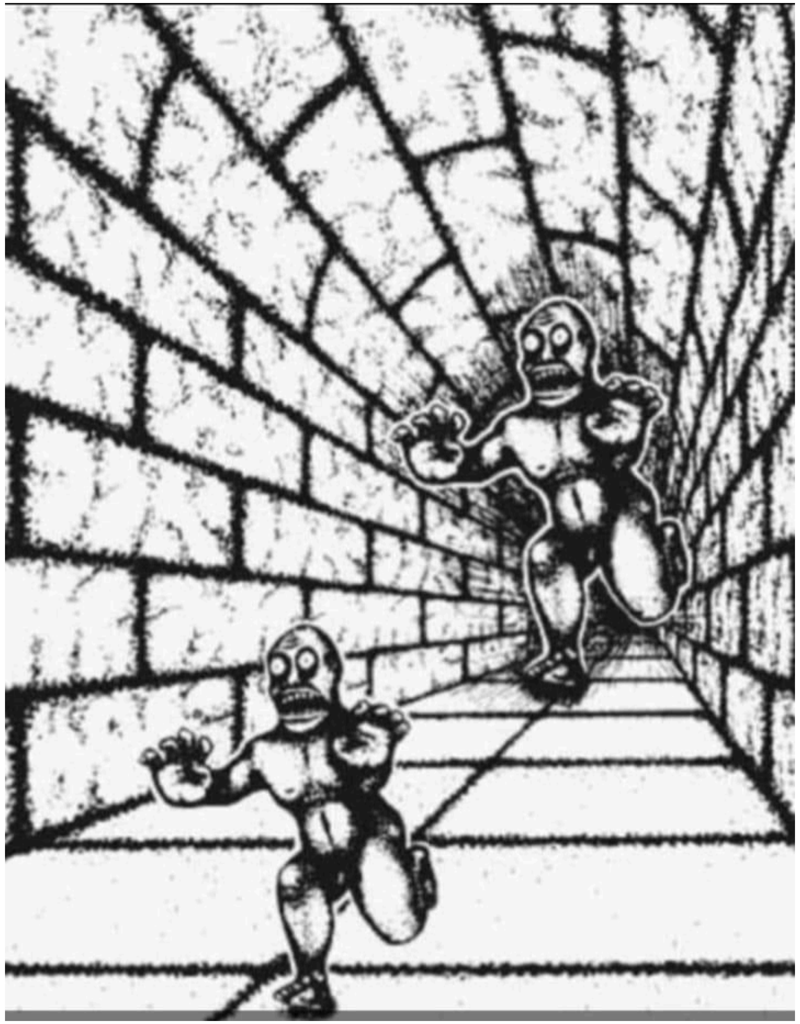
3D Computer Vision

Elliott Wu

We're good at seeing 3D!

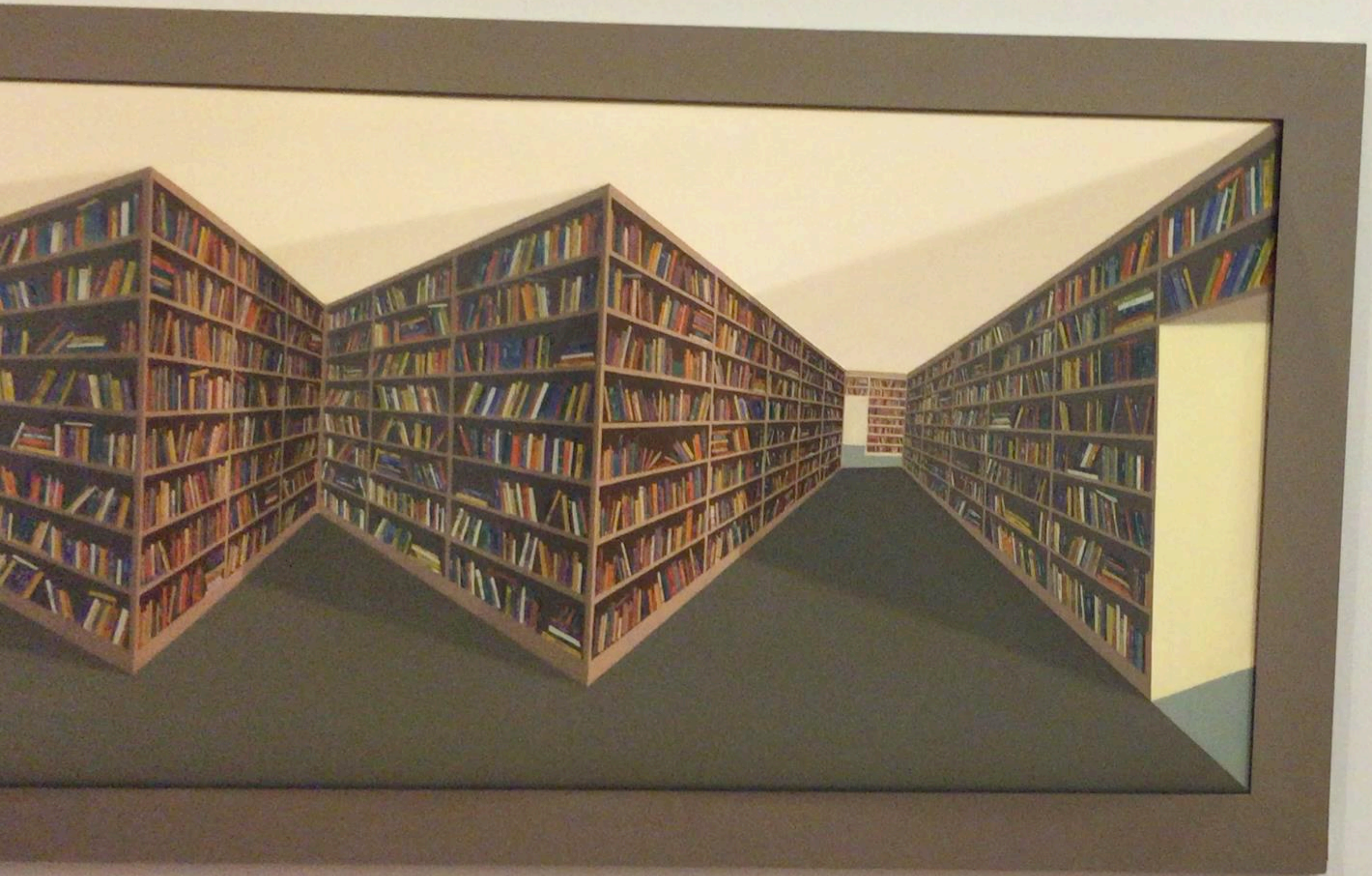


We’re good at seeing 3D!



And sometimes too good..

And sometimes too good..

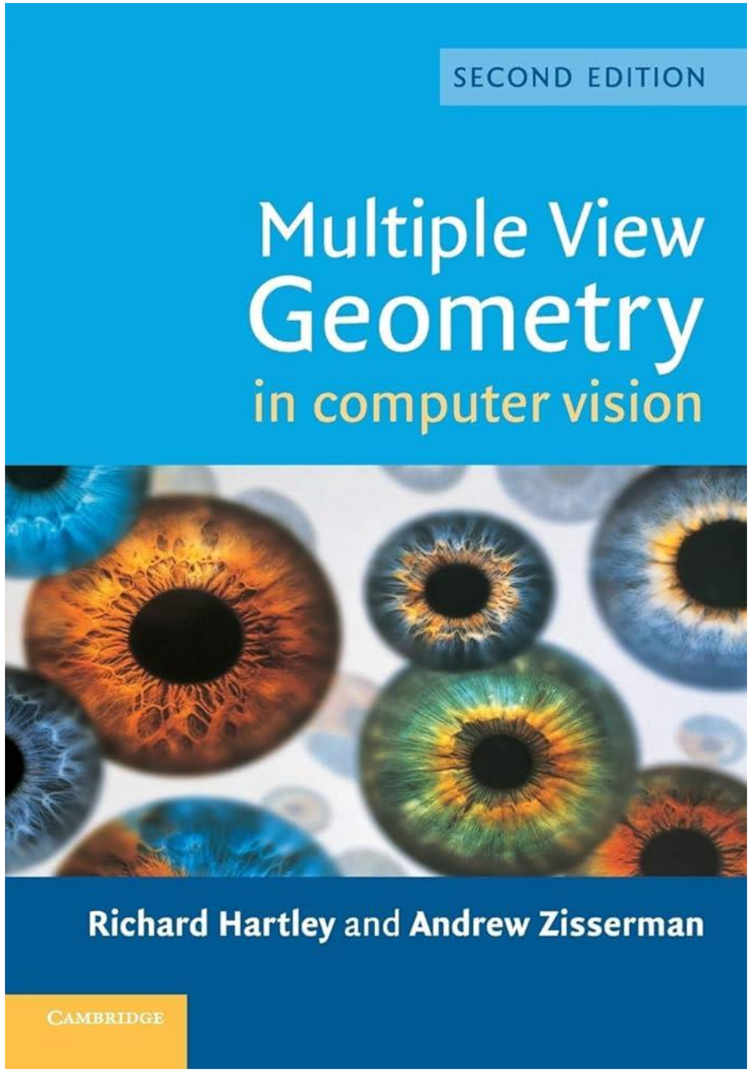


These ‘Reverspective’ paintings are mind-boggling. https://youtu.be/fzZy1FtYsVM

3D Computer Vision – Outline

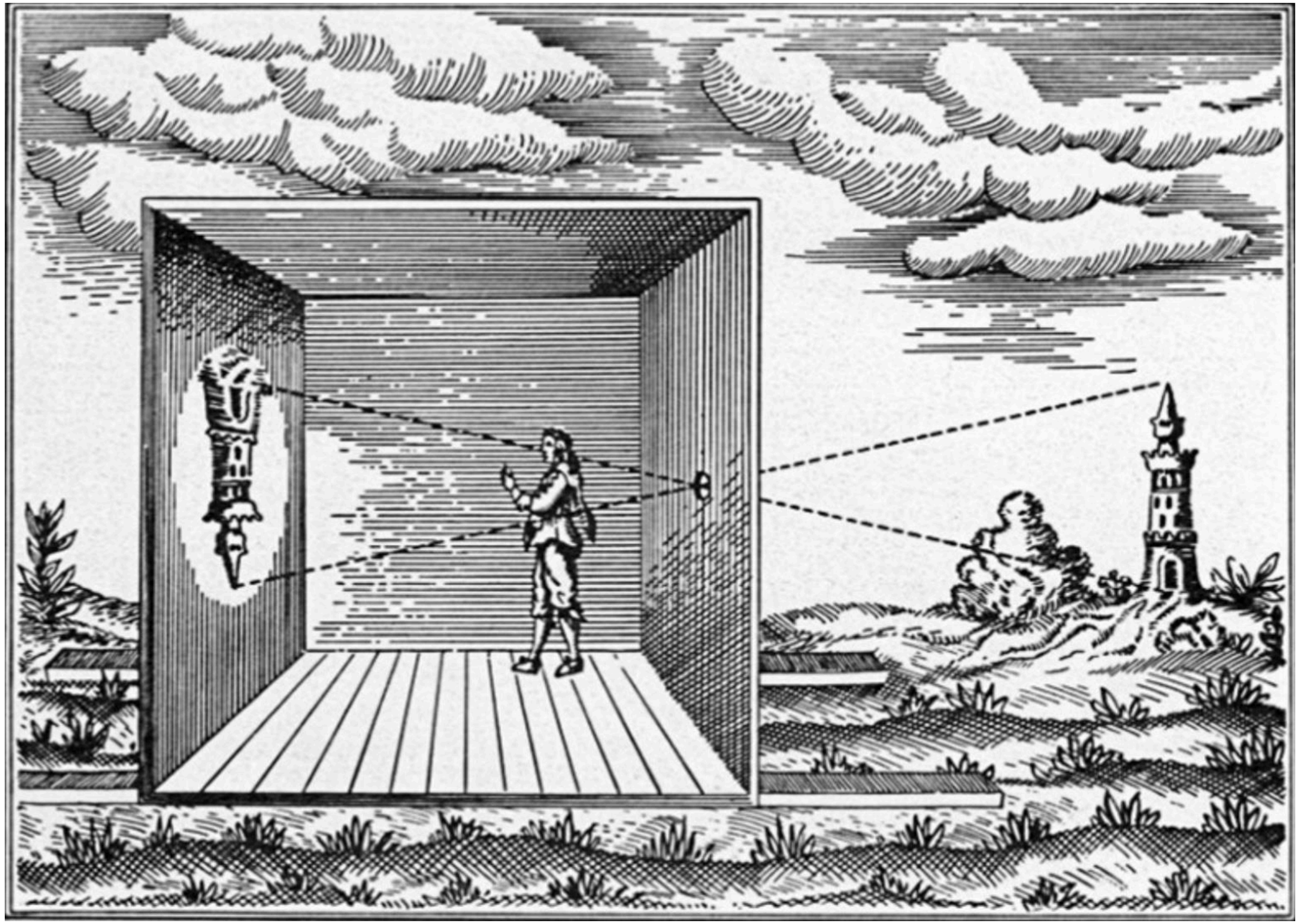
- Part 1 – Camera Model & Projection
- Part 2 – Multi-view Geometry
- Part 3 – 3D Representations & Rendering
- Part 4 – Learning-based 3D Modeling

Multi-view Geometry “Bible”



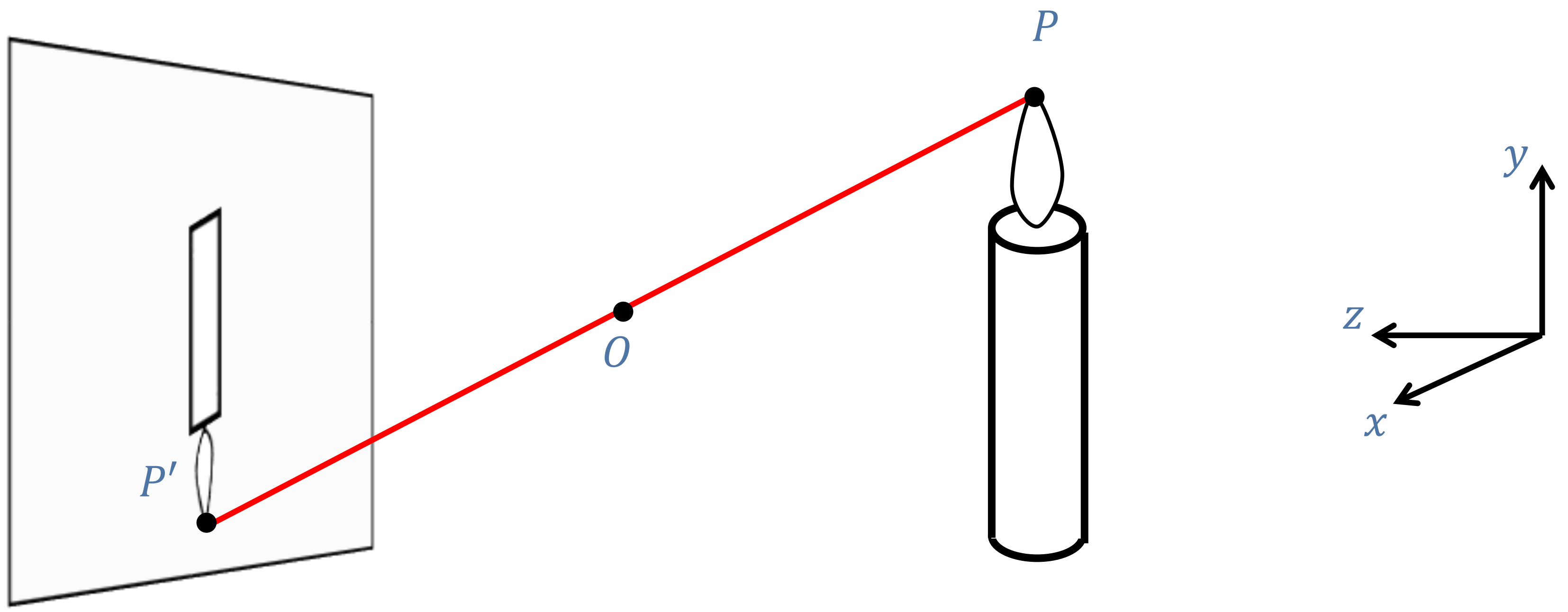
Part 1 – Camera Model & Projection

Image Formation – Camera Obscura



- Basic principle known to Mozi (470-390 BCE), Aristotle (384-322 BCE)
- Drawing aid for artists such as Leonardo da Vinci (1452-1519)

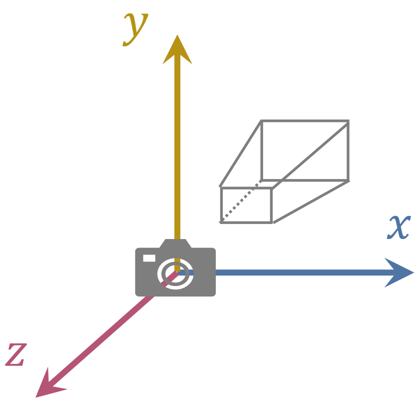
Pinhole Camera



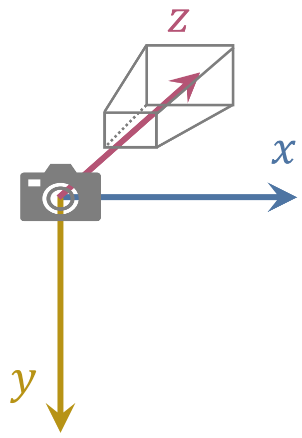
Canonical coordinate system

- The optical center (O) is at the origin
- The z axis is the optical axis perpendicular to the image plane
- The xy plane is parallel to the image plane, x and y axes are horizontal and vertical directions of the image plane

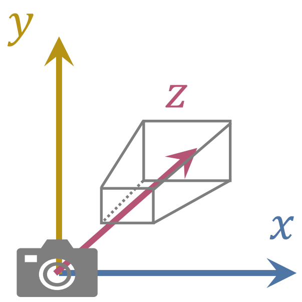
Everybody Agrees… Except on the Axes



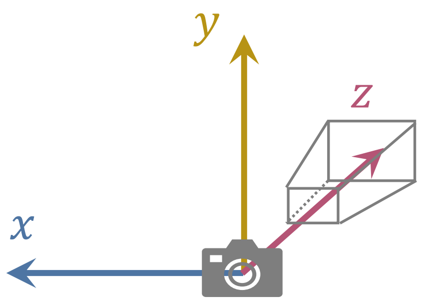
- OpenGL
- Blender
- ARKit
- Three.js
- Nerfstudio
- …



- OpenCV
- Open3D
- COLMAP
- gsplat
- …

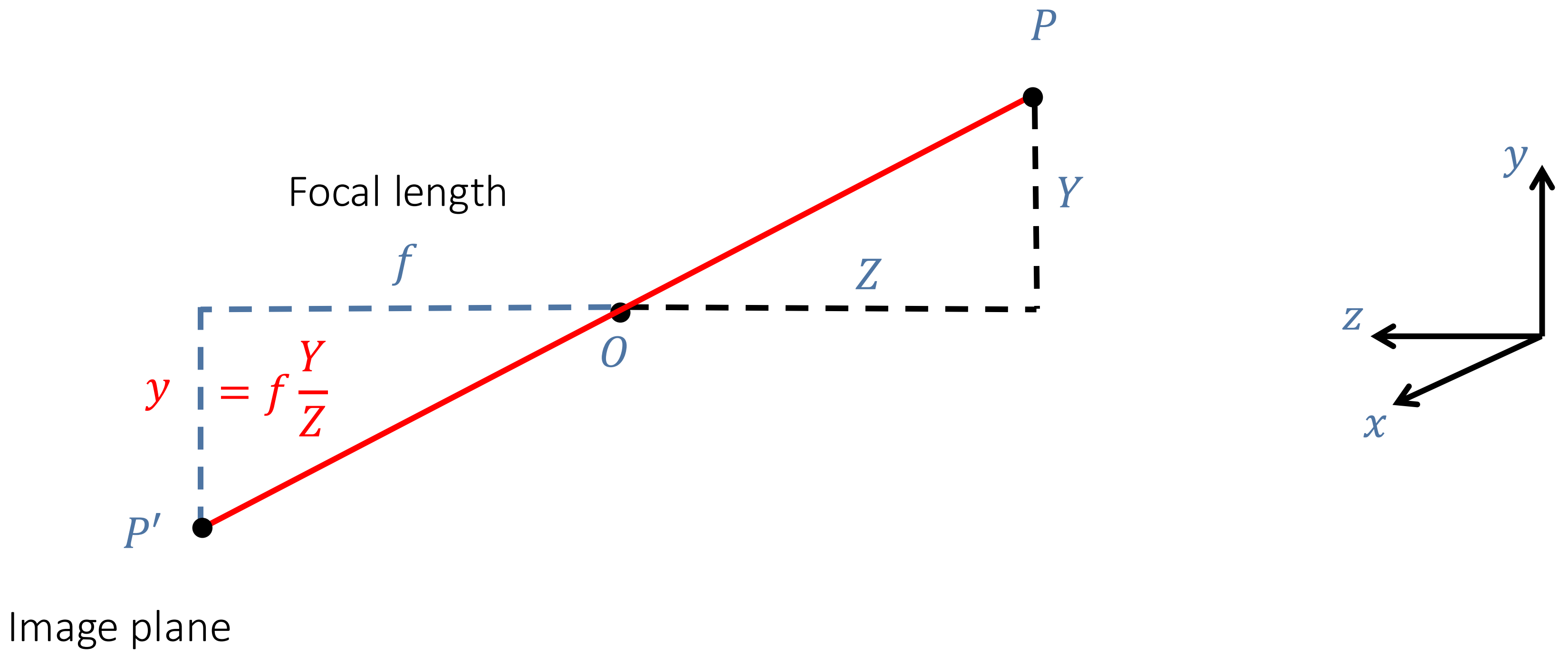


- Unity
- …



- PyTorch3D
- ?

Perspective Projection

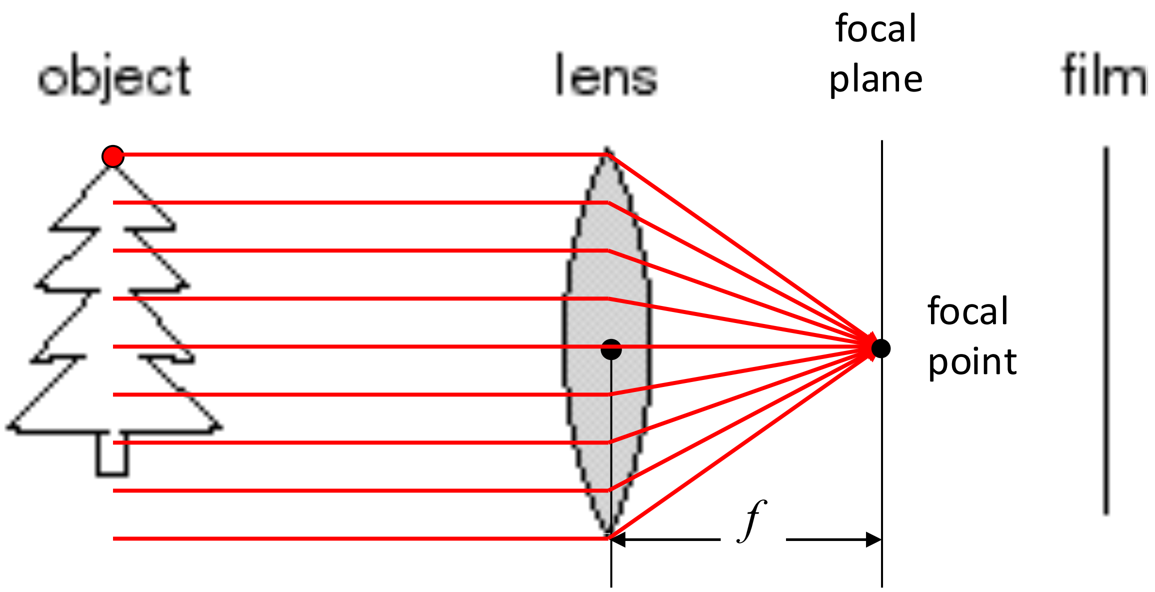


$$(X, Y, Z) \rightarrow \left(f\frac{X}{Z}, f\frac{Y}{Z}\right)$$

3D point 2D point

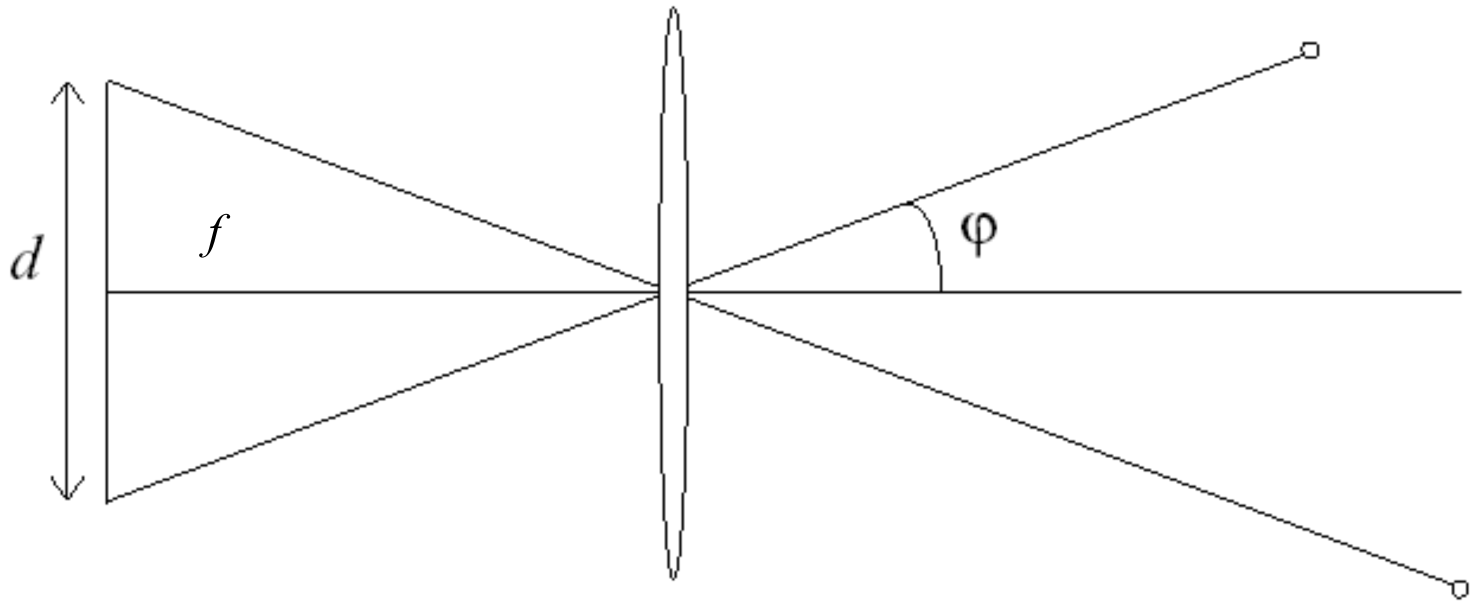
Focal Length of Lenses

- In practice, most cameras use lenses, and the focal length is determined by the physical properties of the lens, such as refraction index, thickness, curvature etc.



Field of View (FOV)

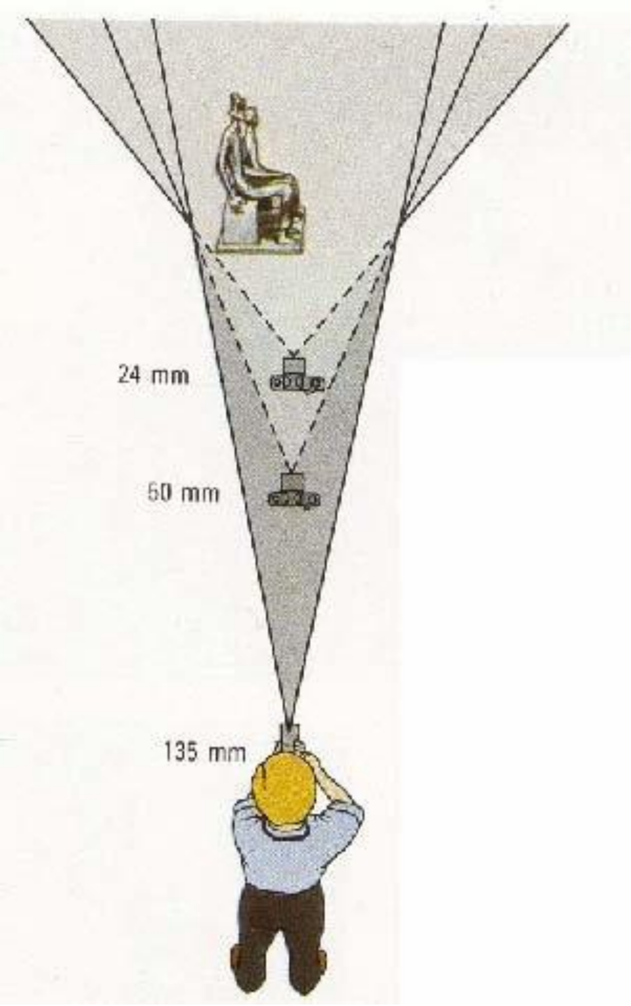
- The field of view is the angular extent of the world observed by the camera.
- Focal length (f), length of the sensor (d):



$$\varphi = \tan^{-1}\frac{d}{2f}$$

- Larger focal length = smaller FOV (assuming a constant sensor size)

Field of View / Focal Length Ambiguity



Large FOV, small f
Camera close to car

Small FOV, large f
Camera far from the car

Field of View / Focal Length Ambiguity

wide-angle

standard

telephoto

- What would happen when reconstructing 3D shapes from a single image with *unknown* focal length/FOV?

Field of View / Focal Length Ambiguity

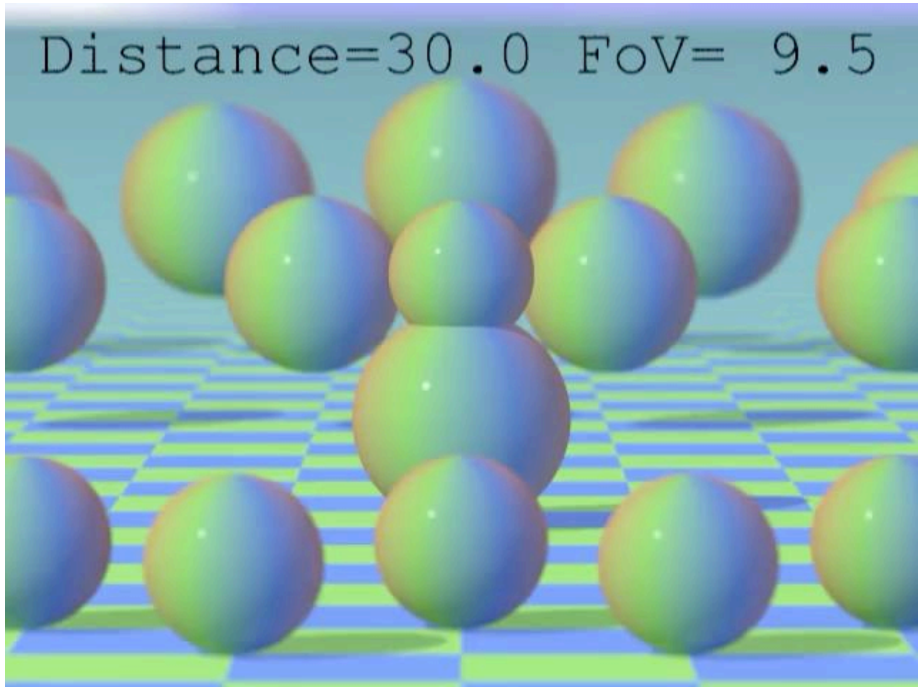
wide-angle standard telephoto

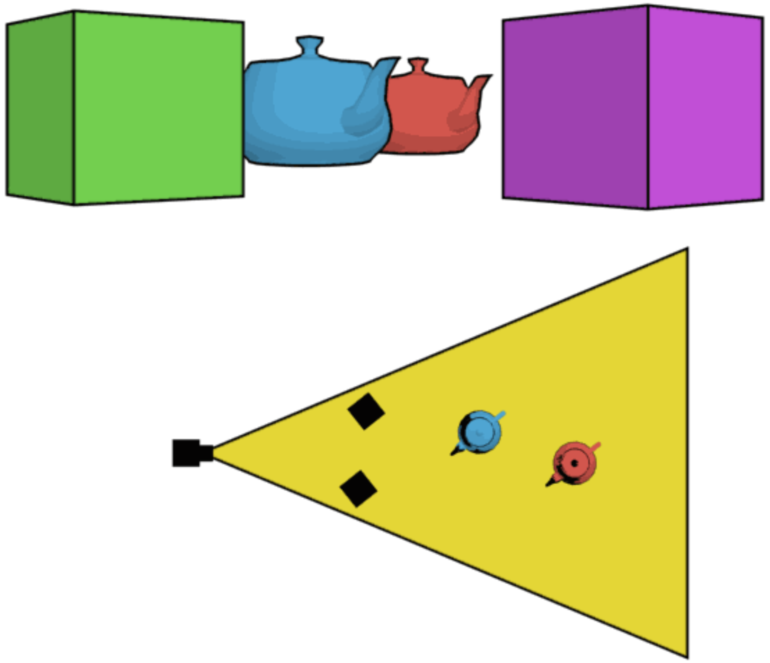
(reconstructed by Hunyuan3D V3.1)

- What would happen when reconstructing 3D shapes from a single image with *unknown* focal length/FOV?

Dolly Zoom Effect

- Continuously adjusting the focal length, while dollying toward or away from the subject.

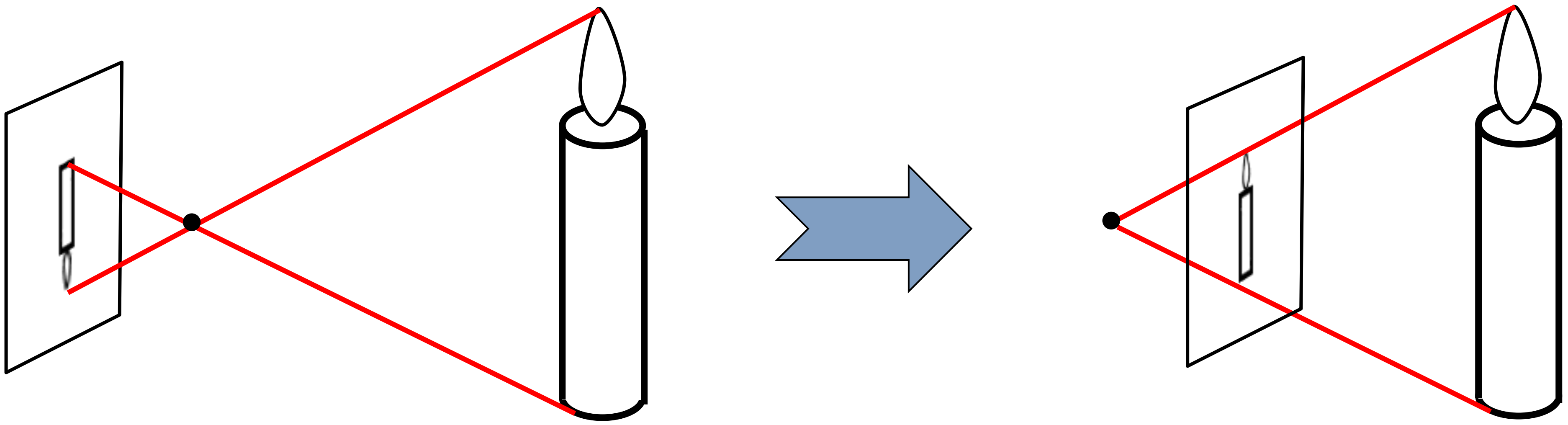




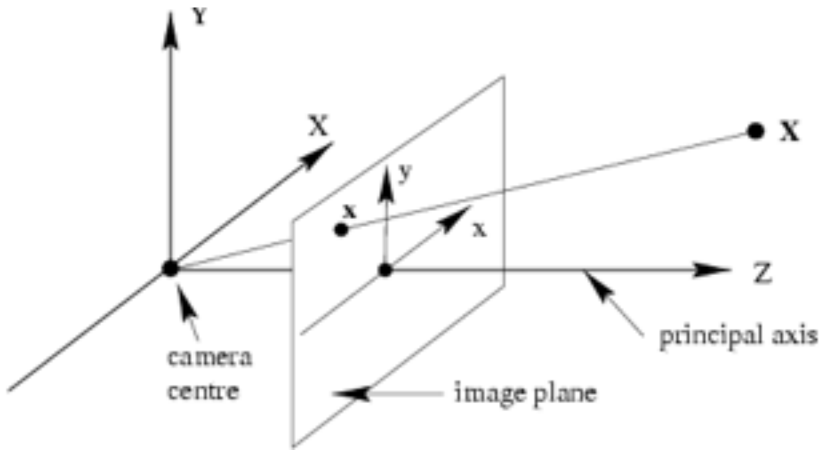
https://en.wikipedia.org/wiki/Dolly_zoom

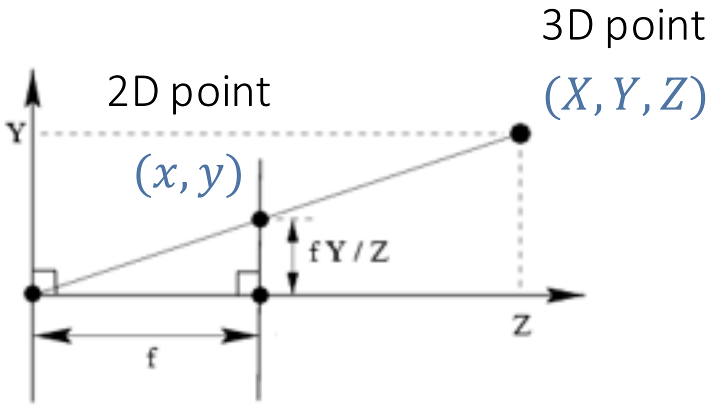
Perspective Projection

- Instead of dealing with an image that is upside down, most of the time we will pretend that the image plane is in front of the camera center.



Perspective Projection





$$(X, Y, Z) \rightarrow (x, y)$$

$$x = f\frac{X}{Z}, y = f\frac{Y}{Z}$$

- Perspective projection is not a linear transformation!
- But most other transformations we will be dealing with are linear.
- Projective geometry provides a handy mathematical tool to unify them.

Homogeneous Coordinates

- To form homogeneous coordinates from Euclidean coordinates, append 1 as the last entry:

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous coordinates of a 2D point (x, y)

$$(X, Y, Z) \Rightarrow \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

homogeneous coordinates of a 3D point (X, Y, Z)

- To convert homogeneous coordinates to Euclidean coordinates, divide by the last entry:

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

$$\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} \Rightarrow (X/W, Y/W, Z/W)$$

- All scalar multiples represent the same point! $\begin{bmatrix} x \\ y \\ w \end{bmatrix} \sim \lambda \begin{bmatrix} x \\ y \\ w \end{bmatrix}, \quad \lambda \neq 0$

Homogeneous Coordinates

2D		3D	
2D point (x, y)	$\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$	3D point (x, y, z)	$\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$
2D line $\mathbf{l}$	$\mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\mathbf{l}^\mathsf{T}\mathbf{x} = ax + by + c = 0$	3D plane $\boldsymbol{\pi}$	$\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ $\boldsymbol{\pi}^\mathsf{T}\mathbf{X} = aX + bY + cZ + d = 0$
2 points $\mathbf{x}_1$, $\mathbf{x}_2$ form a line $\mathbf{l}$	$\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$	3 points $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ form a plane $\boldsymbol{\pi}$	$\boldsymbol{\pi} = \mathbf{x}_1 \times \mathbf{x}_2 \times \mathbf{x}_3$
2 lines $\mathbf{l}_1$, $\mathbf{l}_2$ intersect at point $\mathbf{x}$	$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$	3 planes $\boldsymbol{\pi}_1$, $\boldsymbol{\pi}_2$, $\boldsymbol{\pi}_3$ intersect at point $\mathbf{X}$	$\mathbf{X} = \boldsymbol{\pi}_1 \times \boldsymbol{\pi}_2 \times \boldsymbol{\pi}_3$

Perspective Projection Matrix

- Projection is a matrix multiplication using homogeneous coordinates:

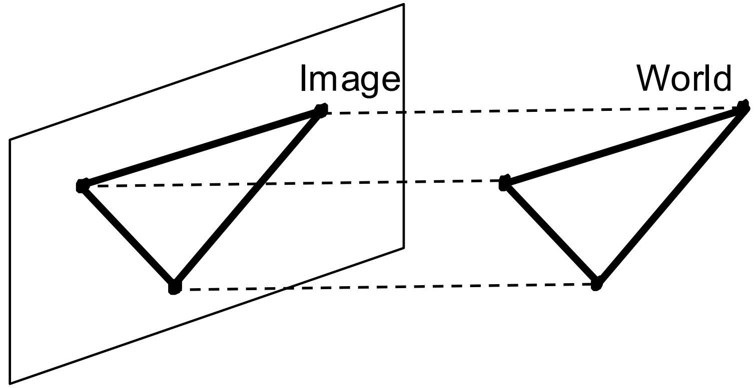
$$\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix} \Rightarrow \left(f\frac{X}{Z}, f\frac{Y}{Z}\right)$$

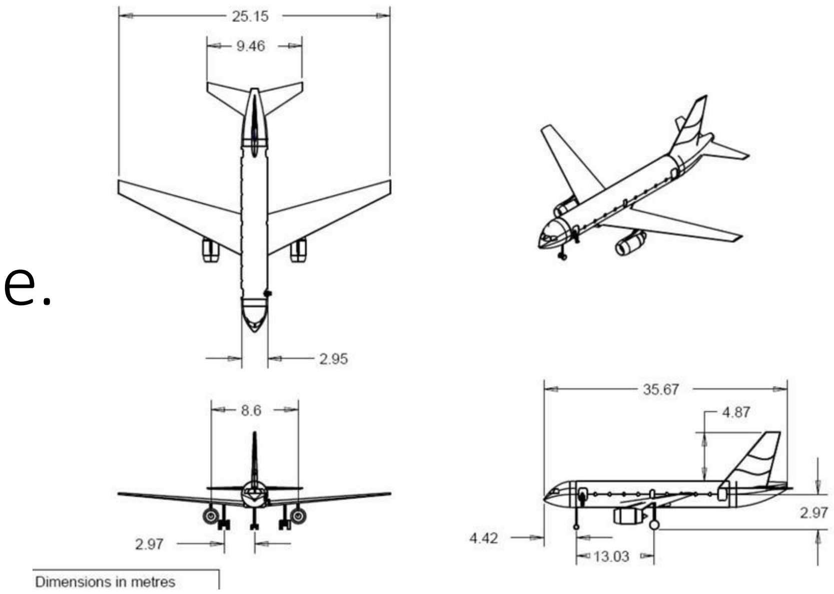
divide by the third coordinate

- What will happen if f is very large?

Orthographic Projection

- A form of *parallel projection* where the projection axis is orthogonal to image plane.





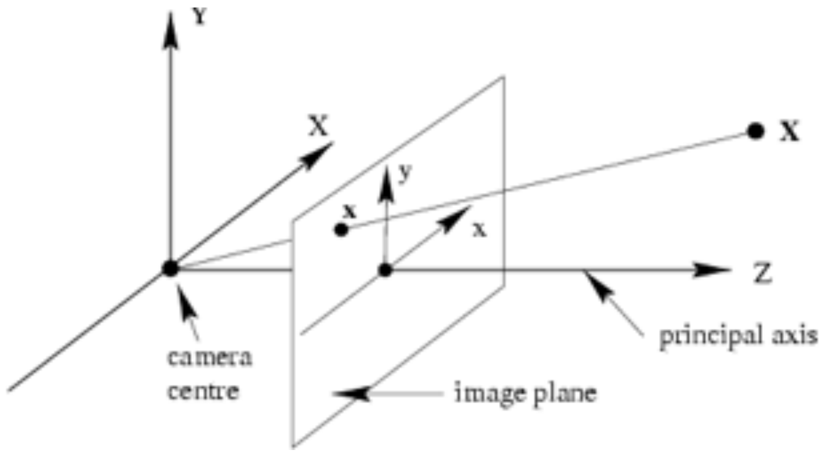
Three-view drawing

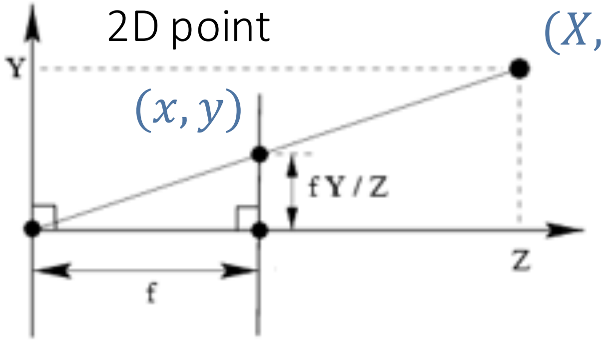
- Assuming projection along the z axis, what's the matrix?

$$\begin{bmatrix} \quad & \quad & \quad & \quad \\ \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \end{bmatrix}$$

Perspective Projection *in Normalized Coordinates*

(assuming camera looking toward z axis and no scaling)





2D point (x, y)

3D point (X, Y, Z)

$$(X, Y, Z) \rightarrow (x, y)$$

$$x = f\frac{X}{Z}, y = f\frac{Y}{Z}$$

$$\underbrace{\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}}_{\boldsymbol{x}} \cong \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \underbrace{\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\boldsymbol{P}} \underbrace{\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}}_{\boldsymbol{X}}$$

Homogeneous coordinates of image point

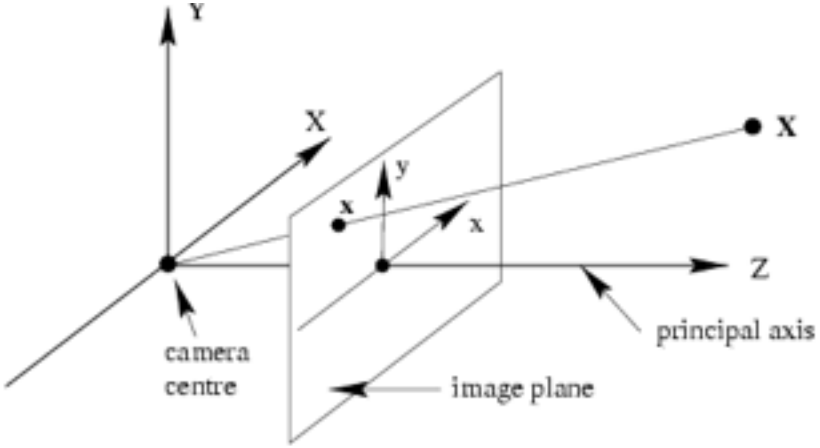
Camera projection matrix

Homogeneous coordinates of 3D point

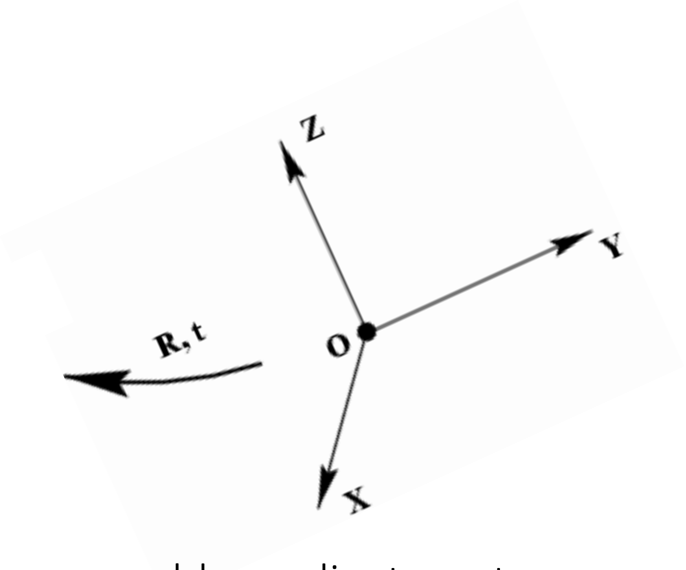
$$\boldsymbol{x} \cong \boldsymbol{P}\boldsymbol{X}$$

Equality up to scale

Camera Calibration



camera coordinate system



world coordinate system

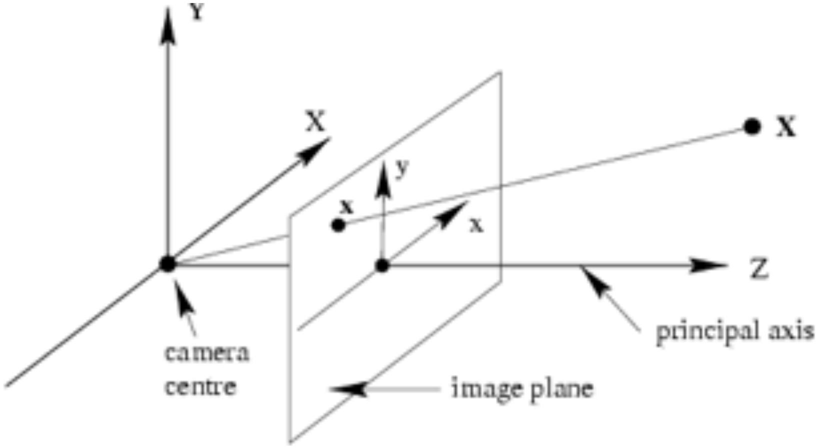
- **Camera calibration**: figuring out transformation from *world* coordinate system to *image* coordinate system

2D point $\boldsymbol{x}$ (3x1) $\cong$ (Camera to pixel coord. trans. matrix $\boldsymbol{K}$ (3x3)) (*Canonical* projection matrix $[\boldsymbol{I} \mid \boldsymbol{0}]$ (3x4)) (World to camera coord. trans. matrix $\begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix}$ (4x4)) (3D point $\boldsymbol{X}$ (4x1))

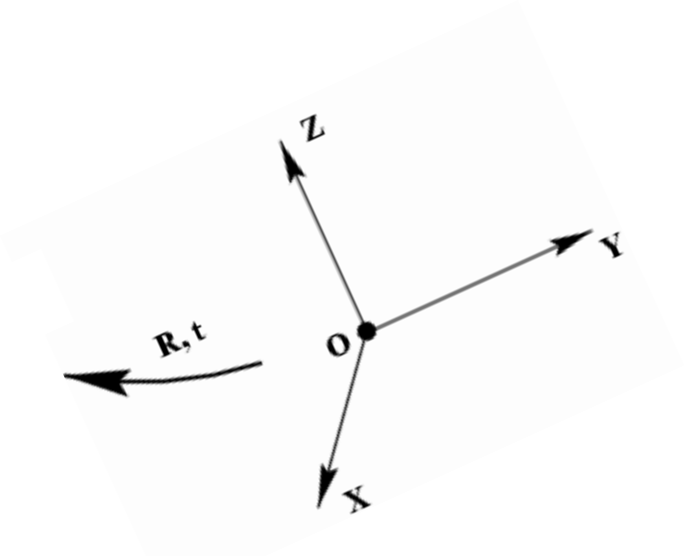
***Intrinsic* camera parameters**: principal point, scaling factors

***Extrinsic* camera parameters**: rotation, translation

Camera Calibration



camera coordinate system



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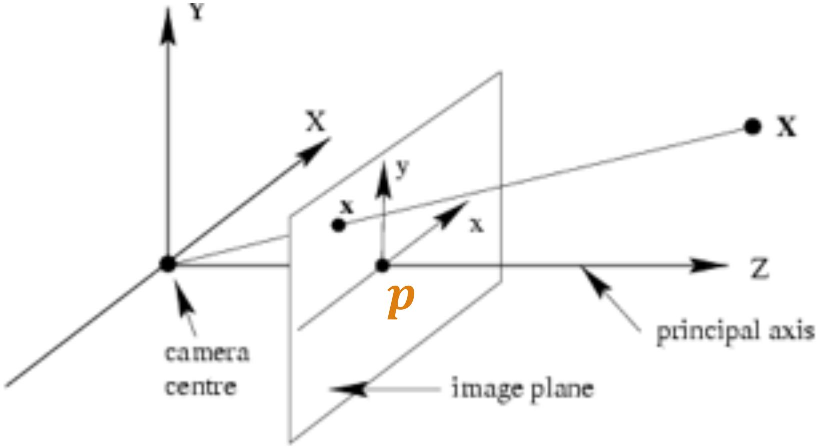
$$\boldsymbol{P} = \boldsymbol{K}[\boldsymbol{R}|\boldsymbol{t}]$$

General camera projection matrix

Intrinsic Parameters – Principal Point



camera coordinate system

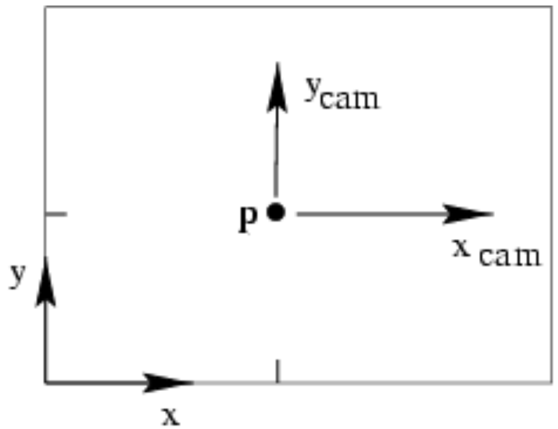
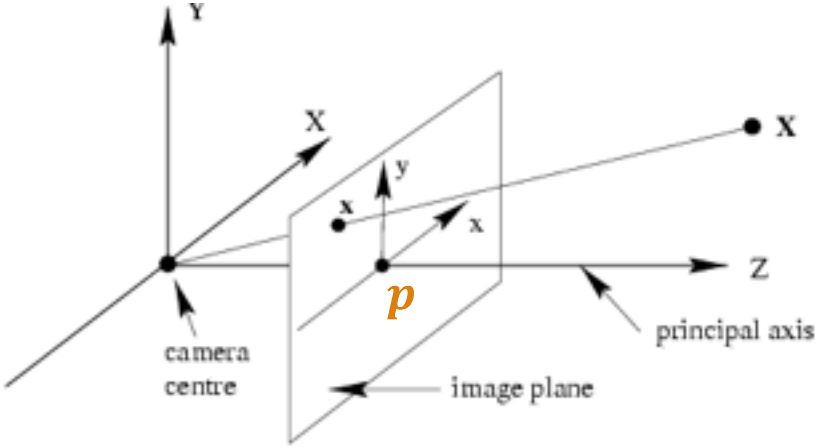


image coordinate system

Principal point (*p*): point where principal axis intersects the image plane

- In the *normalized* coordinate system, the origin is at the principal point.
- In the *image* coordinate system, the origin is in the corner.

Intrinsic Parameters – Principal Point



camera coordinate system

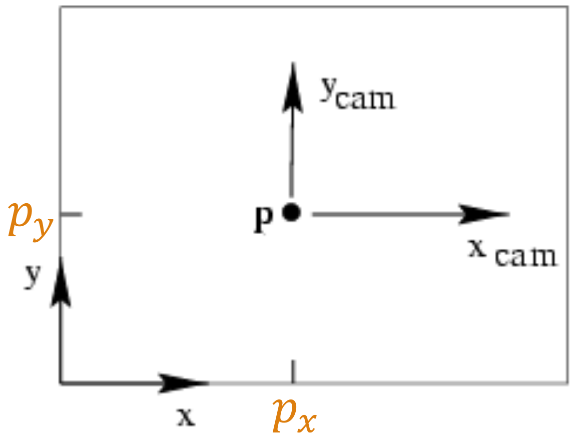


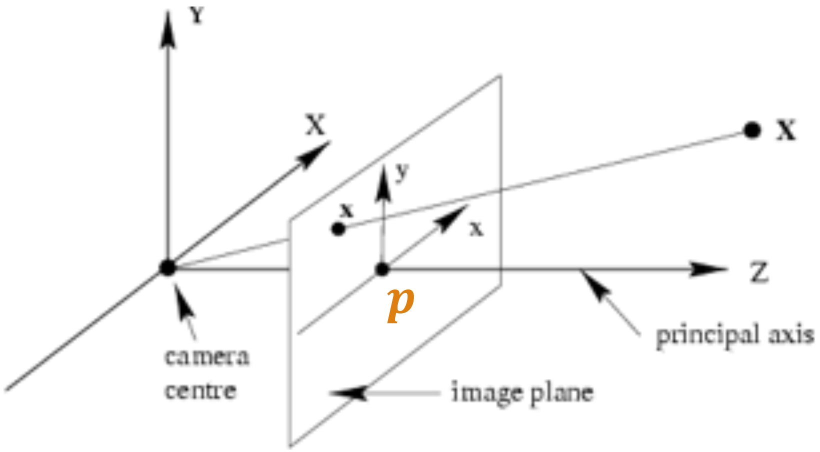
image coordinate system

$$x = f\frac{X}{Z} + p_x, \qquad y = f\frac{Y}{Z} + p_y$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \cong \begin{pmatrix} fX + Zp_x \\ fY + Zp_y \\ Z \end{pmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

calibration matrix ***K***

Intrinsic Parameters – Scaling



camera coordinate system

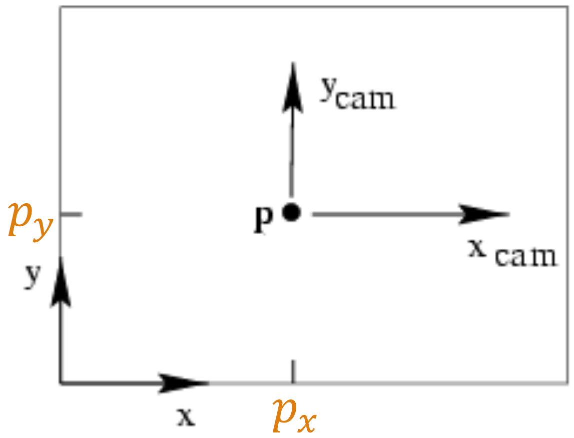


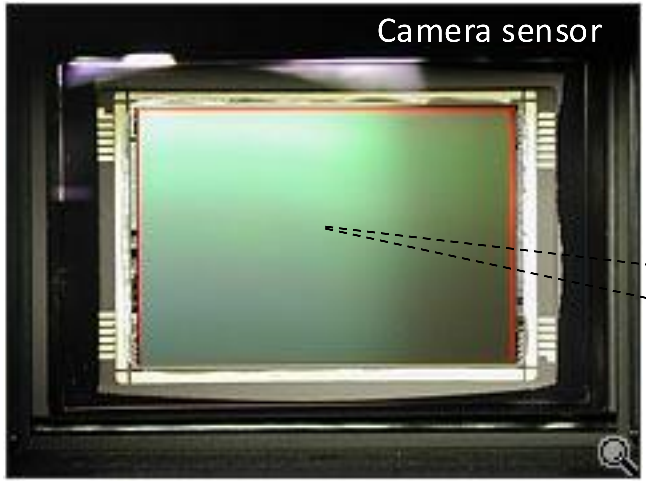
image coordinate system

$$x = f\frac{X}{Z} + p_x, \qquad y = f\frac{Y}{Z} + p_y$$

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \xleftarrow[\text{pixels/m}]{\alpha} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \cong \begin{pmatrix} fX + Zp_x \\ fY + Zp_y \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & p_x & 0 \\ 0 & f & p_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

pixels   m   calibration matrix ***K***

Intrinsic Parameters – Scaling



m_x pixels/m in horizontal direction
m_y pixels/m in vertical direction

Pixel size (m): $\frac{1}{m_x} \times \frac{1}{m_y}$

Scaling factors

$$\begin{bmatrix} m_x & 0 & 0 \\ 0 & m_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pixels/m

Calibration matrix $\boldsymbol{K}$ in metric units

$$\begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

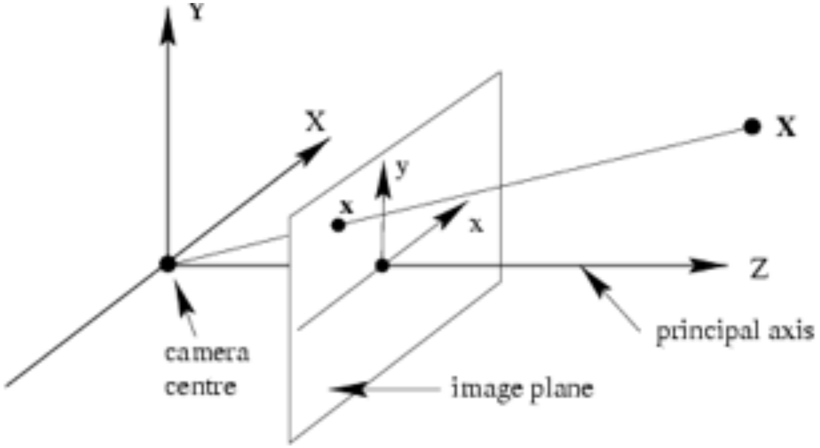
m

Calibration matrix $\boldsymbol{K}$ in pixel units

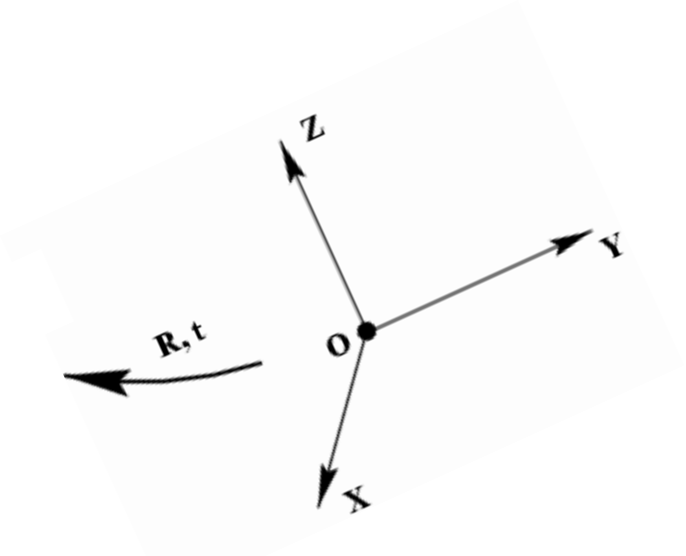
$$= \begin{bmatrix} \alpha_x & 0 & \beta_x \\ 0 & \alpha_y & \beta_y \\ 0 & 0 & 1 \end{bmatrix}$$

pixels

Camera Calibration



camera coordinate system



world coordinate system

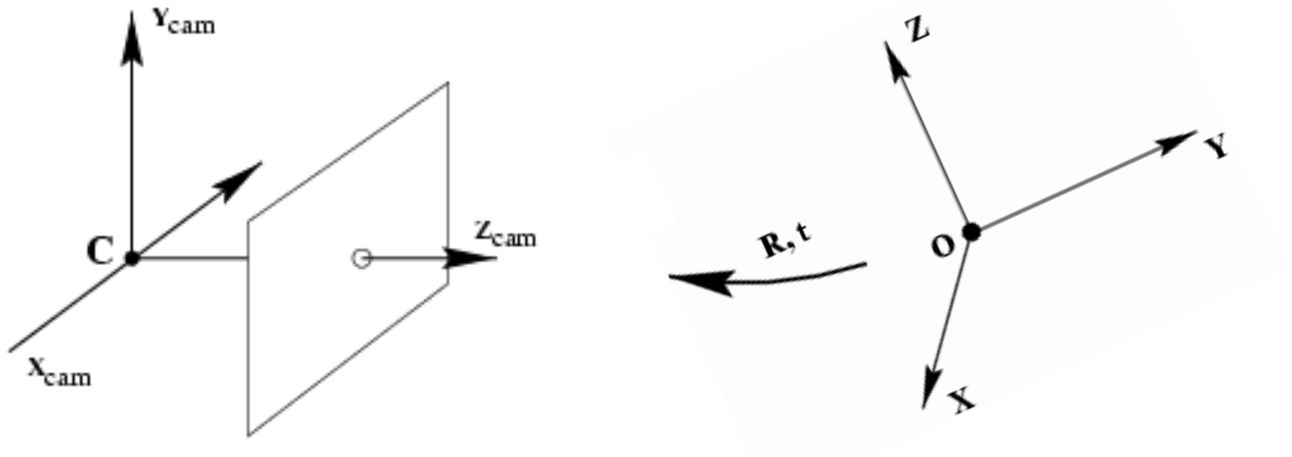
- **Camera calibration**: figuring out transformation from *world* coordinate system to *image* coordinate system

2D point $\boldsymbol{x}$ (3x1) $\cong$ Camera to pixel coord. trans. matrix $\boldsymbol{K}$ (3x3) *Canonical* projection matrix $[\boldsymbol{I} \mid \boldsymbol{0}]$ (3x4) World to camera coord. trans. matrix $\begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix}$ (4x4) 3D point $\boldsymbol{X}$ (4x1)

$$\boldsymbol{P} = \boldsymbol{K}[\boldsymbol{R} | \boldsymbol{t}]$$

General camera projection matrix

Extrinsic Parameters



camera coordinate system   world coordinate system

- In *non-homogeneous* coordinates, the transformation from world to normalized camera coordinate system is given by:

$$\tilde{X}_{\text{cam}} = R(\tilde{X} - \tilde{C}) = R\tilde{X} + t$$

coords. of point in normalized camera frame

3x3 rotation matrix

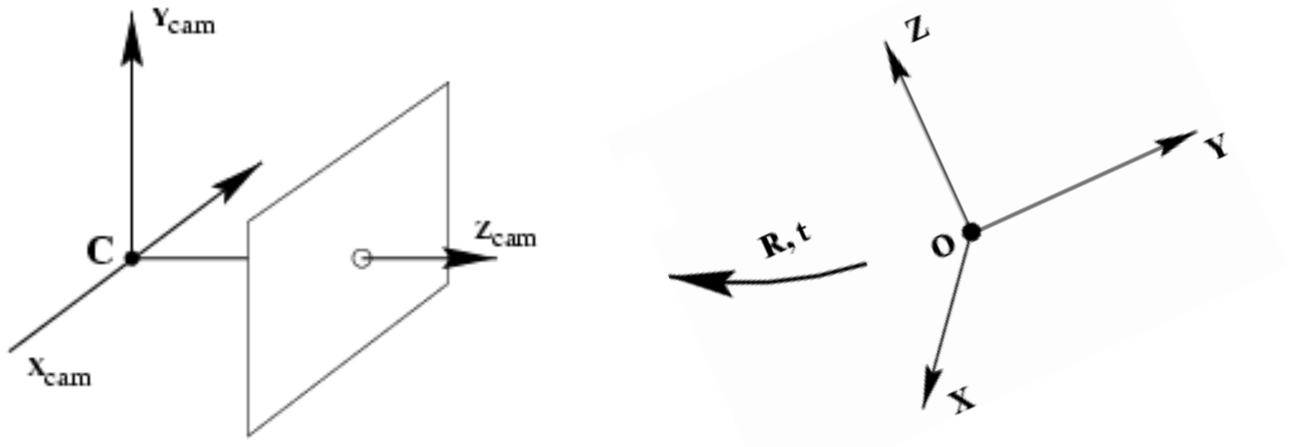
coords. of a point in world frame

coords. of camera center in world frame

W2C: $\tilde{X}_{\text{cam}} = R\tilde{X}_{\text{world}} + t$

C2W: $\tilde{X}_{\text{world}} = R^T\tilde{X}_{\text{cam}} - R^T t$

Extrinsic Parameters



camera coordinate system

world coordinate system

In non-homogeneous coordinates:

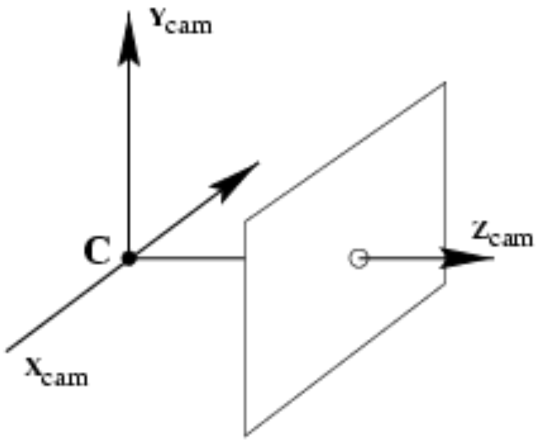
$$\tilde{\boldsymbol{X}}_{\mathrm{cam}} = \boldsymbol{R}\tilde{\boldsymbol{X}} + \boldsymbol{t}$$

In homogeneous coordinates:

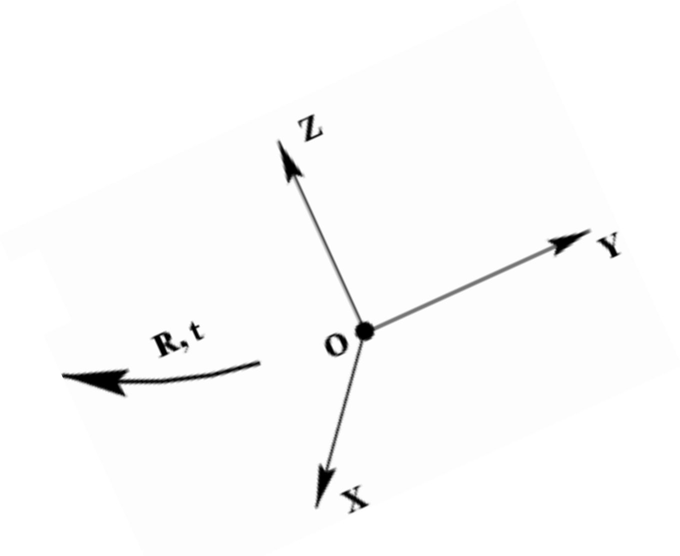
$$\begin{pmatrix} \tilde{\boldsymbol{X}}_{\mathrm{cam}} \\ 1 \end{pmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \begin{pmatrix} \tilde{\boldsymbol{X}} \\ 1 \end{pmatrix}$$

3D transformation matrix (4 x 4)

Extrinsic Parameters



camera coordinate system



world coordinate system

In non-homogeneous coordinates:

$$\widetilde{\boldsymbol{X}}_{\text{cam}} = \boldsymbol{R}\widetilde{\boldsymbol{X}} + \boldsymbol{t}$$

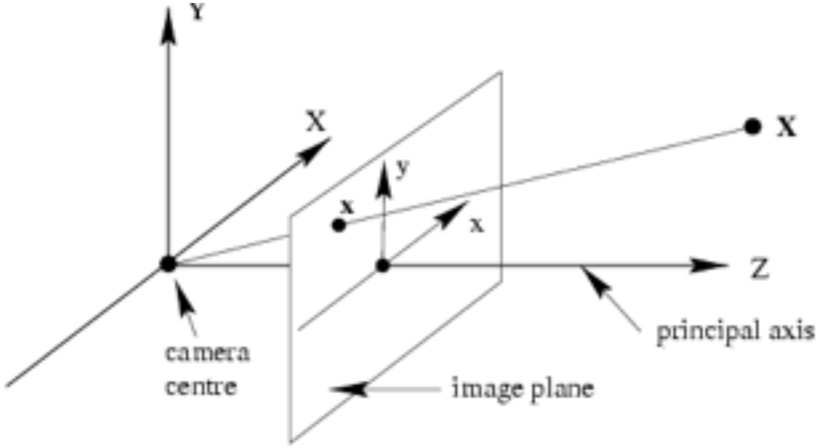
In homogeneous coordinates:

$$\boldsymbol{X}_{\text{cam}} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \boldsymbol{X}$$

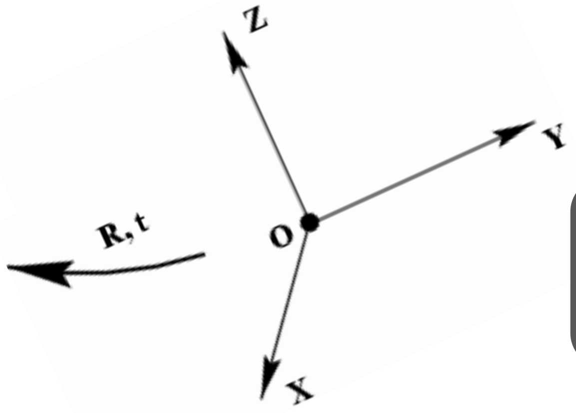
3D transformation matrix (4 x 4)

And then to image coordinates: $\boldsymbol{x} \cong \boldsymbol{K}[\boldsymbol{I}|\boldsymbol{0}]\boldsymbol{X}_{\text{cam}}$

Camera Calibration



camera coordinate system



world coordinate system

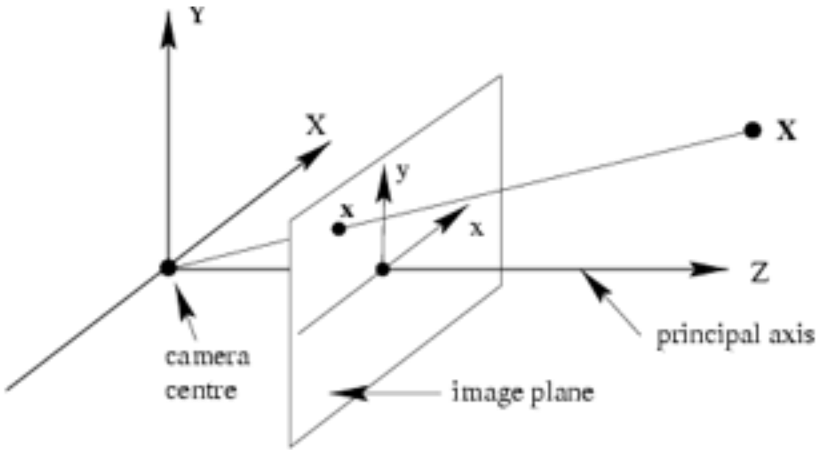
$$\boxed{x \cong K[R|t]X}$$

- **Camera calibration**: figuring out transformation from *world* coordinate system to *image* coordinate system

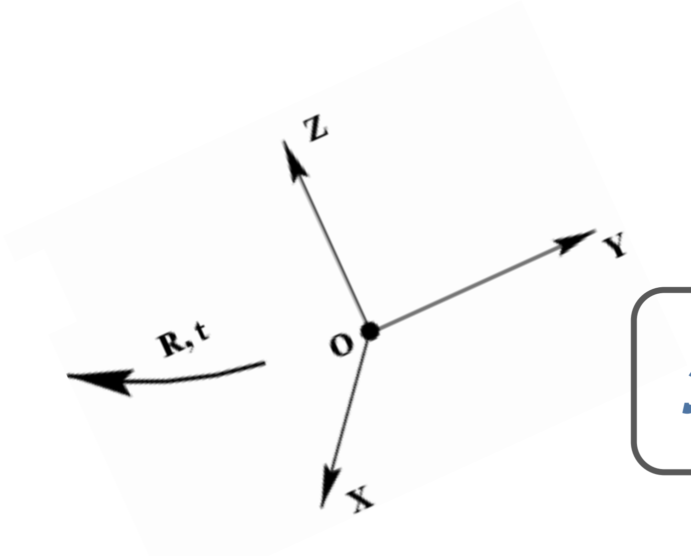
$$\begin{pmatrix}\text{2D point}\\ x\\ (3\text{x}1)\end{pmatrix} \cong \underbrace{\begin{pmatrix}\text{Camera to pixel coord. trans. matrix}\\ K\ (3\text{x}3)\end{pmatrix}\begin{pmatrix}\textit{Canonical}\ \text{projection matrix}\\ [I\,|\,0]\ (3\text{x}4)\end{pmatrix}\begin{pmatrix}\text{World to camera coord. trans. matrix}\\ \begin{bmatrix}R & t\\ 0^T & 1\end{bmatrix}\ (4\text{x}4)\end{pmatrix}}_{P = K[R|t]}\begin{pmatrix}\text{3D point}\\ X\\ (4\text{x}1)\end{pmatrix}$$

General camera projection matrix

Camera Calibration



camera coordinate system



world coordinate system

$$\boxed{\mathbf{x} \cong \mathbf{K}[\mathbf{R}|\mathbf{t}]\mathbf{X}}$$

- **Camera calibration**: figuring out transformation from *world* coordinate system to *image* coordinate system

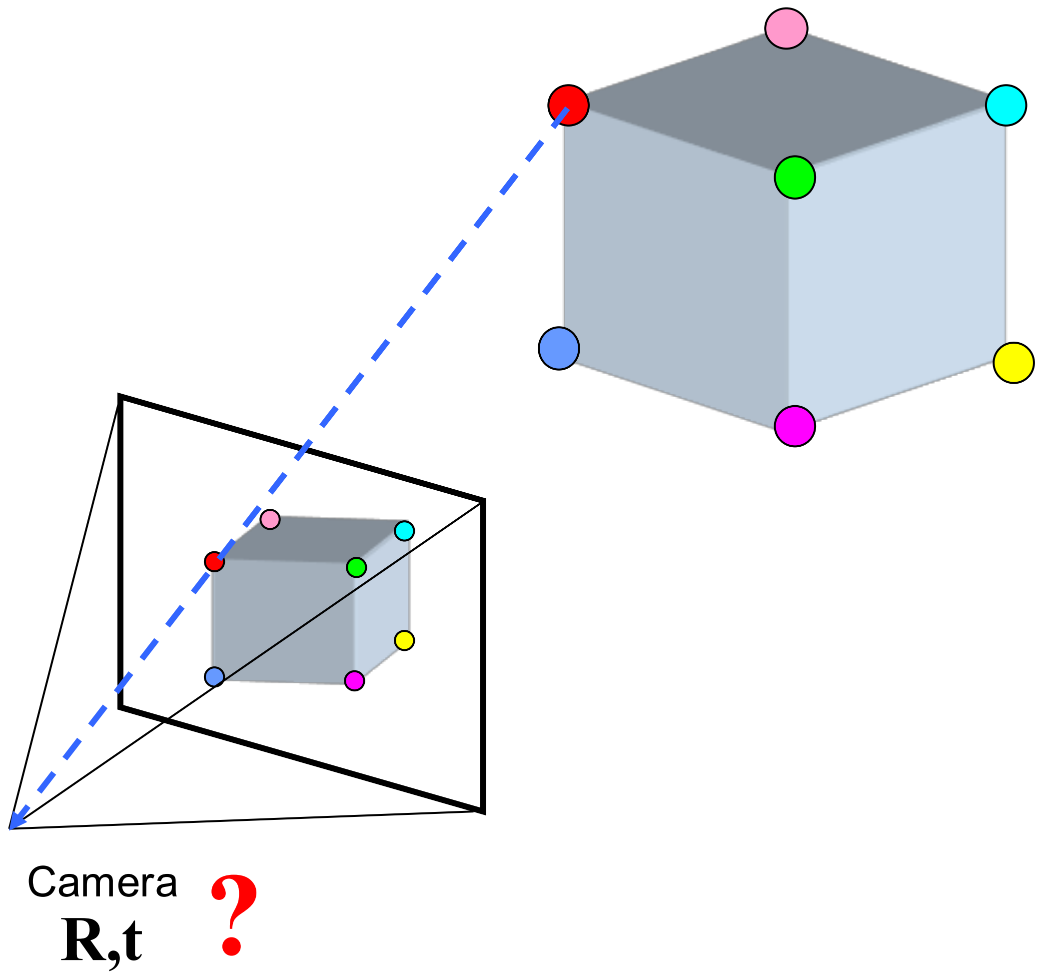
$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \cong \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \leftarrow$$

We could solve this as a linear system, then perform QR decomposition to get $\mathbf{K}$, $\mathbf{R}$, $\mathbf{t}$. (not robust in practice; sensitive to noise)

Slide Source: S. Lazebnik

Perspective-n-Point (PnP)

$$\boldsymbol{x} \cong \boldsymbol{K}[\boldsymbol{R}|\boldsymbol{t}]\boldsymbol{X}$$



Known:

- n 3D points $\boldsymbol{X}$
- Corresponding 2D pixel coordinates $\boldsymbol{x}$
- Camera intrinsics $\boldsymbol{K}$

Solve for:

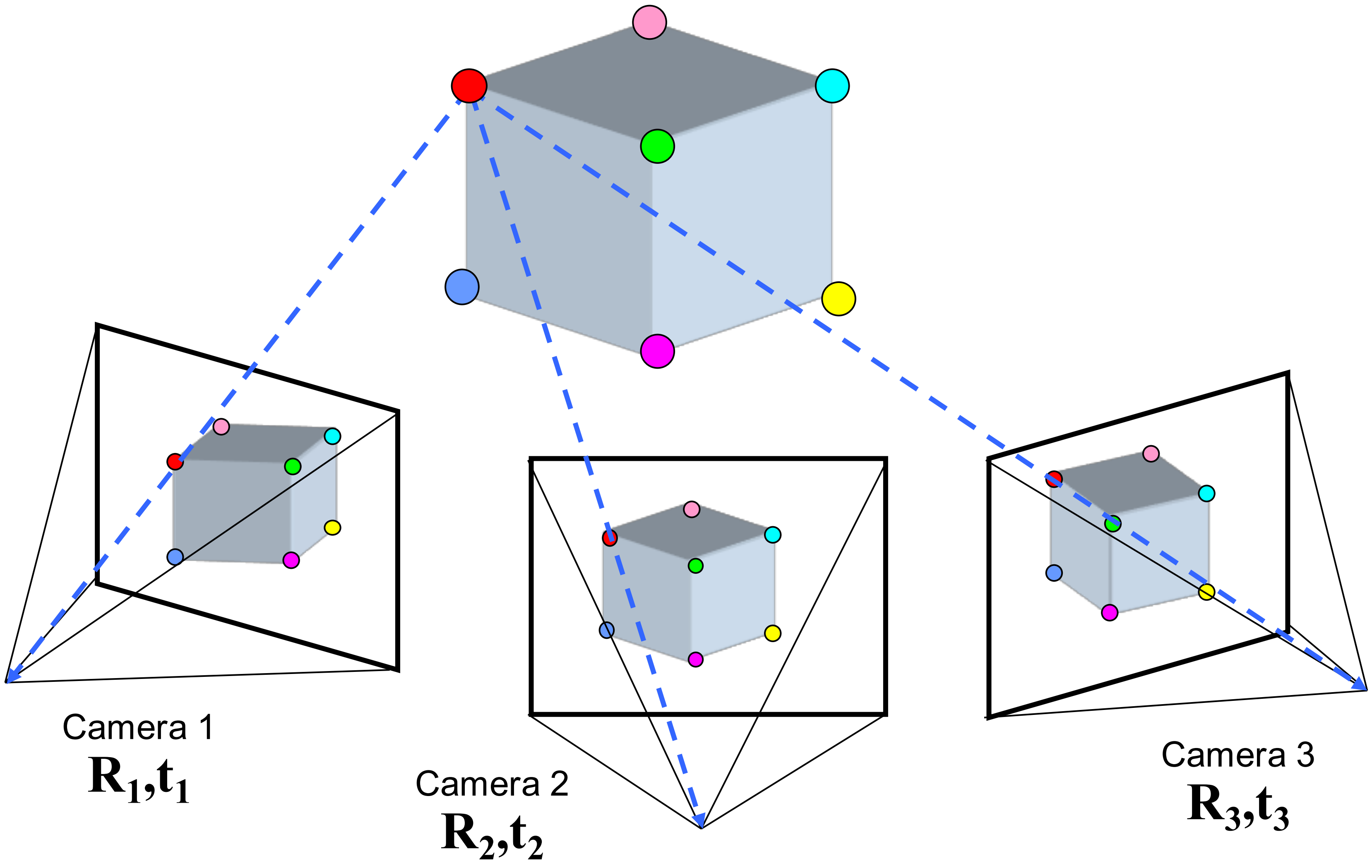
- Camera pose (extrinsics) $\boldsymbol{R}$, $\boldsymbol{t}$

Q: How many points at least do we need?

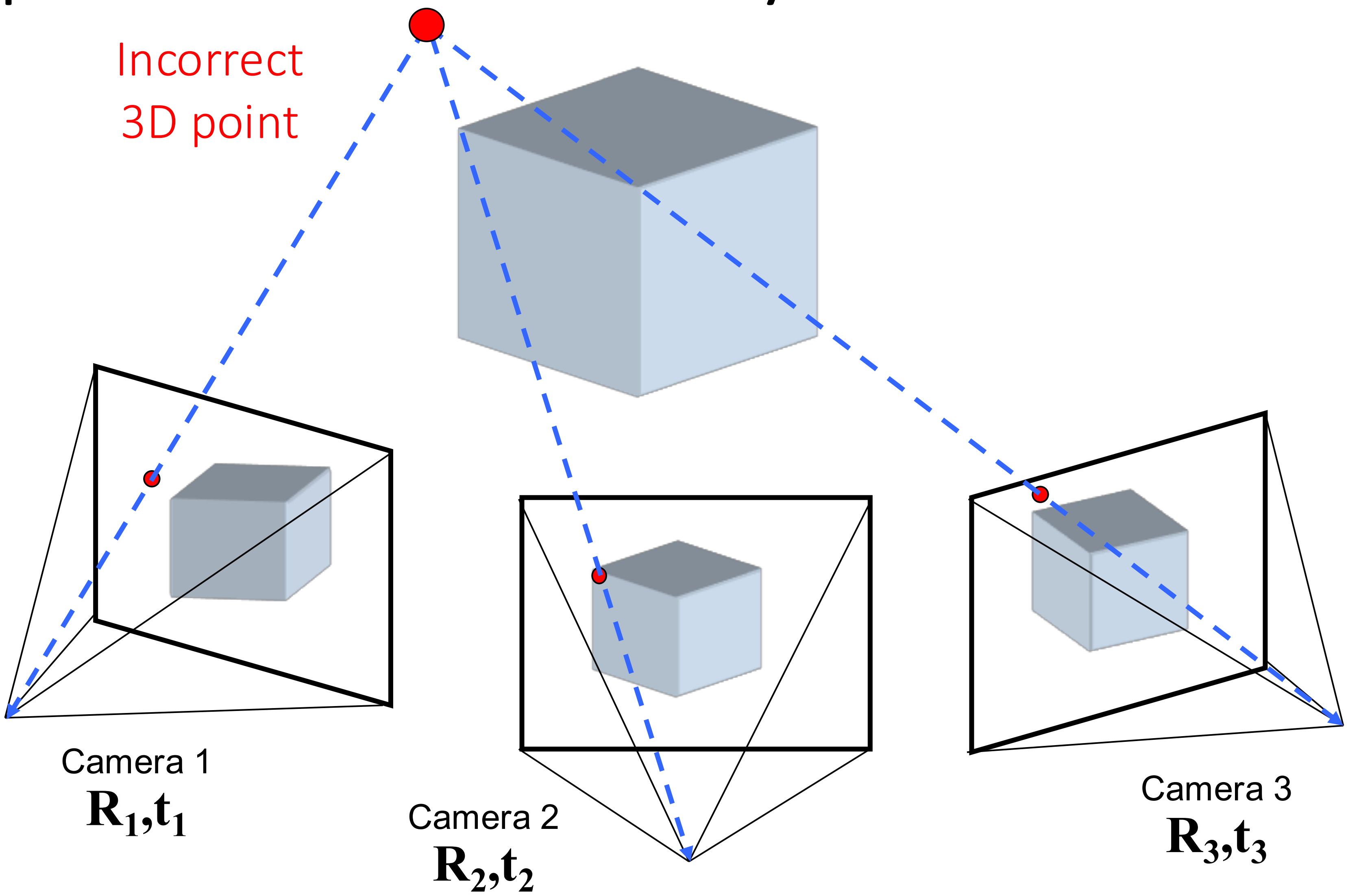
A: n ≥ 3 (in general). We have 6 DoF unknowns $\boldsymbol{R}$, $\boldsymbol{t}$. Each point provides 2 constraints (2 equations for x and y).

Part 2 – Multi-view Geometry

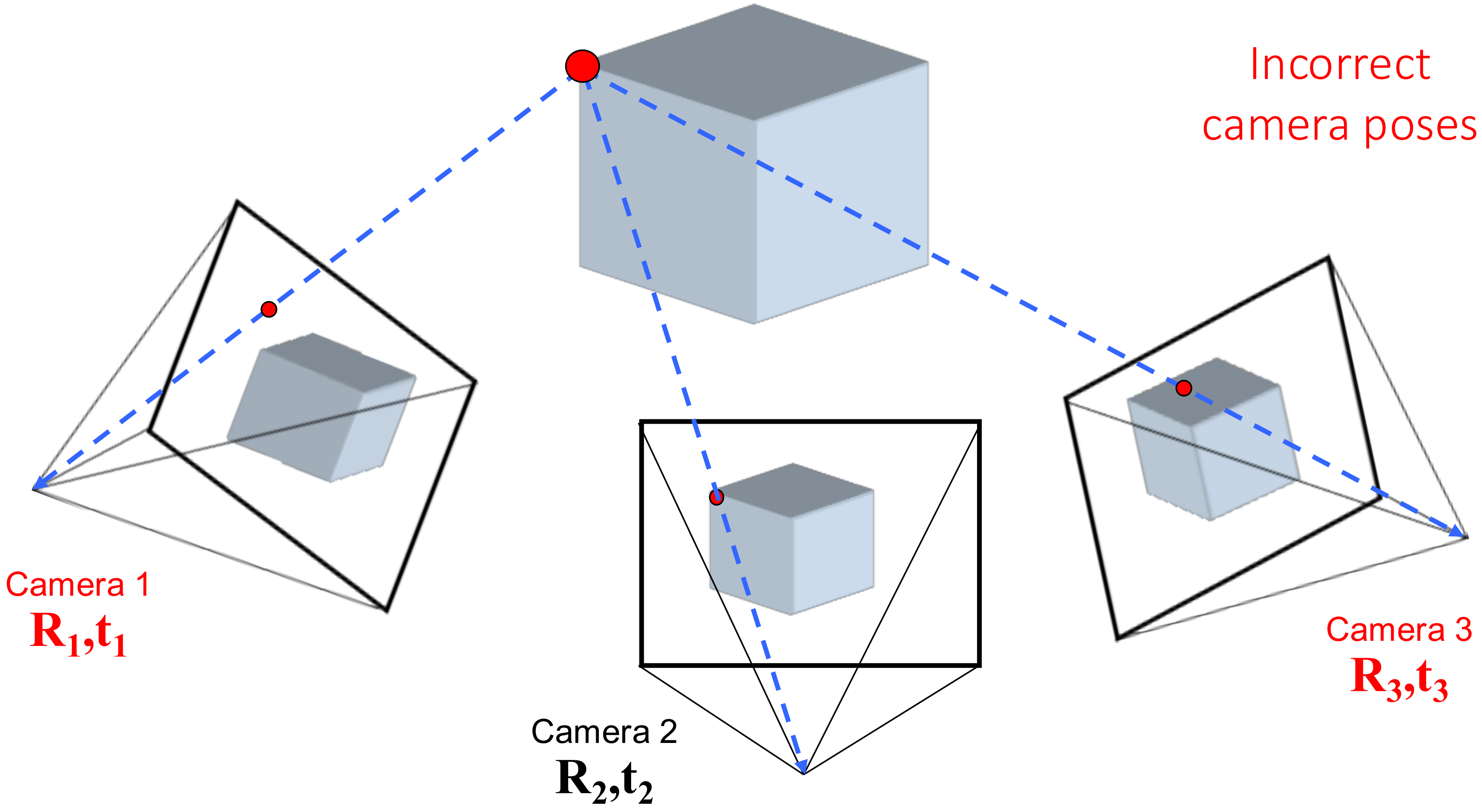
Multiple View Geometry



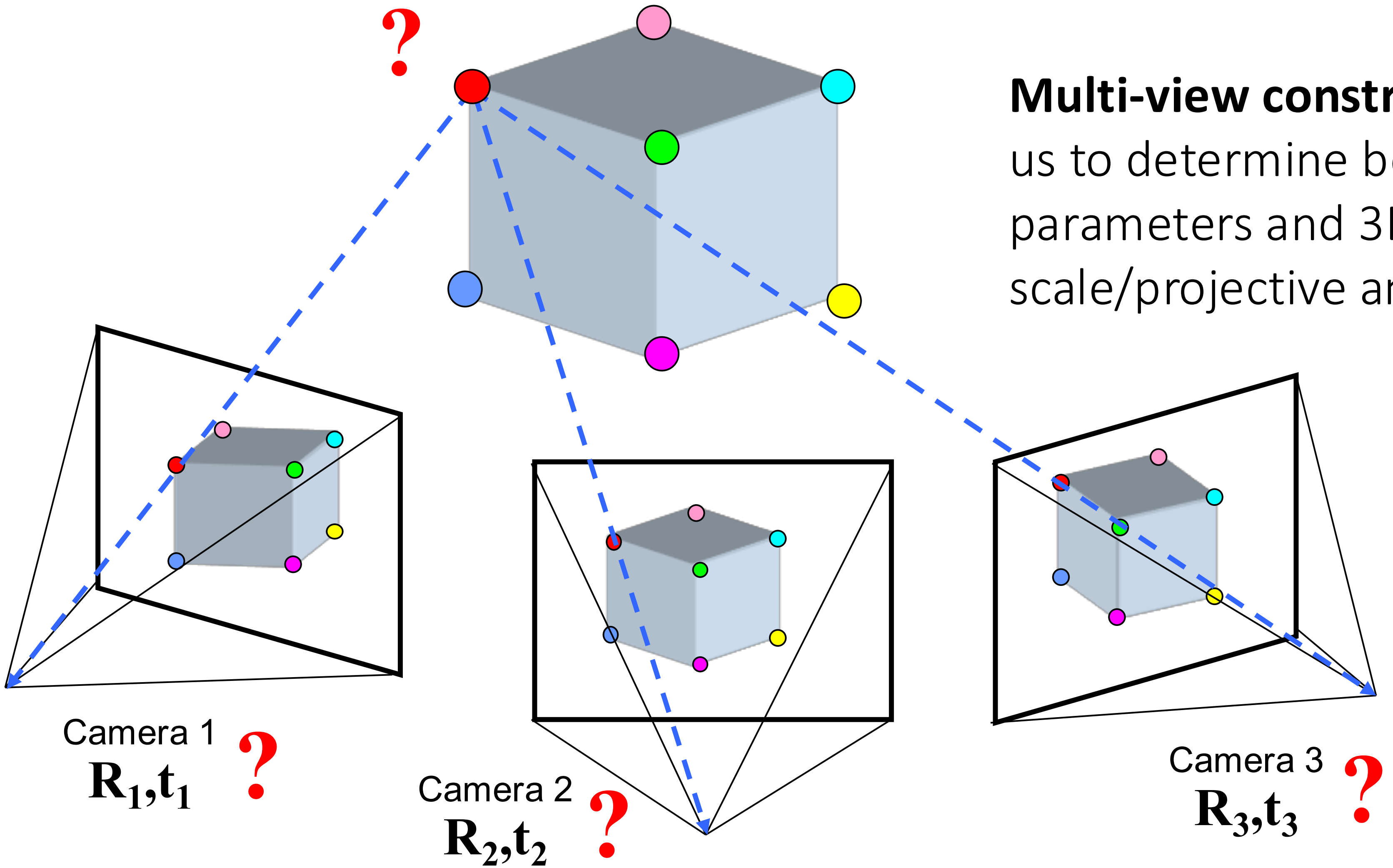
Multiple View Geometry



Multiple View Geometry

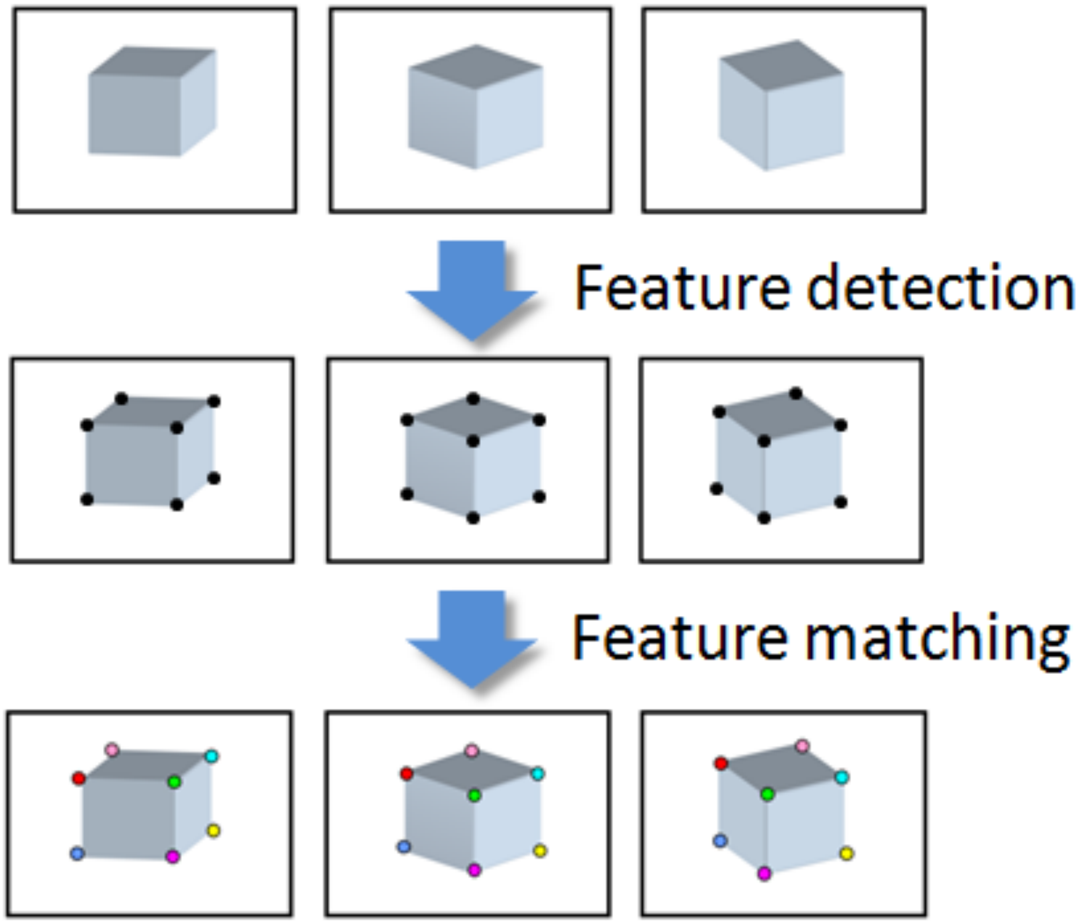


Multiple View Geometry

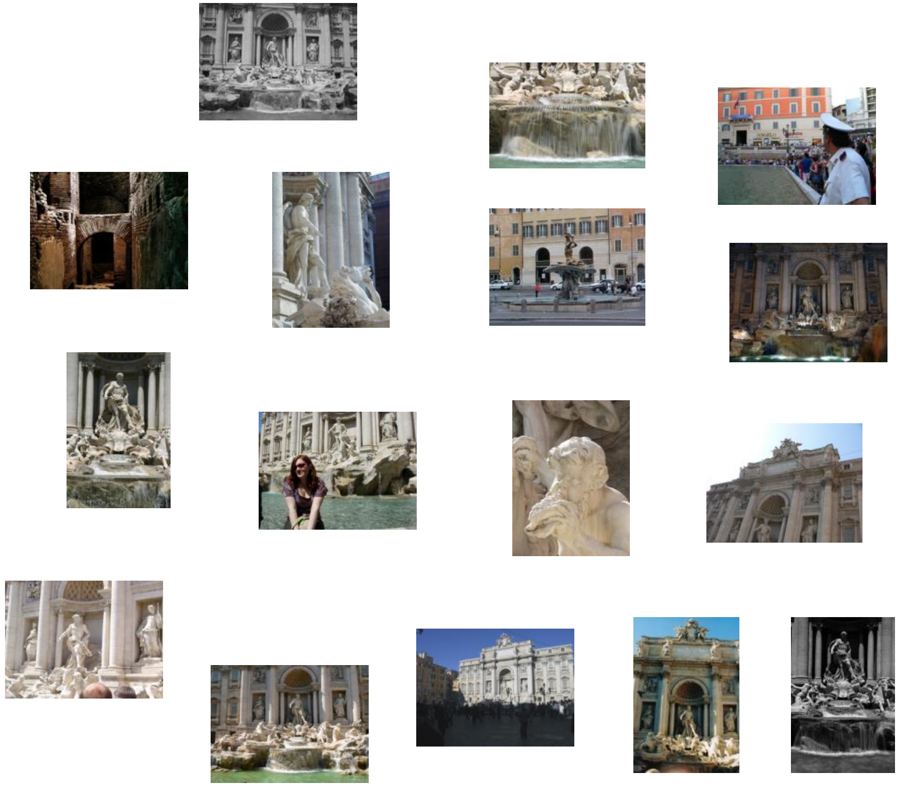


Multi-view constraints allow us to determine both camera parameters and 3D (up to scale/projective ambiguities)

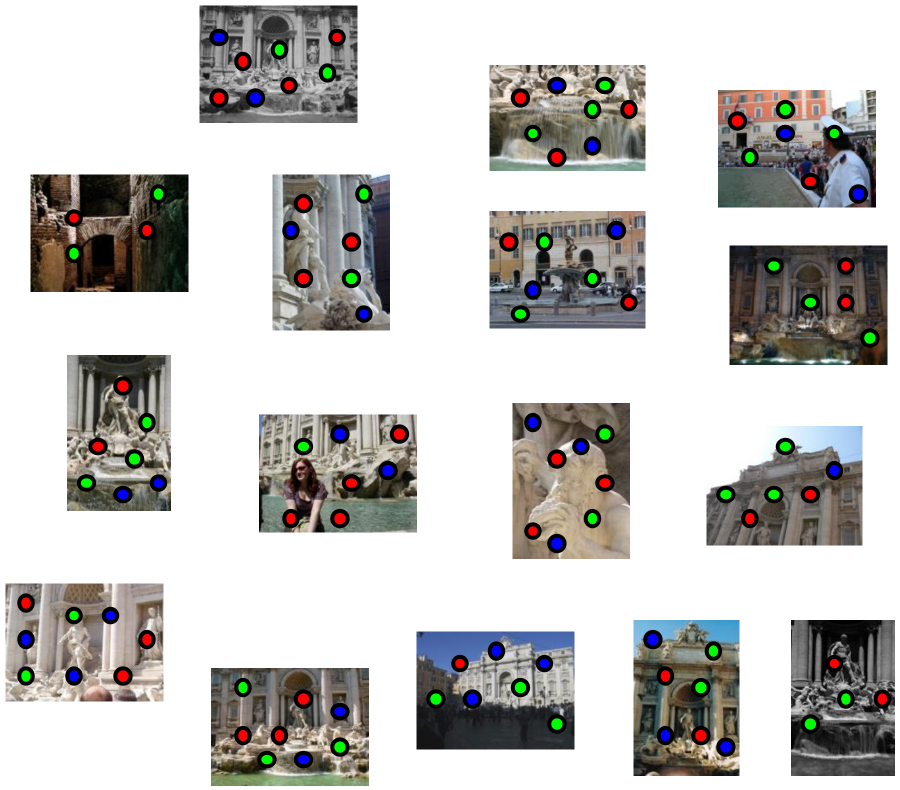
Estimating Correspondences



Estimating Correspondences

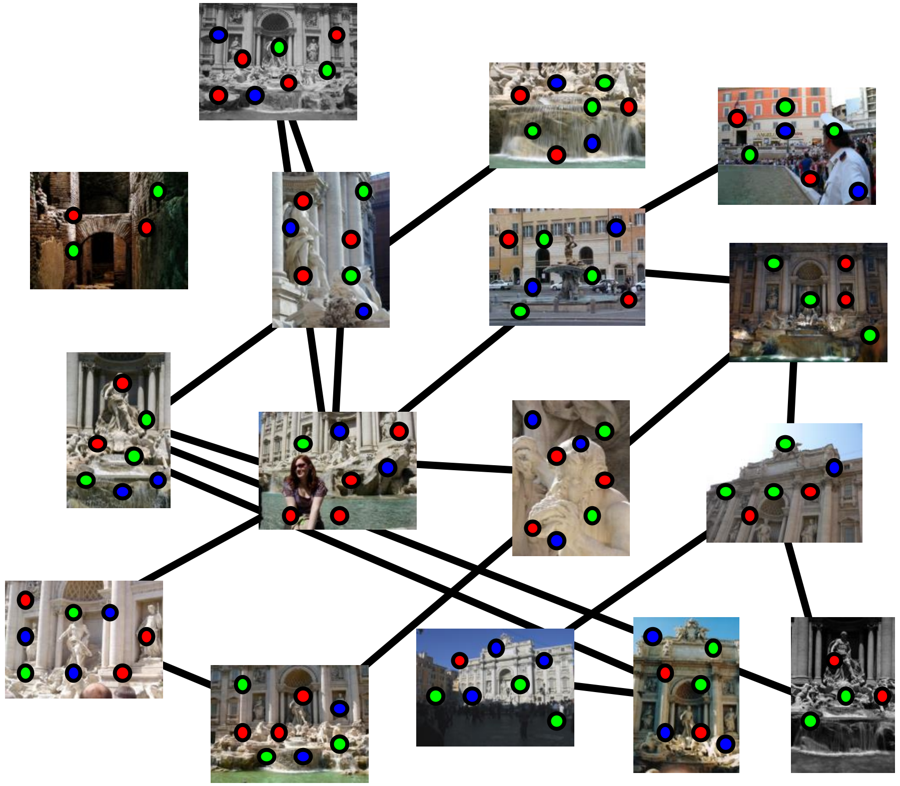


Estimating Correspondences



- **Detect feature points**
  - SIFT descriptors [Lowe, 2004]
  - Learned features – SuperPoint [DeTone, 2018])

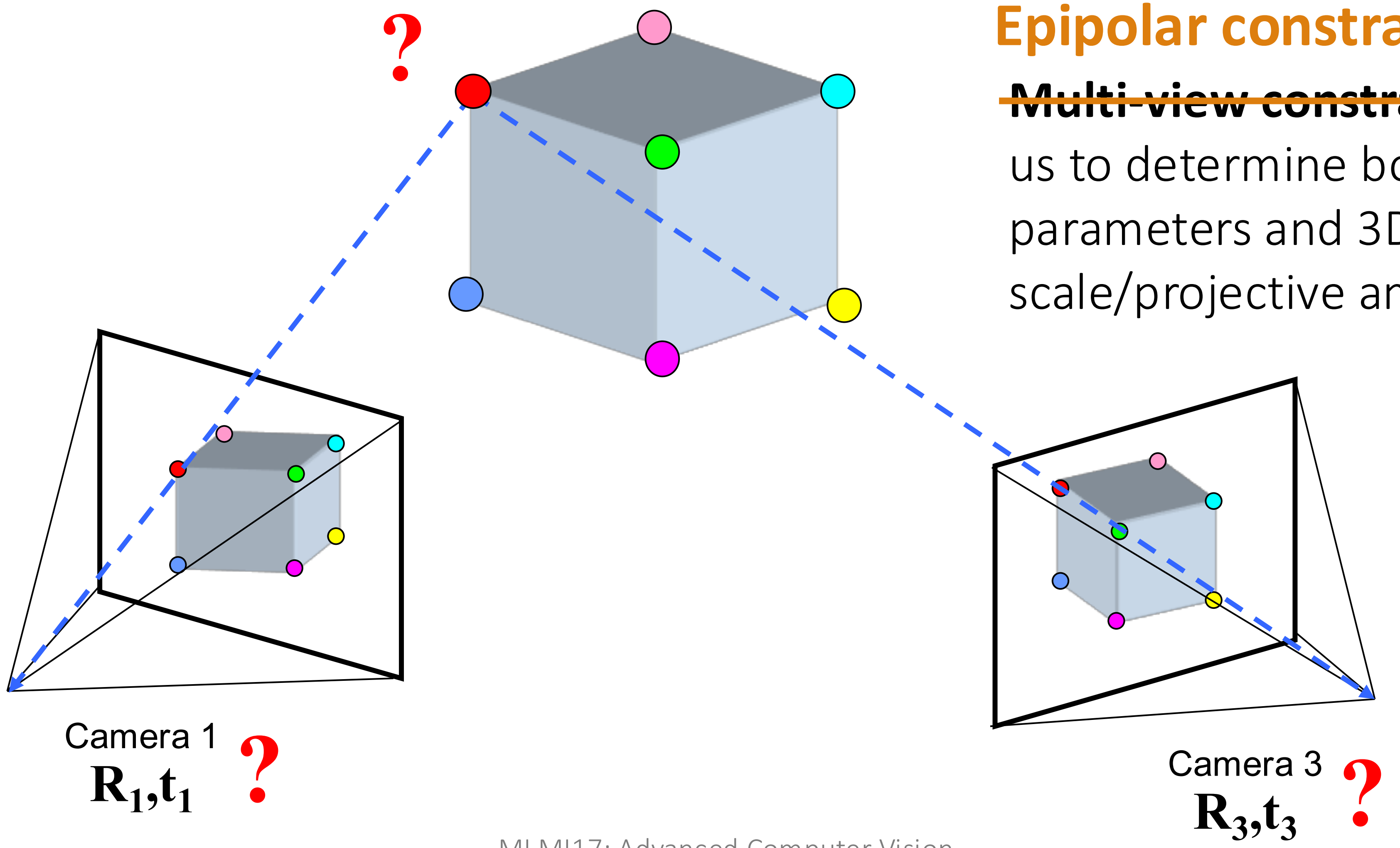
Estimating Correspondences



- **Detect feature points**
  - SIFT descriptors [Lowe, 2004]
  - Learned features – SuperPoint [DeTone, 2018])
- **Match features** between pair of images
  - Use RANSAC to filter outliers
  - Learned matching – SuperGlue [Sarlin, 2020]

Let’s Start with Two Views – Epipolar Geometry

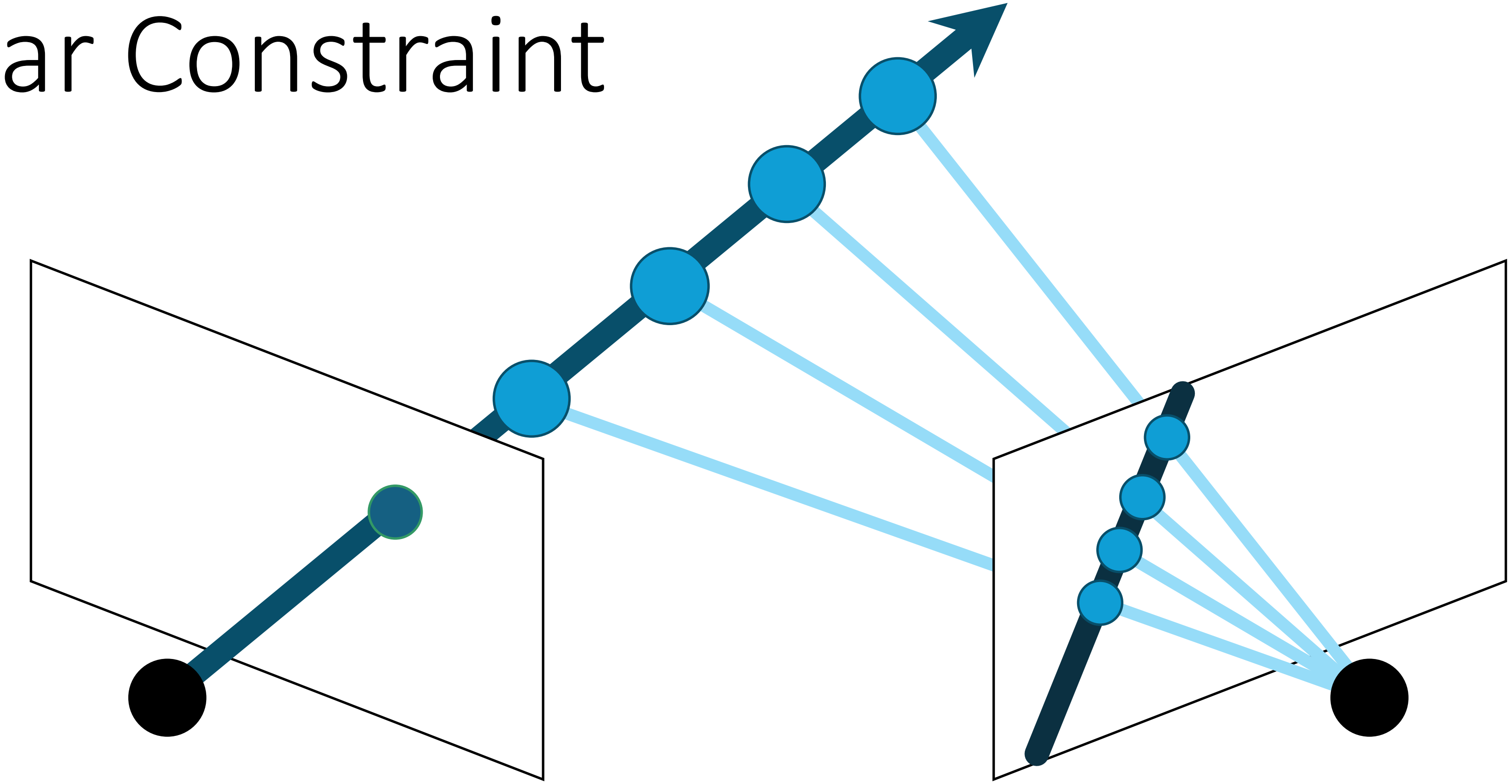
“epi-pole” ≈ upon the pole/pivot



Epipolar constraints

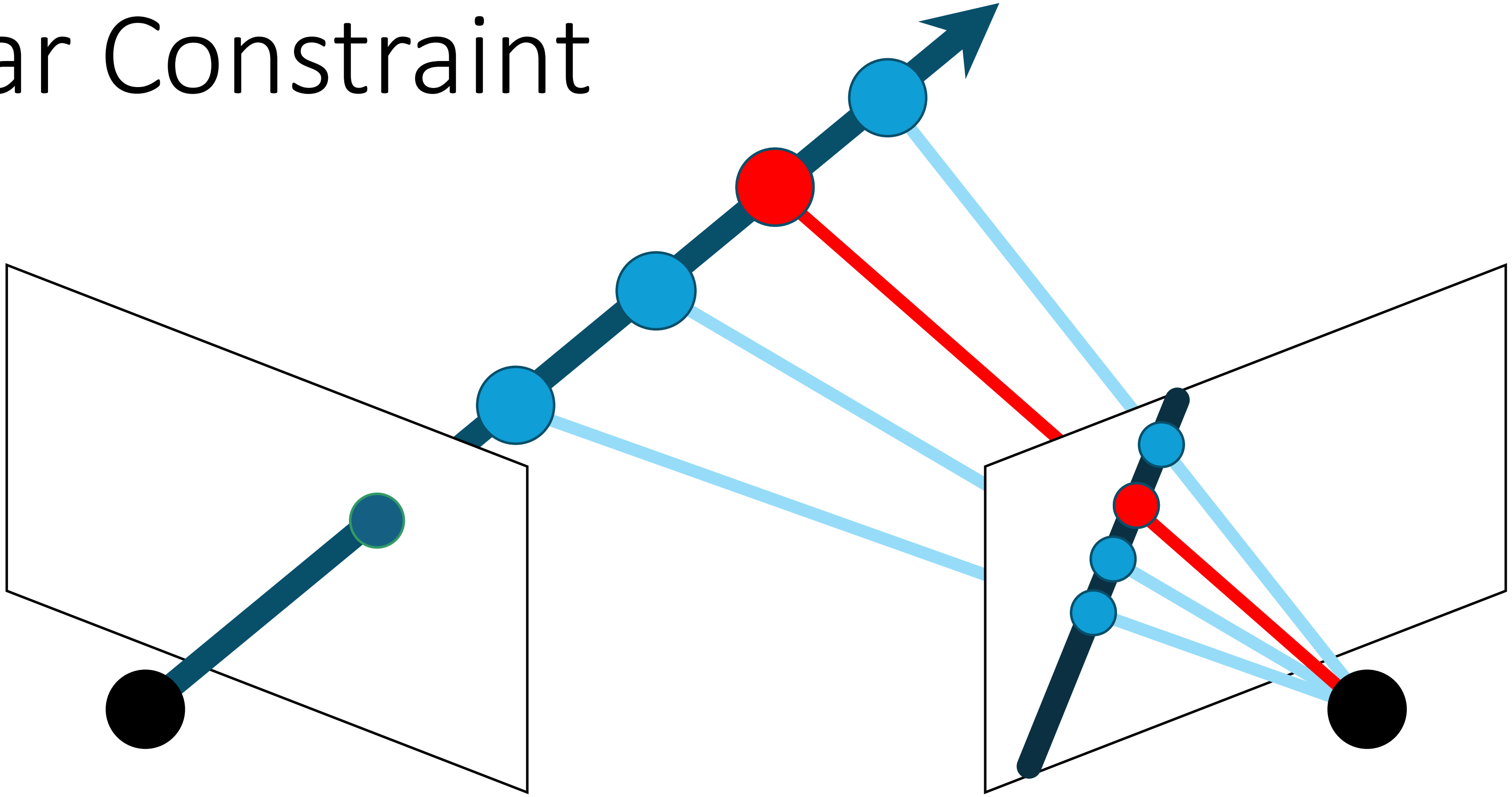
~~Multi-view constraints~~ allow us to determine both camera parameters and 3D (up to scale/projective ambiguities)

Epipolar Constraint



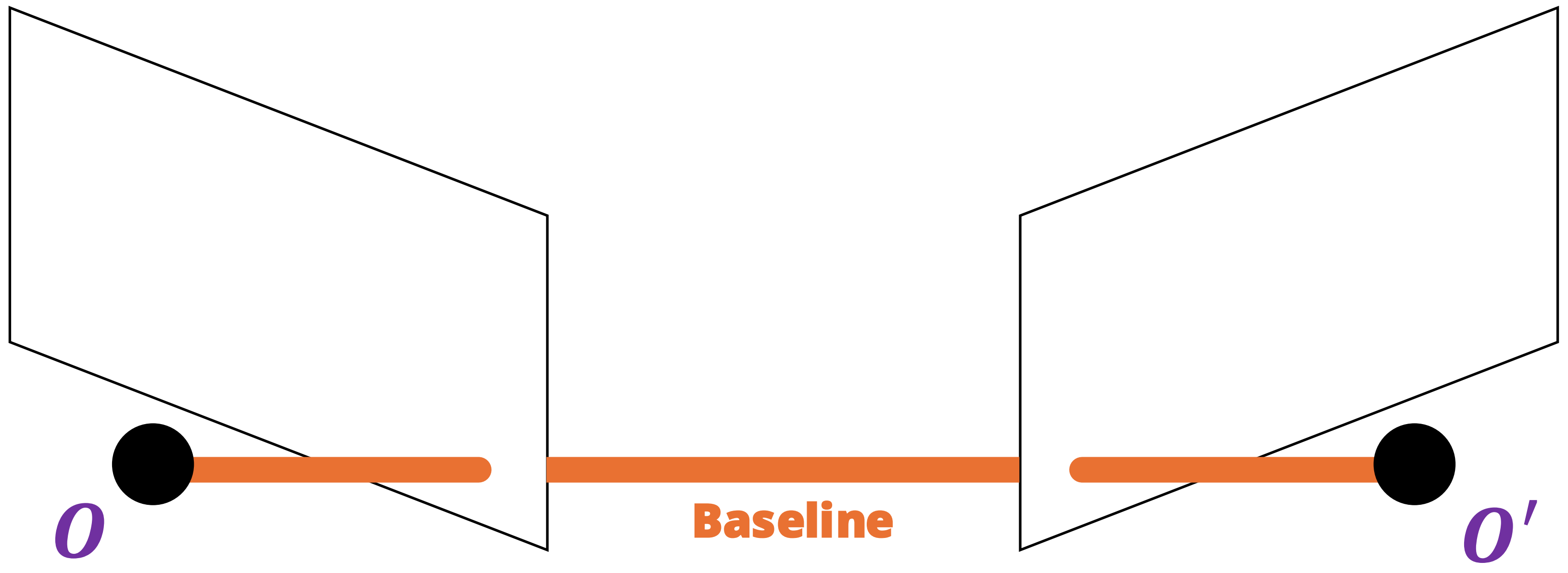
- Projecting points on a ray to another camera forms a **line**, i.e., the corresponding point on the second image *must* lie on this line

Epipolar Constraint



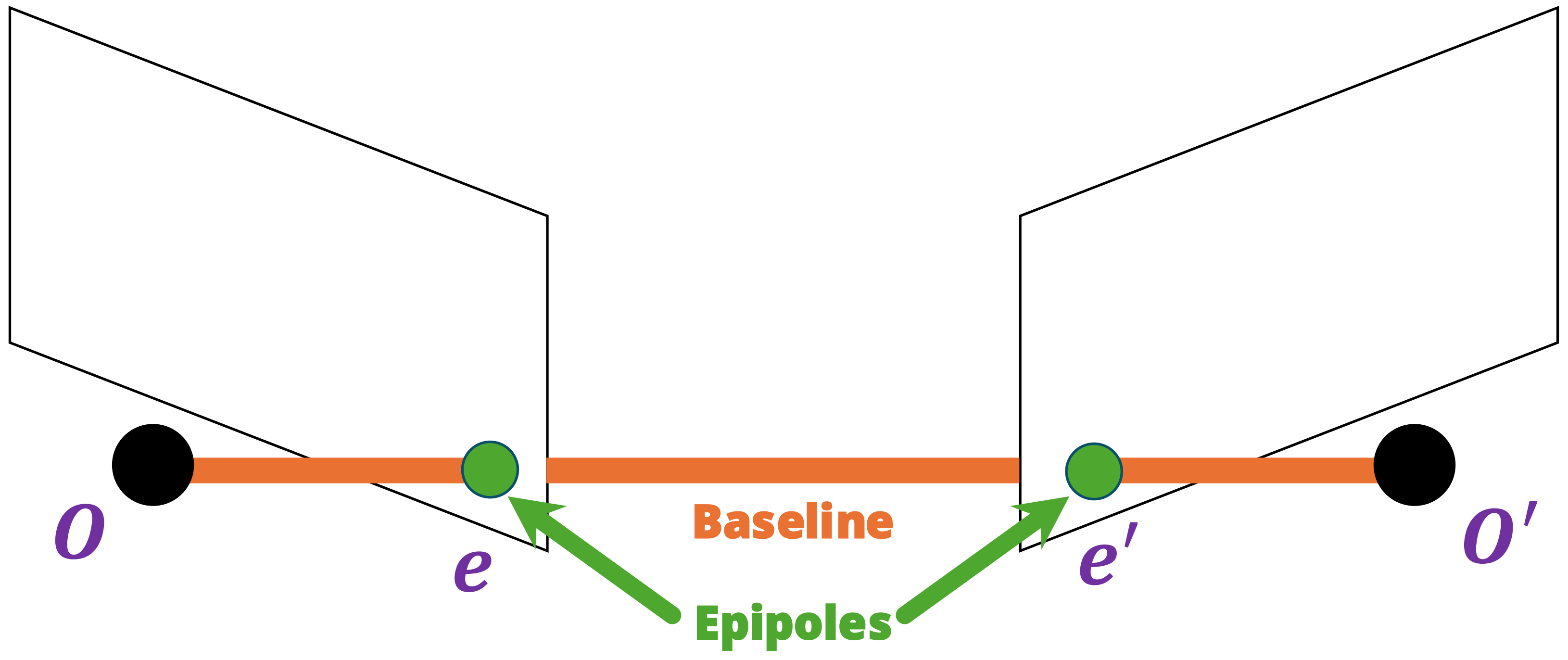
- Projecting points on a ray to another camera forms a **line**, i.e., the corresponding point on the second image *must* lie on this line
- Hence, if the *matched* 2D point is found along this line, we can determine its 3D location

Epipolar Geometry



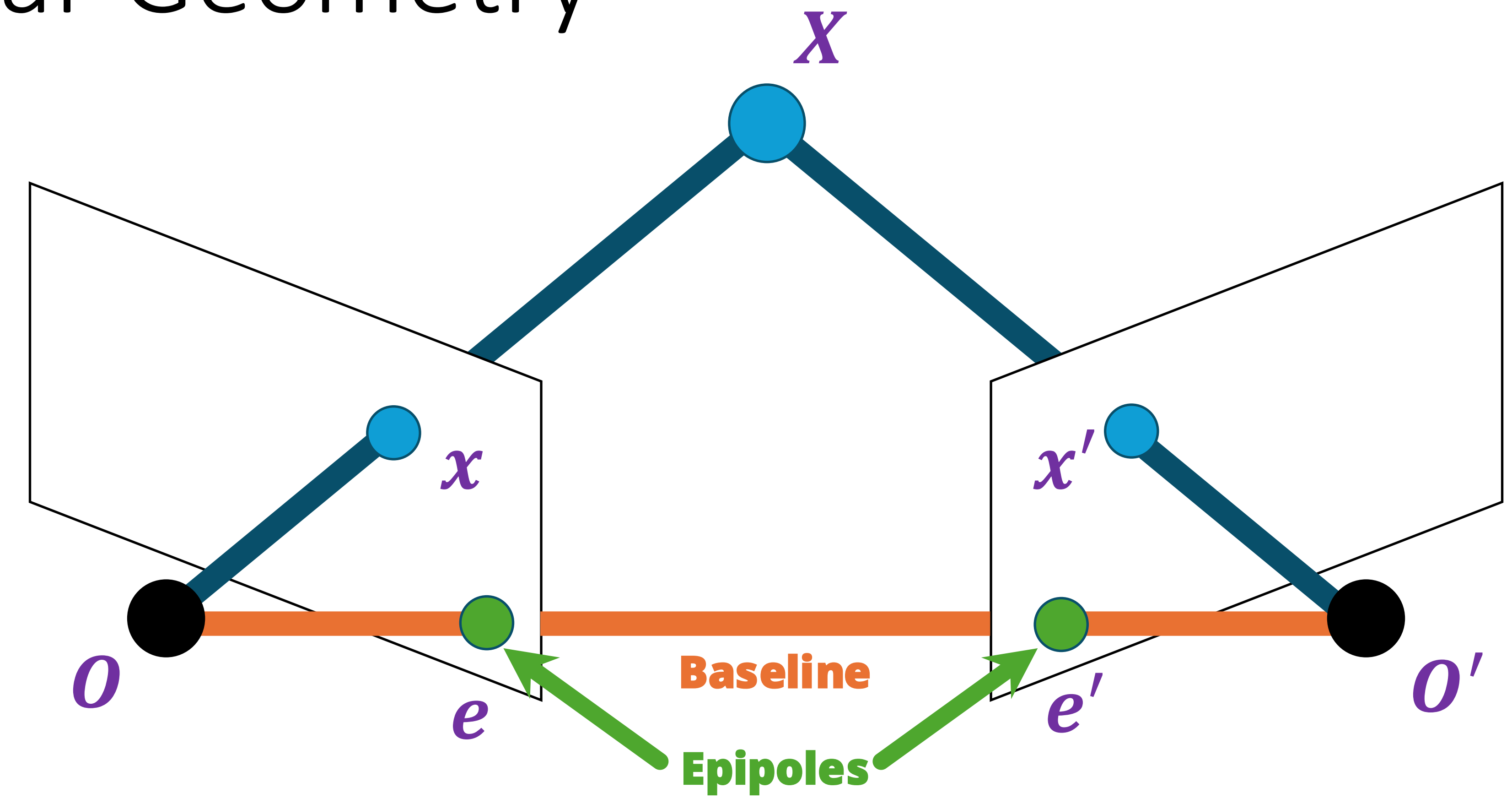
- Suppose we have two cameras with centers O, O'
- The **baseline** is the line connecting the origins

Epipolar Geometry



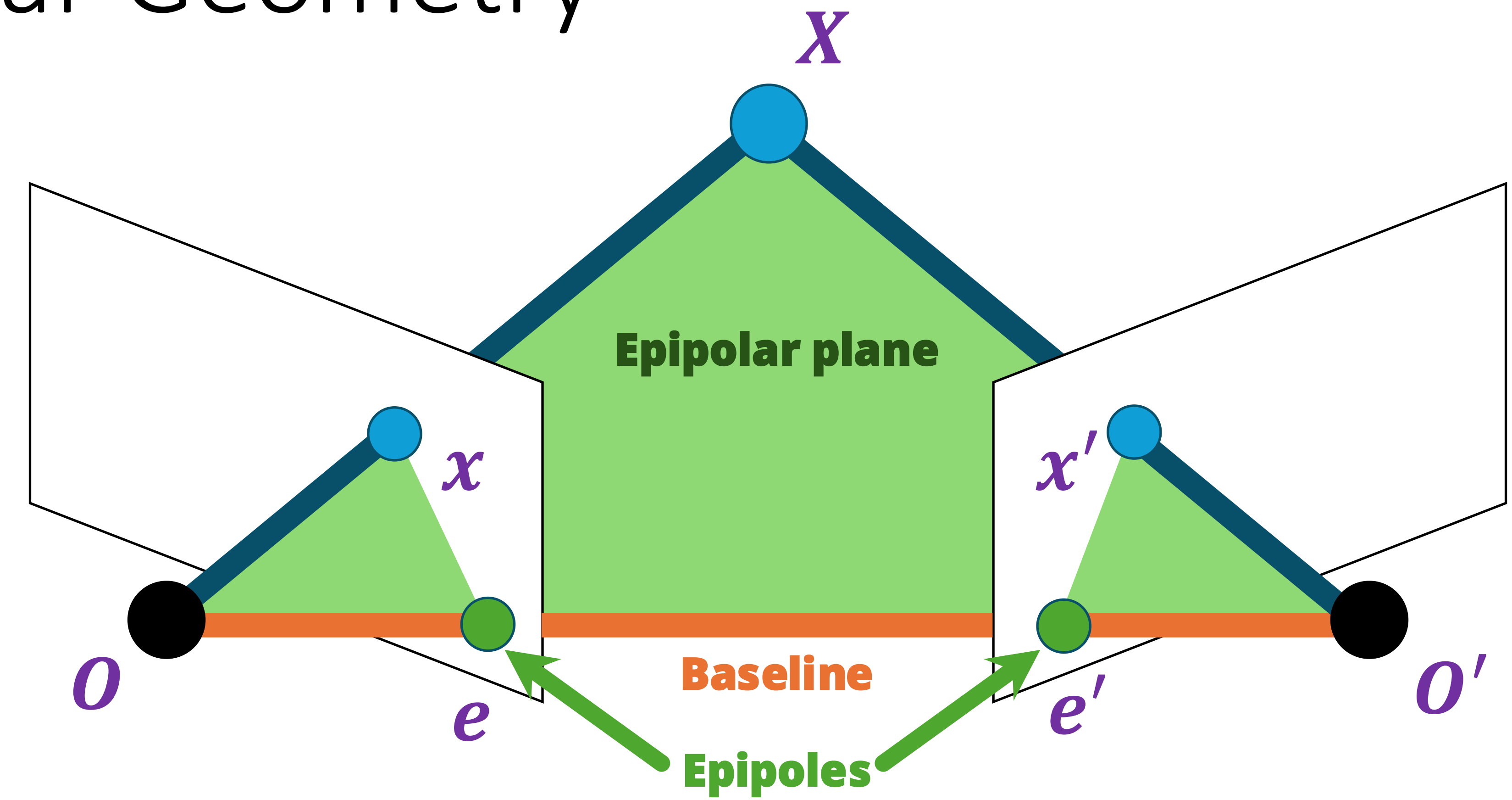
- **Epipoles** e, e' are where the baseline intersects the image planes, or projections of the other camera in each view

Epipolar Geometry



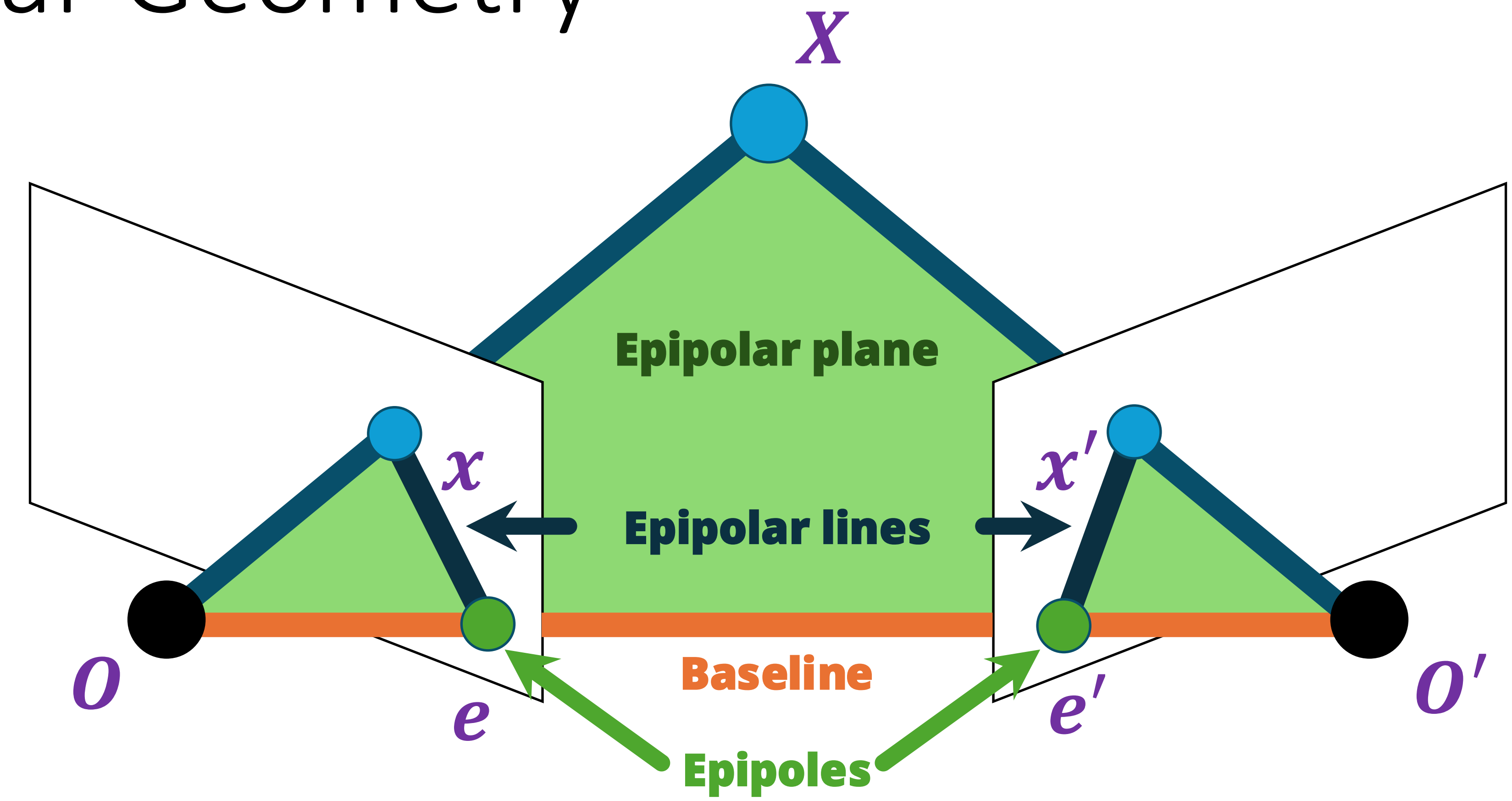
- Consider a **point** X, which projects to x and x'

Epipolar Geometry



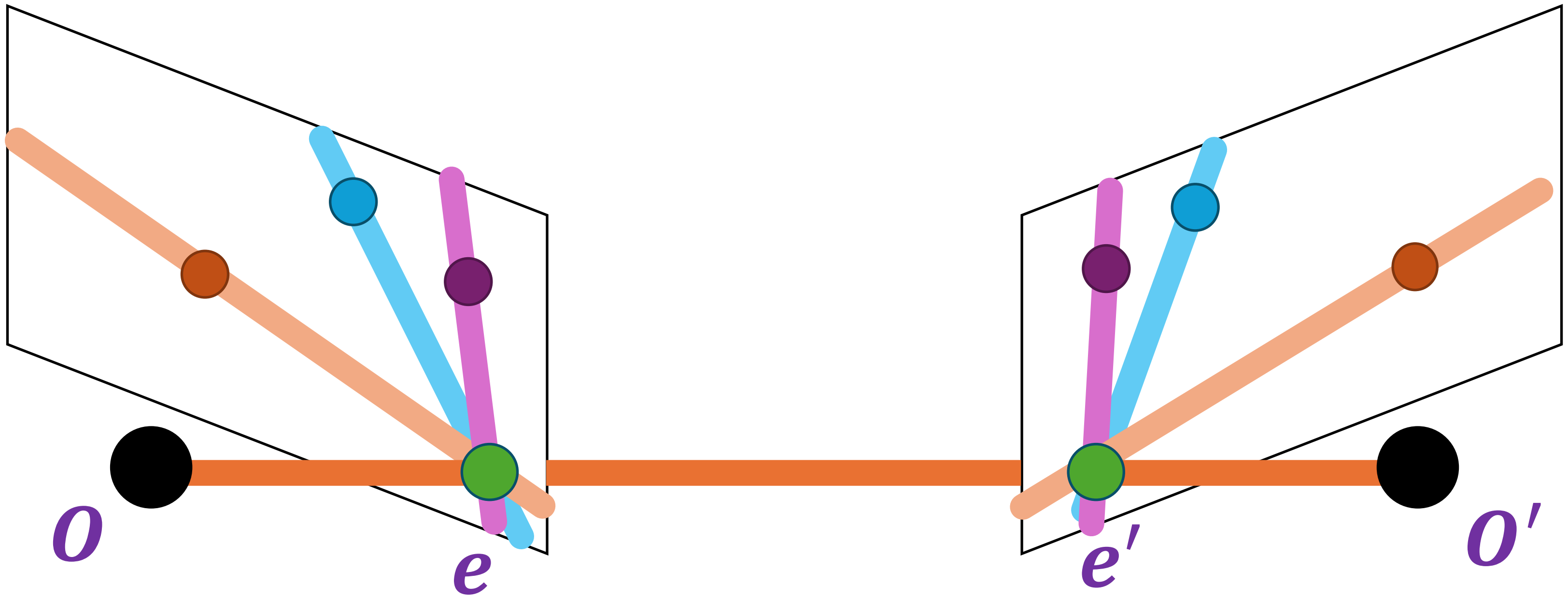
- The plane formed by $\boldsymbol{X}$, $\boldsymbol{O}$, and $\boldsymbol{O'}$ is called an **epipolar plane**
- There is a family of planes passing through $\boldsymbol{O}$ and $\boldsymbol{O'}$

Epipolar Geometry



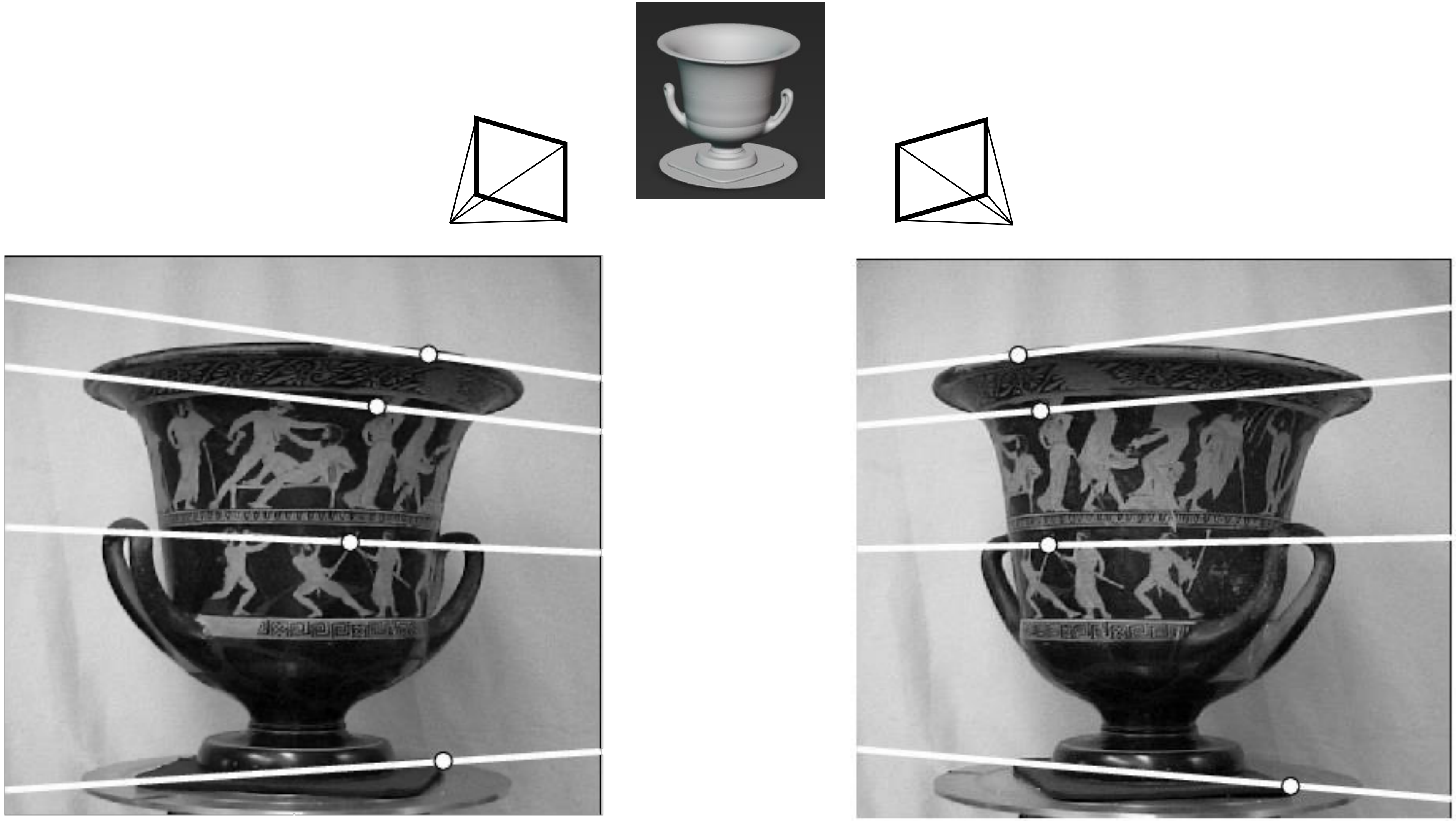
- **Epipolar lines** connect the epipoles to the projections of X
- Equivalently, they are intersections of the epipolar plane with the image planes – thus, they come in matching pairs

Epipolar Geometry

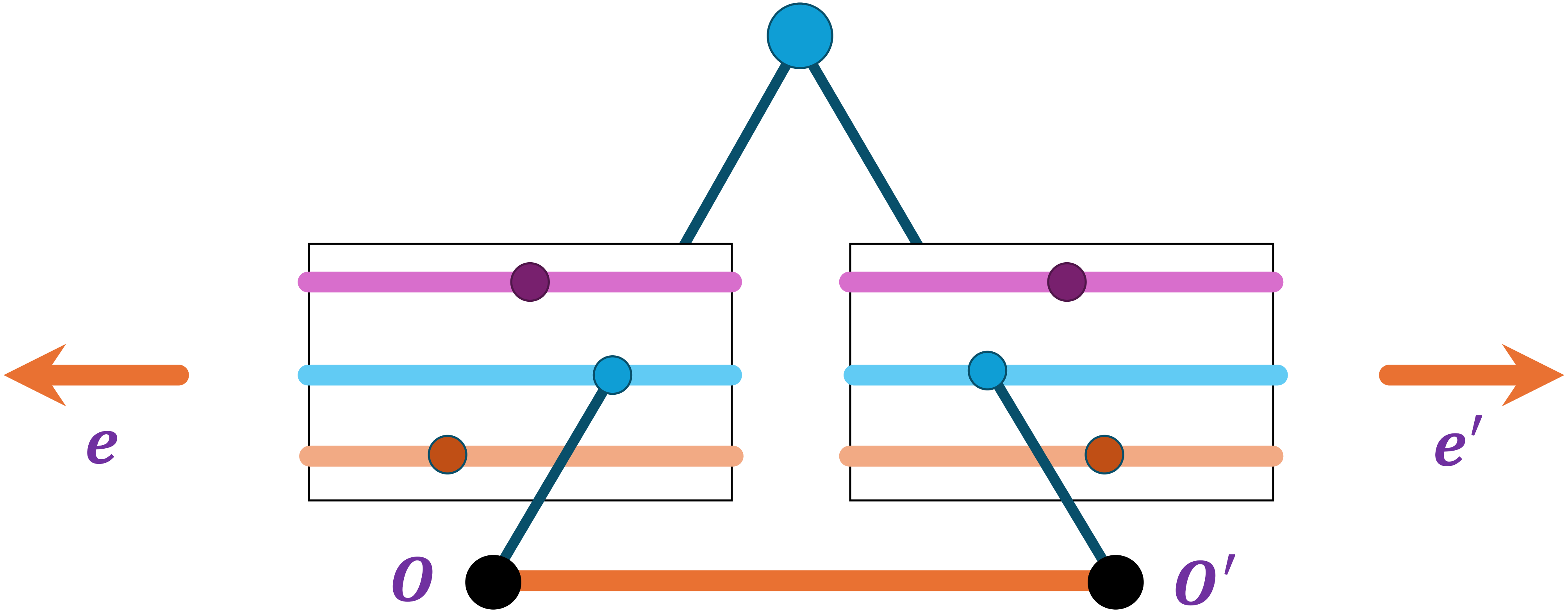


- All **epipolar lines** pass through the **epipoles**
- **Epipoles** can lie outside of the image

Example of Epipolar Lines

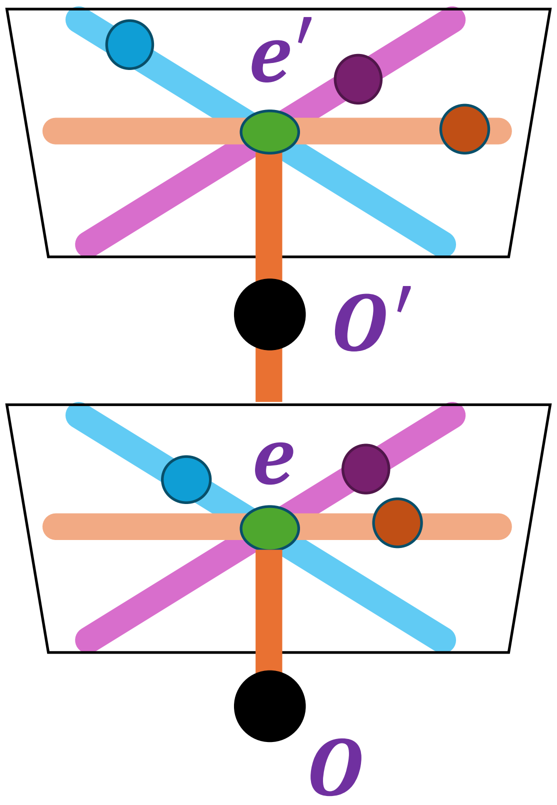


Parallel to Image Plane



- Where are the epipoles and what do the epipolar lines look like?
- Epipoles ***infinitely*** far away, epipolar lines parallel

Perpendicular to Image Plane

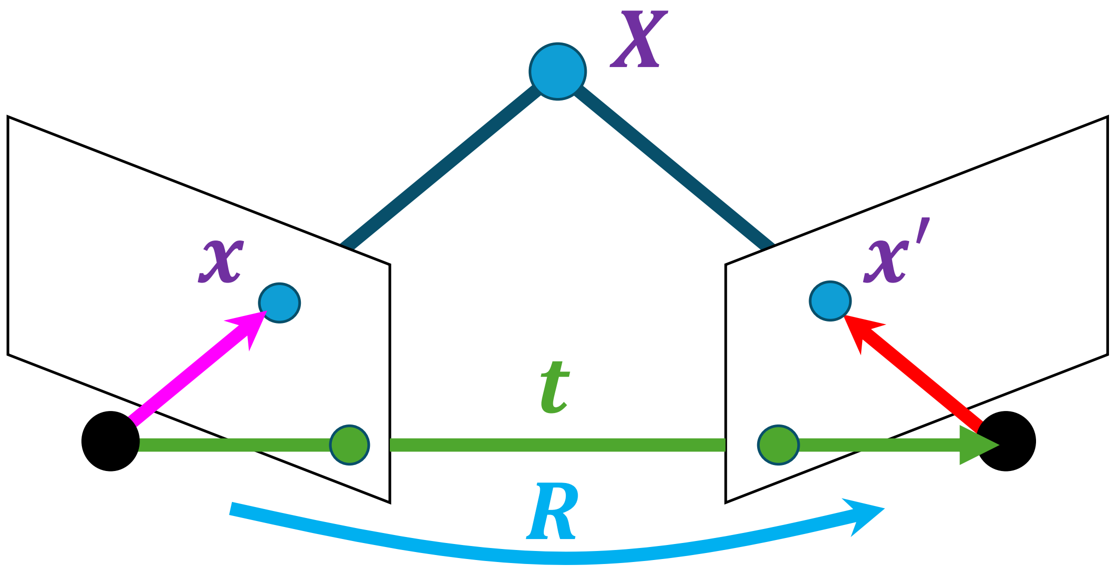


- Where are the epipoles and what do the epipolar lines look like?

- Epipole is “focus of expansion” and coincides with the principal point of the camera
- Epipolar lines go out from principal point

Epipolar Constraint: Calibrated Case

$x \cong K[R|t]X$

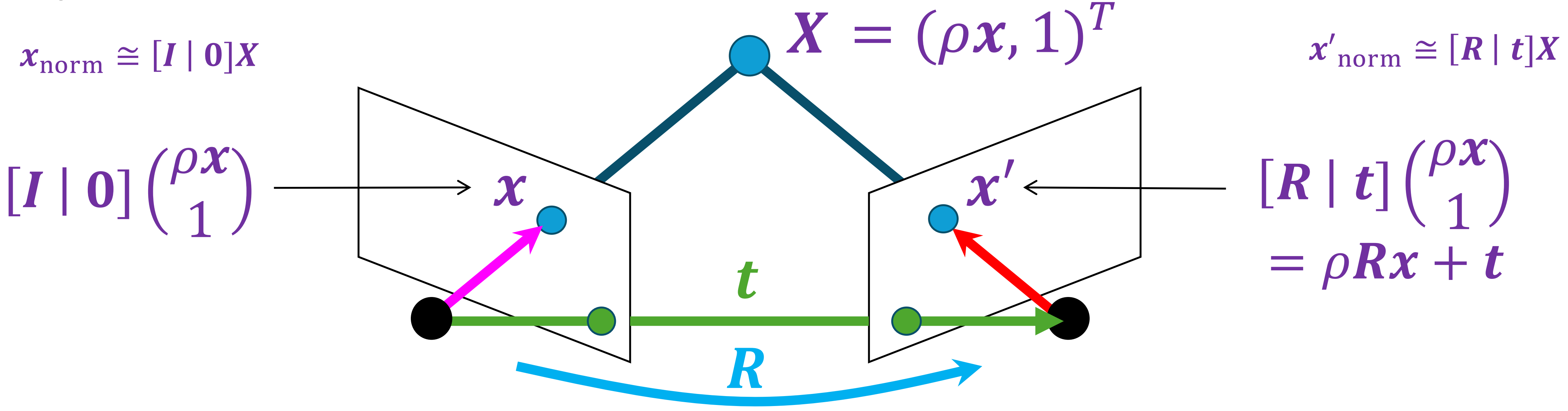


- Assume the intrinsic and extrinsic parameters of the cameras are known (*calibrated*), and world coordinate system is set to that of the first camera
- Then the projection matrices are given by $K[I \mid 0]$ and $K'[R \mid t]$
- We can pre-multiply the projection matrices (and the image points) by the inverse calibration matrices to get *normalized* image coordinates:

$$x_{\text{norm}} = K^{-1} x_{\text{pixel}} \cong [I \mid 0]X, \qquad x'_{\text{norm}} = K'^{-1} x'_{\text{pixel}} \cong [R \mid t]X$$

Epipolar Constraint: Calibrated Case

$\boldsymbol{x} \cong \boldsymbol{K}[\boldsymbol{R}|\boldsymbol{t}]\boldsymbol{X}$



$$\boldsymbol{x}' \cong \rho\boldsymbol{R}\boldsymbol{x} + \boldsymbol{t} \;\Rightarrow\; \boldsymbol{x}' \cdot [\boldsymbol{t} \times (\boldsymbol{R}\boldsymbol{x})] = 0$$

- This means the three vectors $\boldsymbol{x}'$, $\boldsymbol{R}\boldsymbol{x}$, and $\boldsymbol{t}$ are linearly dependent, i.e., lying on the same plane
- This constraint can be written using the *triple product*

Epipolar Constraint: Calibrated Case

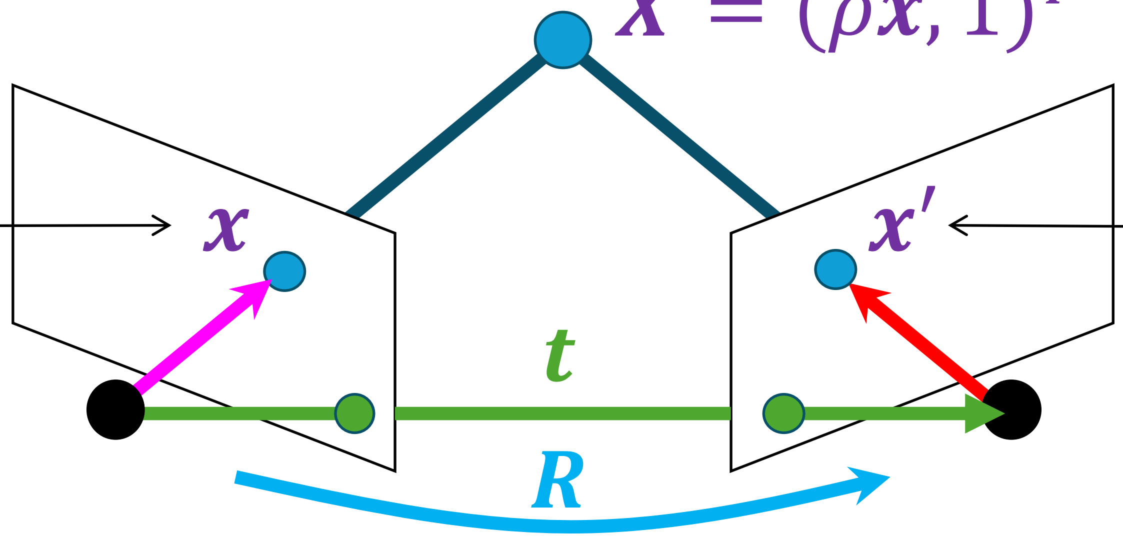
$\boldsymbol{x} \cong \boldsymbol{K}[\boldsymbol{R}|\boldsymbol{t}]\boldsymbol{X}$

$\boldsymbol{X} = (\rho\boldsymbol{x}, 1)^T$

$\boldsymbol{x}_{\text{norm}} \cong [\boldsymbol{I} \mid \boldsymbol{0}]\boldsymbol{X}$

$\boldsymbol{x}'_{\text{norm}} \cong [\boldsymbol{R} \mid \boldsymbol{t}]\boldsymbol{X}$

$[\boldsymbol{I} \mid \boldsymbol{0}]\begin{pmatrix}\rho\boldsymbol{x} \\ 1\end{pmatrix}$



$[\boldsymbol{R} \mid \boldsymbol{t}]\begin{pmatrix}\rho\boldsymbol{x} \\ 1\end{pmatrix} = \rho\boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$

$$\boldsymbol{x}' \cong \rho\boldsymbol{R}\boldsymbol{x} + \boldsymbol{t} \Rightarrow \boldsymbol{x}' \cdot [\boldsymbol{t} \times (\boldsymbol{R}\boldsymbol{x})] = 0 \Rightarrow \boldsymbol{x}'^T[\boldsymbol{t}_\times]\boldsymbol{R}\boldsymbol{x} = 0$$

Recall: $\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = [\boldsymbol{a}_\times]\boldsymbol{b} \Rightarrow$ skew-symmetric matrix: $\boldsymbol{A}^T = -\boldsymbol{A}$

Epipolar Constraint: Calibrated Case

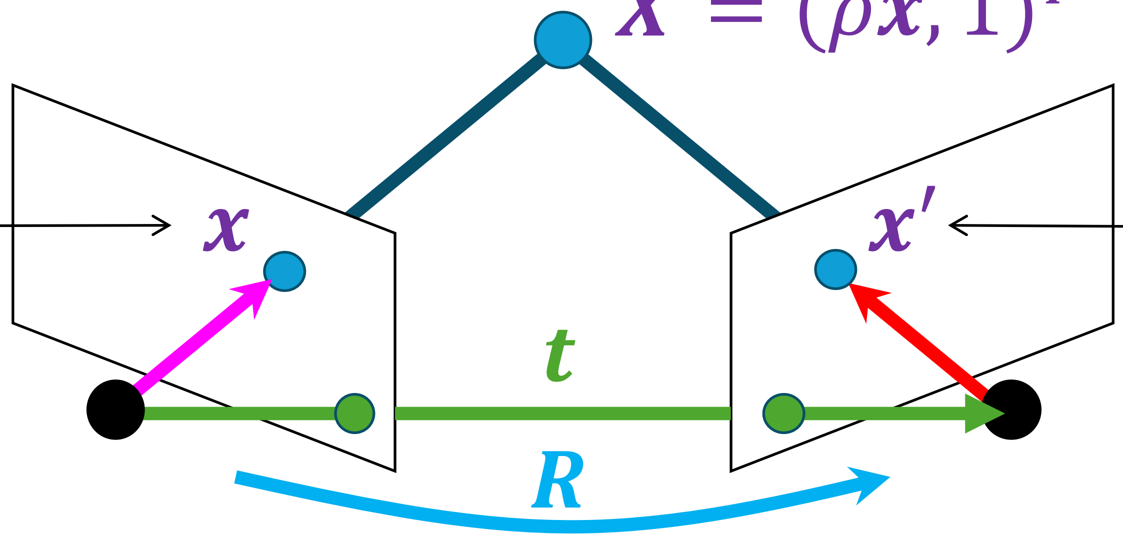
$\boldsymbol{x} \cong \boldsymbol{K}[\boldsymbol{R}|\boldsymbol{t}]\boldsymbol{X}$

$\boldsymbol{x}_{\text{norm}} \cong [\boldsymbol{I} \mid \mathbf{0}]\boldsymbol{X}$

$\boldsymbol{X} = (\rho \boldsymbol{x}, 1)^T$

$\boldsymbol{x}'_{\text{norm}} \cong [\boldsymbol{R} \mid \boldsymbol{t}]\boldsymbol{X}$

$[\boldsymbol{I} \mid \mathbf{0}]\begin{pmatrix}\rho \boldsymbol{x} \\ 1\end{pmatrix}$

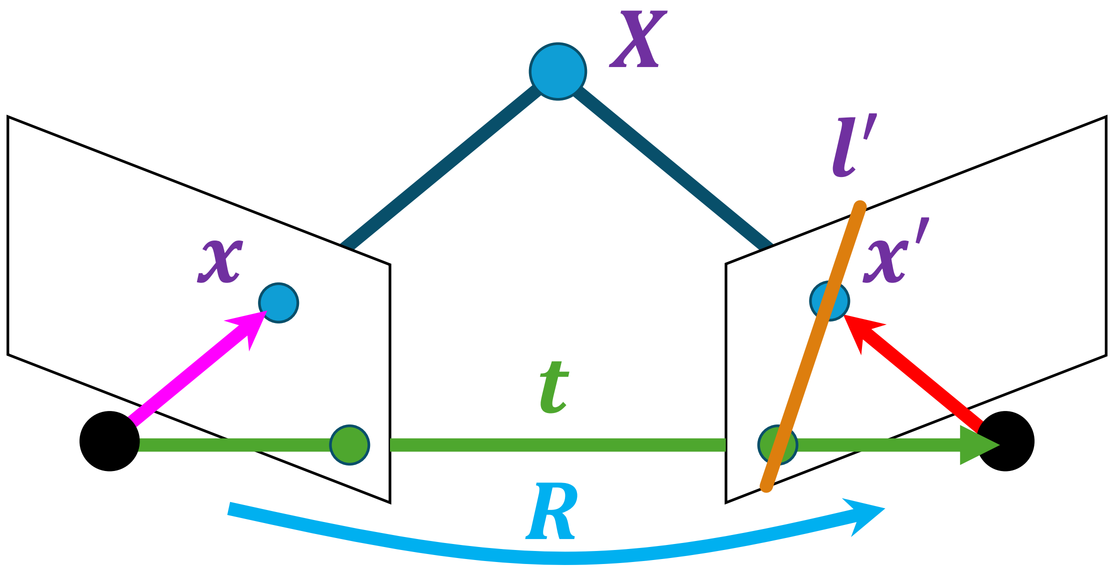


$[\boldsymbol{R} \mid \boldsymbol{t}]\begin{pmatrix}\rho \boldsymbol{x} \\ 1\end{pmatrix} = \rho \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$

$$\boldsymbol{x}' \cong \rho \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t} \;\Rightarrow\; \boldsymbol{x}' \cdot [\boldsymbol{t} \times (\boldsymbol{R}\boldsymbol{x})] = 0 \;\Rightarrow\; \boldsymbol{x}'^T[\boldsymbol{t}_\times]\boldsymbol{R}\boldsymbol{x} = 0 \;\Rightarrow\; \boldsymbol{x}'^T \boldsymbol{E}\boldsymbol{x} = 0$$

Essential Matrix

The Essential Matrix

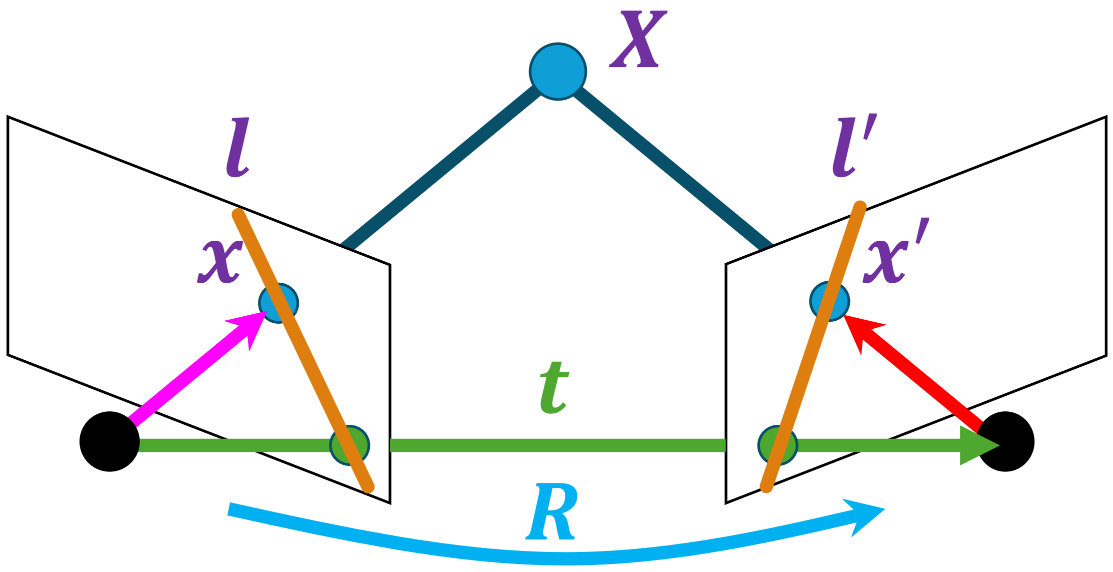


$$\boldsymbol{x}'^T \overbrace{\boxed{\boldsymbol{E}\boldsymbol{x}}}^{\boldsymbol{l}' = \boldsymbol{E}^T\boldsymbol{x}} = 0$$

$$(x', y', 1)\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

- $\boldsymbol{Ex}$ is the **epipolar line** associated with $\boldsymbol{x}$ ($\boldsymbol{l}' = \boldsymbol{Ex}$)
- Recall: a line is given by $ax + by + c = 0$ or $\boldsymbol{l}^T\boldsymbol{x} = 0$ in homogeneous coordinates, where $\boldsymbol{l} = (a, b, c)^T$ and $\boldsymbol{x} = (x, y, 1)^T$
- $\boldsymbol{x}'^T\boldsymbol{Ex} = \boldsymbol{x}'^T\boldsymbol{l}' = 0$ means $\boldsymbol{x}'$ lies on the epipolar line $\boldsymbol{l}'$

The Essential Matrix

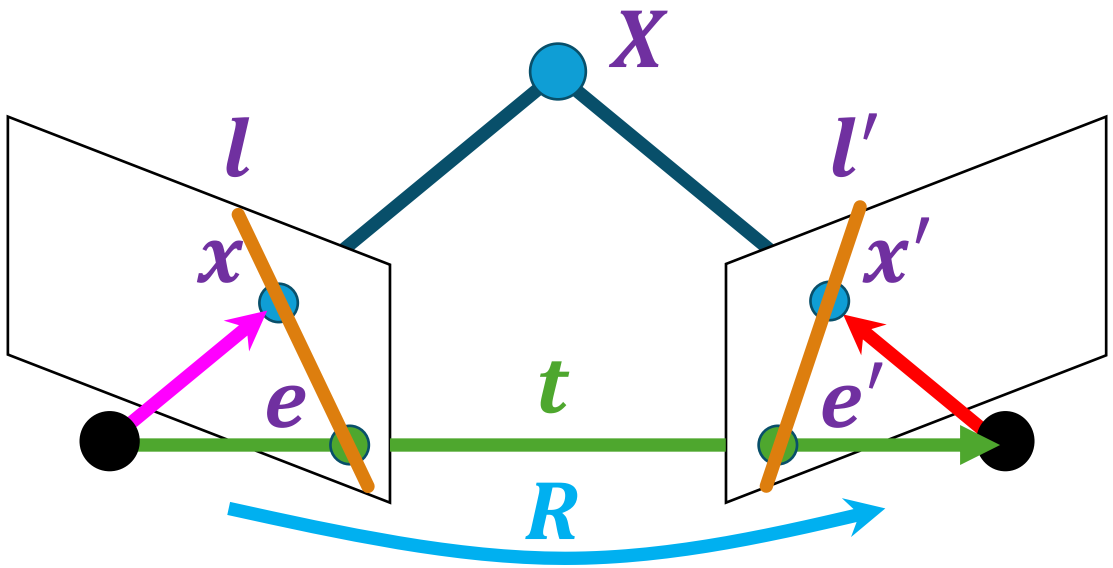


$$\overbrace{\boxed{\boldsymbol{x}'^T \boldsymbol{E}}}^{\boldsymbol{l} = \boldsymbol{E}^T \boldsymbol{x}'} \boldsymbol{x} = 0$$

$$(x', y', 1) \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

- $\boldsymbol{E}\boldsymbol{x}$ is the **epipolar line** associated with $\boldsymbol{x}$ ($\boldsymbol{l}' = \boldsymbol{E}\boldsymbol{x}$)
- $\boldsymbol{x}'^T \boldsymbol{E}\boldsymbol{x} = \boldsymbol{x}'^T \boldsymbol{l}' = 0$ means $\boldsymbol{x}'$ lies on the epipolar line $\boldsymbol{l}'$
- Equivalently, $\boldsymbol{E}^T \boldsymbol{x}'$ is the **epipolar line** associated with $\boldsymbol{x}'$ ($\boldsymbol{l} = \boldsymbol{E}^T \boldsymbol{x}'$)
- $\boldsymbol{x}'^T \boldsymbol{E}\boldsymbol{x} = \boldsymbol{l}\boldsymbol{x} = 0$ means $\boldsymbol{x}$ lies on the epipolar line $\boldsymbol{l}$

The Essential Matrix



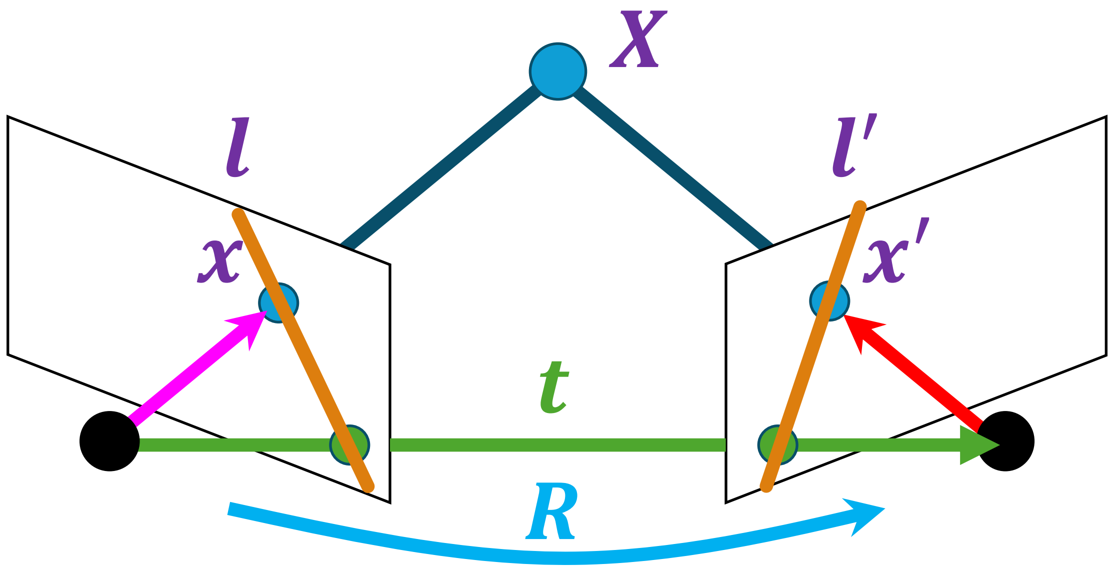
$$x'^T E x = 0$$

$$(x', y', 1)\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

- $\boldsymbol{Ex}$ is the **epipolar line** associated with $\boldsymbol{x}$ ($\boldsymbol{l'} = \boldsymbol{Ex}$)
- $\boldsymbol{E}^T\boldsymbol{x'}$ is the **epipolar line** associated with $\boldsymbol{x'}$ ($\boldsymbol{l} = \boldsymbol{E}^T\boldsymbol{x'}$)
- $\boldsymbol{Ee} = \boldsymbol{0}$ and $\boldsymbol{E}^T\boldsymbol{e'} = \boldsymbol{0}$, where $\boldsymbol{e}$, $\boldsymbol{e'}$ are the epipoles
- $\boldsymbol{E}$ is singular (rank two) and has five degrees of freedom => why?
  - $\boldsymbol{E} = [\boldsymbol{t}_\times]\boldsymbol{R}$! $[\boldsymbol{t}_\times]$ is skew-symmetric; has rank 2
  - $\boldsymbol{R}$: 3 DoF, $\boldsymbol{t}$: 3 DoF, but we lost 1 DoF due to scale; move along $\boldsymbol{t}$ doesn't change $\boldsymbol{l}$

Epipolar Constraint: Uncalibrated

$\boldsymbol{x} \cong \boldsymbol{K}[\boldsymbol{R}|\boldsymbol{t}]\boldsymbol{X}$



$$\boldsymbol{x}'^T \boldsymbol{E} \boldsymbol{x} = 0$$

$$(x', y', 1) \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

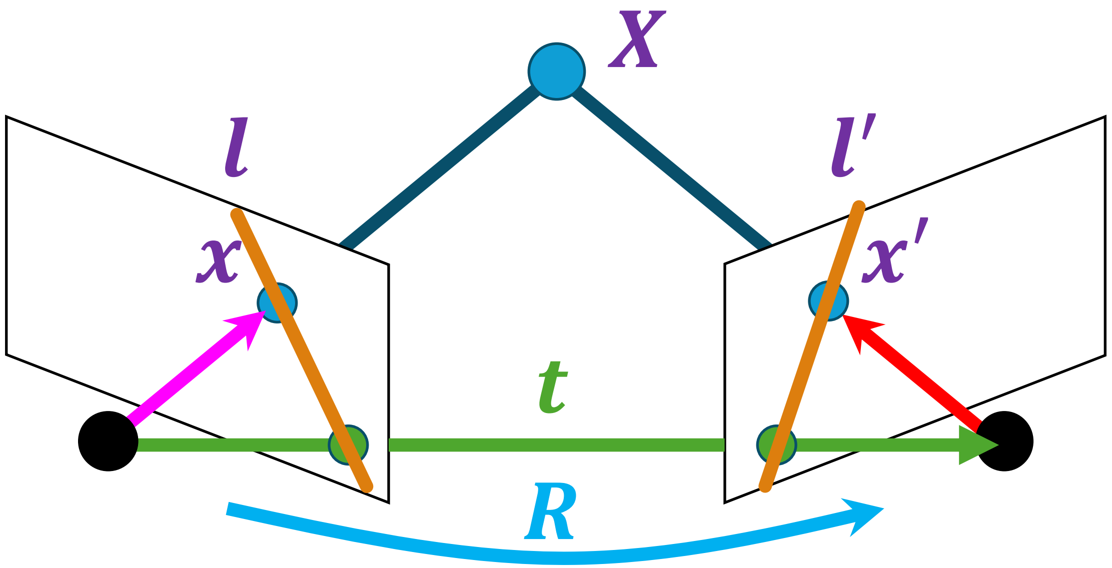
- What if camera intrinsics $\boldsymbol{K}$, $\boldsymbol{K}'$ are unknown?
- We can write the epipolar constraint:

$$\boldsymbol{x}'^T_{\text{norm}} \boldsymbol{E} \boldsymbol{x}_{\text{norm}} = 0$$

where $\boldsymbol{x}_{\text{norm}} = \boldsymbol{K}^{-1}\boldsymbol{x}$, $\boldsymbol{x}'_{\text{norm}} = \boldsymbol{K}'^{-1}\boldsymbol{x}'$

Epipolar Constraint: Uncalibrated

$\boldsymbol{x} \cong \boldsymbol{K}[\boldsymbol{R}|\boldsymbol{t}]\boldsymbol{X}$



$$\boldsymbol{x}'^{T}\boldsymbol{E}\boldsymbol{x} = 0$$

$$(x', y', 1)\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

- What if camera intrinsics $\boldsymbol{K}$, $\boldsymbol{K}'$ are unknown?
- We can write the epipolar constraint:

$$\boldsymbol{x}'^{T}_{\text{norm}}\boldsymbol{E}\boldsymbol{x}_{\text{norm}} = \boldsymbol{x}'^{T}\boxed{\boldsymbol{K}'^{-T}\boldsymbol{E}\boldsymbol{K}^{-1}}\boldsymbol{x} = 0$$

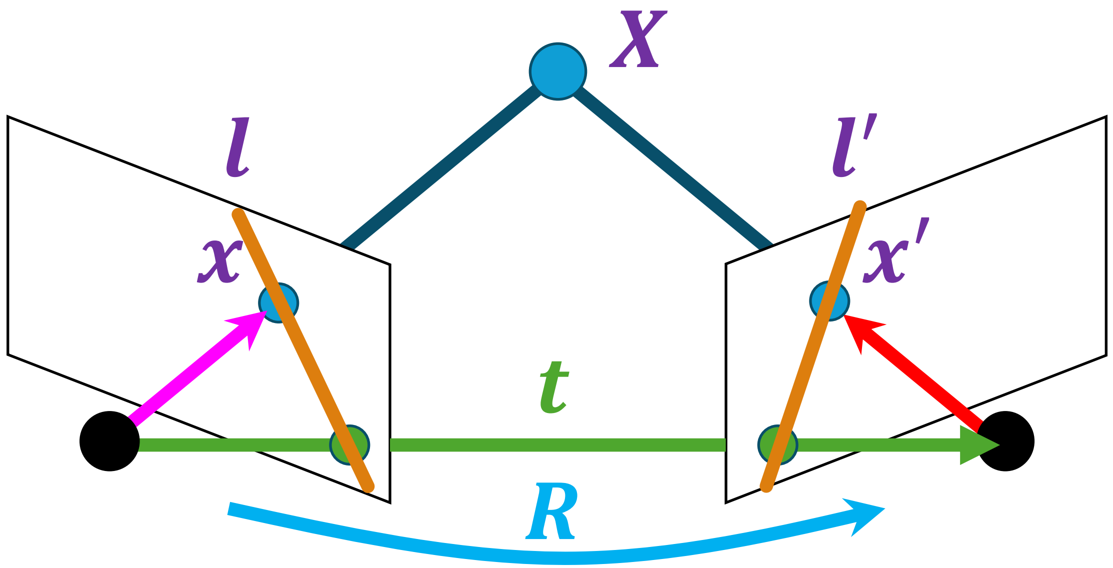
where $\boldsymbol{x}_{\text{norm}} = \boldsymbol{K}^{-1}\boldsymbol{x}$, $\boldsymbol{x}'_{\text{norm}} = \boldsymbol{K}'^{-1}\boldsymbol{x}'$

Fundamental Matrix

$$\boldsymbol{F} = \boldsymbol{K}'^{-T}\boldsymbol{E}\boldsymbol{K}^{-1}$$

The Fundamental Matrix

$$\boldsymbol{x} \cong \boldsymbol{K}[\boldsymbol{R}|\boldsymbol{t}]\boldsymbol{X}$$



$$\boldsymbol{x}'^T \boldsymbol{F} \boldsymbol{x} = 0$$

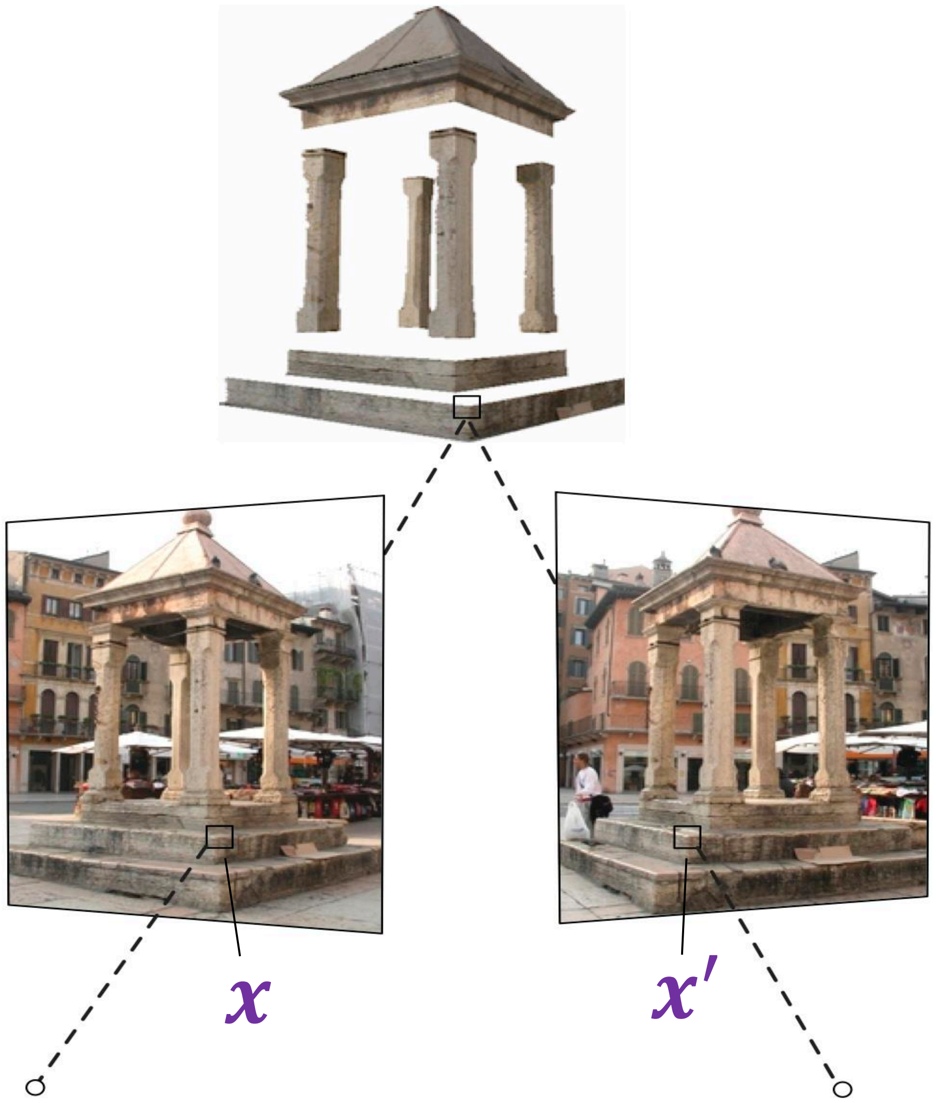
$$(x', y', 1)\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

- $\boldsymbol{F}\boldsymbol{x}$ is the **epipolar line** associated with $\boldsymbol{x}$ ($\boldsymbol{l}' = \boldsymbol{F}\boldsymbol{x}$)
- $\boldsymbol{F}^T\boldsymbol{x}'$ is the **epipolar line** associated with $\boldsymbol{x}'$ ($\boldsymbol{l} = \boldsymbol{F}^T\boldsymbol{x}'$)
- $\boldsymbol{F}\boldsymbol{e} = \boldsymbol{0}$ and $\boldsymbol{F}^T\boldsymbol{e}' = \boldsymbol{0}$, where $\boldsymbol{e}$, $\boldsymbol{e}'$ are the epipoles
- $\boldsymbol{F}$ is singular (rank two) and has seven degrees of freedom => why?
  - $\boldsymbol{F} = \boldsymbol{K}'^{-T}[\boldsymbol{t}_\times]\boldsymbol{R}\boldsymbol{K}^{-1}$! $[\boldsymbol{t}_\times]$ is skew-symmetric; has rank 2
  - 9 elements, but we lost 1 DoF to scale, and 1 DoF to rank constraint

How Can We Use the Epipolar Constraint?

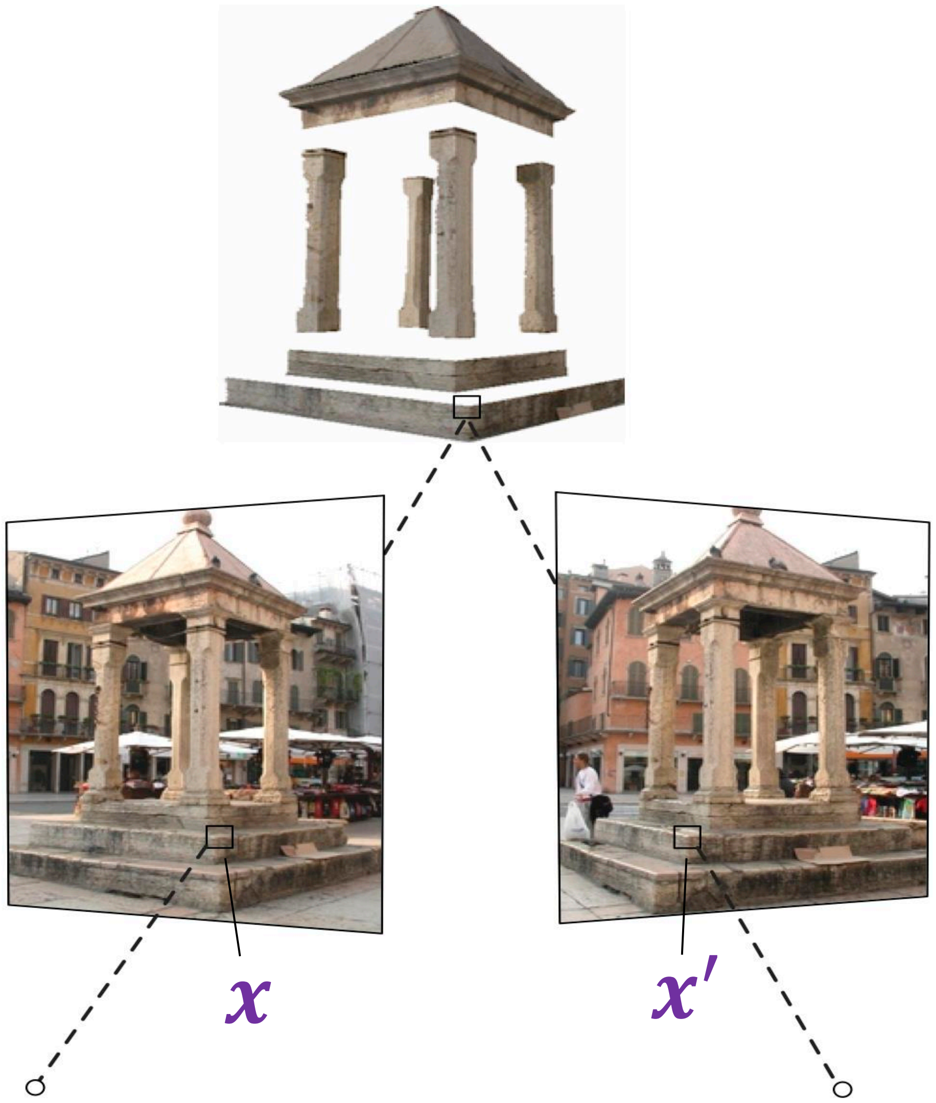
- **Given:** $\boldsymbol{F}$, $\boldsymbol{x}$, $\boldsymbol{x}'$
- **Q:** does there exist a 3D point that projects to $\boldsymbol{x}$ and $\boldsymbol{x}'$?
- **A:** Yes, if $\text{residual}(\boldsymbol{x}'^T \boldsymbol{F} \boldsymbol{x})$ is sufficiently low

- Note: the interpretation of $\text{residual}(\boldsymbol{x}'^T \boldsymbol{F} \boldsymbol{x})$ is the distance (geometric or algebraic) between $\boldsymbol{x}$ and $\boldsymbol{l} = \boldsymbol{F}^T \boldsymbol{x}'$, or $\boldsymbol{x}'$ and $\boldsymbol{l}' = \boldsymbol{F}\boldsymbol{x}$



How Can We Use the Epipolar Constraint?

- **Given:** $\boldsymbol{F}$
- **Q:** how do we find $\boldsymbol{R}$, $\boldsymbol{t}$?
- **A:**
  - **Step 0:** estimate $\boldsymbol{K}$, $\boldsymbol{K}'$ if not known (self-calibration)
  - **Step 1:** compute $\boldsymbol{E} = \boldsymbol{K}'^{T}\boldsymbol{F}\boldsymbol{K}$
  - **Step 2:** since $\boldsymbol{E} = [\boldsymbol{t}_\times]\boldsymbol{R}$, perform SVD on $\boldsymbol{E} = \boldsymbol{U\Sigma V}^T$. We have 4 solutions $(\boldsymbol{R}_1, \pm\boldsymbol{t})$, $(\boldsymbol{R}_2, \pm\boldsymbol{t})$, where $\boldsymbol{t} = \boldsymbol{U}[:,3]$, $\boldsymbol{R}_1 = \boldsymbol{UWV}^T$, $\boldsymbol{R}_2 = \boldsymbol{UW}^T\boldsymbol{V}^T$, and $\boldsymbol{W}$ is the matrix that rotates 90° about z-axis.
  - **Step 3:** pick the one that gives 3D points in front of both cameras (cheirality check).



How to Estimate the Fundamental Matrix?

- **Given:** correspondences $\boldsymbol{x_i} = (x_i, y_i, 1)^T$ and $\boldsymbol{x_i'} = (x_i', y_i', 1)^T$
- **Constraint:** $\boldsymbol{x}'^T \boldsymbol{F} \boldsymbol{x} = 0$

$$(x', y', 1) \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \Rightarrow (x'x, x'y, x', y'x, y'y, y', x, y, 1) \begin{pmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{pmatrix} = 0$$

How to Estimate the Fundamental Matrix?

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$$\underbrace{\begin{bmatrix} x'_1 x_1 & x'_1 y_1 & x'_1 & y'_1 x_1 & y'_1 y_1 & y'_1 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_n x_n & x'_n y_n & x'_n & y'_n x_n & y'_n y_n & y'_n & x_n & y_n & 1 \end{bmatrix}}_{\boldsymbol{A}} \begin{pmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{pmatrix} = \boldsymbol{0}$$

Homogeneous least squares to find $\boldsymbol{f}$:

$$\arg\min_{\|\boldsymbol{f}\|=1} \|\boldsymbol{A}\boldsymbol{f}\|_2^2 \quad \Rightarrow \quad \text{Least eigenvector of } \boldsymbol{A}^T\boldsymbol{A}$$

This is known as the "*eight-point algorithm*" introduced by Christopher Longuet-Higgins (1981).

Small Trick to Enforce Rank-2 Constraint

- We know $\boldsymbol{F}$ must be singular/rank 2. How do we force that?
- **Solution:** take SVD of the initial estimate and throw out the smallest singular value

$$\boldsymbol{F}_{\text{init}} = \boldsymbol{U\Sigma V}^T$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \cancel{\sigma_3} \end{bmatrix} \longrightarrow \boldsymbol{\Sigma}' = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{F} = \boldsymbol{U\Sigma' V}^T$$

The Fundamental Matrix Song

By Daniel Wedge (2008): https://danielwedge.com/fmatrix/

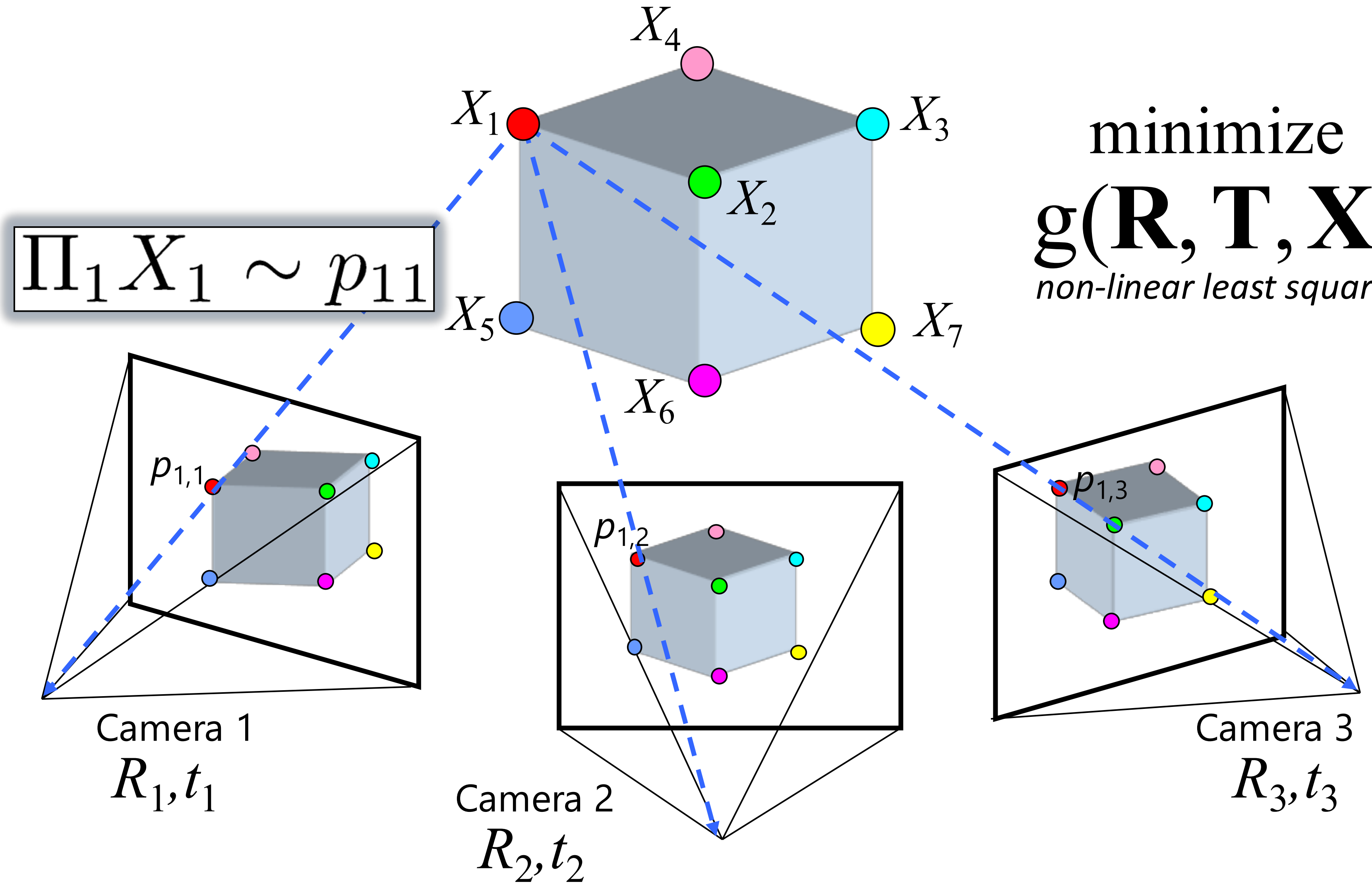
The Fundamental Matrix Song – Live!

Daniel Wedge and the CVPR house band at CVPR 2023 in Vancouver

More Than Two Views?

- The geometry of three views is described by a 3 x 3 x 3 tensor called the *trifocal tensor*
- The geometry of four views is described by a 3 x 3 x 3 x 3 tensor called the *quadrifocal tensor*
- After this it starts to get complicated…
- Or we can pose it as an optimization problem

Structure from Motion (SfM)



minimize

$$\mathrm{g}(\mathbf{R}, \mathbf{T}, \mathbf{X})$$

non-linear least squares

Bundle Adjustment

- Minimize reprojection errors:

$$g(\mathbf{X},\mathbf{R},\mathbf{T}) = \sum_{i=1}^{m}\sum_{j=1}^{n} \underbrace{w_{ij}}_{\substack{\textit{indicator variable}\text{:} \\ \text{is point } i \text{ visible in image } j\text{ ?}}} \cdot \left\| \underbrace{\mathbf{P}(\mathbf{x}_i,\mathbf{R}_j,\mathbf{t}_j)}_{\substack{\textit{predicted} \\ \text{image location}}} - \underbrace{\begin{bmatrix} u_{i,j} \\ v_{i,j} \end{bmatrix}}_{\substack{\textit{observed} \\ \text{image location}}} \right\|^2$$

- Optimized using non-linear least squares, e.g., Levenberg-Marquardt algorithm
- Susceptible to local minima; requires careful initialization

Incremental Structure from Motion

- Photo Tourism (Snavely et al., SIGGAPH'06)
- Trevis Fountain, Rome
  - 466 Internet photos
  - > 100,000 3D points
  - Very large optimization problem

Practical SfM Tools

- COLMAP (by Schönberger et al.): https://colmap.github.io/
- nerfstudio COLMAP Python wrapper: https://docs.nerf.studio/quickstart/custom_dataset.html

- Still, with thousands of images, it can take many hours or even days!
- There's a family of methods optimized for efficiency and continuous streams, often referred to as *Simultaneous Localization and Mapping* (**SLAM**), particularly useful for robot navigation for instance
  - One classic, widely-used system is *ORB-SLAM* (Mur-Artal et al., 2015)

- We will discuss learning-based approaches later

Part 1&2 Summary – Multi-view Geometry

- Camera model and projection
  - Homogeneous coordinates
  - Pinhole camera, perspective projection $\boldsymbol{x} \cong \boldsymbol{K}[\boldsymbol{R}|\boldsymbol{t}]\boldsymbol{X}$
  - Intrinsics $\boldsymbol{K}$, extrinsics $[\boldsymbol{R}|\boldsymbol{t}]$
  - Camera calibration
- Epipolar geometry
  - Epipolar plane, epipolar lines
  - Essential matrix $\boldsymbol{E}$ => $\boldsymbol{x}_{\text{norm}}'^{T}\boldsymbol{E}\boldsymbol{x}_{\text{norm}} = 0$ (calibrated)
  - Fundamental matrix $\boldsymbol{F}$ => $\boldsymbol{x}'^{T}\boldsymbol{E}\boldsymbol{x} = 0$ (uncalibrated)
- Structure from Motion (SfM)
  - Bundle adjustment