Phenomenology of Particle Physics

Solutions to the Exercises

Chapters 1-17

Chapters 18-31 in preparation

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Contents

Codes and other resources associated with the textbook are available at
https://github.com/CambridgeUniversityPress/Phenomenology-Particle-Physics

Errata to the printed book (May 2022) is regularly updated at
https://github.com/CambridgeUniversityPress/Phenomenology-Particle-Physics/blob/master/Errata.pdf

1 Introduction and notation

1.1 Discrete transformation

Derive Table 1.3.

Solution:

Discrete transformation and the transformation properties of given physical observables have been introduced in Sections 1.4 and 1.5 of the book. Below we derive in details the transformation properties of the important observables listed in Table 1.3 of the book.

(a) Position:

$$P(\vec{x}) = -\vec{x}\,; \qquad T(\vec{x}) = +\vec{x}\,; \qquad C(\vec{x}) = +\vec{x} \tag{1.1}$$

(b) Velocity:

$$P(\vec{v}) = P(\mathrm{d}\vec{x}/\mathrm{dt}) = -\mathrm{d}\vec{x}/\mathrm{dt} = -\vec{v}\,; \qquad T(\vec{v}) = T(\mathrm{d}\vec{x}/\mathrm{dt}) = d\vec{x}/(-\mathrm{dt}) = -\vec{v}\,; \qquad C(\vec{v}) = +\vec{v} \tag{1.2}$$

(c) Linear momentum: $\vec{p} = m\vec{v}$ same behavior as $\vec{v}$.

$$P(\vec{p}) = -\vec{p}; \qquad T(\vec{p}) = -\vec{p}; \qquad C(\vec{p}) = +\vec{p} \tag{1.3}$$

(d) Angular momentum:

$$P(\vec{L}) = P(\vec{x}\times\vec{p}) = (-\vec{x})\times(-\vec{p}) = +\vec{L}\,; \qquad T(\vec{L}) = T(\vec{x}\times\vec{p}) = \vec{x}\times(-\vec{p}) = -\vec{L}\,; \qquad C(\vec{L}) = +\vec{L} \tag{1.4}$$

(e) Spin: same behavior as angular momentum.

$$P(\vec{\sigma}) = +\vec{\sigma}\,; \qquad T(\vec{\sigma}) = -\vec{\sigma}\,; \qquad C(\vec{\sigma}) = +\vec{\sigma} \tag{1.5}$$

(f) Helicity: it is defined as $\vec{\sigma}\cdot\vec{p}/|\vec{p}| = \vec{\sigma}\cdot\hat{p}$. Hence:

$$P(h) = P(\vec{\sigma}\cdot\hat{p}) = P(\vec{\sigma})\cdot P(\hat{p}) = \vec{\sigma}\cdot(-\hat{p}) = -h\,; \qquad T(h) = (-\vec{\sigma})\cdot(-\hat{p}) = +h\,; \qquad C(h) = \vec{\sigma}\cdot\hat{p} = +h \tag{1.6}$$

(f) Electric field: As shown in Section 1.5 of the book, we use Gauss's law $\nabla\vec{E}\,(\vec{x},t) = \rho\,(\vec{x},t)$ to deduce the transformation properties of the electric field. Consequently, using $P(\nabla) = -\nabla$,

$$P\left(\nabla\vec{E}\,(\vec{x},t) = \rho\,(\vec{x},t)\right) \qquad \overset{P}{\Longrightarrow} \qquad -\nabla\vec{E}'\,(-\vec{x},t) = \rho'\,(-\vec{x},t) = \rho\,(-\vec{x},t) \tag{1.7}$$

which yields the original equation of electromagnetism if $\vec{E}' = P(\vec{E}) = -\vec{E}$. Under time reversal we have:

$$T\left(\nabla\vec{E}\,(\vec{x},t) = \rho\,(\vec{x},t)\right) \qquad \overset{T}{\Longrightarrow} \qquad \nabla\vec{E}'\,(\vec{x},-t) = \rho'\,(\vec{x},-t) = \rho\,(\vec{x},-t) \tag{1.8}$$

So $T(\vec{E}) = +\vec{E}$. Under charge conjugation, charges change sign and so does the electric field:

$$C(\vec{E}) = -\vec{E} \tag{1.9}$$

(g) Magnetic field: We use Ampère's law:

$$\nabla \times \vec{B}\left(\vec{x}, t\right) = \mu_0 \vec{j}\left(\vec{x}, t\right) + \frac{1}{c^2}\frac{\partial \vec{E}\left(\vec{x}, t\right)}{\partial t} \tag{1.10}$$

Under parity transformation, we obtain using $P(\nabla) = -\nabla$, $P(\vec{j}) = -\vec{j}$ and $P(\vec{E}) = -\vec{E}$:

$$-\nabla \times \vec{B}'\left(-\vec{x}, t\right) = -\mu_0 \vec{j}\left(-\vec{x}, t\right) - \frac{1}{c^2}\frac{\partial \vec{E}\left(-\vec{x}, t\right)}{\partial t} \tag{1.11}$$

hence, we obtain the original Ampère's law with $P(\vec{B}) = \vec{B}$. Under time reversal operator, we note that $T(\vec{j}) = -\vec{j}$, $T(\vec{E}) = \vec{E}$, and $T(\partial t) = -\partial t$, and Ampère's law becomes:

$$\nabla \times \vec{B}'\left(\vec{x}, -t\right) = -\mu_0 \vec{j}\left(\vec{x}, -t\right) - \frac{1}{c^2}\frac{\partial \vec{E}\left(\vec{x}, -t\right)}{\partial t} \tag{1.12}$$

hence, we obtain the original Ampère's law with $T(\vec{B}) = -\vec{B}$. Under charge conjugation, currents change direction, hence

$$C(\vec{B}) = -\vec{B} \tag{1.13}$$

The rest of the table are products of the already deduced transformations.

1.2 Symmetries and conservation laws

Noether's theorem states that for each continuous symmetry, there is a conserved quantity. What is the conserved quantity associated with a Lorentz boost?

Solution:

We have seen that for every continuous symmetry there exists a conservation quantity and a conservation law. Conserved quantities are crucial to describe the behavior of a physical system. They are at the basis of good quantum numbers. In order to answer to the question, we can think of all the variables that we use to describe a particle: (1) its rest mass, (2) its energy, (3) its momentum, (4) its angular momentum, (5) its spin, etc. We proceed by elimination. We know that the conservation of energy is related to the time-translation invariance. Similarly, the conservation of momentum is related to the space-translation invariance. Angular momentum is related to rotations. Spin is an internal degree of freedom. So it looks like the rest mass is a good candidate for being associated to the Lorentz boost transformation. We can convince ourselves by considering first the rotation. Invariance leads to conservation of angular momentum which can be written as (e.g for its z coordinate)

$$xp_y - yp_x = \text{const} \tag{1.14}$$

Lorentz boosts can be seen as rotations in the space-time Minkowski space (see Appendix D of the book). Hence, we can write

$$tp_x - xE = t\gamma mv - x\gamma m = \gamma m(tv - x) = \text{const} \tag{1.15}$$

This shows that the rest mass is the conserved quantity. A more rigorous derivation can be done in the context of field theory.

1.3 Local and global gauge symmetries

Gauge symmetries play a fundamental role in particle physics.

(a) Develop Table 1.4 by expanding on each symmetry and its conserved charge. Discuss the long-range force associated with each local gauge symmetry.

(b) Gravitation is a long-range force. Is there an associated conserved charge? What would be the local gauge symmetry?

Solution:

(a) Long-range forces are those that fall off inversely proportional to the square of the distance between the interacting bodies (Gauss law), while short-range forces fall off faster than that.

1. **Electric charge** e**:** The most stringent experimental tests of the conservation of the electric charge are the non-observation of the following decays (see M. Agostini et al. (Borexino Collaboration), "Test of Electric Charge Conservation with Borexino", *Phys. Rev. Lett.* 115, 231802 `http://doi.org/10.1088/1742-6596/888/1/012193` and Eric B. Norman, John N. Bahcall, and Maurice Goldhaber, "Improved limit on charge conservation derived from ^{71}Ga solar neutrino experiments", *Phys. Rev.* D 53, 4086, `http://doi.org/10.1103/PhysRevD.53.4086`):

$$\tau(e^- \to \nu_e \gamma) > 6.6 \times 10^{28}\,\text{yr} \quad \text{and} \quad \frac{\Gamma\,(n \to p + \nu_e + \bar{\nu}_e)}{\Gamma\,(n \to p + e^- + \bar{\nu}_e)} \leqslant 8 \times 10^{-27} \tag{1.16}$$

As is discussed in the book, the electric charge is the conserved charge associated to the $U(1)$ gauge symmetry. The Noether theorem states that a conserved quantity is to be expected as a result of an invariance. Here we interpret the electric charge as the conserved quantity under the global $U(1)$ invariance. The $U(1)$ **local gauge transformation** acts on the wavefunction of the system in the following way:

$$\Psi \to \Psi e^{-ie\lambda(x)} \tag{1.17}$$

where the parameter λ depends on the space-time variable x^μ. This is equation (10.96) of the book. The requirement of local gauge invariance requires the law of motion to be modified by the so-called covariant derivative, as explained in Section 24.2 of the book. The replacement of the partial derivative by the covariant derivative adds a term in the Lagrangian that exactly describes the long-range electromagnetic force, mediated by the photon. Local $U(1)$ gauge invariance also constrains the photon to be massless (see Section 24.2 of the book) which, as well, is verified experimentally (see D. D. Ryutov, "Using Plasma Physics to Weigh the Photon," Plasma Phys. Control. Fusion **49**, B429 (2007) `http://doi.org/10.1088/0741-3335/49/12B/S40`):

$$m_\gamma < 1 \times 10^{-18}\ \text{eV} \tag{1.18}$$

2. **Lepton number** L**:** The total lepton L is defined as $L = L_e + L_\mu + L_\tau$. By analogy with the electric charge, the conservation of L implies a global $U(1)$ symmetry of the type

$$\Psi \to \Psi e^{-iL\alpha} \tag{1.19}$$

where α is a **global parameter**. This symmetry cannot acceptably be local, since its locality would, as in the case of the electric charge, produce a long-range force associated to L, which is not observed. The most promising way for probing lepton number violation is the search for neutrinoless double-beta decay of atomic nuclei. So far it has not been observed experimentally. The limit of the lifetimes are in the range, depending on the isotope (See Table 32.1 of the book for the references):

$$T^{0\nu}_{1/2} > (0.35 - 18) \times 10^{25}\ \text{yr} \qquad (90\%\text{C.L.}) \tag{1.20}$$

Model of neutrino masses predict this process to occur, although present bound indicate that it is very rare. This is discussed in Chapters 31 and 32 of the book. If neutrinoless double-beta decay is observed, then we will know that the global symmetry L is not exact.

3. **Baryon number** B**:** For the same reasons as for the lepton, this symmetry cannot acceptably be local and is considered as global. Many experiments have searched for B-violating transitions, but no positive signal has been identified so far. Proton decay would be the most relevant violation of B, as it would imply the instability of matter. The current lower bound on the proton lifetime is (see M. Anderson *et al.* [SNO+], "Search for invisible modes of nucleon decay in water with the SNO+ detector," *Phys. Rev. D* **99**, no.3, 032008 (2019) `http://doi.org/10.1103/PhysRevD.99.032008` [arXiv:1812.05552 [hep-ex]]):

$$\tau(p \to \text{invisible}) > 3.6 \times 10^{29} \text{ yr} \qquad (90\%\text{C.L.}) \tag{1.21}$$

which is many orders of magnitude larger than the age of the Universe of 13.8×10^9 years. Stronger limits have been set for particular decay modes, such as (see A. Takenaka *et al.* [Super-Kamiokande], "Search for proton decay via $p \to e^+\pi^0$ and $p \to \mu^+\pi^0$ with an enlarged fiducial volume in Super-Kamiokande I-IV," *Phys. Rev. D* ***102**, no.11, 112011 (2020)* `http://doi.org/10.1103/PhysRevD.102.112011` [arXiv:2010.16098 [hep-ex]]):

$$\tau(p \to e^+\pi^0) > 2.4 \times 10^{34} \text{ yr} \qquad (90\%\text{C.L.}) \tag{1.22}$$

See Table 2.1 in **Ex. 2.2**ăfor more decay modes.

Grand Unified Theories predict the violation of the baryon and lepton quantum numbers and if experimentally verified then baryon number conservation will not be exact, although true to a high degree. Note that baryon number violation is also one of the necessary Sakharov conditions for baryogenesis (see Section 30.1 of the book). So there are several reasons to believe that the global symmetry B cannot be exact.

- **Weak hypercharge and weak isospin:** The weak hypercharge Y and the weak isospin are fundamental quantities at the basis of the $SU(2) \times U(1)_Y$ local gauge symmetry of the electroweak theory (see Chapter 25 of the book). Although they are exact symmetries, they are "spontaneously broken" at low energies (see Chapter 24 of the book) via a mechanism called the Brout–Englert–Higgs mechanism (see Section 25.6 of the book).

- **Color** $SU(3)$**:** The $SU(3)_C$ color symmetry is the fundamental local symmetry of quantum chromodynamics, which describes the strong interaction between fundamental quarks and gluons (see Chapter 18 of the book). The strong force is mediated by 8 massless gluons, which themselves carry color–anticolors pairs. Although gluons are massless (as a consequence of the gauge symmetry, just like in the case of the photon), we experimentally know that there is no long-range strong force. This is understood in the context of color confinement, which is discussed in Chapter 18.13 of the book, and the fact that a color singlet gluon does not exist in Nature.

- **Quark flavor** F**:** Quark flavor is certainly not an exact fundamental symmetry, since quarks have very different rest masses, ranging from a few MeV up to $\simeq$ 170 GeV in the case of the top quark. However, strong and electromagnetic forces preserve the quark flavor. Differently, the charged-current weak interactions generate transitions among the different quark species. One can introduce the flavor quantum number F, defined to be $+1$ for positively charged quarks (u, c, t) and -1 for quarks with negative charges (d, s, b). The tree-level weak transitions satisfy a $\Delta F = \Delta Q$ rule (originally called $\Delta S = \Delta Q$ rule – see Chapter 23 of the book) where ΔQ denotes the change in electric charge of the relevant hadrons. The strongest tests on this conservation law have been obtained in kaon decays such as (see R. P. Ely, G. Gidal, V. Hagopian, G. E. Kalmus, K. Billing, F. W. Bullock, M. J. Esten, M. Govan, C. Henderson and W. L. Knight, *et al.* "Study of K_{e4} decays," Phys. Rev. **180**, 1319-1330 (1969) `http://doi.org/10.1103/PhysRev.180.1319`):

$$Br(K^+ \to \pi^+\pi^+e^-\bar{\nu_e}) < 7 \times 10^{-7} \qquad (95\%\text{C.L.}) \tag{1.23}$$

The $\Delta F = \Delta Q$ rule can be violated through quantum loop contributions giving rise to flavor-changing neutral-current transitions (FCNC). Owing to the GIM mechanism(see Section 23.13 of the book), processes of this type are very suppressed. Experimental evidence for FCNC suppression can for example be observed in the following decays (see D. Ambrose *et al.* [BNL E871], "First observation of the rare decay mode $K_L^0 \to e^+e^-$" *Phys. Rev. Lett.* **81**, 4309-4312 (1998) `http://doi.org/10.1103/PhysRevLett.81.4309` [arXiv:hep-ex/9810007 [hep-ex]]):

$$Br(K_L^0 \to \mu^+\mu^-) = (6.84 \pm 0.11) \times 10^{-9} \qquad \text{and} \qquad Br(K_L^0 \to e^+e^-) = (8.7^{+5.7}_{-4.1}) \times 10^{-12} \tag{1.24}$$

(b) **Gravity:** The theories of the electromagnetic, strong and weak forces are based on "gauge" invariance under a group of local transformations. On the other hand, the theory of gravity is presently described by General Relativity (GR). The principle of equivalence identifies gravity as the inertial force in an accelerating system. Therefore, there is no gravity in an inertial frame and the equation of motion is described by Newton's law $\mathrm{d}p^\mu/\mathrm{d}t = 0$ or its extension to special relativity $\mathrm{d}p^\mu/\mathrm{d}\tau = 0$, where τ is the proper time. In order to transform this equation to an accelerated frame, the equation of motion should be expressed in terms of the covariant derivative and hence becomes:

$$\frac{Dp^\mu}{\mathrm{d}\tau} = 0 \tag{1.25}$$

Therefore, according to GR, a particle always moves on a straight line (geodesic) in space-time. Gravity, rather than a dynamical phenomenon between two masses, is replaced by a geometry of space-time determined by the energy-momentum tensor $T^{\mu\nu}$. Einstein's equation, relating gravity and matter, is defined as:

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi G T^{\mu\nu} \tag{1.26}$$

where $R^{\mu\nu}$ and $R = (g^{\mu\nu}R_{\mu\nu})$ is the Ricci tensor. It is derived from Riemann's curvature tensor (or Riemann–Christoffel tensor) $R^\lambda_{\mu\nu\rho}$ that describes the curvature of space-time in the following way:

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu} \tag{1.27}$$

where $R^\rho_{\mu\rho\nu}$ can be written using Christoffel symbols:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{1.28}$$

Note the mathematical similarity with Eq. (24.49) of the book. The Christoffel symbols are an array of numbers describing a metric connection. Einstein's equation states that the energy distribution in space-time determines the curvature of space-time.

GR has its own invariance group. Insofar as one understands "gauge theory" to mean a theory in which the physics is invariant under a certain group of transformations, one is tempted to construct GR as a gauge theory (see S. Weinstein, "Gravity and Gauge Theory", Philosophy of Science Vol. 66, Supplement. Proceedings of the 1998 Biennial Meetings of the Philosophy of Science Association. Part I: Contributed Papers (Sep., 1999), pp. S146-S155). In this context, the symmetry group would be the Poincaré group. This is understood by the fact that the Minkowski space is homogeneous (hence invariant under translation) and isotropic and relativistic (hence invariant under rotations and Lorentz boosts). So, formally gravity can be seen as a gauge theory. However, we must not forget that GR is a classical theory while the electroweak and strong theories are not, so one cannot make a direct comparison.

Noethers procedure for constructing conserved symmetry currents should provide a universal method, which in particular may be used to derive energy-momentum (pseudo) tensors in GR. However, the Noether procedure for GR is not without ambiguity. It is extensively discussed in the literature.

1.4 Fundamental symmetries

The Particle Data Group (PDG) regularly produces a Review of Particle Physics. It contains, among others, a summary of all known properties of subatomic particles and their interactions. It is available online at `http://pdg.lbl.gov/`. *Use the PDG review to describe which symmetry forbids the following elementary reactions or, on the contrary, which interaction governs the decays:*

1. $\Sigma^0 \to \Lambda + \pi^0$
2. $\Delta^+ \to p + \pi^0$
3. $p \to e^+ + \gamma$
4. $K^- \to \pi^- + \pi^0$
5. $\Sigma^0 \to \Lambda + \gamma$
6. $\Xi^0 \to p + \pi^-$
7. $\pi^0 \to \gamma + \gamma$
8. $\eta \to \gamma + \gamma$
9. $\Sigma^- \to n + \pi^-$
10. $\mu^- \to e^- + \bar{\nu}_e$
11. $\Xi^- \to \Lambda + \pi^-$
12. $\Sigma^- \to n + e + \bar{\nu}_e$

Solution:

We can use the PDG homepage, to check if any symmetries would be broken by the decays. Furthermore, if the decays are allowed, we can think about which interaction is responsible, e.g. by using the decay widths from the PDG tables. The results are summarized in Table 1.1.

Reaction	condition	comment
1. $\Sigma^0 \to \Lambda + \pi^0$	forbidden	energy conservation
2. $\Delta^+ \to p + \pi^0$	allowed	strong interaction
3. $p \to e^+ + \gamma$	forbidden	baryon and lepton number conservation
4. $K^- \to \pi^- + \pi^0$	allowed	weak interaction
5. $\Sigma^0 \to \Lambda + \gamma$	allowed	electromagnetic interaction
6. $\Xi^0 \to p + \pi^-$	allowed	weak interaction
7. $\pi^0 \to \gamma + \gamma$	allowed	electromagnetic interaction
8. $\eta \to \gamma + \gamma$	allowed	electromagnetic interaction
9. $\Sigma^- \to n + \pi^-$	allowed	weak interaction
10. $\mu^- \to e^- + \bar{\nu}_e$	forbidden	(flavorwise) lepton number conservation
11. $\Xi^- \to \Lambda + \pi^-$	allowed	weak interaction
12. $\Sigma^- \to n + e + \bar{\nu}_e$	allowed	weak interaction

Table 1.1 Table of reaction discussed in Ex. 1.4.

1.5 Natural units

Express your age, your mass, your height, and your body surface in natural units.

Solution:

We use the values derived in Section 1.8 and Table 1.6 of the book.

(a) your age (assuming 25 years old):

$$(25\,\text{yr})\frac{365 \times 24 \times (3600\,\text{s})}{1\,\text{yr}}(1.5 \times 10^{24}\,\text{GeV}^{-1}) = 1.2 \times 10^{33}\,\text{GeV}^{-1} \tag{1.29}$$

(b) your weight (assuming 60 kg):

$$F = mg = (60\text{ kg})(9.81\,\text{ăms}^{-2}) = (590\text{ N})\frac{\text{GeV}^2}{8.119 \times 10^5\text{ N}} = 7 \times 10^{-4}\,\text{GeV}^2 \tag{1.30}$$

(c) your height (assuming 1.7 m):

$$h = (1.7 \text{ m})(5 \times 10^{15}\,\text{GeV}^{-1}\text{m}^{-1}) = 8.5 \times 10^{15}\,\text{GeV}^{-1} \tag{1.31}$$

(d) your body surface (assuming 1 m^2):

$$\begin{aligned} A &= \frac{(1\,\text{m}^2)}{(\hbar c)^2} = \frac{(10^{30}\,\text{fm}^2)}{(197\,\text{MeV fm})^2} = 2.6 \times 10^{33}\,\text{GeV}^{-2} = (2.6 \times 10^{33}\,\text{GeV}^{-2})(0.389\,\text{mb GeV}^2) \\ &= 2 \times 10^{33} \text{ mb} \end{aligned} \tag{1.32}$$

The point of this exercise was to show that natural units are not very "natural" for our everyday's life!

1.6 Natural units

Derive Table 1.6.

Solution:

In order to derive the conversion factor, the trick is to reinsert the proper constants $\hbar$, c and consequently $\hbar c$ which are set to one in the natural units. This is done in similar way as for Eqs. (1.28), (1.29) and (1.30) of the book. Namely:

$$\begin{aligned} 1 \text{ m} &= \frac{10^{15} \text{ fm}}{\hbar c} \approx \frac{10^{15} \text{ fm}}{200\,\text{MeV fm}} \approx 5.067\,731 \ldots \times 10^{15} \text{ GeV}^{-1} \\ 1 \text{ s} &= \frac{1 \text{ s}}{\hbar} \approx \frac{1 \text{ s}}{6.582 \times 10^{-22} \text{ MeV s}} \approx 1.519\,267 \times 10^{24} \text{ GeV}^{-1} \\ m &= 1 \text{ GeV/c}^2 = \frac{10^9 e\text{J}}{c^2 \frac{\text{m}^2}{\text{s}^2}} = \frac{1.602177 \times 10^{-10}\text{J}}{(299792458)^2 \frac{\text{m}^2}{\text{s}^2}} = 1.7826619 \times 10^{-27} \text{ kg} \end{aligned} \tag{1.33}$$

Once this is done, rather than computing these values by hand, we implement the relations into a PYTHON program. The listing is show in the following. The code is also available in our GITHUB[1].

```
import math

# charge in Coulomb (exact)
e=1.602176634e-19

# Planck h J.s (exact)
h=6.62607015e-34

# speed of light m/s (exact)
c= 299792458

hbar = h/(2*math.pi)
hbar_MeVs = hbar/e/1e6
hbar_GeVs = hbar/e/1e9

print("hbar = ", hbar, " J*s")
print("hbar = ", hbar_MeVs, " MeV*s")

print ("————————————")
```

1 https://github.com/CambridgeUniversityPress/Phenomenology-Particle-Physics

```python
hbarc = hbar*c
print("hbar*c = ", hbarc, " J*m")
hbarc_MeVfm = hbar*c/e/1e6*1e15
print("hbar*c = ", hbarc_MeVfm, " MeV*fm")
hbarc_GeVfm = hbarc_MeVfm/1e3

print ("-----------")

onemeter_GeVinverse = 1e15/(hbarc_GeVfm)
print("1 meter = ", "{:.7e}".format(onemeter_GeVinverse), "GeV^-1")

onesecond_GeVinverse = 1/hbar_GeVs
print("1 second = ", "{:.7e}".format(onesecond_GeVinverse), "GeV^-1")

print ("------- TABLE 1.6 -------")
mass_oneGeV_in_kg = 1e9*e/(c**2)
length_oneGeVexpominusone_in_m = hbarc_GeVfm*1e-15
time_oneGeVexpominusone_in_s = hbar_GeVs
energy_oneGeV_in_J = e*1e9
momentum_oneGeV_in_kgms = e*1e9/c
force_oneGeV_in_N = (e*1e9)**2/hbar/c
area_oneGeV_in_m2 = (hbarc_MeVfm/1e3/1e15)**2
area_oneGeV_in_mb = area_oneGeV_in_m2*1e3/1e-28

print("Mass: 1 GeV = ", "{:.7e}".format(mass_oneGeV_in_kg), "kg")
print("Length: 1 GeV^-1 = ", "{:.7e}".format(length_oneGeVexpominusone_in_m), "m")
print("Time: 1 GeV^-1 = ", "{:.7e}".format(time_oneGeVexpominusone_in_s), "s")
print("Energy 1 GeV = ", "{:.7e}".format(energy_oneGeV_in_J), "J")
print("Momentum 1 GeV = ", "{:.7e}".format(momentum_oneGeV_in_kgms), "kg m/s")
print("Force: 1 GeV^2 = ", "{:.7e}".format(force_oneGeV_in_N), "N")
print("Cross-section: 1 GeV^-2 = ", "{:.7f}".format(area_oneGeV_in_mb), "mb")

print ("-----------")

e_field_oneGeV2 = force_oneGeV_in_N/e
b_field_oneGeV2 = e_field_oneGeV2/c

print("E-field: 1 GeV^2 = ", "{:.7e}".format(e_field_oneGeV2), "V/m")
print("B-field: 1 GeV^2 = ", "{:.7e}".format(b_field_oneGeV2), "T")
```

Running the code gives the following output which can be checked to coincide with the values in Table 1.6 of the book:

```
hbar =  1.0545718176461565e-34  J*s
hbar =  6.582119569509067e-22  MeV*s
-----------
hbar*c =  3.1615267734966903e-26  J*m
hbar*c =  197.3269804593025  MeV*fm
-----------
1 meter =  5.0677307e+15 GeV^-1
1 second =  1.5192674e+24 GeV^-1
------- TABLE 1.6 -------
Mass: 1 GeV =  1.7826619e-27 kg
Length: 1 GeV^-1 =  1.9732698e-16 m
```

```
Time: 1 GeV^-1 =  6.5821196e-25 s
Energy 1 GeV =  1.6021766e-10 J
Momentum 1 GeV =  5.3442860e-19 kg m/s
Force: 1 GeV^2 =  8.1193997e+05 N
Cross-section: 1 GeV^-2 =  0.3893794 mb
-----------
E-field: 1 GeV^2 =  5.0677307e+24 V/m
B-field: 1 GeV^2 =  1.6904130e+16 T
```

1.7 Natural units

What is the typical kinetic energy of a served tennis ball expressed in electronvolt? What is the total energy (including rest mass) of that tennis ball? What is the velocity of a proton with the same kinetic energy? Compare the energy to that of a proton in the Large Hadron Collider (LHC).

Solution:

The speed of the tennis ball is 263 km/h which corresponds to 73 m/s. The kinetic energy of this tennis ball of mass $m_0 = 0.058$ kg is

$$T = \frac{1}{2} m_0 v^2 \approx 155 \text{ J} \approx 10^{21} \text{ eV} \tag{1.34}$$

The rest mass energy of the tennis ball is:

$$E = m_0 c^2 \approx 5 \times 10^{15} \text{ J} \approx 3 \times 10^{34} \text{ eV} \tag{1.35}$$

The Lorentz gamma factor of a proton with the same kinetic energy is:

$$T = (\gamma - 1) m_p c^2 \quad \Longrightarrow \quad \gamma = \frac{T}{m_p c^2} + 1 \approx \frac{10^{21} \text{ eV}}{10^9 \text{ eV}} + 1 \approx 10^{12} \tag{1.36}$$

where we used $m_p c^2 \approx 1$ GeV. The velocity of the proton can be found with $\gamma = 1/\sqrt{1-\beta^2}$ so

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2} \frac{1}{\gamma^2} \tag{1.37}$$

where we used $(1+x)^{(1/2)} \simeq 1 + x/2 + \ldots$. Hence,

$$\beta \approx 1 - \frac{1}{2 \times 10^{24}} \approx 1 \tag{1.38}$$

We can compare this to the performance of the LHC collider which accelerates protons to the highest energy. In this case, the energy of a proton in the LHC is 7 TeV or 7×10^{12} eV. This is much less than the kinetic energy of the tennis ball. However, the main difference is that this energy is stored into a single proton! In fact, in the tennis ball, the energy stored per nucleon (proton or neutron) is only

$$\frac{T}{N_A(\text{nucleons/mol}) \times m(\text{g})} \approx \frac{(10^{21} \text{ eV})}{(6.02 \times 10^{23})(58)} \approx 3 \times 10^{-5} \text{ eV/nucleon.} \tag{1.39}$$

2 Basic Concepts

2.1 Law of radioactive decay

A weak radioactive source is observed for a period of 60 s. In this period of time, 1000 decays are detected. What is the estimation of the decay rate of the source and the error from this measurement?

Solution:

The decay rate or activity ($A = -\mathrm{d}N/\mathrm{d}t$) is given by the number of decays measured per second. Note that they are related to the decay constant Γ by the radioactive decay law:

$$A(t) = -\frac{\mathrm{d}N}{\mathrm{d}t} = \Gamma N(t) \tag{2.1}$$

Since we have a weak source, we expect the lifetime to be much larger than the period of time in which the decays were measured. If we would repeat this measurement several times we would observe fluctuations due to the statistical nature of the process. The probability of observing N decays during a certain period of time is given by a Poisson distribution where the standard deviation is given by $\sqrt{M}$, being M the average number of decays in that period (see Appendix G of the book for its derivation).

In this case we have just one measurement and thus the error on the number of decays measured would be $\sqrt{1000}$. Therefore, we can estimate the decay rate and its error as:

$$\left(1000 \pm \sqrt{1000}\right)/60 = 16.7 \pm 0.5 \text{ decays/s} \tag{2.2}$$

which is an error of 3%.

2.2 Proton decay

Proton decays are suppressed by the global baryon number symmetry. Check in the Particle Data Group's Review what is the current experimental bound on its lifetime. Assuming its lifetime is given by this bound, how many protons would be needed to obtain on average one decay per year. How many cubic meters of water does that represent (count also the protons in oxygen)?

Solution:

The Particle Data group distinguishes between potential decays of the proton into identified final states and the generic "invisible" decay of a proton. The experimental results for identified final states are reported in Table 2.1. The lower bound for the proton lifetime into an invisible mode is given by (see M. Anderson *et al.* [SNO+], "Search for invisible modes of nucleon decay in water with the SNO+ detector," *Phys. Rev. D* **99**, no.3, 032008 (2019) `http://doi.org/10.1103/PhysRevD.99.032008` [arXiv:1812.05552 [hep-ex]]):

$$\tau(p \to \text{invisible}) > 3.6 \times 10^{29} \text{ yr} \qquad (90\%\text{C.L.}) \tag{2.3}$$

Decay channel	τ (90%C.L.)	events observed	expected background
$\tau(p \to e^+\pi^0)$	2.4×10^{34} yr	0	0.59
$\tau(p \to \mu^+\pi^0)$	1.6×10^{34} yr	1	0.94
$\tau(p \to \bar{\nu} + K^+)$	5.9×10^{33} yr	0	1.0

Table 2.1 Limits for the proton lifetime in different channels.

Since the experimental limits on the lifetime are much larger than 1 year, we can simply estimate the probability for a proton to decay as 1 yr$/\tau$. More explicitly, for one decay, we have:

$$N_0 - 1 = N(t) = N_0 e^{-t/\tau} \approx N_0 \left(1 - \frac{t}{\tau}\right) \qquad \Longrightarrow \qquad N_0 \approx \frac{\tau}{t} \tag{2.4}$$

Hence, the number of proton needed to obtain on the order of one decay per year is simply given by:

$$N_0 \approx \frac{\tau}{1 \text{ yr}} \approx 2.4 \times 10^{34} \quad \text{protons} \tag{2.5}$$

where we have taken the most stringent experimental limit.

Since 1 g or equivalently 1 cm^3 of water corresponds to 1/18 mol of H_2O, the density n of protons in water is:

$$n = (2+8) \times \left(\frac{1}{18} \frac{\text{mol}}{\text{cm}^3}\right) \times (6 \times 10^{23}\,\text{mol}^{-1}) \simeq 3.3 \times 10^{23} \text{ cm}^{-3} \tag{2.6}$$

The required volume of water is then given by:

$$V = \frac{N_0}{n}\text{ă} = \frac{2.4 \times 10^{34}}{3.3 \times 10^{23} \text{ cm}^{-3}} = 7.3 \times 10^{10} \text{ cm}^3 = 7.3 \times 10^4 \text{ m}^3 \tag{2.7}$$

or equivalently 73 kton of water.

2.3 Rutherford scattering

Derive the classical trajectory of the α particle scattered by a point-like non-recoiling nucleus. Show that it can be expressed as:

$$\frac{1}{r} = \frac{1}{b}\sin\varphi + \frac{d_0}{2b^2}(\cos\varphi - 1) \quad \text{where} \quad d_0 \equiv \frac{\alpha z Z}{E_K} \tag{2.8}$$

where the position of the α particle along its trajectory is given in polar coordinates by (r, φ). E_K is the kinetic energy of the incoming particle and b the impact parameter.

Solution:

The geometry of the problem is shown in Figure 2.1. The interaction between the scattering center (charge Ze) and the projectile (charge ze) is given by the Coulomb potential:

$$U(r) = \frac{zZe^2}{4\pi\epsilon_0 r} = \alpha\frac{zZ}{r} \tag{2.9}$$

The mechanical energy is conserved and the motion is in a plane:

$$E = \frac{1}{2}mv^2 + U(r) = \text{constant} \tag{2.10}$$

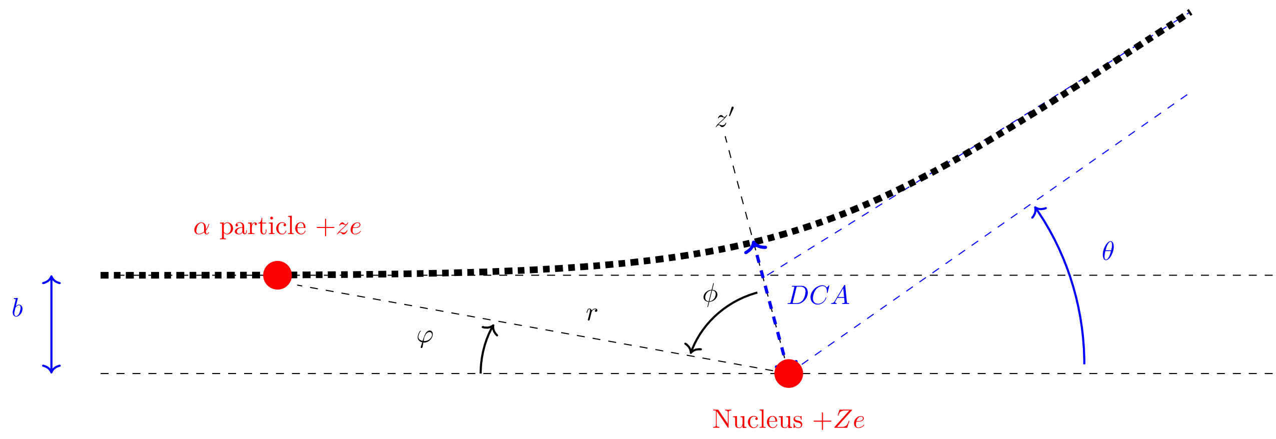

Figure 2.1 Definition of the kinematical quantities relevant for the large-angle scattering experiment of Rutherford. The picture depicts the large-angle deviation due to a single atom. The impact parameter is b (before the collision) and the resulting scattering angle is θ. The position of the α particle along its trajectory is given by (r, φ). The distance of closest approach is defined by DCA along an axis z'.

The initial energy is just the initial kinetic energy of the particle since the potential vanishes for $r \to \infty$. Thus:

$$E = \frac{1}{2} m v_0^2 \equiv E_K \tag{2.11}$$

where v_0 is the initial velocity of the α particle. Using polar coordinates, $\vec{r} = (r \cos \phi, r \sin \phi)$ we can calculate v^2:

$$v^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}t} \right)^2 + r^2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}t} \right)^2 \tag{2.12}$$

Then we can write the equation for the energy as:

$$E = \frac{1}{2} m \left(\frac{\mathrm{d}r}{\mathrm{d}t} \right)^2 + \frac{1}{2} m r^2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}t} \right)^2 + U(r) = E_K \tag{2.13}$$

Similarly, since the angular momentum is constant for a central force, we have:

$$L = m r^2 \frac{\mathrm{d}\phi}{\mathrm{d}t} = m v_0 b \tag{2.14}$$

where b is the impact parameter, and also:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{L}{m r^2} \qquad \Longrightarrow \qquad \mathrm{d}t = \frac{m r^2 \mathrm{d}\phi}{L} \tag{2.15}$$

Substituting this expression in Eq. (2.13), we obtain

$$E = \frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \underbrace{\frac{L^2}{2 m r^2} + U(r)}_{U_{eff}(r)} = E_K \tag{2.16}$$

where since L is a constant, we recognize the effective potential U_{eff} and the fact that the expression is only function of r. Therefore, we can solve for $\mathrm{d}r/\mathrm{d}t$:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t} \right)^2 = \frac{2}{m} \left(E_K - U(r) \right) - \frac{L^2}{m^2 r^2} \tag{2.17}$$

We now want to eliminate the time variable to find the trajectory in the (r, ϕ) plane, so using Eq. (2.15), we find:

$$\mathrm{d}t = \frac{dr}{\sqrt{\frac{2}{m}\left(E_K - U(r)\right) - \frac{L^2}{m^2 r^2}}} = \frac{mr^2}{L} d\phi \tag{2.18}$$

Consequently:

$$\mathrm{d}\phi = \frac{L\mathrm{d}r}{mr^2\sqrt{\frac{2}{m}\left(E_K - U(r)\right) - \frac{L^2}{m^2 r^2}}} = \frac{L\mathrm{d}r}{r^2\sqrt{2mE_K - 2mU(r) - \frac{L^2}{r^2}}} \tag{2.19}$$

We now need to integrate the equation of motion for the Coulomb potential:

$$\int \mathrm{d}\phi = \int \frac{L\mathrm{d}r}{r^2\sqrt{2mE_K - 2m\frac{\alpha z Z}{r} - \frac{L^2}{r^2}}} \tag{2.20}$$

We define $A \equiv m\alpha z Z/L$. We have:

$$\begin{aligned}
\frac{L}{r^2\sqrt{2mE_K - 2m\frac{\alpha z Z}{r} - \frac{L^2}{r^2}}} &= \frac{L}{r^2\sqrt{2mE_K - \frac{2AL}{r} - \frac{L^2}{r^2} + A^2 - A^2}} \\
&= \frac{L}{r^2\sqrt{2mE_K + A^2 - \left(\frac{2AL}{r} + \frac{L^2}{r^2} + A^2\right)}} \\
&= \frac{L}{r^2\sqrt{2mE_K + A^2 - \left(\frac{L}{r} + A\right)^2}} = \frac{L}{r^2\sqrt{B^2 - \left(\frac{L}{r} + A\right)^2}}
\end{aligned} \tag{2.21}$$

where $B \equiv \sqrt{2mE_K + A^2}$. We now recall that

$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos(x) = -\frac{1}{\sqrt{1-x^2}} \qquad \Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}r}\arccos\left(\frac{A + \frac{L}{r}}{B}\right) = -\frac{1}{\sqrt{1 - \left(\frac{A+\frac{L}{r}}{B}\right)^2}} \cdot \left(-\frac{L}{Br^2}\right) \tag{2.22}$$

Hence:

$$\frac{\mathrm{d}}{\mathrm{d}r}\arccos\left(\frac{A + \frac{L}{r}}{B}\right) = \frac{L}{Br^2\sqrt{1 - \left(\frac{A+\frac{L}{r}}{B}\right)^2}} = \frac{L}{r^2\sqrt{B^2 - \left(A + \frac{L}{r}\right)^2}} \quad \square \tag{2.23}$$

Consequently, we have shown that the integral of the equation of motion yields:

$$\phi - \phi_0 = \arccos\left(\frac{A + \frac{L}{r}}{B}\right) = \arccos\left(\frac{A\left(1 + \frac{L}{Ar}\right)}{A\sqrt{\frac{2mE_K}{A^2} + 1}}\right) \tag{2.24}$$

or

$$\cos(\phi - \phi_0) = \left(\frac{\left(1 + \frac{L}{Ar}\right)}{\sqrt{\frac{2mE_K}{A^2} + 1}}\right) \qquad \Longrightarrow \qquad \frac{L}{Ar} = \left(\sqrt{\frac{2mE_K}{A^2} + 1}\right)\cos(\phi - \phi_0) - 1 \tag{2.25}$$

Finally, we find:

$$r = \frac{L/A}{\left(\sqrt{\frac{2mE_K}{A^2} + 1}\right)\cos(\phi - \phi_0) - 1} = \frac{L/A}{e\cos(\phi - \phi_0) - 1} \tag{2.26}$$

If we choose r to be minimum when $\phi = 0$ as is the case for the axes shown in Figure 2.1, then $\phi_0 = 0$. So finally, we simply have:

$$\frac{L/A}{r} = e\cos(\phi) - 1 \tag{2.27}$$

where $A \equiv m\alpha zZ/L$ and $e \equiv \sqrt{\frac{2mE_K}{A^2} + 1}$. Since L is a constant of motion for a central force, so are A and e.

We should now express the trajectory as a function of the angles φ and θ. Geometrically, we note that the axis z' divides the angle $\pi - \theta$ in two parts. Hence, the angles φ and ϕ subtend the bisector of $\pi - \theta$. Accordingly:

$$\varphi - \phi = \frac{\pi - \theta}{2} \qquad \Longrightarrow \qquad \phi = \varphi - \frac{\pi - \theta}{2} \tag{2.28}$$

The range of the angles go as follows:

$$\varphi \in [0, \pi - \theta] \qquad \Longrightarrow \qquad \phi \in \left[-\frac{\pi - \theta}{2}, \pi - \theta - \frac{\pi - \theta}{2}\right] = \left[-\frac{\pi - \theta}{2}, +\frac{\pi - \theta}{2}\right] \tag{2.29}$$

We can now use trigonometry to find:

$$\cos\phi = \cos\left(\varphi - \frac{\pi - \theta}{2}\right) = \cos\left(\varphi - \frac{\pi}{2}\right)\cos\frac{\theta}{2} - \sin\left(\varphi - \frac{\pi}{2}\right)\sin\frac{\theta}{2} = \sin\varphi\cos\frac{\theta}{2} + \cos\varphi\sin\frac{\theta}{2} \tag{2.30}$$

Eq. (2.27) then becomes:

$$\frac{L/A}{r} = e\cos\frac{\theta}{2}\sin\varphi + e\sin\frac{\theta}{2}\cos\varphi - 1 \tag{2.31}$$

We now consider the trajectory of the particle before it scatters off the nucleus. We note that for $r \to \infty$, we have either $\varphi \to 0$ or $\varphi \to \pi - \theta$. These give us a relationship between e and the scattering angle θ, namely for $\varphi = 0$:

$$0 = e\sin\frac{\theta}{2} - 1 \qquad \Longrightarrow \qquad e = \frac{1}{\sin\frac{\theta}{2}} \tag{2.32}$$

Alternatively, for $\varphi = \pi - \theta$, we find:

$$\begin{aligned} 0 &= e\cos\frac{\theta}{2}\sin(\pi - \theta) + e\sin\frac{\theta}{2}\cos(\pi - \theta) - 1 = e\left(\cos\frac{\theta}{2}\sin(\theta) - \sin\frac{\theta}{2}\cos(\theta)\right) - 1 \\ &= e\left(2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2} - 1\right)\right) - 1 \\ &= e\left(2\cos^2\frac{\theta}{2}\sin\frac{\theta}{2} - 2\cos^2\frac{\theta}{2}\sin\frac{\theta}{2} + \sin\frac{\theta}{2}\right) - 1 \qquad \Longrightarrow \qquad e = \frac{1}{\sin\frac{\theta}{2}} \end{aligned} \tag{2.33}$$

Thus Eq. (2.31) can be written as:

$$\frac{L/A}{r} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\sin\varphi + (\cos\varphi - 1) = \cot\frac{\theta}{2}\sin\varphi + (\cos\varphi - 1) \tag{2.34}$$

With the help of Eq. (2.14), we note that

$$E_K = \frac{1}{2}mv_0^2 = \frac{L^2}{2mb^2} \qquad \Longrightarrow \qquad \frac{L}{A} = \frac{L^2}{m\alpha zZ} = \frac{2mb^2E_K}{m\alpha zZ} = \frac{2b^2}{d_0} \tag{2.35}$$

where

$$d_0 \equiv \frac{\alpha zZ}{E_K} \tag{2.36}$$

We will see in **Ex. 2.4** that d_0 is the distance of closest approach to the nucleus for head-on collisions. We can now use the definitions of e and A to simplify the trajectory further. We recall:

$$e^2 \equiv \frac{2mE_K}{A^2} + 1 = \frac{1}{\sin^2\frac{\theta}{2}} \qquad \Longrightarrow \qquad \frac{1}{\sin^2\frac{\theta}{2}} - 1 = \frac{2mE_K L^2}{(m\alpha z Z)^2} = \frac{2mE_K^2 2mb^2}{(m\alpha z Z)^2} = \left(\frac{2E_K b}{\alpha z Z}\right)^2 \tag{2.37}$$

Hence:

$$\frac{1-\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = \cot^2\frac{\theta}{2} = \left(\frac{2E_K b}{\alpha z Z}\right)^2 \qquad \Longrightarrow \qquad b = \frac{\alpha z Z}{2E_K}\cot\frac{\theta}{2} = \frac{d_0}{2}\cot\frac{\theta}{2} \tag{2.38}$$

Finally, using Eq. (2.34), the trajectory of the α particle can be expressed as:

$$\frac{1}{r} = \frac{d_0}{2b^2}\cot\frac{\theta}{2}\sin\varphi + \frac{d_0}{2b^2}(\cos\varphi - 1) \qquad \Longrightarrow \qquad \frac{1}{r} = \frac{1}{b}\sin\varphi + \frac{d_0}{2b^2}(\cos\varphi - 1) \tag{2.39}$$

where we used Eq. (2.35) to replace L/A.

2.4 Distance of closest approach in Rutherford scattering

*We consider the classical Rutherford scattering problem (see **Ex. 2.3**).*

(a) Show that the distance of closest approach for a head-on collision is given by:

$$d_0 = \frac{zZ\alpha}{E_K} \tag{2.40}$$

(b) Show that the relation between the impact parameter b and the scattering angle simplifies to:

$$b = \frac{d_0}{2}\cot\left(\frac{\theta}{2}\right) \tag{2.41}$$

(c) Now consider the general case, and calculate the distance of closest approach d as a function of the impact parameter b. Show that it can be expressed as:

$$b^2 = d(d - d_0) \tag{2.42}$$

(d) Assume the kinetic energy of the α particle is 130 MeV, impinging on a ^{208}Pb target. What is d_0? Compute d for $b = 1, 4, 7,$ and 15 fm.

(e) When the distance of closest approach is in the range of the nuclear force, we expect a deviation from the Rutherford cross-section. Assuming that the ^{208}Pb nucleus has a radius of $R = 12.5$ fm, compute the necessary kinetic energy of the α particles to start seeing such deviations.

Solution:

(a) In a head-on collision the minimum distance is achieved when the velocity is zero and, thus, the total energy is given by the potential energy. From energy conservation we have:

$$E_K = \frac{1}{2}mv_0^2 = \alpha\frac{zZ}{d_0} \qquad \Longrightarrow \qquad d_0 = \frac{\alpha z Z}{E_K} \tag{2.43}$$

(b) As discussed in **Ex. 2.3**, the angles in Figure 2.1 are related by the following equation:

$$\phi = \varphi - \frac{\pi - \theta}{2} \tag{2.44}$$

The range of φ is given by $\varphi \in [0, \pi - \alpha]$. After the scattering, as $r \to \infty$, we have that $\varphi \to \pi - \theta$. Starting from the Eq. (2.39) of the trajectory, we find in this case:

$$0 = \frac{1}{b}\sin\varphi + \frac{d_0}{2b^2}(\cos\varphi - 1) \qquad \Longrightarrow \qquad \sin\varphi = \frac{d_0}{2b}(1 - \cos\varphi) \tag{2.45}$$

Hence:

$$\frac{\sin\varphi}{1-\cos\varphi} = \frac{2\sin\varphi/2\cos\varphi/2}{1 - 1 + 2\sin^2\varphi/2} = \frac{\cos\varphi/2}{\sin\varphi/2} = \cot\varphi/2 = \frac{d_0}{2b} \tag{2.46}$$

It follows that:

$$\frac{2b}{d_0} = \tan\left(\frac{\varphi}{2}\right) = \tan\left(\frac{\pi-\theta}{2}\right) = \cot\left(\frac{\theta}{2}\right) \qquad \Longrightarrow \qquad b = \frac{d_0}{2}\cot\left(\frac{\theta}{2}\right) \tag{2.47}$$

(c) The distance to the closest approach d for any collision can be obtained from Eq. (2.27) when $\phi \to 0$:

$$\phi = 0 \qquad \Longrightarrow \qquad \frac{1}{d} = \frac{A(e-1)}{L} \tag{2.48}$$

We find that:

$$\begin{aligned}
\frac{A(e-1)}{L} &= \frac{A}{L}\left(\sqrt{\frac{2mE_K}{A^2}+1} - 1\right) = \sqrt{\frac{2mE_K}{L^2} + \frac{A^2}{L^2}} - \frac{A}{L} = \sqrt{\frac{1}{b^2} + \left(\frac{A}{L}\right)^2} - \frac{A}{L} \\
&= \sqrt{\frac{1}{b^2} + \left(\frac{d_0}{2b^2}\right)^2} - \frac{d_0}{2b^2} = \frac{1}{d}
\end{aligned} \tag{2.49}$$

Consequently:

$$\begin{aligned}
\sqrt{\frac{1}{b^2} + \left(\frac{d_0}{2b^2}\right)^2} = \frac{1}{d} + \frac{d_0}{2b^2} \qquad &\Longrightarrow \qquad \frac{1}{b^2} + \left(\frac{d_0}{2b^2}\right)^2 = \left(\frac{1}{d} + \frac{d_0}{2b^2}\right)^2 \\
&\Longrightarrow \qquad \frac{1}{b^2} = \frac{1}{d^2} + \frac{d_0}{d}\frac{1}{b^2}
\end{aligned} \tag{2.50}$$

So, finally:

$$\frac{1}{b^2}\left(1 - \frac{d_0}{d}\right) = \frac{1}{d^2} \qquad \Longrightarrow \qquad b^2 = d^2\left(1 - \frac{d_0}{d}\right) = d^2 - dd_0 = d\,(d - d_0) \tag{2.51}$$

(d) The kinetic energy is set to $E_K = 130$ MeV. For an α particle colliding against lead (Pb) we have $z = 2$ and $Z = 82$. Hence, the distance of closest approach for head-on collisions is equal to:

$$d_0 = \frac{\alpha z Z \hbar c}{E_k} \approx \frac{(2)(82)(1/137)}{130\text{ MeV}}(197\text{ MeV fm}) \approx 1.8\text{ fm} \tag{2.52}$$

In order to compute d for various impact parameters b, we set:

$$b^2 = d^2 - dd_0 \qquad \Longrightarrow \qquad d = \frac{1}{2}\left(d_0 \pm \sqrt{d_0^2 + 4b^2}\right) \tag{2.53}$$

b (fm)	θ	d (fm)
1	83°	2
4	25°	5
7	15°	8
15	7°	16

Table 2.2 The scattering angle and distance of closest approach for various impact parameters b in collisions of 130 MeV α particles on lead.

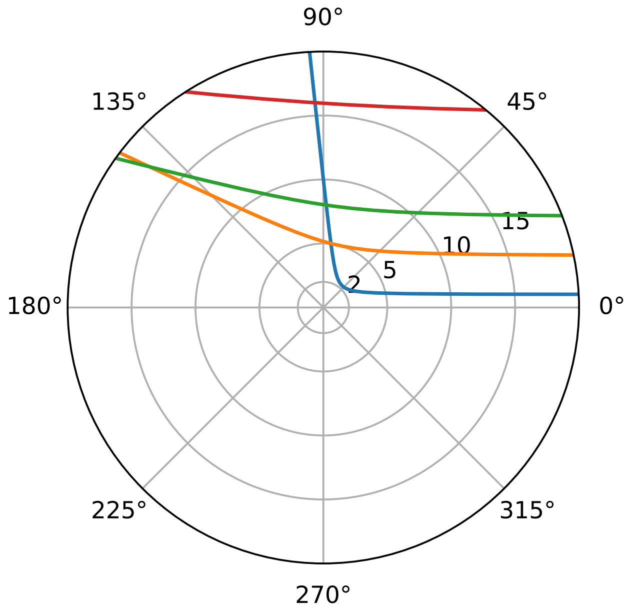

Figure 2.2 The corresponding trajectories of the α particle for $b = 1, 4, 7, 15$ fm.

Only the solution with the plus sign is relevant. For each impact paramter, we can also compute the scattering angle θ:

$$b = \frac{d_0}{2}\cot\left(\frac{\theta}{2}\right) \qquad \Longrightarrow \qquad \theta = 2\arctan\left(\frac{d_0}{2b}\right) \tag{2.54}$$

The results are tabulated in Table 2.2 and the corresponding trajectories are plotted in Figure 2.2.

(e) The radius of the lead nucleus is assumed to be $R = 12.5$ fm. Let's assume we want to see a deviation in the Rutherford scattering above a scattering angle θ. We have:

$$b = \frac{d_0}{2}\cot\left(\frac{\theta}{2}\right) \qquad \text{and} \qquad b^2 = d^2 - dd_0 \tag{2.55}$$

Hence:

$$d_0^2\frac{\cot^2\left(\frac{\theta}{2}\right)}{4} + dd_0 - d^2 = 0 \qquad \Longrightarrow \qquad d_0 = 2d\frac{-1 \pm \sqrt{1+\cot^2\left(\frac{\theta}{2}\right)}}{\cot^2\left(\frac{\theta}{2}\right)} \tag{2.56}$$

We keep the solution with the plus sign. Let's assume that we seek a deviation for a scattering angle θ above 60°. We have:

$$\theta = 60^\circ \qquad \text{and} \qquad \cot^2\left(\frac{\theta}{2}\right) = 3 \tag{2.57}$$

Then for $d = R$, we have:

$$d_0 = 2R\frac{-1+\sqrt{4}}{3} = \frac{2}{3}R = \frac{\alpha z Z}{E_{k,min}} \qquad \Longrightarrow \qquad E_{k,min} = \frac{3\alpha z Z}{2R}\hbar c \approx 28 \text{ MeV} \tag{2.58}$$

The energy of the α particles in the original experiment of Geiger and Marsden was limited to a little over 5 MeV by the available alpha decay sources. They did not find the nuclear radius: they only put an upper limit on the size of the nucleus.

2.5 Origin of β particles

Before the discovery of the neutron, early discussions contemplated the possibility that atomic nuclei are composed of electrons and protons. Using Heisenberg's uncertainty principle, estimate the minimal energy of electrons bound inside a nucleus assumed to have a radius smaller than 10 fm. Is this consistent with an observed β decay spectrum?

Solution:

Heisenberg's uncertainty principle states that

$$\Delta p \Delta x \geq \frac{\hbar}{2} \tag{2.59}$$

where x and p are the position and momentum operators. We can use this relation to find the minimal uncertainty on the kinetic energy of an electron confined in a region of dimension $\Delta x \approx 10$ fm:

$$c\Delta p \Delta x \geq \frac{\hbar c}{2} \qquad \Longrightarrow \qquad c\Delta p \geq \frac{\hbar c}{2\Delta x} \gtrsim \frac{(197 \text{ MeV fm})}{2(10 \text{ fm})} \approx 10 \text{ MeV} \tag{2.60}$$

By comparing this minimal value with the typical energies of β particles, we clearly conclude that the observation of is not compatible with a model where electrons are living inside the nucleus and are ejected out of the it. In fact, we know that β particles are emitted via the weak decay where a neutron (resp. a proton) is converted into a proton (resp. a neutron)

2.6 Order of magnitude of cross-sections

Compare the typical cross-sections for (a) pp interactions, (b) γp interactions, and (c) νp interactions. What can you say about the differences between them?

Solution:

A compilation of hadronic and γp cross-sections is reproduced in Figure 2.3. Neutrino- and antineutrino–nucleon cross-sections are reported in Figure 2.4. We use these data to estimate typical cross-sections for the various processes. For instance, for $p_{lab} \simeq 10$ GeV we have:

$$\sigma_{pp} \approx 40\,\text{mb}, \qquad \sigma_{\gamma p} \approx 0.1\,\text{mb}, \qquad \sigma_{\nu p} \approx (10^{-38}\,\text{cm}^2/\text{GeV})(10 \text{ GeV}) = 10^{-8}\,\text{mb}$$

or, considering only the order of magnitudes:

$$\sigma_{pp} : \sigma_{\gamma p} : \sigma_{\nu p} \approx 1 : 10^{-3} : 10^{-10} \tag{2.61}$$

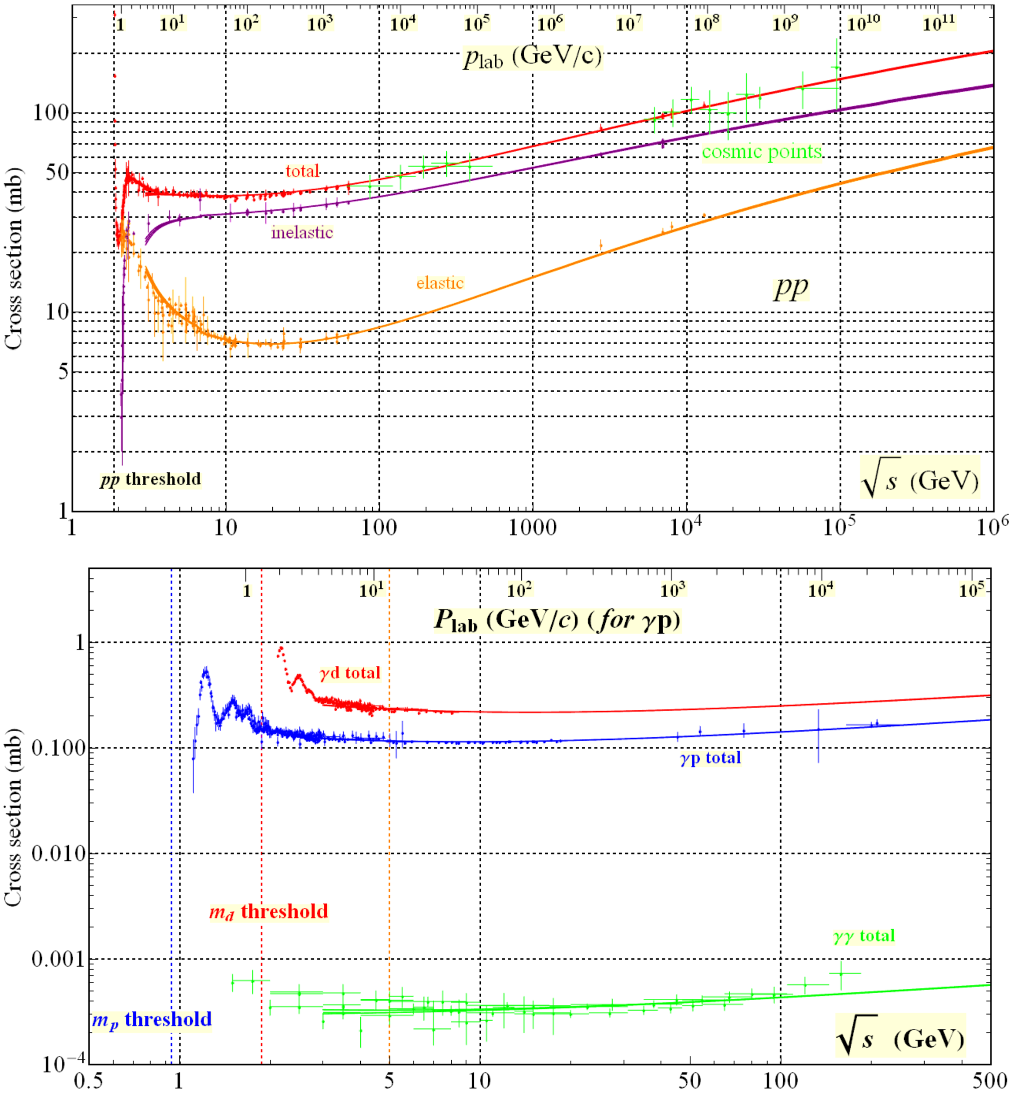

Figure 2.3 A compilation of hadronic and γp cross-sections taken from Ref. P.A. Zyla et al. (Particle Data Group), Prog. Theor. Exp. Phys. 2020, 083C01 (2020).

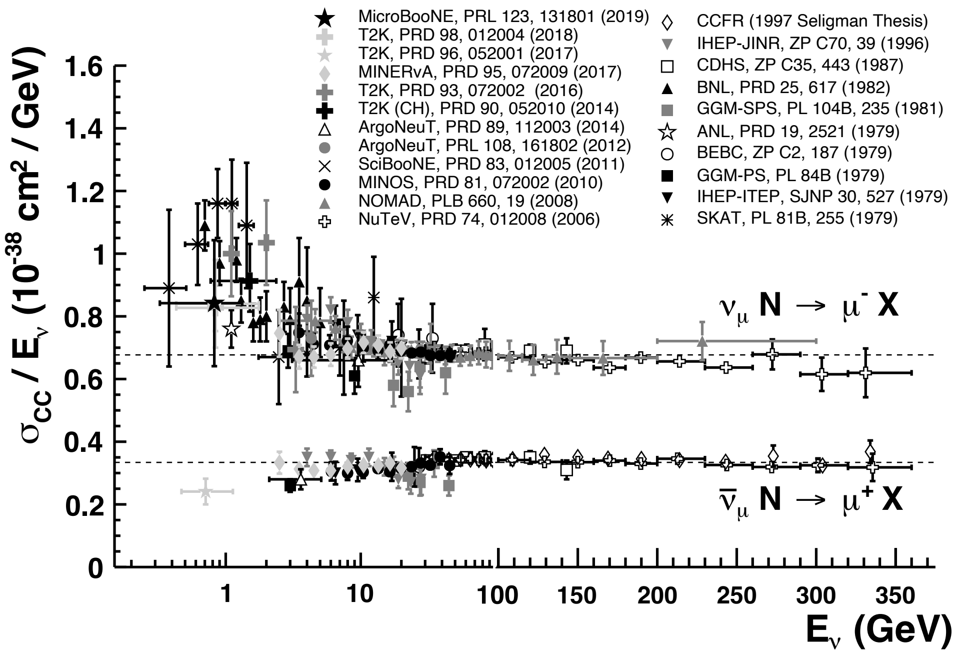

Figure 2.4 A compilation of neutrino cross-sections taken from `https://pdg.lbl.gov/2020/reviews/rpp2020-rev-nu-cross-sections.pdf`.

The pp cross-section is the largest as it is mediated by the strong force. The γp process is an electromagnetic process. While neutrinos interact only weakly (via the weak interaction). The strong and the electromagnetic forces are mediated by massless gauge bosons: the gluon and the photon. The difference in strength is given by an intrinsic difference between the couplings. These latter are respectively described by α_s and α. At the chosen p_{lab}, the center-of-mass energy is approximately 5 GeV. The values of the coupling constants are then: $\alpha_s(Q = 5\ GeV) \approx 0.2$ (see Figure 18.16 of the book) and $\alpha \approx 1/137$. Their ratio is:

$$\frac{\alpha}{\alpha_s} \approx 3.6 \times 10^{-2} \qquad \Longrightarrow \qquad \left(\frac{\alpha}{\alpha_s}\right)^2 \approx 10^{-3} \tag{2.62}$$

So, the difference of coupling constants is able to explain the majority of the difference between σ_{pp} and $\sigma_{\gamma p}$. The weak interaction is mediated by the massive gauge bosons $W^{\pm}$ and Z^0. The apparent weakness of the weak interaction at low energies actually comes from the heaviness of the gauge bosons, while the weak coupling constant is of the same order as the electromagnetic coupling constant. See Section 23.3 of the book for more details.

2.7 Mean free path of neutrinos in water

A neutrino with an energy of a few mega-electronvolts has an interaction cross-section with free protons of about $\sigma_{\nu p} \approx 10^{-43}\,\mathrm{cm}^2$. Compute the mean free path of such neutrinos in water.

Solution:

The mean free path λ is given by (equation 2.32 of the book):

$$\lambda = \frac{1}{\rho\sigma} \tag{2.63}$$

where ρ is the total number of target centers in the target volume and σ the cross-section. In this case we need to calculate the density of free protons per volume in water:

$$\rho = \frac{N_{\text{freep}}}{V} = 2\frac{N_A \times \rho_{H_2O}}{A_{H_2O}} = 2\frac{6.022 \cdot 10^{23}\text{mol}^{-1} \times 1\text{g/cm}^3}{18\text{g/mol}} = 6.7 \cdot 10^{22}\text{cm}^{-3} \tag{2.64}$$

Then, the mean free path of neutrinos in water is:

$$\lambda = \frac{1}{\rho\sigma_{\nu p}} = \frac{1}{6.7 \cdot 10^{22} \times 10^{-43}} = 1.5 \cdot 10^{20}\text{cm} = 1.5 \cdot 10^{18}\text{m} \tag{2.65}$$

To have an idea of what this distance represents we can convert it to light years (the distance that light travels in one year):

$$\lambda = 1.5 \cdot 10^{18}\text{m}\frac{1\text{ ly}}{365 \times 24 \times 3600 \times 3 \cdot 10^8\text{ m}} = 1.5 \cdot 10^{18}\text{m}\frac{1\text{ ly}}{9.46 \cdot 10^{15}\text{m}} = 159\text{ ly} \tag{2.66}$$

2.8 Neutron interactions

Estimate the mean free path of (a) a thermal, (b) a fast, and (c) a high-energy neutron in water.

Solution:

The mean free path λ is defined in Eq. (2.70) of the book as:

$$\lambda = \frac{1}{\rho\sigma} \tag{2.67}$$

where ρ is the density of scatterers or in our case the number of water molecules per unit volume. We hence find that:

$$\rho = \frac{\rho_{H_2O}(\text{g cm}^{-3}) \times N_A(\text{mol}^{-1})}{A(\text{g mol}^{-1})} \approx 3.3 \times 10^{22}\text{ cm}^{-3} \tag{2.68}$$

where we used the atomic number of water $A = 18$.

(a) **Thermal neutrons**: The energy of thermal neutrons can be taken as $E \approx kT \approx 3 \times 10^{-2}$ eV where we set $T = 293$ K. From Figure 2.28 from the book, we can estimate that the cross-section in water is about:

$$E \approx kT \approx 3 \times 10^{-2}\text{ eV} \quad \Longrightarrow \quad \sigma \approx 100\text{ barns} = 10^2 \times 10^{-24}\text{ cm}^2 = 10^{-22}\text{ cm}^2 \tag{2.69}$$

Consequently, the mean free path of thermal neutron in water can be estimated to be in the range of:

$$\lambda = \frac{1}{(3.3 \times 10^{22}\text{ cm}^{-3})(10^{-22}\text{ cm}^2)}\text{ă} \approx 0.30\,\text{cm} \tag{2.70}$$

(b) **Fast neutrons**: they are in the range of MeV. For definiteness, we take $E = 2$ MeV. From Figure 2.28 from the book, we have in this case

$$E = 2\text{ MeV} \quad \Longrightarrow \quad \sigma \approx 10\text{ barns} \quad \Longrightarrow \quad \lambda \approx 3\,\text{cm} \tag{2.71}$$

(c) **High-energy neutrons**: We note that above 1 GeV of energy, the energy for proton-neutron interaction is almost flat with $\sigma_{np} \approx 40$ mb. The cross-section on nuclei can be estimated to be:

$$\sigma_{n-nuclei} \approx 2\pi \left(1.3A^{1/3} - 0.6\right)^2 \text{ fm}^2 \approx 44\,\text{fm}^2 \text{ for } A = 16 \tag{2.72}$$

Since 1 $\text{fm}^2 = 10^{-26}\text{cm}^2$ (or 10 mb per water molecule), we find that the mean free path of high-energy neutron in water can be estimated to be:

$$\lambda = \frac{1}{(3.3 \times 10^{22}\,\text{cm}^{-3})(10^{-26}\ \text{cm}^2)}\text{ă} \approx 3 \times 10^3\,\text{cm} = 30\ \text{m} \tag{2.73}$$

2.9 GEANT4 simulation

Using the GEANT4 framework, set up the simulation of muons crossing 50 slabs of iron, each 2 cm thick. Shooting muons of 1 GeV, study the effect of multiple scattering. Do you see δ-rays? What happens if you increase the energy of the muons, to say, 10 GeV, 50 GeV, 200 GeV?

Solution:

In the book, we have seen that charged particles crossing matter will lose energy due to multiple collisions with the atomic components of the target material. Assuming that the atomic nuclei are much more massive than the incident charged particle, individual scattering with one of those will be governed by the Rutherford scattering cross-section. Hence, most collisions will result in a small angular deflection in the forward direction. The cumulative is a random net deflection as it traverses the target medium.

Stochastic processes can be conveniently addressed using Monte-Carlo techniques (see Appendix H of the book). Here, we use the program available on GitHub[1]in the corresponding folder for this chapter. The program is called ExampleB1. The program is based on the GEANT4 framework, introduced in Section 2.22 of the book. A simulation is a sort of "virtual reality". Simulation is used both to help designing detectors during R&D phase and understanding the response of the detector for the physics studies. To create such virtual reality we need to model the particle-matter interactions, geometry and materials in order to propagate elementary particles into the detector. We need also to describe the sensitivity of the detector for generating raw data.

In the following, we recreate in the simulation our 50 slabs of iron, each 2 cm thick and we will shoot muons of different energies at it. We will follow the physics predictions of GEANT4.

Main program: The main program initializes several part of the GEANT4 framework and calls the specific parts of the code that define our geometry and the specific parameters of our example. As an analogy of the real experiment, GEANT4 has a "run" manager. A run is composed of events that describe the simulation of one event. We will consider a run composed of 1 million muons crossing the 50 slabs of iron. Each crossing muon will be stored in an event. The main program initializes the run manager, define the class to call to define the geometry ("G4DetectionConstructionB1"), initializes the physics (in our case we take a predefined list of physics processes called "QBBC"), and then define the class to be called for the initialization of the required actions ("G4ActionInitializationB1"). And then it goes with run manager ("runManager->Initialize()").

```
int main(int argc, char **argv)
{
  // construct the default run manager
  G4RunManager* runManager = new G4RunManager;

  // Detector construction
  runManager->SetUserInitialization(new G4DetectorConstructionB1());
```

1 https://github.com/CambridgeUniversityPress/Phenomenology-Particle-Physics

```
  // Physics list
  G4VModularPhysicsList* physicsList = new QBBC;
  physicsList->SetVerboseLevel(1);
  runManager->SetUserInitialization(physicsList);

  // User action initialization
  runManager->SetUserInitialization(new G4ActionInitializationB1());

  // initialize G4 kernel
  runManager->Initialize();

  (...)
}
```

Class G4DetectorConstructionB1: The geometry class G4DetectorConstructionB1 creates our virtual world. It starts with a box called "world" with dimensions 20 m × 20 m × 20 m filled with the material "Air" (note that GEANT4 uses half-lengths to define geometries). This box is then placed at the origin of our physical world. One then creates a slab with dimensions 1 m × 1 m × 2 cm made of iron. We place 50 of these slabs in a logical volume called "Detector". This detector is placed into the logical world. Finally, we create a volume called "Counter" which is placed after the 50 slabs in order to count exiting muons. That's it! We create our virtual world.

```
G4VPhysicalVolume* G4DetectorConstructionB1::Construct()
{
  G4bool checkOverlaps = false;

  G4double world_hx = 10.0*m;
  G4double world_hy = 10.0*m;
  G4double world_hz = 10.0*m;

  // Get nist material manager
  G4NistManager* nist = G4NistManager::Instance();

  G4Box* worldBox
     = new G4Box("World", world_hx, world_hy, world_hz);

  G4Material* world_mat = nist->FindOrBuildMaterial("G4_AIR");

  G4LogicalVolume* logicalWorld
    = new G4LogicalVolume(worldBox, world_mat, "World");

  G4VisAttributes* worldVA = new G4VisAttributes;
  worldVA -> SetVisibility(false);
  logicalWorld -> SetVisAttributes(worldVA);

  G4VPhysicalVolume* physWorld =
    new G4PVPlacement(0,                    //no rotation
                      G4ThreeVector(),      //at (0,0,0)
                      logicalWorld,         //its logical volume
                      "World",              //its name
                      0,                    //its mother  volume
                      false,                //no boolean operation
                      0,                    //copy number
                      checkOverlaps);       //overlaps checking
```

```
  G4Box* DetBox
     = new G4Box("Slab", 0.5*m, 0.5*m, 1.0*cm);

  G4Material* det_mat = nist->FindOrBuildMaterial("G4_Fe");

  G4LogicalVolume* logicaldet
    = new G4LogicalVolume(DetBox, det_mat, "Detector");

  G4Colour brown(0.7, 0.4, 0.1);
  G4VisAttributes* DetVisAttributes = new G4VisAttributes(brown);
  logicaldet -> SetVisAttributes(DetVisAttributes);

  for(int i=0;i<50;i++) {
    G4double zpos = i*2.0*cm;
    new G4PVPlacement(0,                     //no rotation
                      G4ThreeVector(0,0,zpos),
                      logicaldet,            //its logical volume
                      "Detector",            //its name
                      logicalWorld,          //its mother volume
                      false,                 //no boolean operation
                      0,                     //copy number
                      checkOverlaps);        //overlaps checking
  };

  G4Box* CounterBox
     = new G4Box("Counter", 0.5*m, 0.5*m, 1.0*cm);

  G4Material* counter_mat = nist->FindOrBuildMaterial("G4_AIR");

  G4LogicalVolume* logicalcounter
    = new G4LogicalVolume(CounterBox, counter_mat, "Counter");

  G4double zpos = 100*cm;
  new G4PVPlacement(0,                       //no rotation
                    G4ThreeVector(0,0,zpos),
                    logicalcounter,          //its logical volume
                    "Counter",               //its name
                    logicalWorld,            //its mother volume
                    false,                   //no boolean operation
                    0,                       //copy number
                    checkOverlaps);          //overlaps checking

  return physWorld;
}
```

Class G4ActionInitializationB1: After the creation of our virtual world, the initialization continues with the setting up of various user actions, which will be called at particular moments in our simulation. We define an action to (a) create the generator of primary particles (in our case the muon of a given energy), (b) an action to be called at specific moments of the run, (c) an action to be called at specific moments of each event, (d) an action to be called at each "step" in the simulation of the track of our muon.

```
void G4ActionInitializationB1::Build() const
{
  SetUserAction(new G4PrimaryGeneratorActionB1());
  SetUserAction(new G4RunAction());
```

```
  SetUserAction(new G4EventAction());
  SetUserAction(new G4SteppingAction());
}
```

Class G4PrimaryGeneratorActionB1: At beginning of processing, an event contains primary particles. These primaries are pushed into a list and processed one by one. When the list becomes empty, processing of an event is over. The class G4PrimaryGeneratorActionB1 in our example in creates a single muon with momentum oriented in the z-axis direction and an energy of 5 GeV.

```
G4PrimaryGeneratorActionB1::G4PrimaryGeneratorActionB1():
                        G4VUserPrimaryGeneratorAction(), fParticleGun(0)
{
  G4int n_particle = 1;
  fParticleGun  = new G4ParticleGun(n_particle);

  // default particle kinematic
  G4ParticleTable* particleTable = G4ParticleTable::GetParticleTable();
  G4String particleName;
  G4ParticleDefinition* particle
    = particleTable->FindParticle(particleName="mu-");
  fParticleGun->SetParticleDefinition(particle);
  fParticleGun->SetParticleMomentumDirection(G4ThreeVector(0.,0.,1.));
  fParticleGun->SetParticleEnergy(5.*GeV);
}

void G4PrimaryGeneratorActionB1::GeneratePrimaries(G4Event* anEvent)
{
  //this function is called at the beginning of each event
  //
  G4double x0 = 0;
  G4double y0 = 0;
  G4double z0 = -0.5*m;
  fParticleGun->SetParticlePosition(G4ThreeVector(x0,y0,z0));
  fParticleGun->GeneratePrimaryVertex(anEvent);
}
```

Class G4RunAction: As an analogy of the real experiment, a run of GEANT4 starts with “Beam On”. Conceptually, a run is a collection of events which share the same detector conditions. In our example, we create an ntuple which will contain relevant variables for each event that will be stored as a sequence in a file called “simple_simu.root”.

```
G4RunAction::G4RunAction() : G4UserRunAction()
{
  fNtupleMgr = new G4NtupleManager("simple_simu.root");
}
```

Class G4NtupleManager: The G4NtupleManager creates our ntuple. We decide to store 8 observables per event. They are: (1) the initial muon energy ("EnergyAtEmission"), (2) the muon energy after it has traversed the 50 slabs of iron ("EnergyAfterCAL"), (3) how many δ-rays have been produced by the muon ("DeltaRayCounter"), (4) the initial transverse position of the muon ("XInit","Yinit"), (5) the final transverse position of the muon ("XFinal", "Yfinal"), and (6) the angle of the muon with respect to the z-axis at the exit ("OutAngle").

```
void G4NtupleManager::Book()
{
  G4AnalysisManager* analysisMgr = G4AnalysisManager::Instance();
```

```
    analysisMgr->OpenFile(fFileName);
    analysisMgr->CreateNtuple("ntuple", "ntuple");
    analysisMgr->CreateNtupleDColumn("EnergyAtEmission");
    analysisMgr->CreateNtupleDColumn("EnergyAfterCAL");
    analysisMgr->CreateNtupleIColumn("DeltaRayCounter");
    analysisMgr->CreateNtupleDColumn("XInit");
    analysisMgr->CreateNtupleDColumn("YInit");
    analysisMgr->CreateNtupleDColumn("XFinal");
    analysisMgr->CreateNtupleDColumn("YFinal");
    analysisMgr->CreateNtupleDColumn("OutAngle");
    analysisMgr->FinishNtuple();
}
```

Class G4EventAction: This class defines the actions to be called at specific moments of each event. We define the "EndOfEventAction" which is called at the end of the processing of one event. We use it to fill our ntuple with the results of the simulation of one muon crossing through the slabs.

```
void G4EventAction::EndOfEventAction(const G4Event*)
{
    G4AnalysisManager* analysisMgr = G4AnalysisManager::Instance();

    G4PrimaryGeneratorActionB1* primary =
    (G4PrimaryGeneratorActionB1*) G4RunManager::GetRunManager()->
                                  GetUserPrimaryGeneratorAction();

    G4double initialEnergy = primary->GetParticleGun()->GetParticleEnergy()/GeV;

    // Fill the ntuple
    analysisMgr->FillNtupleDColumn(0, initialEnergy);
    analysisMgr->FillNtupleDColumn(1, fFinalEnergy);
    analysisMgr->FillNtupleIColumn(2, fDeltaRayCount);
    analysisMgr->FillNtupleDColumn(3, fVecInit.x());
    analysisMgr->FillNtupleDColumn(4, fVecInit.y());
    analysisMgr->FillNtupleDColumn(5, fVecFinal.x());
    analysisMgr->FillNtupleDColumn(6, fVecFinal.y());
    analysisMgr->FillNtupleDColumn(7, fAngle);
    analysisMgr->AddNtupleRow();
}
```

Class G4SteppingAction: This class defines the action to be called at every "step" of the simulation of the muon. It is the most complicated one. As the muon track is propagated through the medium, it can undergo several processes. At each step we check if the muon is inside the Detector volume. If it is and it underwent a process called "muIoni", which involves the emission of a δ-ray, we increase a counter calle fDeltaRayCount. On the other hand, if our muon has reached the volume Counter, we store its final kinetic energy, its final position in the transverse plane, and the relevant scattering angle. Once the muon has reached the counter, we "kill" it, since we obtained the information we wanted and there is no point continuing to process it.

```
void G4SteppingAction::UserSteppingAction(const G4Step* aStep)
{
    G4PhysicalVolumeStore* storePV = G4PhysicalVolumeStore::GetInstance();
    G4EventAction* evtAction = (G4EventAction*)
                   G4RunManager::GetRunManager()->GetUserEventAction();

    // The particle track
    G4Track* aTrack = aStep->GetTrack();
```

```
  // The volumes
  G4VPhysicalVolume* prePhysVol = aStep->GetPreStepPoint()
                ->GetTouchableHandle()->GetVolume();
  G4VPhysicalVolume* postPhysVol = aStep->GetPostStepPoint()
                ->GetTouchableHandle()->GetVolume();

  // Process
  const G4VProcess* process = aStep->GetPostStepPoint()
                ->GetProcessDefinedStep();

  // Primary track only
  if (aTrack->GetParentID() == 0)
    {
      // In detector
      if (prePhysVol == storePV->GetVolume("Detector")
          ||
          postPhysVol == storePV->GetVolume("Detector"))
        {
          // Delta electron along muon ionisation path
          if (process->GetProcessName() == "muIoni")
                        evtAction->fDeltaRayCount += 1;
        }

      // In counter module
      if (prePhysVol != storePV->GetVolume("Counter")
          &&
          postPhysVol == storePV->GetVolume("Counter"))
        {
          evtAction->fFinalEnergy = aTrack->GetKineticEnergy()/GeV;
          evtAction->fVecFinal.setX(aTrack->GetPosition().x());
          evtAction->fVecFinal.setY(aTrack->GetPosition().y());

          // Calculate outgoing angle
          G4ThreeVector finalMom = aTrack->GetMomentumDirection();
          G4ThreeVector initMom = aTrack->GetVertexMomentumDirection();
          evtAction->fAngle = finalMom.angle(initMom);
          aTrack->SetTrackStatus(fStopAndKill);
        }
    }
}
```

Running B1 in interactive mode: In order to run the code in interactive mode, we add line to instantiate G4UIExecutive as shown in the code below:

```
int main(int argc, char **argv)
{

 (...)
   // Instantiation and initialization of the Visualization Manager
#if G4VIS_USE
   G4VisManager* visManager = new G4VisExecutive;
   // G4VisExecutive can take a verbosity argument - see /vis/verbose guidance.
   // G4VisManager* visManager = new G4VisExecutive("Quiet");
   visManager->Initialize();
#endif
```

```
   G4UIExecutive* ui = new G4UIExecutive(argc, argv);
   ui->SessionStart();

   (...)
}
```

We use the B1.mac to setup the interactive environment:

```
/vis/open OGLI
/vis/drawVolume
/vis/scene/add/trajectories smooth rich
/vis/modeling/trajectories/create/drawByParticleID
/vis/modeling/trajectories/create/drawByCharge
/vis/filtering/trajectories/create/particleFilter
/vis/filtering/trajectories/particleFilter-0/add nu_e
/vis/filtering/trajectories/particleFilter-0/add nu_mu
/vis/filtering/trajectories/particleFilter-0/add anti_nu_e
/vis/filtering/trajectories/particleFilter-0/add anti_nu_mu
/vis/filtering/trajectories/particleFilter-0/invert true

/vis/viewer/set/viewpointThetaPhi -90 0
/gun/particle mu+
/gun/energy 1 GeV
/run/beamOn 1
```

We then use the B1.mac macro to start an interactive session:

```
$ ./exampleB1
```

Once the graphical window opens, we type "/control/execute B1.mac". The 3D view of the event appears. We can use the mouse to rotate the view. The result of a simulation of one 1 GeV muon is shown in Figure 2.5. We note that the muon also suffers from energy losses (see Section 2.17 of the book) and ranges out before reaching the end of the 50 slabs and stops inside the detector. The simulation of several positive muons entering the detector is report in Figure 2.6. The stochastic nature of the multiple scattering is clearly visible.

Running B1 in batch mode:

We use the B1batch.mac macro to shoot 1 million positive muons of a given energy:

```
/gun/particle mu+
/gun/energy 50 GeV
/run/beamOn 1000000
```

To run the code, we use the Unix command:

```
$ ./exampleB1 B1batch.mac
```

The output is written in the ntuple which can be analysed with the ROOT macro discussed in the following.

ROOT macro analyse.C: In order to analyse the ntuples generated by the GEANT4 simulation, we use the CERN ROOT package (`https://root.cern`). ROOT is an open-source data analysis framework used by high energy physics and others. It emables the analysis and visualization of large amounts of data. It comes with a C++ interpreter, ideal for fast prototyping. We list in the following a Macro which is essentially a small piece of C++ code, that we use to analyse the ntuple and plot an histogram of the angle of the exiting muons.

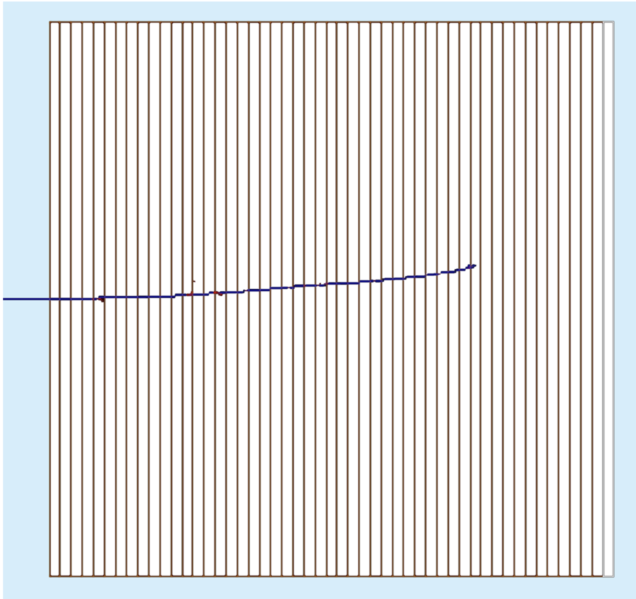

Figure 2.5 Simulation of a 1 GeV positive muon entering a detector composed of 50 2 cm thick iron slabs. The main muon track is visible in blue. The short red tracks represent δ-rays (electrons).

```
void analyse() {

  TFile *f10 = new TFile("simple_simu_10GeV.root");
  TFile *f50 = new TFile("simple_simu_50GeV.root");

  TNtuple *nt10 = (TNtuple*)f10->Get("ntuple");
  cout <<"nt 10 GeV filled with "<<nt10->GetEntries()<<" entries"<<endl;
  TNtuple *nt50 = (TNtuple*)f50->Get("ntuple");
  cout <<"nt 50 GeV filled with "<<nt50->GetEntries()<<" entries"<<endl;

  TH1F * AA = new TH1F("AA","Muons through 50 2cm-thick slabs of iron", 100, 0., 0.2);
  AA->GetXaxis()->SetTitle("Angle at exit (rad)");
  AA->GetYaxis()->SetTitle("Number of events");
  AA->SetMaximum(5e5);
  AA->SetFillColor(kRed);
  AA->SetFillStyle(3003);

  TH1F * BB = new TH1F("BB","Angle at output", 100, 0., 0.2);
  BB->SetFillColor(kGreen);
  BB->SetFillStyle(3003);

  nt10->Draw("OutAngle>>AA");
  nt50->Draw("OutAngle>>BB");

  gStyle->SetOptStat(0);
  AA->Draw();
  BB->Draw("same");
```

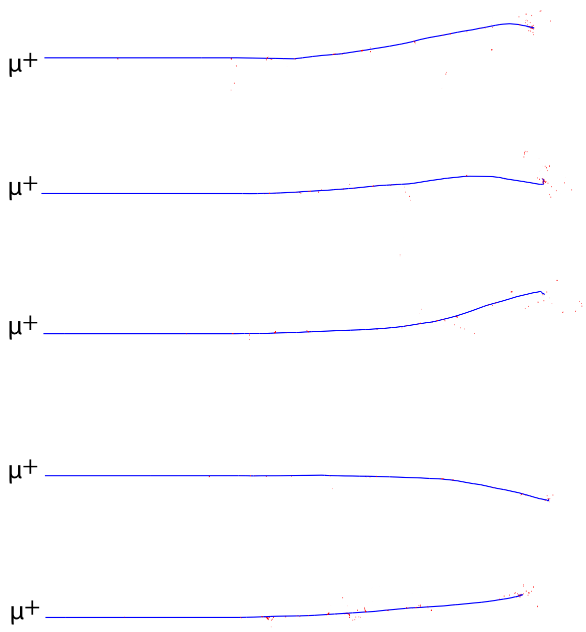

Figure 2.6 Simulation of several positive muons entering the detector. The stochastic nature of the multiple scattering is clearly visible.

```
  gPad->SetLogy();

  auto legend = new TLegend(0.6,0.7,0.9,0.9);
  legend->AddEntry("AA","10 GeV \\mu","f");
  legend->AddEntry("BB","50 GeV \\mu","f");
  legend->Draw();
}
```

The Macro can be run with the Unix command:

```
$ root -L analyse.C
```

The output is shown in Figure 2.7. The two histograms show the distribution of the scattering angle of 1 million muons after traversing the 50 2 cm thick slabs of iron. We note that the distribution has a dominant part which is the result of many small scatters through the medium. The distributions also contain a so-called non-Gaussian tail, visible at large angles, where the muons underwent a single very large scatter as they closely approached the scattering centers and suffered a Rutherford-like scattering.

Ignoring the effect of large angle scatters, the Molière theory states that the scattering angle distribution at the exit of the target of a given thickness x is distributed as a Gaussian with RMS given by Eqs. (2.144) and (2.145) of the book:

$$\theta_0 = \frac{13.6\ \mathrm{MeV}}{\beta c p} z \sqrt{\frac{x}{X_0}} \left[1 + 0.038 \ln \left(\frac{x z^2}{X_0 \beta^2} \right) \right] \tag{2.74}$$

where

$$\theta_0 = \theta_{plane}^{RMS} = \frac{1}{\sqrt{2}} \theta_{space}^{RMS} \tag{2.75}$$

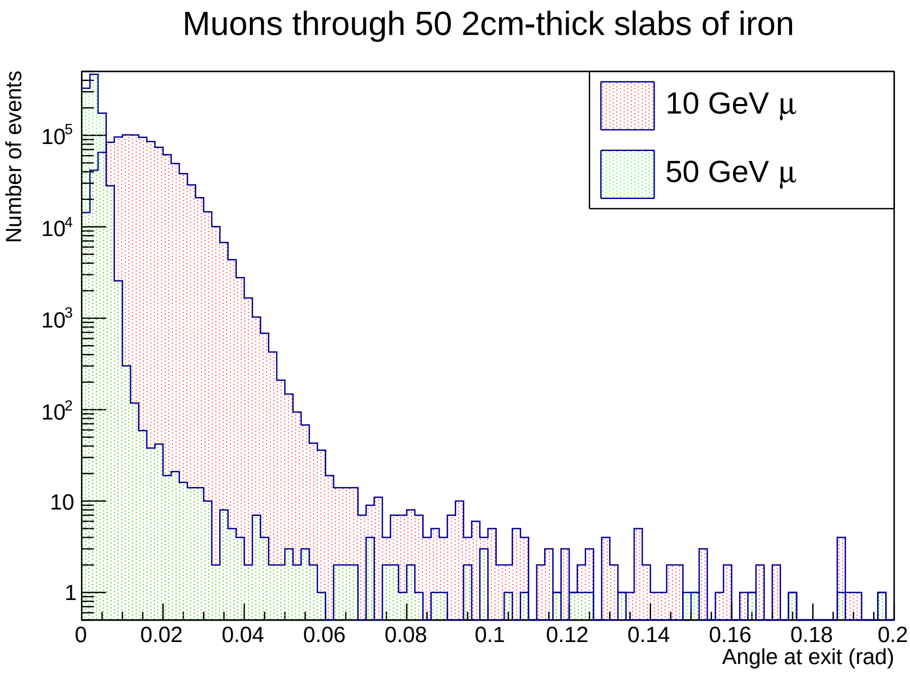

Figure 2.7 Distribution of the scattering angle of 1 million muons after traversing 50 2 cm thick slabs of iron, simulated with GEANT4. Non-Gaussian tails (large angle scatters) are visible at large angles.

and $p, \beta c$, and z are the momentum, velocity, and charge of the incident particle. X_0 is the **radiation length** of the target material, defined in Eq. (2.130) of the book. See Table 2.1 of the book for some values.

The results obtained with the GEANT4 simulations are compared to the predictions of Eq. (2.74) in Table 2.3. They compare well.

Momentum p	Average angle at exit (GEANT4)	θ^{RMS}_{space} (Eq. (2.74))
10 GeV	15 mrad	16.7 mrad
50 GeV	2.8 mrad	3.3 mrad
200 GeV	0.7 mrad	0.8 mrad

Table 2.3 Comparison of GEANT4 simulations and the analytical prediction of the scattering angle after fifty 2 cm thick iron slabs.

3 Overview of Accelerators and Detectors

3.1 Synchrotron radiation

A stored charged particle loses energy by synchrotron radiation because it is subjected to an acceleration.

1. *Show that the average power radiated in a circular ring is given by:*

$$P_{rad} \simeq \frac{2e^2 c \gamma^4}{3R^2} \qquad (cgs\ unit) \tag{3.1}$$

2. *Show that the radiated energy per turn is:*

$$E_{turn} = \frac{4\pi}{3} \frac{e^2 \gamma^4}{R} \qquad (cgs\ unit) \tag{3.2}$$

3. *Show that at an equivalent energy, electrons will radiate much more than protons.*

Solution:

1. The relativistic Larmor formula expresses the total radiated power by a relativistic particle subject to an acceleration and is given by[1]:

$$P_{rad} = \frac{2e^2}{3c} \gamma^6 \left(|\dot{\vec{\beta}}|^2 - (\vec{\beta} \times \dot{\vec{\beta}})^2 \right) \qquad \text{(cgs unit)} \tag{3.3}$$

where $\vec{\beta}$ is the particle velocity. For a circular motion, we have that $\vec{\beta} \perp \dot{\vec{\beta}}$ and hence the above equation reduces to:

$$P_{rad} = \frac{2e^2}{3c} \gamma^6 |\dot{\vec{\beta}}|^2 (1 - |\vec{\beta}|^2) = \frac{2e^2}{3c} \gamma^4 |\dot{\vec{\beta}}|^2 \tag{3.4}$$

Since $|\dot{\vec{\beta}}| = |\dot{\vec{v}}|/c = v^2/(cR)$, we write:

$$P_{rad} = \frac{2e^2}{3c^3} \gamma^4 |\dot{\vec{v}}|^2 = \frac{2e^2 c}{3R^2} \gamma^4 \beta^4 \tag{3.5}$$

In the ultra relativistic limit, $\beta \to 1$:

$$P_{rad} \simeq \frac{2e^2 c}{3R^2} \gamma^4 (\text{cgs unit}) = \frac{2e^2 c}{3(4\pi\epsilon_0)R^2} \gamma^4 (\text{SI unit}) = \frac{e^2 c}{6\pi\epsilon_0 R^2} \gamma^4 (\text{SI unit}) \tag{3.6}$$

1 The full derivation of this formula can be obtained using Classical Electrodynamics, see e.g. Jackson, *Classical Electrodynamics*, 3rd Edition, Wiley (1998).

2. The energy loss per turn is given by the integral:

$$E_{turn} = \oint_C dt\ P_{rad} = P_{rad} \oint dt = P_{rad} \frac{2\pi R}{c} = \frac{4\pi e^2}{3R}\gamma^4 \qquad \text{(cgs unit)} \tag{3.7}$$

where we use that the period is defined such that $T = 2\pi R/c$.

3. The power emitted is proportional to the inverse mass squared of the accelerated particle such that $P \sim \gamma^4 = (E/m)^4$. For a similar energy we find that:

$$\left(\frac{\gamma_e}{\gamma_p}\right)^4 = \left(\frac{m_p}{m_e}\right)^4 \simeq 10^{13} \tag{3.8}$$

Light particles emit much more power than heavy particles! The energy radiated, E_{turn}, is lost during each turn and the particles must be re-accelerated by the same amount for the beam to remain at a constant energy.

For electron-positron colliders, synchrotron radiation becomes a limiting factor. For example, the CERN LEP collider has a radius $R = 4.3$ km, hence for 100 GeV electrons, we find $\gamma \simeq 2 \times 10^5$ and the loss per turn (in SI units we must add a $4\pi\epsilon_0$ in the denominator) is:

$$\begin{aligned} E_{turn} &= \frac{4\pi e^2}{3(4\pi\epsilon_0)R}\gamma^4 = \frac{e^2}{3\epsilon_0 R}\gamma^4 = \frac{(1.602 \times 10^{-19}\,\mathrm{C})^2}{3(8.85 \times 10^{-12}\,\mathrm{Fm}^{-1})}\frac{\gamma^4}{R} \\ &\approx (10^{-27}\,\mathrm{Jm})\frac{\gamma^4}{R} \approx 3.72 \times 10^{-10}\,\mathrm{J} \approx 2.3\ \mathrm{GeV} \end{aligned} \tag{3.9}$$

In comparison, for the LHC in the same LEP tunnel, protons with an energy of 7 TeV have $\gamma \simeq 7 \times 10^3$ and hence

$$E_{turn} \approx 5.6 \times 10^{-16}\,\mathrm{J} \approx 3.5\ \mathrm{keV} \tag{3.10}$$

which is negligible.

3.2 Energy stored in the LHC beams.

Verify the claim that for a total current of 0.5 A, the total energy stored in the LHC beams is 360 MJ. In this case, what is the number of protons per bunch?

Solution:

The period of the protons around the LHC ring of radius $R = 4.3$ km is (assuming the protons travels at the speed of light):

$$\text{Period: } T \approx \frac{2\pi R}{c} = \frac{27 \times 10^3\,\mathrm{m}}{c} \approx 9 \times 10^{-5}\,\mathrm{s} \approx 90\,\mu\mathrm{s} \tag{3.11}$$

The total charge stored in the LHC is therefore:

$$Q = It = (0.5\,\mathrm{A})(9 \times 10^{-5}\,\mathrm{s}) = 4.5 \times 10^{-5}\,\mathrm{C} \tag{3.12}$$

This charge corresponds to

$$N_p = \frac{Q}{e} \approx 2.8 \times 10^{14}\,\text{protons} \tag{3.13}$$

The total energy stored for 8 TeV beams is therefore:

$$E = N_p \times (8\ \mathrm{TeV}) \approx (2.8 \times 10^{14})(1.28 \times 10^{-6}\,\mathrm{J}) \approx 3.6 \times 10^8\,\mathrm{J} \tag{3.14}$$

Since there are 2'800 bunches in total, each bunch has 10^{11} protons.

3.3 ILC luminosity

The foreseen ILC collider needs to reach a luminosity up to 5×10^{34} cm^{-2} s^{-1}. Estimate the transverse dimensions of the bunches.

Solution:

There are several scenarios for the ILC. We consider the baseline scenario, where an electron beam accelerated along a straight line to energy of 125 GeV collides head-on with a positron beam similarly accelerated in the opposite direction, resulting in collision energy of 250 GeV. Each beam consists of 5 trains per second (i.e., the repetition rate is 5 Hz) where one train is roughly 1 ms long consisting of 1312 bunches. One bunch contains 2×10^{10} particles. A bunch is very flat and long: its size (rms) at the collision point is σ_y high, σ_x wide, and 300 μm long. For technical reasons, one has $\sigma_y \ll \sigma_x$.

Equation (3.7) of the book gives the estimate for the luminosity of a collider based on its parameters:

$$\mathcal{L} = \frac{fnN_1N_2}{4\pi\sigma_x\sigma y} = \frac{fnN^2}{4\pi\sigma_x\sigma_y} \qquad \Longrightarrow \qquad \sigma_x\sigma_y = \frac{fnN^2}{4\pi\mathcal{L}} \tag{3.15}$$

where we set $N_1 = N_2 = N$. We find that:

$$\sigma_x\sigma_y \approx \frac{(5\,\text{Hz})(1312)(2\times10^{10})^2}{4\pi(5\times10^{34}\ \text{cm}^{-2}\ \text{s}^{-1})} \approx 4\times10^{-12}\ \text{cm}^2 = 400\ \text{nm}^2 \tag{3.16}$$

This is a very small beam spot! The actual target sizes for the bunch sizes at the collision points are 7.7 nm high and 516 nm wide. This gives an area of 4000 nm^2. So, our choice of baseline parameters cannot reach a luminosity of 5×10^{34} cm^{-2} s^{-1}.

We however note that the formula we used yields the so-called "geometrical" luminosity. In practice, one has to include additional effects that increase the luminosity of a collider. One writes:

$$\mathcal{L} = H_D\frac{fnN^2}{4\pi\sigma_x\sigma_y} \tag{3.17}$$

where H_D is a factor that contains the impact of beam–beam forces and other relevant effects. The beams are so dense that they generate very strong electromagnetic fields. In an electron–positron collider they focus each other. This so-called pinch effect reduces the effective beam size and leads to an increase in luminosity. The factor H_D is typically in the order of $1.5 - 2.5$! If we plug in the ILC baseline parameters, we then get:

$$\mathcal{L} = H_D\frac{fnN^2}{4\pi\sigma_x\sigma_y} = (2.5)\frac{(5)(1312)(2\times10^{10})^2}{4\pi(7.7\,\text{nm})(516\,\text{nm})} \approx 1.3\times10^{34}\ \text{cm}^{-2}\ \text{s}^{-1} \tag{3.18}$$

In conclusion, a luminosity of 5×10^{34} cm^{-2} s^{-1} will require a further increase in the number of bunches and also a further reduction of the beam spot size.

3.4 Probing the Planck scale

The direct study of the quantum nature of the gravitational force would require reaching the Planck scale defined by $\Lambda_{Pl} \simeq \sqrt{\hbar c/G}$. What kind of collider using a reasonable extrapolation of current technologies would be necessary to reach such an energy frontier?

Solution:

The Planck scale is given by

$$\begin{aligned}\Lambda_{Pl} &\simeq \sqrt{\frac{\hbar c}{G}} = \sqrt{\frac{197\,\text{MeV fm}}{6.67\times10^{-11}\,\text{J m kg}^{-2}}} \approx \sqrt{\frac{197\,\text{MeV fm}}{4.17\times10^{23}\,\text{eV fm kg}^{-2}}} \\ &\approx 2.2\times10^{-8}\,\text{kg} = 2\times10^{9}\,\text{J/c}^2\text{ă} = 1.2\times10^{28}\,\text{eV/c}^2 = 1.2\times10^{19}\,\text{GeV/c}^2\end{aligned} \tag{3.19}$$

We can consider both circular and linear colliders.

1. Circular collider: In order to reach a center-of-mass energy $\sqrt{s} \simeq 1.2 \times 10^{19}$ GeV, one would need two colliding beams of $E_b = 6 \times 10^{18}$ GeV. The LHC dipole magnets reach a field of 8 Tesla. In the context of the next generation FCC-hh collider, CERN is developing new dipole magnets of 16 Tesla. Let's assume that one can reach 20 Tesla. To calculate the curvature of a particle in this magnetic field, we use Eq. (3.1) of the book, which represents the cyclotron condition:

$$p = \gamma m v = eBR \qquad \Longrightarrow \qquad R = \frac{\gamma m v}{eB} = \frac{E_b}{ecB} \tag{3.20}$$

   Hence:

$$R = \frac{6 \times 10^{18}\,\text{GeV}}{ec(20\ \text{T})} \approx \frac{10^9\ \text{J}}{(1.602 \times 10^{-19}\,\text{C})(3 \times 10^8 \text{m s}^{-1})(20\ \text{kg s}^{-2}\ \text{A}^{-1})} \approx 10^{18}\,\text{m} \approx 1000\,\text{ly} \tag{3.21}$$

   This radius is comparable to the size of our galaxy.

2. Linear collider: the SLC collider at SLAC reached an accelerating gradient of 20 MeV/m using RF cavities. The ILC and the CLIC studies foresee respectively 31.5 MeV/m and 100 MeV/m also using RF cavities. A new technique for acceleration is provided by plasma wakefield acceleration (PWFA). These can be broken down into laser wakefield acceleration and beam wakefield acceleration. Both rely on plasma as a medium, but to drive the wake, one technique uses lasers while the other uses a beam of particles. If this technique is mastered, it would have huge potential. For example, an energy gain of more than 40 GeV was achieved using the SLAC SLC beam (42 GeV) in just 85 cm using a plasma wakefield accelerator!

   Let us assume that one reaches an acceleration gradient 100 GeV/m. We need to accelerate our beams to $E_b = 6 \times 10^{18}$ GeV. Hence the length of each leg of the linear accelerator should be:

$$L \simeq \frac{6 \times 10^{18}\,\text{GeV}}{100\ \text{GeV/m}} = 6 \times 10^{16}\ \text{m} \approx 6\ \text{ly} \tag{3.22}$$

   This is comparable to the distance between two stellar systems.

Directly probing the Planck scale using accelerators to study center-of-mass energies close to the Planck scale is clearly not practical! And we haven't discussed the issue of the luminosity.

3.5 Neutrino collider

Is it possible to build a neutrino collider to study $\nu + \bar{\nu} \to Z^0$ production?

Solution:

Using Eq. (26.95) of the book, we can estimate the cross-section at the Z^0 pole:

$$\sigma^0(\nu\bar{\nu} \to Z^0 \to f\bar{f}) \simeq 12\pi \frac{\Gamma_{\nu\nu}\Gamma_{ff}}{M_Z^2 \Gamma_Z^2} \tag{3.23}$$

The total cross-section (including also decay to neutrinos – this won't be seen but here we are just interested in order of magnitudes) is then found replacing Γ_{ff} by Γ_Z. We get:

$$\begin{aligned} \sigma^0(\nu\bar{\nu} \to Z^0) &\simeq 12\pi \frac{\Gamma_{\nu\nu}}{M_Z^2 \Gamma_Z} \approx 12\pi \frac{0.17\,\text{GeV}}{(91.2\,\text{GeV})^2 (2.5\,\text{GeV})} \approx 3 \times 10^{-4}\,\text{GeV}^{-2} \approx 1.2 \times 10^{-4}\,\text{mb} \\ &\approx 120\,\text{nb} \approx 1.2 \times 10^{-31}\,\text{cm}^2 \end{aligned} \tag{3.24}$$

where we used the values from Eqs. (26.77) and (26.79) of the book, and also that 1 nb = $10^{-33}\,\mathrm{cm}^2$.

If we aim at having one Z^0 event per second ($R = 1$ Hz), the requirement for the luminosity of the collider is then

$$R = \mathcal{L}\sigma \qquad \Longrightarrow \qquad \mathcal{L}\breve{a} = \frac{R}{\sigma} \approx \frac{1\,\mathrm{event/s}}{1.2\times 10^{-31}\,\mathrm{cm}^2} = 8\times 10^{30}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1} \tag{3.25}$$

The luminosity of a collider can be estimated from the geometrical parameters of the beams using Eq. (3.7) of the book:

$$\mathcal{L} = fn\frac{N_1N_2}{4\pi\sigma_x\sigma_y} \tag{3.26}$$

Once a neutrino beam is created, it cannot be focalized since it is composed of neutral particles. The parent meson should be focalized before they decay. The dimension of the decay tunnel and the kinematics of the meson decays defines the neutrino beam divergence. Let's assume for the sake of discussion that we succeeded in creating a neutrino beam with transverse dimensions such that

$$\sigma_x \approx \sigma_y \approx 100\,\mathrm{cm} \tag{3.27}$$

We then get:

$$fnN_1N_2 = 4\pi\sigma_x\sigma_y\mathcal{L} \approx 10^{36}\,\mathrm{s}^{-1} \tag{3.28}$$

Neutrinos can be produced by a pulse of high-energy protons hitting a target every second. So, $f = 1$ Hz, and we have:

$$N_1N_2 = N^2 \approx \frac{10^{36}\,\mathrm{s}^{-1}}{(1\,\mathrm{Hz})} = 10^{36} \qquad \Longrightarrow \qquad N \approx 10^{18}\,\nu \tag{3.29}$$

The secondary jet of particles produced by the proton-target interactions will contain mesons (e.g., π and K) that will decay into neutrinos. For definiteness, let's assume that the protons have an energy of 120 GeV and that the efficiency for producing neutrinos with an energy in the region of $M_Z/2 \approx 45$ GeV is:

$$\eta \approx 10^{-6}\,\nu/\mathrm{proton} \tag{3.30}$$

(this is a wild guess!) Then, each pulse of accelerated proton should contain:

$$N_p = \frac{N}{\eta} = 10^{24}\,\mathrm{protons/pulse} \tag{3.31}$$

Typical accelerators currently operating for neutrino beams generation are able to accelerate $10^{13} - 10^{14}$ protons per pulse. So we are very far from being able to construct a neutrino collider to study the Z^0! Also the efficiency η for producing the desired neutrinos could be even much less than what we assumed so it would make the feasibility of the collider even more impractical.

4 Non-relativistic Quantum Mechanics

4.1 Electron polarization

Discuss methods to experimentally determine the polarization of an electron.

Solution:

An electron is said to be polarized if its two spin states are not equally probable (after one measures its spin). An ensemble of electrons is said to be polarized if the electron spins have a preferential orientation so that there exists a direction for which the two possible spin states are not equally populated. If all spins have the same direction one has the extreme case of a totally polarized ensemble of electrons. If not all, but only a majority of the spins has the same direction, the ensemble is called partially polarized. See Section 4.14 of the book.

Electron beams can be polarized by scattering, and the angular distribution of scattered electrons depends on the state of polarization of the incident beam. These effects can be treated by the Dirac equation discussed in Chapter 8 of the book, which is the basic equation for describing the electron, including its spin and its relativistic behavior.

In order to experimentally determine the polarization, one uses an electron polarimeter. The main idea is to find an appropriate scattering process which has a spin-dependent probability. The beam polarization may be measured by observing a sufficient number of electrons scattered by the spin-dependent interaction. The polarization $\mathcal{P}$ăis then given by:

$$\mathcal{P} = \frac{N_+ - N_-}{N_+ + N_-} \tag{4.1}$$

where $N_{+(-)}$ are the measured counting rate associated with the change of polarization.

For electrons, the useful scattering processes involve (see e.g., C. K. Sinclair, "Electron beam polarimetry," AIP Conf. Proc. **451**, no.1, pp. 23-39 (1998) `http://doi.org/10.1063/1.57045`):

1. Mott scattering, i.e., Coulomb scattering by heavy nuclei, is used in practice for low energy electrons ($\sim$50 keV to few MeV); it uses the spin-orbit coupling of the electron in the field of the nucleus (see e.g., F.B. Dunning, "Mott electron polarimetry", Nucl. Instr. and Meth. A347 pp. 152-160 (1994) `http://doi.org/10.1016/0168-9002(94)91872-4`).

2. Møller scattering, i.e., Coulomb scattering from other polarized electrons, for electron energies above 100 MeV and has been used up to 50 GeV;

3. Compton scattering, i.e., Coulomb scattering from polarized photons.

• **Møller scattering.** The differential cross-section of the the pure QED $e^-e^- \to e^-e^-$ process can precisely calculated as a function of the polarizations of the two electrons. The effect of polarization is most easily studied in the center-of-mass system of the two electrons. We note that, if in the laboratory system the 4-momenta of the electrons are given by $p_1^\mu = (E, \vec{p})$ and $p_2^\mu = (m_e, 0)$, we have (see Section 5.8 of the book):

$$(p_1 + p_2)^2 = p_1^2 + p_2^2 + 2\vec{p}_1 \cdot \vec{p}_2 = 2m_e^2 + 2m_e E = 2m_e(m_e + E) \tag{4.2}$$

Hence, the energy of the electrons in the center-of-mass system is:

$$(2E^\star)^2 = 2m_e(m_e + E) \qquad \Longrightarrow \qquad E^\star = \sqrt{\frac{m_e(m_e+E)}{2}} \tag{4.3}$$

Similarly, the Lorentz boost of the electrons in the center-of-mass is equal to:

$$\gamma^\star = \frac{E^\star}{m_e} \qquad \Longrightarrow \qquad \mathcal{G} \equiv (\gamma^\star)^2 = \frac{m_e(m_e+E)}{2m_e^2} = \frac{\gamma+1}{2} \tag{4.4}$$

In the center-of-mass system, the differential scattering cross-section can be expressed as (see e.g., B. Wagner, H. G. Andresen, K. H. Steffens, W. Hartmann, W. Heil and E. Reichert, "A Moller polarimeter for CW and pulsed intermediate-energy electron beams," Nucl. Instrum. Meth. A **294**, pp. 541-548 (1990) `http://doi.org/10.1016/0168-9002(90)90296-I`):

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^\star}\right) = \left(\frac{\mathrm{d}\sigma_0}{\mathrm{d}\Omega^\star}\right)\left[1 + \sum_{j,k=x,y,z} a_{jk} P_j^{\mathrm{B}} P_k^{\mathrm{T}}\right] \tag{4.5}$$

where P_j^B are the components of the polarization of the incoming electron, and P_k^T are the components of the target. The unpolarized cross-section was first calculated in C. Møller, "Zur Theorie des Durchgangs schneller Elektronen durch Materie.", Ann. Phys. 406, pp. 531-585 (1932) `http://doi.org/10.1002/andp.19324060506`. It can be written:

$$\left(\frac{\mathrm{d}\sigma_0}{\mathrm{d}\Omega^\star}\right) = \frac{\alpha^2}{4m^2\mathcal{G}(\mathcal{G}-1)^2\sin^4\theta^\star} a_0 \tag{4.6}$$

where $\theta^\star$ is the scattering angle in the center-of-mass, and

$$a_0 = (2\mathcal{G}-1)^2\left(4 - 3\sin^2\theta^\star\right) + (\mathcal{G}-1)^2\left(4+\sin^2\theta^\star\right)\sin^2\theta^\star \tag{4.7}$$

Let us define the momentum of the incoming electron as $\vec{p}$ and that of a scattered electron as $\vec{k}$. Then, the coordinate system is conveniently chosen as follows:

$$\hat{z} = \vec{p}/|\vec{p}|, \qquad \hat{y} = \vec{p}\times\vec{k}/|\vec{p}\times\vec{k}|, \qquad \hat{x} = \hat{y}\times\hat{z} \tag{4.8}$$

So, the z axis points in the direction of momentum of the incoming electron, the $y-z$ plane is formed from the momenta of the incoming and scattered electron, and x is perpendicular to the other two axes. We note that under parity the vectors transform as follows:

$$\hat{z} \quad\longrightarrow\quad -\hat{z}, \qquad \hat{y} \quad\longrightarrow\quad \hat{y}, \qquad \hat{x} \quad\longrightarrow\quad -\hat{x} \tag{4.9}$$

Hence, parity conservation implies that

$$a_{xy} = a_{yx} = a_{zy} = a_{yz} = 0 \tag{4.10}$$

The non-vanishing terms can be found in Eq. (6.43) of H.A. Olsen, "Application of quantum electrodynamics", Springer Tracts in Modern Physics, Volume 44. Springer Tracts in Modern Physics, vol 44 (1968) `https://doi.org/10.1007/BFb0045483`:

$$\begin{aligned}
a_{zz} &= \sin^2\theta^\star\left[\left(\mathcal{G}^2-1\right)\sin^2\theta^\star - (2\mathcal{G}-1)(4\mathcal{G}-3)\right]/a_0 \\
a_{yy} &= \sin^2\theta^\star\left[(\mathcal{G}-1)^2\sin^2\theta^\star - (4\mathcal{G}-3)\right]/a_0 \\
a_{xx} &= -\sin^2\theta^\star\left[\left(\mathcal{G}^2-1\right)\sin^2\theta^\star + (2\mathcal{G}-1)\right]/a_0 \\
a_{xz} &= a_{zx} = -\left(\sin^2\theta^\star\right)\sqrt{\mathcal{G}}\,(\mathcal{G}-1)\,(\sin 2\theta^\star)/a_0
\end{aligned} \tag{4.11}$$

For electron energies above 100 MeV, we have $\gamma \gtrsim 200$, hence $\mathcal{G} \gg 1$. Since $a_0 \propto \mathcal{G}^2$, the factors $a_{xz} = a_{zx}$ vanish, and:

$$\begin{aligned}
a_{zz} &\approx \frac{\sin^2\theta^\star\left[\mathcal{G}^2\sin^2\theta^\star - 8\mathcal{G}^2\right]}{4\mathcal{G}^2\left(4-3\sin^2\theta^\star\right)+\mathcal{G}^2\left(4+\sin^2\theta^\star\right)\sin^2\theta^\star} = \frac{\sin^2\theta^\star\left(\sin^2\theta^\star-8\right)}{4\left(4-3\sin^2\theta^\star\right)+\left(4+\sin^2\theta^\star\right)\sin^2\theta^\star} \\
&= -\frac{\sin^2\theta^\star\left(\sin^2\theta^\star-8\right)}{(\sin^2\theta^\star-4)^2} = -\frac{\sin^2\theta^\star(7+\cos^2\theta^\star)}{(3+\cos^2\theta^\star)^2} \qquad \longrightarrow -\frac{7}{9}\text{ for }\theta^\star = 90^\circ \\
a_{yy} &= -\,a_{xx} \approx \frac{\sin^2\theta^\star\left[\mathcal{G}^2\sin^2\theta^\star\right]}{a_0} = \frac{\sin^4\theta^\star}{(3+\cos^2\theta^\star)^2} \qquad \longrightarrow \frac{1}{9}\text{ for }\theta^\star = 90^\circ
\end{aligned} \tag{4.12}$$

These results are plotted as a function of the scattering angle in the center-of-mass in Figure 4.1. The maximum

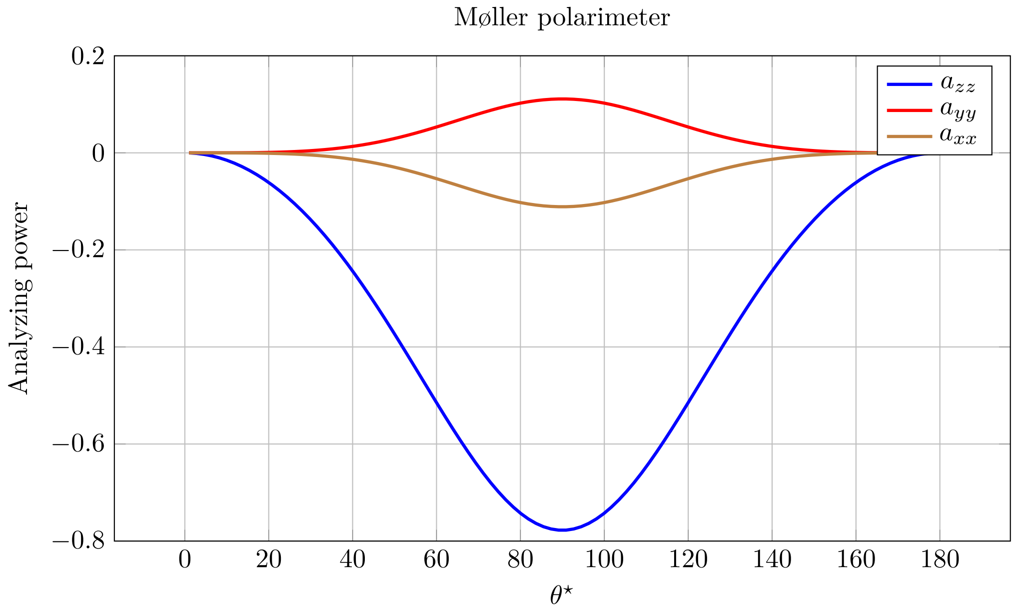

Figure 4.1 Components of the Møller asymmetry versus the scattering angle in the center-of-mass.

sensitivity is obtained for $\theta^\star = 90^\circ$. The maximum analyzing power for scattering longitudinally polarized electrons on longitudinally polarized electrons is 7/9. Møller scattering of transversely polarized electrons can be used to analyze transverse beam polarization; although in this case the maximum analyzing power is only 1/9. Hence, in practice, the Møller is most adequate to measure the longitudinal polarization, but is not as practical for measuring the transverse polarization of electrons.

At the maximum analyzing power, the beam and target electrons are each scattered through $\theta^\star = 90^\circ$ in the center-of-mass system. When we do the Lorentz transformation from the center-of-mass system to the laboratory system, the result is two electrons with equal energies (each having half of the incident beam energy), moving at equal and opposite small angles to the incident beam direction in the scattering plane. It can be shown that the center-of-mass scattering angles and the angle in the laboratory frame are related by:

$$\tan\theta_i^\star/2 = \tan\theta_i\sqrt{\frac{E+m_e}{2m_e}} \tag{4.13}$$

Hence, for $\theta_i^\star = 90^\circ$, we have $\theta_1 = \theta_2 \equiv \theta$, and

$$\tan\theta = \sqrt{\frac{2m_e}{E+m_e}} \approx 6^\circ \quad \text{for } E = 100\text{ MeV} \tag{4.14}$$

This is a small angle.

A typical Møller polarimeter (see Figure 4.2) is composed of a thin ferromagnetic magnetized foil (target) and a set of quadrupoles magnets which are used to separate both the recoiling target and scattered electrons from the beam electrons. Two detectors are symmetrically placed around the beam axis downstream the target to detect the produced electrons in coincidence and the maximum of the sensitivity. The undeflected beam electrons are brought to a dump.

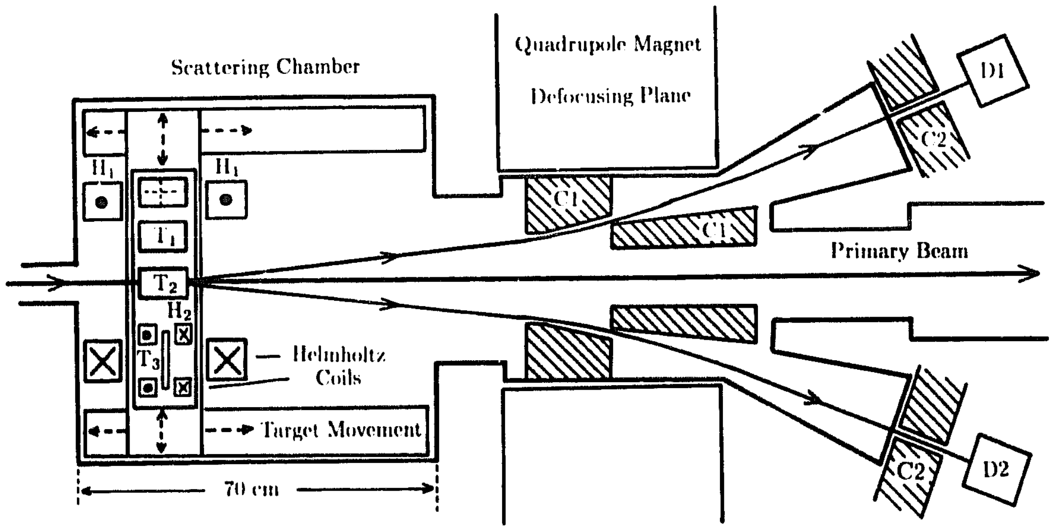

Figure 4.2 Møller polarimeter. Taken from B. Wagner, H. G. Andresen, K. H. Steffens, W. Hartmann, W. Heil and E. Reichert, "A Moller polarimeter for CW and pulsed intermediate-energy electron beams," Nucl. Instrum. Meth. A **294**, pp. 541-548 (1990) `http://doi.org/10.1016/0168-9002(90)90296-I`

• **Compton scattering.** In these polarimeters, polarized photons from a laser beam are backscattered by the beam electrons, as shown in Figure 4.3. Compton scattering may be used to analyze either transverse or longitudinal electron polarization. To analyze transverse polarization, circularly polarized laser light is scattered off the polarized electrons. The backscattered photon rate has a $\cos\phi$ dependence, where ϕ is the azimuthal angle of the backscattered photon with respect to the polarization direction of the electron. This azimuthal dependence is usually measured by observing an up-down asymmetry, with respect to the scattering plane, in the backscattered counting rate. The various asymmetries in Compton scattering of polarized photons from polarized electrons are worked out in full detail in F.W. Lipps, H.A. Tolhoek, "Polarization phenomena of electrons and photons. I: General method and application to Compton scattering," Physica 20 (1954) 85-98 `http://doi.org/10.1016/S0031-8914(54)80018-8` and F.W. Lipps, H.A. Tolhoek, "Polarization phenomena of electrons and photons. II: Results for Compton scattering," Physica 20 (1954) 395-405 `http://doi.org/10.1016/S0031-8914(54)80054-1`.

The first demonstration of a Compton polarimeter, used to measure the transverse polarization of the circulating positron beam in the SPEAR storage ring is reported in D.B. Gustavson, J.J. Murray, T.J. Phillips, R.F. Schwitters, C.K. Sinclair, J.R. Johnson, R. Prepost, D.E. Wiser, "A backscattered laser polarimeter e^+e^- storage rings", Nucl. Instrum. and Meth., 165 (1979) 177-186, `https://doi.org/10.1016/0029-554X(79)90268-4`.

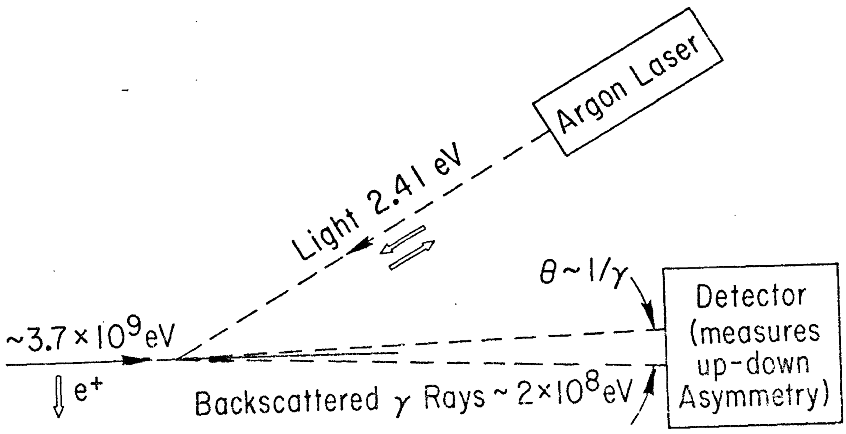

Figure 4.3 Principle of a backscattered Compton polarimeter for electrons (or positrons). Taken from D.B. Gustavson, J.J. Murray, T.J. Phillips, R.F. Schwitters, C.K. Sinclair, J.R. Johnson, R. Prepost, D.E. Wiser, "A backscattered laser polarimeter e^+e^- storage rings", Nucl. Instrum. and Meth., 165 (1979) 177-186, `https://doi.org/10.1016/0029-554X(79)90268-4`

4.2 Photon polarization

Discuss methods to experimentally determine the polarization of a photon.

Solution:

For radio waves or light within the infrared, visible or UV spectrum, one uses standard polarizers' techniques. Here we are interested in X-ray or γ-ray energies where the single particle behavior of the photons dominate. As discussed in Section 10.4 of the book, we must regard a linearly polarized light as carried by a quantum which has an even chance of having plus or minus one Bohr unit as its angular momentum. Elliptically polarized light would be similarly regarded as characterized by unequal chances of possessing spins with the two alternative signs. This is allowed by the fundamental principle of superposition of quantum states.

Several physical processes such as the photoelectric effect, Thomson scattering, Compton scattering, and electron-positron pair production can be used to measure photon linear polarization. Polarimeters based on the photoelectric effect and Thomson scattering are used at very low energies. Compton polarimeters are commonly used for energies from 50 keV to a few MeV. However, they become inefficient at a photon energy of 100 MeV because kinematical suppression, and proposals of using electron-positron pair creation have been made.

We illustrate the Compton scattering $e^-\gamma \to e^-\gamma$ method (see Section 11.16 of the book). Similar to the case of electrons discussed in **Ex. 4.1**, we can write the spin-dependent differential Compton cross-section as:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_0}{d\Omega}\left(1 + P_\gamma^L A_L + P_\gamma^C P_e A_C\right) \tag{4.15}$$

where $d\sigma_0/d\Omega$ is the Klein-Nishina equation for unpolarized photons scattered on free electrons, given in Eq. (11.269) of the book. P_γ^L, P_γ^C and P_e are the linear, circular and longitudinal polarizations of the initial beam photon and the target electron respectively. The analysing powers are given by (see e.g., V. Gharibyan, K. Flöttmann, G. Kube and K. Wittenburg, "Vector Polarimeter for Photons in keV-MeV Energy Range," `http://doi.org/10.18429/JACoW-IBIC2015-MOPB012`):

$$A_L = \frac{\sin^2\theta}{k'/k + k/k' - \sin^2\theta} \qquad \text{and} \qquad A_C = \frac{(k/k' - k'/k)\cos\theta}{k'/k + k/k' - \sin^2\theta} \tag{4.16}$$

where we used the notation of Section 11.16 of the book. The relation between the outgoing photon energy k' and the incoming photon energy k is given by Eq. 11.270 of the book:

$$k' = \frac{k}{1 + (k/m_e)(1 - \cos\theta)} \qquad \Longrightarrow \qquad k/k' = 1 + (k/m_e)(1 - \cos\theta) \tag{4.17}$$

The asymmetries are relative differences of the cross-sections at $P_\gamma^L = \pm$ and $P_\gamma^C = \pm 1$ for a 100% target electron polarization, respectively for the linear and circular photon polarizations. They are plotted in Figure 4.4.

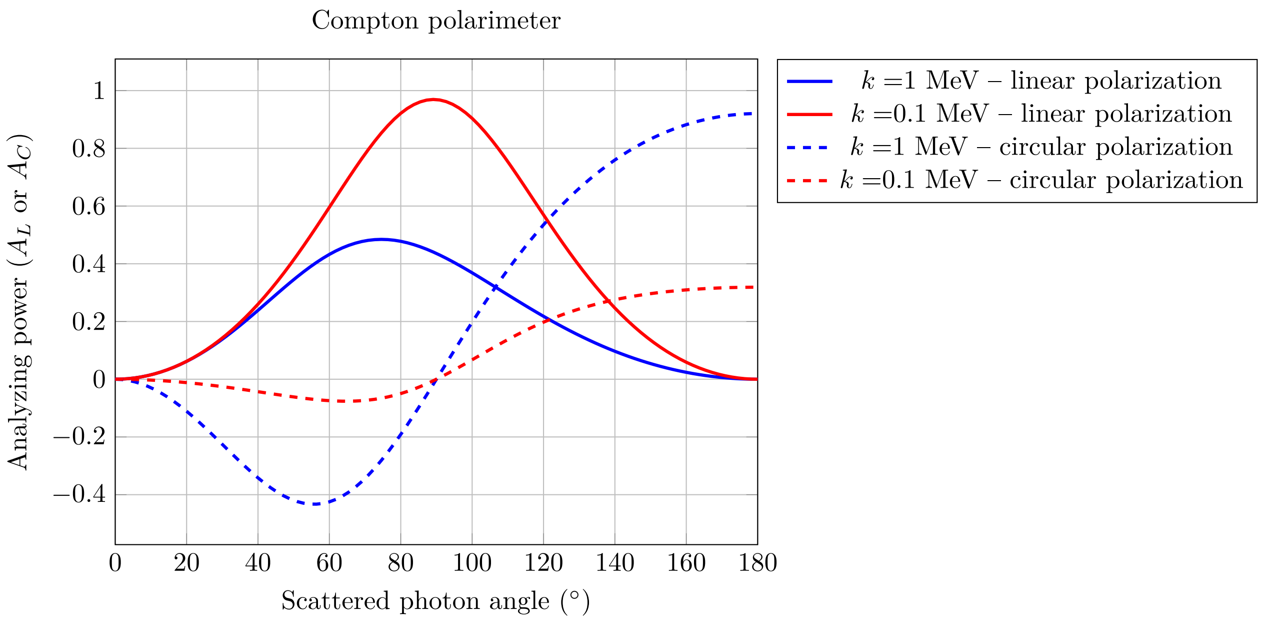

Figure 4.4 Compton asymmetry versus the scattering angle of the scattered photon, for incoming photon energies $k = 0.511$ MeV and 1 MeV.

4.3 The linear harmonic oscillator

The Hamiltonian operator for the linear harmonic oscillator is given by:

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$$

where the commutator of the operators p and x is $[x, p] = i$. Thus, H is the sum of two squares of operators, and we decompose this sum – analogous to the sum of two positive numbers $\alpha^2 + \beta^2 = (\alpha + i\beta)(\alpha - i\beta)$ – into two Hermitian adjoint operators:

$$a = \sqrt{\frac{m\omega}{2}}x + \frac{i}{\sqrt{2m\omega}}p; \quad a^\dagger = \sqrt{\frac{m\omega}{2}}x - \frac{i}{\sqrt{2m\omega}}p$$

(a) Show that the commutator of the operators a and $a^\dagger$ is $[a, a^\dagger] = 1$.

(b) Show that the Hamiltonian operator can be expressed in the following way through the operators a and $a^\dagger$:

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right) = \omega\left(N + \frac{1}{2}\right) \quad \text{with } N \equiv a^\dagger a = N^\dagger$$

(c) $|\lambda\rangle$ is an eigenstate of the Hermitian operator N with eigenvalue λ. Show that the states $a|\lambda\rangle$ and $a^\dagger|\lambda\rangle$ are also eigenstates of N with eigenvalues $(\lambda-1)$ and $(\lambda+1)$, respectively.

(d) Show that the spectrum of eigenvalues of N corresponds to the integer numbers $n \geq 0$, i.e., the Hamiltonian operator H has the eigenvalue spectrum

$$E_n = \omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$$

Solution:

(a) For clarity, let us define $\alpha = \sqrt{m\omega/2}$ and $\beta = 1/\sqrt{2m\omega}$. Then developping the commutator gives:

$$\begin{aligned}
[a, a^\dagger] &= [\alpha x + i\beta p, \alpha x - i\beta p] = \underbrace{[\alpha x, \alpha x]}_{=0} + [\alpha x, -i\beta p] + [i\beta p, \alpha x] - \underbrace{[i\beta p, -i\beta p]}_{=0} \\
&= -2i\alpha\beta[x, p] = 2\alpha\beta = 2/\sqrt{4} = 1
\end{aligned} \tag{4.18}$$

(b) By inspection, we find that:

$$a = \alpha x + i\beta p, \quad a^\dagger = \alpha x - i\beta p \qquad \Longrightarrow \qquad x = \frac{a + a^\dagger}{2\alpha} \quad \text{and} \quad p = \frac{a - a^\dagger}{2i\beta} \tag{4.19}$$

Plugging x and p in the Hamiltonian, we can explicitly develop:

$$\begin{aligned}
H &= \frac{1}{2m}\left(\frac{a - a^\dagger}{2i\beta}\right)^2 + \frac{1}{2}m\omega^2\left(\frac{a + a^\dagger}{2\alpha}\right)^2 \\
&= \frac{1}{2m}\left(\frac{1}{2i\beta}\right)^2 (a^2 - aa^\dagger - a^\dagger a + a^{\dagger 2}) + \frac{1}{2}m\omega^2\left(\frac{1}{2\alpha}\right)^2 (a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}) \\
&= F_1 a^2 + F_2 a^{\dagger 2} + F_3 aa^\dagger + F_4 a^\dagger a
\end{aligned} \tag{4.20}$$

Let us explicitly calculate the factors F_i. We note that:

$$\begin{aligned}
F_1 = F_2 &= \frac{1}{2m}\left(\frac{1}{2i\beta}\right)^2 + \frac{1}{2}m\omega^2\left(\frac{1}{2\alpha}\right)^2 = \frac{1}{2}\left(-\frac{2m\omega}{4m} + \frac{2m\omega^2}{4m\omega}\right) = 0 \\
F_3 = F_4 &= -\frac{1}{2m}\left(\frac{1}{2i\beta}\right)^2 + \frac{1}{2}m\omega^2\left(\frac{1}{2\alpha}\right)^2 = \frac{1}{2}\left(\frac{2m\omega}{4m} + \frac{2m\omega^2}{4m\omega}\right) = \frac{1}{2}\omega
\end{aligned} \tag{4.21}$$

Hence we obtain the desired form for the Hamiltonian:

$$H = \frac{1}{2}\omega\left(aa^\dagger + a^\dagger a\right) = \frac{1}{2}\omega\left(aa^\dagger + aa^\dagger + 1\right) = \omega\left(a^\dagger a + \frac{1}{2}\right) = \omega\left(N + \frac{1}{2}\right) \tag{4.22}$$

where we used $[a, a^\dagger] = 1$ and introduced the number operator $N \equiv a^\dagger a$.

(c) Let us first calculate following commutators, using that $[AB, C] = A[B, C] + [A, C]B$:

$$\begin{aligned} [N, a] &= [a^\dagger a, a] = a^\dagger [a, a] + [a^\dagger, a]a = -[a, a^\dagger]a = -a \\ [N, a^\dagger] &= [a^\dagger a, a^\dagger] = a^\dagger [a, a^\dagger] + [a^\dagger, a^\dagger]a = a^\dagger [a, a^\dagger] = a^\dagger \end{aligned} \tag{4.23}$$

Starting from $N|\lambda\rangle = \lambda|\lambda\rangle$, we explicitly find:

$$\begin{aligned} Na|\lambda\rangle &= ([N, a] + aN)|\lambda\rangle = a(N-1)|\lambda\rangle = (\lambda - 1)a|\lambda\rangle \\ Na^\dagger|\lambda\rangle &= ([N, a^\dagger] + a^\dagger N)|\lambda\rangle = a^\dagger(N+1)|\lambda\rangle = (\lambda + 1)a^\dagger|\lambda\rangle \end{aligned} \tag{4.24}$$

which is the desired result. This implies that the operator a acts on a state $|\lambda\rangle$ to produce a state $|\lambda - 1\rangle$ (up to a multiplicative constant) and similarily for the operator $a^\dagger$ acting on $|\lambda\rangle$ to produce a state $|\lambda + 1\rangle$. They are according labelled “lowering” and “raising” operators.

(d) From the action of the number operator N on a state $|\lambda\rangle$ we have:

$$H|\lambda\rangle = \omega\left(N + \frac{1}{2}\right)|\lambda\rangle = \omega\left(\lambda + \frac{1}{2}\right)|\lambda\rangle \tag{4.25}$$

The allowed values of λ are determined given that by definition:

$$\langle\lambda|N|\lambda\rangle = \lambda\langle\lambda|\lambda\rangle = \langle\lambda|a^\dagger a|\lambda\rangle = (a|\lambda\rangle)^\dagger a|\lambda\rangle = ||a|\lambda\rangle||^2 \geq 0 \quad \Longrightarrow \quad \lambda = \frac{||a|\lambda\rangle||^2}{\langle\lambda|\lambda\rangle} \geq 0 \tag{4.26}$$

Hence, we conclude that λ is an integer number n where $n \geq 0$.

4.4 The photoelectric effect

In the electric dipole approximation, the time-dependent potential describing the interaction of an electron, with momentum $\vec{p}$, and an external field $\vec{A}(\vec{x}, t)$ is given by

$$V(\vec{x}, t) = \frac{e}{m}\vec{A}(\vec{x}, t) \cdot \vec{p}$$

Let us consider a hydrogen atom in his fundamental state – $(n, l, m) = (1, 0, 0)$, orbital $1s$ – and an external field of the form $\vec{A}(\vec{x}, t) = \vec{A}_0 \cos\left(\vec{k} \cdot \vec{x} - \omega t\right)$ impinging on the atom, and compute the probability of the electron being ejected using Fermi's Golden Rule.

(a) Using the time-dependent perturbation theory at first order, show that the potential describing the ejection of an electron can be written as:

$$\widetilde{V}(\vec{x}, t) = \frac{e}{2m} e^{i\vec{k}\cdot\vec{x}} \vec{A}_0 \cdot \vec{p} e^{-i\omega t} = \widetilde{V}(\vec{x}) e^{-i\omega t}$$

Hint: *Explicitly develop the integrand of the first-order coefficient (the integration over space is omitted)*

$$a_f^{(1)}(t) = -i \int_0^t \mathrm{d}t' \, \langle f|\widetilde{V}(\vec{x}, t)|i\rangle e^{i(E_f - E_i)t'}$$

(b) Calculate the transition probability using Fermi's Golden Rule. Recall that the hydrogen atom wave function for the $1s$ state is given by $\psi(\vec{x}) = \langle\vec{x}|1, 0, 0\rangle = e^{-r/a_0}/\sqrt{\pi a_0^3}$, where $r = |\vec{x}|$ and a_0 is the Bohr radius, and that the final-state electron vector in momentum space, $|\vec{p}_f\rangle \sim |\vec{k}_f\rangle$, can be expressed in

position space as a continuous sum over the vectors $|\vec{x}\rangle$.

Hint: *To compute the integral, think about how the term $e^{i\vec{k}\cdot\vec{x}}$ varies on the domain of integration and use the following relations:*

$$\begin{aligned}\int \mathrm{d}^3\vec{x}\ e^{-i\vec{k}_f\cdot\vec{x}}\nabla\left(e^{-r/a_0}\right) &= -\int \mathrm{d}^3\vec{x}\ \nabla\left(e^{-i\vec{k}_f\cdot\vec{x}}\right)e^{-r/a_0}\\ \int \mathrm{d}r\ re^{-r/a_0}\sin(k_f r) &= 2a_0^3\frac{k_f}{(1+a_0^2k_f^2)^2}\end{aligned} \tag{4.27}$$

(c) In which direction is the electron most likely to be ejected?

Solution:

(a) In the photoelectric effect, incoming light causes an atom to eject an electron. We consider the simplest possible scenario: the atom is hydrogen in its ground state. We use the non-relativistic perturbation theory described in Section 4.16 of the book. The time-dependent perturbation theory is discussed in Section 4.16.2 of the book. The transition probability per unit time can be derived from Eq. (4.182) of the book to be:

$$W_{fi}\breve{a} = (2\pi)|V_{fi}|^2\delta(E_f - E_i - \omega) \tag{4.28}$$

In the born approximation, Eq. (4.168) of the book tells us that (setting $T = 0$ for simplicity):

$$a_f^{(1)}(t) \approx -i\int_0^t \mathrm{d}t' \int \mathrm{d}^3\vec{x}u_f^*(\vec{x})V(\vec{x},t)u_i(\vec{x})e^{i(E_f-E_i)t'} = -i\int_0^t \mathrm{d}t'\langle f|V(\vec{x},t')|i\rangle e^{i(E_f-E_i)t'} \tag{4.29}$$

We take the incoming electromagnetic wave to have the following vector potential:

$$\vec{A}(\vec{x},t) = \vec{A}_0\cos\left(\vec{k}\cdot\vec{x} - \omega t\right) \tag{4.30}$$

The interaction Hamiltonian is given by the principle of minimal substitution described in Section 4.10 of the book, hence replacing $\vec{p} \longrightarrow \vec{p} - q\vec{A}$. Consequently, we find that the Hamiltonian contains a new term (the interacting potential) of the form:

$$-\frac{q}{2m}\left(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}\right) = -\frac{q}{m}\vec{A}\cdot\vec{p} = \frac{e}{m}\vec{A}_0\cdot\vec{p}\cos\left(\vec{k}\cdot\vec{x} - \omega t\right) \tag{4.31}$$

This interaction term is called the dipole approximation. We can then write using perturbation theory:

$$\begin{aligned}a_f^{(1)}(t) &= -i\int_0^t dt'\ \langle f|\frac{e}{m}\vec{A}_0\cdot\vec{p}\cos\left(\vec{k}\cdot\vec{x} - \omega t'\right)|i\rangle e^{i(E_f-E_i)t'}\\ &= -i\frac{e}{2m}\int_0^t dt'\ \langle f|\vec{A}_0\cdot\vec{p}\left(e^{i(\vec{k}\cdot\vec{x}-\omega t')} + e^{-i(\vec{k}\cdot\vec{x}-\omega t')}\right)|i\rangle e^{i(E_f-E_i)t'}\\ &= -i\frac{e}{2m}\int_0^t dt'\ \left(\langle f|e^{i\vec{k}\cdot\vec{x}}\vec{A}_0\cdot\vec{p}|i\rangle e^{i(E_f-(E_i+\omega))t'} + \langle f|e^{-i\vec{k}\cdot\vec{x}}\vec{A}_0\cdot\vec{p}|i\rangle e^{i((E_f+\omega)-E_i)t'}\right)\end{aligned} \tag{4.32}$$

Two terms appear in the integral. The first term refers to the absorption of a photon (the initial state energy is $E_i + \omega$) and the ejection of an electron, whereas the second term corresponds to the capture of an electron and the emission of a photon (the final energy state energy is $E_f + \omega$). Keeping only the first term we can deduce that in the case of the ejection of an electron, we have:

$$a_f^{(1)}(t) = -i\int_0^t dt'\ \langle f|\widetilde{V}(\vec{x},t')|i\rangle e^{i(E_f-E_i)t'} \quad \text{where} \quad \widetilde{V}(\vec{x},t) \equiv \frac{e}{2m}e^{i\vec{k}\cdot\vec{x}}\vec{A}_0\cdot\vec{p}e^{-i\omega t} = \widetilde{V}(\vec{x})e^{-i\omega t} \tag{4.33}$$

(b) We make the assumption that the final state electron is a plane wave state $|\vec{p}_f\rangle = |\vec{k}_f\rangle \propto e^{i\vec{k}_f\cdot\vec{x}}$. The most straightforward way of handling the plane wave states is to confine the whole system to an extremely large cubical box of side, and impose periodic boundary conditions, as was done in the case of elastic scattering discussed in Section 4.17 of the book:

$$|\vec{k}_f\rangle = \frac{1}{L^{3/2}} e^{i\vec{k}_f\cdot\vec{x}} \tag{4.34}$$

The ground state wave function for hydrogen is

$$|1,0,0\rangle = \sqrt{\frac{1}{\pi a_0^3}} e^{-r/a_0} \qquad \text{where} \quad r = |\vec{x}| \tag{4.35}$$

where a_0 is the Bohr radius.

The matrix element entering Fermis Golden Rule is therefore:

$$\begin{aligned} \widetilde{V}_{fi} &= \langle \vec{k}_f | \frac{e}{2m} e^{i\vec{k}\cdot\vec{x}} \vec{A}_0 \cdot \vec{p} | 1,0,0\rangle = \int \mathrm{d}^3\vec{x} \frac{1}{L^{3/2}} e^{-i\vec{k}_f\cdot\vec{x}} \frac{e}{2m} e^{i\vec{k}\cdot\vec{x}} \vec{A}_0 \cdot (-i\nabla) \sqrt{\frac{1}{\pi a_0^3}} e^{-r/a_0} \\ &= -i \frac{e}{2m} \frac{1}{L^{3/2}} \sqrt{\frac{1}{\pi a_0^3}} \vec{A}_0 \cdot \int \mathrm{d}^3\vec{x} e^{-i\vec{k}_f\cdot\vec{x}} e^{i\vec{k}\cdot\vec{x}} \nabla e^{-r/a_0} \end{aligned} \tag{4.36}$$

We now note that typical work functions of metals range between 2 and 5 eV, i.e. the equivalent photon wavelength is in the range 250 to 620 nm. The typical wavelength of the photon is then much bigger than the scale of the atom, given by a_0. Hence, the term $e^{i\vec{k}\cdot\vec{x}} \simeq 1$ where the wave function of the atom is non vanishing, and we can drop that term. We now notice that:

$$\int \mathrm{d}^3\vec{x} e^{-i\vec{k}_f\cdot\vec{x}} \, \nabla e^{-r/a_0} = \underbrace{\int \mathrm{d}^3\vec{x} \, \nabla \left(e^{-i\vec{k}_f\cdot\vec{x}} e^{-r/a_0} \right)}_{=0} + i \int \mathrm{d}^3\vec{x} \, \vec{k}_f e^{-i\vec{k}_f\cdot\vec{x}} e^{-r/a_0} \tag{4.37}$$

The integral is now a Fourier transform of the hydrogen ground state wave function!

$$\widetilde{V}_{fi} = \frac{e}{2m} \frac{1}{L^{3/2}} \sqrt{\frac{1}{\pi a_0^3}} \vec{A}_0 \cdot \vec{k}_f \int \mathrm{d}^3\vec{x} \, e^{-i\vec{k}_f\cdot\vec{x}} e^{-r/a_0} \tag{4.38}$$

Let us choose the $\hat{z}$-axis in the direction of $\vec{k}_f$, such that $\vec{k}_f \cdot \vec{x} = k_f r \cos\theta$ and hence:

$$\begin{aligned} \int \mathrm{d}^3\vec{x} \, e^{-i\vec{k}_f\cdot\vec{x}} e^{-r/a_0} &= \int r^2 \mathrm{d}r \, \mathrm{d}\cos\theta \, \mathrm{d}\phi \; e^{-ik_f r\cos\theta} e^{-r/a_0} = 2\pi \int \mathrm{d}r \; r^2 e^{-r/a_0} \int_1^{-1} \mathrm{d}\cos\theta \; e^{-ik_f r\cos\theta} \\ &= 2\pi \int \mathrm{d}r \; r^2 e^{-r/a_0} \frac{1}{-ik_f r} \left(e^{-ik_f r} - e^{ik_f r} \right) = 2\pi \int \mathrm{d}r \; r^2 e^{-r/a_0} \frac{1}{-ik_f r} \left(-2i\sin(k_f r) \right) \\ &= \frac{4\pi}{k_f} \int \mathrm{d}r \; r e^{-r/a_0} \sin(k_f r) = \frac{4\pi}{k_f} 2a_0^3 \frac{k_f}{(1+a_0^2 k_f^2)^2} \\ &= \frac{8\pi}{a_0} \frac{1}{(a_0^{-2} + k_f^2)^2} \end{aligned} \tag{4.39}$$

Finally, the matrix element becomes:

$$\widetilde{V}_{fi} = \frac{e}{2m} \frac{1}{L^{3/2}} \sqrt{\frac{1}{\pi a_0^3}} \vec{A}_0 \cdot \vec{k}_f \left(\frac{8\pi}{a_0} \frac{1}{(a_0^{-2} + k_f^2)^2} \right) \tag{4.40}$$

The transition rate is found using Fermi's Golden Rule by squaring the matrix element $\widetilde{V}_{fi}$ and hence we obtain:

$$W_{fi} = 2\pi \left| \frac{e}{2m} \frac{1}{L^{3/2}} \sqrt{\frac{1}{\pi a_0^3}} \left(\frac{8\pi}{a_0} \frac{1}{(a_0^{-2} + k_f^2)^2} \right) (\vec{A}_0 \cdot \vec{k}_f) \right|^2 \delta(E_f - E_i - \omega) \tag{4.41}$$

To detect the ejected electron, we will have a detector sensitive to some small solid angle and not to some precise value of $\vec{k}_f$. As in the case described in Section 4.17 of the book, we must multiply this result by the density of the final states $\rho(E_f)$. The value of this latter is given in Eq. (4.192) of the book. Hence:

$$W_{fi} = 2\pi \left| \frac{e}{2m} \frac{1}{L^{3/2}} \sqrt{\frac{1}{\pi a_0^3}} \left(\frac{8\pi}{a_0} \frac{1}{(a_0^{-2} + k_f^2)^2} \right) (\vec{A}_0 \cdot \vec{k}_f) \right|^2 L^3 \left(\frac{1}{2\pi} \right)^3 m k_f \mathrm{d}\Omega \tag{4.42}$$

We note first that the L^3 terms cancel, reassuringly, our result cannot depend on the size of the box chosen for the plane wave states. Regrouping the other factors, we finally find:

$$W_{fi} = \frac{4 m k_f}{\pi a_0^5} \left(\frac{e}{m} \right)^2 \left(\frac{1}{a_0^{-2} + k_f^2} \right)^4 (\vec{A}_0 \cdot \vec{k}_f)^2 \mathrm{d}\Omega \tag{4.43}$$

(c) The transition rate is angle-dependent, since $W_{fi} \propto (\vec{A}_0 \cdot \vec{k}_f)^2$. The electric field can be computed for the potential Eq. (4.30).

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} ă = -\frac{\partial}{\partial t} \left(\vec{A}_0 \cos\left(\vec{k} \cdot \vec{x} - \omega t \right) \right) = \omega \vec{A}_0 \sin\left(\vec{k} \cdot \vec{x} - \omega t \right) \tag{4.44}$$

Hence, the electric field is parallel to the vector potential. The ejection is most likely parallel to the electric field. Since $\vec{k} \perp \vec{E}$, we have $\vec{A}_0 \perp \vec{k}$, the transition probability is maximal for $\vec{k}_f \perp \vec{k}$. In conclusion, the direction of emission of the electron is most likely perpendicular to the direction of propagation of the wave and in the direction of the electric field (i.e., parallel to its polarization).

4.5 Born approximation in three-dimensional elastic scattering

Compute the differential and total cross-section for a particle scattering off the following potentials in the first-order approximation:

(a) $V(r) = V_0 e^{-r/R}$

(b) $V(r) = V_0 e^{-r^2/R^2}$

Solution:

(a) The Born expression for the differential cross-section for scattering off a given potential V has been given in Eq. (4.197) of the book:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{m}{2\pi} \right)^2 \left| \int \mathrm{d}^3\vec{x} V(\vec{x}) e^{i\vec{q}\cdot\vec{x}} \right|^2 \qquad \text{(first order)} \tag{4.45}$$

where $\vec{q} = \vec{p}_i - \vec{p}_f$ is the change of momentum during the scattering.

Let us consider the first potential $V(r) = V_0 e^{-r/R}$. We have:

$$\begin{aligned}\int \mathrm{d}\vec{x}\; V(r) e^{i\vec{q}\cdot\vec{x}} &= V_0 \int \mathrm{d}\vec{r}\; e^{-r/R} e^{i\vec{q}\cdot\vec{r}} = V_0 \int \mathrm{d}r r^2 e^{-r/R} \int \mathrm{d}\phi \int \;\mathrm{d}\cos\theta\, e^{iqr\cos\theta} \\ &= \frac{4\pi}{q} V_0 \int \mathrm{d}r\; r e^{-r/R} \sin(qr) = \frac{4\pi V_0}{q} \frac{2R^3 q}{(1+R^2q^2)^2} \\ &= 8\pi V_0 \frac{R^3}{(1+R^2q^2)^2}\end{aligned} \tag{4.46}$$

which is essentially the derivation done in Eq. (4.39). Thus the differential cross-section reads:

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi}\right)^2 \left| 8\pi V_0 \frac{R^3}{(1+R^2q^2)^2} \right|^2 = \frac{16 m^2 R^6 V_0^2}{(1+q^2R^2)^4} \equiv \frac{\sigma_0}{(1+q^2R^2)^4} \tag{4.47}$$

In this expression, the angular dependence enters through its relation with the change of momentum. We use the result of Eq. (4.214) of the book:

$$q^2 \equiv (\vec{q})^2 = (\vec{p_i} - \vec{p_f})^2 = \vec{p_i}^{\,2} + \vec{p_f}^{\,2} - 2\vec{p_i}\cdot\vec{p_f} = 2p^2(1-\cos\theta) = 4p^2 \sin^2\left(\frac{\theta}{2}\right) \tag{4.48}$$

The total cross-section is obtained by integrating over the full solid angle. This integral is actual equivalent to integrating over all possible momentum transfers. Indeed, the maximum momentum transfer is $q^2_{max} = 4p^2 = (2p)^2$, which occurs for a backscatter at $\theta = 180°$. On the other hand, $q^2 \to 0$ when $\theta \to 0$. We set:

$$x \equiv \cos\theta \qquad \Longrightarrow \qquad \mathrm{d}x = -\sin\theta \mathrm{d}\theta \qquad \Longrightarrow \qquad \int d\Omega = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \mathrm{d}\theta = 2\pi \int_{-1}^{1} \mathrm{d}x \tag{4.49}$$

In addition, we find:

$$q = \sqrt{2}p(1-x)^{1/2} \qquad \Longrightarrow \qquad \frac{\mathrm{d}q}{\mathrm{d}x} = \frac{\sqrt{2}}{2} p (1-x)^{-1/2}(-1) \qquad \Longrightarrow \qquad \mathrm{d}x = -\frac{\sqrt{2}p\sqrt{1-x}}{p^2}\mathrm{d}q = -\frac{q\mathrm{d}q}{p^2} \tag{4.50}$$

Hence:

$$\int d\Omega = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\cos\theta = 2\pi \int_{-1}^{1} \mathrm{d}\cos\theta = -\frac{2\pi}{p^2} \int_{2p}^{0} dq\; q = \frac{2\pi}{p^2} \int_0^{2p} dq\; q \tag{4.51}$$

We are now ready to perform the integration of the total cross-section:

$$\begin{aligned}\sigma(p) &= \frac{2\pi\sigma_0}{p^2} \int_0^{2p} \mathrm{d}q\; q \frac{1}{(1+q^2R^2)^4} = \frac{2\pi\sigma_0}{p^2} \int \mathrm{d}q\; \frac{1}{6R^2} \frac{\mathrm{d}}{\mathrm{d}q} \left(-\frac{1}{(1+q^2R^2)^3}\right) \\ &= \frac{\pi\sigma_0}{3p^2R^2} \left[-\frac{1}{(1+q^2R^2)^3}\right]_0^{2p} = \frac{\pi\sigma_0}{3p^2R^2} \left(1 - \frac{1}{(1+4p^2R^2)^3}\right)\end{aligned} \tag{4.52}$$

Finally, we obtain:

$$\sigma(p) = \frac{16\pi m^2 R^4 V_0^2}{3p^2} \left(1 - \frac{1}{(1+4p^2R^2)^3}\right) \tag{4.53}$$

(b) We proceed in a similar fashion:

$$\begin{aligned}\int \mathrm{d}^3\vec{x}\; V(r) e^{i\vec{q}\cdot\vec{x}} &= V_0 \int \mathrm{d}^3\vec{x}\; e^{-r^2/R^2} e^{i\vec{q}\cdot\vec{x}} = 2\pi V_0 \int \mathrm{d}r\; r^2 e^{-r^2/R^2} \int_{-1}^{1} \mathrm{d}\cos\theta\, e^{iqr\cos\theta} \\ &= 2\pi V_0 \int \mathrm{d}r\; r^2 e^{-r^2/R^2} \frac{(e^{iqr} - e^{-iqr})}{iqr} = 2\pi V_0 \int \mathrm{d}r\; r^2 e^{-r^2/R^2} \frac{2\sin(qr)}{q}\end{aligned} \tag{4.54}$$

We use online resources (e.g., http://www.wolframalpha.com) to compute:

$$\int_0^\infty \mathrm{d}r\ r^2 e^{-r^2/R^2} \sin(qr) = \frac{1}{4}\sqrt{\pi} q R^3 e^{-q^2R^2/4} \tag{4.55}$$

Hence:

$$\int \mathrm{d}^3\vec{x}\ V(r) e^{i\vec{q}\cdot\vec{x}} = \frac{4\pi V_0}{q}\frac{1}{4}\sqrt{\pi} q R^3 e^{-q^2R^2/4} = \pi\sqrt{\pi} V_0 R^3 e^{-\frac{q^2R^2}{4}} \tag{4.56}$$

Thus the differential cross-section reads:

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi}\right)^2 \left|\pi\sqrt{\pi} V_0 R^3 e^{-\frac{q^2R^2}{4}}\right|^2 = \left(\frac{\sqrt{\pi} m V_0 R^3}{2}\right)^2 e^{-\frac{q^2R^2}{2}} \equiv \sigma_0 e^{-\frac{q^2R^2}{2}} \tag{4.57}$$

The total cross-section is obtained by integration as before:

$$\begin{aligned}\sigma(p) &= \sigma_0 \int_0^{2p} \mathrm{d}q\ q e^{-q^2R^2/2} = \sigma_0 \int \mathrm{d}q\ \left(-\frac{1}{R^2}\right)\frac{d}{dq}(e^{-q^2R^2/2}) = \sigma_0\left(-\frac{1}{R^2}\right)\left[e^{-q^2R^2/2}\right]_0^{2p} \\ &= \frac{\sigma_0}{R^2}(1 - e^{-2p^2R^2})\end{aligned} \tag{4.58}$$

And finally our result is:

$$\sigma(p) = \left(\frac{\sqrt{\pi} m V_0 R^2}{2}\right)^2 (1 - e^{-2p^2R^2}) \tag{4.59}$$

5 Relativistic Formulation and Kinematics

5.1 Lorentz transformations

(a) Derive the following relations for general Lorentz transformations (which include boosts, rotations, and reflections):

$$\Lambda_\mu{}^\nu \;=\; g_{\mu\rho}\,\Lambda^\rho{}_\lambda\,g^{\lambda\nu}$$
$$\Lambda^\mu{}_\nu\,\Lambda_\mu{}^\rho \;=\; \Lambda_\mu{}^\nu\,\Lambda^\mu{}_\rho \;=\; \delta^\rho_\nu$$

Hint: Use the fact that the inner product of two four-vectors is necessarily Lorentz-invariant.

(b) Prove the following relations for the determinants.

For general Lorentz transformations:

$$\det\,(\Lambda^\mu{}_\nu) \;=\; \pm 1$$

For proper Lorentz transformations (boosts and rotations):

$$\det\,(\Lambda^\mu{}_\nu) \;=\; +1$$

For space reflection and time reversal:

$$\det\,(\Lambda^\mu{}_\nu) \;=\; -1$$

Solution:

(a) We recall that x^μ is a contravariant four-vector, while x_μ is a covariant four-vector of our flat Minkowski space. The are related by the metric tensor, as introduced in Eq. (5.5) of the book:

$$x_\mu = g_{\mu\nu}x^\nu \quad \text{and} \quad x^\mu = g^{\mu\nu}x_\nu \tag{5.1}$$

where $g = diag(1, -1, -1, -1)$. The inner product can be expressed with the metric tensor as:

$$x \cdot y = x^\mu y_\mu = g_{\mu\nu}x^\mu y^\nu \tag{5.2}$$

The **Lorentz group** consists of the linear transformations of space-time conserving the inner product (note the position of the Lorentz indices on Λ!). For a contravariant and covariant four-vector we have:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \text{and} \qquad x'_\mu = \Lambda_\mu{}^\nu x_\nu \tag{5.3}$$

where Λ is a 4×4 matrix. The components of the matrix are all real and dimensionless. They are independent of x^μ. We seek the relation between $\Lambda^\mu{}_\nu$ and $\Lambda_\nu{}^\mu$. On one hand, we find:

$$x'_\mu = g_{\mu\nu}x'^\nu = g_{\mu\nu}\Lambda^\nu{}_\alpha x^\alpha \tag{5.4}$$

On the other:

$$x'_\mu = \Lambda_\mu{}^\nu x_\nu = \Lambda_\mu{}^\nu g_{\nu\alpha} x^\alpha \tag{5.5}$$

Consequently, since this relation must be valid for any four-vector x:

$$g_{\mu\nu}\Lambda^\nu{}_\alpha = \Lambda_\mu{}^\nu g_{\nu\alpha} \tag{5.6}$$

We now multiply this result by $g^{\alpha\beta}$ on the r.h.s. to find the desired first relation:

$$g_{\mu\nu}\Lambda^\nu{}_\alpha g^{\alpha\beta} = \Lambda_\mu{}^\nu g_{\nu\alpha} g^{\alpha\beta} = \Lambda_\mu{}^\nu \delta^\beta_\nu = \Lambda_\mu{}^\beta \tag{5.7}$$

where δ^α_β is the four dimensional Krönecker delta.

We now prove the second relation. We have:

$$x' \cdot y' = x'_\mu y'^\mu = \Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta x_\alpha y^\beta \quad \text{and} \quad x' \cdot y' = x'^\mu y'_\mu = \Lambda^\mu{}_\alpha \Lambda_\mu{}^\beta x^\alpha y_\beta \quad \Longrightarrow \quad \Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta = \Lambda^\mu{}_\alpha \Lambda_\mu{}^\beta \tag{5.8}$$

Since the inner product between two vectors is a conserved quantity in a Lorentz transformation, we have that:

$$(x\cdot y)' = x' \cdot y' = x'_\mu y'^\mu = \Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta x_\alpha y^\beta \quad \text{and} \quad (x\cdot y)' = x \cdot y = x_\alpha y^\alpha = \delta^\alpha_\beta x_\alpha y^\beta \tag{5.9}$$

Hence:

$$\Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta = \Lambda^\mu{}_\alpha \Lambda_\mu{}^\beta = \delta^\alpha_\beta \tag{5.10}$$

(b) We start with the relation of orthogonality (Lorentz condition) in matrix form, given in Appendix D of the book in Eq. (D.12):

$$g = \Lambda^T g \Lambda \quad \Longrightarrow \quad \det(g) = \det\big(\Lambda^T g \Lambda\big) = \det\big(\Lambda^T\big)\det(g)\det(\Lambda) \tag{5.11}$$

By definition $\det(g) = -1$ and $\det(A^T) = \det(A)$ for any squared matrix A. We then find:

$$\det(\Lambda^T g\Lambda) = \det(\Lambda)\det(g)\det(\Lambda) = -\det{}^2(\Lambda) \qquad \Longrightarrow \qquad \det{}^2(\Lambda) = 1 \tag{5.12}$$

which implies that for a general Lorentz transformation, we have $\det(\Lambda) = \pm 1$.

The explicit matrix forms of the Lorentz transformation are listed in Table 5.2 of the book. We consider a boost transformation in the $\hat{x}$ direction. We have that:

$$\det\left(\Lambda^\mu{}_\nu(\beta_x)\right) = \det\begin{pmatrix} \gamma & -\gamma\beta_x & 0 & 0 \\ -\gamma\beta_x & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \gamma^2 + \gamma\beta(-\gamma\beta) = \gamma^2(1-\beta^2) = 1 \tag{5.13}$$

for which the determinant is explicitly calculated as:

$$\det(\Lambda^\mu{}_\nu(\beta_x)) = \gamma\cdot\gamma + \gamma\beta(-\gamma\beta) = \gamma^2(1-\beta^2) = 1 \tag{5.14}$$

A similar result is obtained for boosts along the $\hat{y}$ and $\hat{z}$ direction. For a rotation along the $\hat{x}$ axis we have that:

$$\det\Lambda^\mu{}_\nu(R_x) = \det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} = 1\cdot 1\cdot(\cos^2\theta + \sin^2\theta) = 1 \tag{5.15}$$

A similar result is obtained for rotations around the $\hat{y}$ and $\hat{z}$ axis. For discrete transformations such as reflection and time reversal we have:

$$\det\Lambda^\mu{}_\nu(P) = \det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -1, \qquad \det\Lambda^\mu{}_\nu(T) = \det\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -1 \tag{5.16}$$

5.2 Invariant mass

For the decay $R \to 1 + 2$, show that the mass of the particle R can be reconstructed from the daughters as:

$$m_R^2 = m_1^2 + m_2^2 + 2E_1E_2(1 - \beta_1\beta_2 \cos\theta_{12}) \tag{5.17}$$

where β_1 and β_2 are the velocities of particles 1 and 2, E_1 and E_2 their energies, and θ_{12} is the angle between them. In the ultra-relativistic case, one has simply:

$$m_R^2 = 2E_1E_2(1 - \cos\theta_{12}) \tag{5.18}$$

Solution:

Let us consider the decay process $R \to 1 + 2$. For a particle R initially at rest, we define following four-momenta:

$$p_R = (m_R, \vec{0}), \qquad p_1 = (E_1, \vec{p}_1), \qquad p_2 = (E_2, \vec{p}_2) \tag{5.19}$$

Consequently:

$$\begin{aligned} p_R^2 = m_R^2 = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 &= m_1^2 + m_2^2 + 2(E_1E_2 - \vec{p}_1 \cdot \vec{p}_2) \\ &= m_1^2 + m_2^2 + 2(E_1E_2 - |\vec{p}_1||\vec{p}_2| \cos\theta_{12}) \\ &= m_1^2 + m_2^2 + 2E_1E_2(1 - |\vec{\beta}_1||\vec{\beta}_2| \cos\theta_{12}) \end{aligned} \tag{5.20}$$

where θ_{12} is the angle between the two momenta of the final state particles. In the ultra-relativistic limit, we have $\beta_i \to 1$ and we can neglect the rest masses. Thus the above formula reduces to:

$$m_R^2 = 2E_1E_2(1 - \cos\theta_{12}) \tag{5.21}$$

which is the desired result.

5.3 Antiproton kinematical threshold

Antiprotons can be produced via the following reaction:

$$p + p \to p + p + p + \bar{p} \tag{5.22}$$

Find the minimum proton energy needed in the laboratory frame if one of the initial protons is at rest.

Solution:

We are in the fixed target configuration as shown in Figure 5.2 of the book. We write the 4-momenta of the two initial-state protons labelled A and B, relative the center-of-mass system:

$$p_A^\mu = (E_A, \vec{p}_A) \qquad p_B^\mu = (M_p, \vec{0}) \tag{5.23}$$

where M_p is the rest mass of the proton and we have for A that:

$$E_A^2 = \vec{p}_A^{\,2} + M_p^2 \tag{5.24}$$

We compute the total energy–momentum squared which gives us a result as a function of E_A:

$$p^2 = (p_A + p_B)^2 = E_A^2 + 2M_pE_A + M_p^2 - \vec{p}_A = 2M_p^2 + 2M_pE_A \tag{5.25}$$

We now consider the final state composed of three protons and one anti-proton. The kinematical threshold corresponds to the situation where all four particles are at rest and the total energy of the final state is just equal to the sum of the rest masses of end products. Any increase of momenta would imply a kinetic term that would increase the total energy. Hence, relative to the center-of-mass system of the final state particles we have that the total energy–momentum four-vector squared is equal to:

$$(p^\star)^2 = (4M_p, \vec{0})^2 = (4M_p)^2 = 16M_p^2 \tag{5.26}$$

where we assumed that the antiproton rest mass is equivalent to the proton rest mass. Although we calculate this value in the center-of-mass frame, the inner product of two four-vectors is an invariant under Lorentz transformation. Hence, it's value is the same in the lab frame! We can then write the condition that the initial state have enough energy to kinematically create the final state. We hence have:

$$p^2 > 16M_p^2 \quad \Longrightarrow \quad 2M_p^2 + 2M_pE_A > 16M_p^2 \quad \Longrightarrow \quad 2M_p + 2E_A > 16M_p \tag{5.27}$$

So finally:

$$E_A \equiv T_p + M_p > 7M_p \quad \Longrightarrow \quad T_A > 6M_p \tag{5.28}$$

where T_A is the kinetic energy of the initial-state proton A. Numerically, we find $T_A \approx 5.6$ GeV.

5.4 General formula for differential cross-section of the scattering process $1 + 2 \to 3 + 4$

Show that the differential cross-section in the center-of-mass system, keeping the rest masses of the four particles, can be expressed as:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CMS} = \frac{1}{64\pi^2 s}\sqrt{\frac{\lambda\left(s, m_3^2, m_4^2\right)}{\lambda\left(s, m_1^2, m_2^2\right)}}|\mathcal{M}|^2 \tag{5.29}$$

where $\lambda(x, y, z)$ is the Källén function.

Solution:

The relativistic form of the differential cross-section for a $2 \to 2$ scattering process is given in Eq. (5.117) of the book. Assuming that the particles are distinguishable, we have:

$$\mathrm{d}\sigma = \frac{1}{F}\prod_{i=3,4}\frac{\mathrm{d}^3\vec{p}_i}{(2\pi)^3 2E_i} \times (2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)|\mathcal{M}(12 \to 34)|^2 \tag{5.30}$$

In the center-of-mass system (CMS) the 2-body differential phase-space, $d\Pi_2$, is given by Eq. (5.139) of the book:

$$\mathrm{d}\Pi_2 = \frac{1}{(2\pi)^2}\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}\delta^4(p_1 + p_2 - p_3 - p_4) = \mathrm{d}\Omega\frac{1}{16\pi^2}\frac{p_f}{E_\star} \tag{5.31}$$

where $E_\star = \sqrt{s}$ is the CMS energy. The final state momentum $\vec{p}_f$ is defined as $\vec{p}_f = \vec{p}_3 = -\vec{p}_4$ and its magnitude is given in Eq. (5.133) of the book:

$$p_f = \frac{\sqrt{(s - m_3^2 - m_4^2)^2 - 4m_3^2m_4^2}}{2\sqrt{s}} = \frac{\sqrt{\lambda(s, m_3^2, m_4^2)}}{2\sqrt{s}} \tag{5.32}$$

where $\lambda(a, b, c)$ is the Källén function. The manifestly invariant flux factor F is defined in Eq. (5.121) of the book. We first note that:

$$s \equiv (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2p_1 \cdot p_2 \qquad \Longrightarrow \qquad (p_1 \cdot p_2)^2 = \frac{1}{4}(s - m_1^2 - m_2^2)^2 \tag{5.33}$$

Therefore:

$$\begin{aligned}
F^2 &= 16\left((p_1 \cdot p_2)^2 - m_1^2 m_2^2\right) = 16\left(\frac{1}{4}\left(s - (m_1^2 + m_2^2)\right)^2 - m_1^2 m_2^2\right) \\
&= \frac{16}{4}\left(s^2 + (m_1^2 + m_2^2)^2 - 2s(m_1^2 + m_2^2) - 4m_1^2 m_2^2\right) \\
&= 4(s^2 + m_1^4 + m_2^4 - 2m_1^2 m_2^2 - 2sm_1^2 - 2sm_2^2) = 4\lambda(s, m_1^2, m_2^2) \qquad \Longrightarrow \qquad F = 2\sqrt{\lambda(s, m_1^2, m_2^2)}
\end{aligned} \tag{5.34}$$

Regrouping all the term, we find the relativistic differential cross-section in the CMS for a $2 \to 2$ scattering process:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CMS} = \frac{1}{16\pi^2 F}\frac{p_f}{\sqrt{s}} = \frac{1}{64\pi^2 s}\sqrt{\frac{\lambda\left(s, m_3^2, m_4^2\right)}{\lambda\left(s, m_1^2, m_2^2\right)}}|\mathcal{M}|^2 \tag{5.35}$$

5.5 Opening angles

We consider the decay of a particle of mass m and momentum $\vec{p}$ into two daughter particles.

(a) Determine the minimum and maximum opening angle between the two daughter particles in the case they are both massless.

(b) Determine the minimum and maximum opening angle between the two daughter particles in the case they have masses m_1 and m_2, respectively.

(c) Let us assume that a calorimeter is able to separate the electromagnetic showers induced by high-energy photons when they are separated by an angle larger than 10°. What is the maximum energy of the π^0 which can be reconstructed as a pair of distinct photons in such a calorimeter?

Solution:

(a) This case has been treated in Section 5.10 of the book. In the center-of-mass frame, the two massless particles are emitted back-to-back, with momenta and energies $p^\star = E^\star = m/2$. The emission angle of one of the photons in the center-of-mass system is given by $\theta^\star$. The second photon is therefore emitted along the angle $\theta^\star + \pi$. According to Eqs. (5.80) and (5.82) of the book, we have:

$$E_{1,2} = \gamma\frac{m}{2}(1 \pm \beta_L \cos\theta^\star), \quad p_{1,||} = \gamma\frac{m}{2}\left(\beta_L + \cos\theta^\star\right) \quad \text{and} \quad p_{1,T} = p^\star_{1,T} = \frac{m}{2}\sin\theta^\star \tag{5.36}$$

The opening angle in the laboratory frame is given by Eq. (5.87) of the book:

$$1 - \cos\alpha = \frac{2}{\gamma^2(1 - \beta_L^2\cos^2\theta^\star)} \tag{5.37}$$

By noting that $1 - \cos\alpha = 2\sin^2(\alpha/2)$, we get:

$$\sin^2(\alpha/2) = \frac{1}{2}\frac{2}{\gamma^2(1 - \beta_L^2\cos^2\theta^\star)} \quad \Longrightarrow \quad \sin(\alpha/2) = \frac{1}{\gamma\sqrt{1 - \beta_L^2\cos^2\theta^\star}} \tag{5.38}$$

Over the range $-1 \leq \cos\theta^\star \leq 1$, the minimum is found for $\cos\theta^\star = 0$. Hence, the minimum opening angle α_m in the laboratory frame is given by:

$$\sin\left(\frac{\alpha_m}{2}\right) = \frac{1}{\gamma} \tag{5.39}$$

which is Eq. (5.88) of the book. For small opening angle (i.e. large Lorentz factors), we can approximate $\sin x \approx x$ and then rewrite the result as:

$$\alpha_m \approx \frac{2}{\gamma} \tag{5.40}$$

(b) Momentum is conserved hence $\vec{p}_1^\star = -\vec{p}_2^\star$ and so is energy, therefore:

$$E_1^\star + E_2^\star = \sqrt{p_1^{\star 2} + m_1^2} + \sqrt{p_2^{\star 2} + m_2^2} = m \tag{5.41}$$

These equations can be solved as in Section 5.15 of the book to yield Eqs. (5.133) and (5.135):

$$E_1^\star = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}, \qquad E_2^\star = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}}, \qquad |\vec{p}_1^\star| = |\vec{p}_2^\star| = \frac{\lambda^{1/2}\left(s, m_1^2, m_2^2\right)}{2\sqrt{s}} \tag{5.42}$$

where $\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx = (x - y - z)^2 - 4yz$, We can now consider the Lorentz boost from the center-of-mass frame to the laboratory frame. What we really care about are the velocities of the particles in the center-of-mass frame, i.e. $\beta_i^\star = |\vec{p}_i^\star|/E_i^\star$ $(i = 1, 2)$. For example, for particle 1:

$$\begin{aligned} p_{1,T} &= p_{1,T}^\star = p_1^\star \sin\theta^\star \\ p_{1,||} &= \gamma(\beta_L E_1^\star + p_{1,||}^\star) = \gamma\left(\beta_L p_1^\star/\beta_1^\star + p_1^\star \cos\theta^\star\right) = \gamma p_1^\star \left(\beta_L/\beta_1^\star + \cos\theta^\star\right) \end{aligned} \tag{5.43}$$

The angle of particle in the laboratory frame is then given by:

$$\tan\theta_1 = \frac{p_{1,T}}{p_{1,||}} = \frac{p_1^\star \sin\theta^\star}{\gamma p_1^\star \left(\beta_L/\beta_1^\star + \cos\theta^\star\right)} = \frac{\sin\theta^\star}{\gamma\left(\beta_L/\beta_1^\star + \cos\theta^\star\right)} \tag{5.44}$$

and similarly for particle 2:

$$\tan\theta_2 = \frac{p_{2,T}}{p_{2,||}} = -\frac{\sin\theta^\star}{\gamma\left(\beta_L/\beta_2^\star - \cos\theta^\star\right)} \tag{5.45}$$

We can now use the following trigonometric relation to find the opening angle $\alpha \equiv \theta_1 - \theta_2$:

$$\tan(\alpha) = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} \tag{5.46}$$

The numerator is simply:

$$\begin{aligned} \tan\theta_1 - \tan\theta_2 &= \frac{\sin\theta^\star}{\gamma}\left(\frac{1}{\beta_L/\beta_1^\star + \cos\theta^\star} + \frac{1}{\beta_L/\beta_2^\star - \cos\theta^\star}\right) \\ &= \frac{\sin\theta^\star}{\gamma}\left(\frac{\beta_L/\beta_2^\star - \cos\theta^\star + \beta_L/\beta_1^\star + \cos\theta^\star}{(\beta_L/\beta_1^\star + \cos\theta^\star)(\beta_L/\beta_2^\star - \cos\theta^\star)}\right) \\ &= \frac{\beta_L \sin\theta^\star}{\gamma}\left(\frac{1/\beta_2^\star + 1/\beta_1^\star}{(\beta_L/\beta_1^\star + \cos\theta^\star)(\beta_L/\beta_2^\star - \cos\theta^\star)}\right) \end{aligned} \tag{5.47}$$

The denominator is given by:

$$\begin{aligned} 1 + \tan\theta_1 \tan\theta_2 &= 1 + \frac{-\sin^2\theta^\star}{\gamma^2\left(\beta_L/\beta_1^\star + \cos\theta^\star\right)\left(\beta_L/\beta_2^\star - \cos\theta^\star\right)} \\ &= \frac{\gamma^2\left(\beta_L/\beta_1^\star + \cos\theta^\star\right)\left(\beta_L/\beta_2^\star - \cos\theta^\star\right) - \sin^2\theta^\star}{\gamma^2\left(\beta_L/\beta_1^\star + \cos\theta^\star\right)\left(\beta_L/\beta_2^\star - \cos\theta^\star\right)} \end{aligned} \tag{5.48}$$

Therefore,

$$\begin{aligned}
\tan(\alpha) &= \frac{\beta_L \sin\theta^\star}{\gamma}\left(\frac{1/\beta_2^\star + 1/\beta_1^\star}{(\beta_L/\beta_1^\star + \cos\theta^\star)(\beta_L/\beta_2^\star - \cos\theta^\star)}\right)\frac{\gamma^2\left(\beta_L/\beta_1^\star + \cos\theta^\star\right)\left(\beta_L/\beta_2^\star - \cos\theta^\star\right)}{\gamma^2\left(\beta_L/\beta_1^\star + \cos\theta^\star\right)\left(\beta_L/\beta_2^\star - \cos\theta^\star\right) - \sin^2\theta^\star} \\
&= \gamma\beta_L \sin\theta^\star\left(\frac{1/\beta_2^\star + 1/\beta_1^\star}{\gamma^2\left(\beta_L/\beta_1^\star + \cos\theta^\star\right)\left(\beta_L/\beta_2^\star - \cos\theta^\star\right) - \sin^2\theta^\star}\right) \qquad (5.49)
\end{aligned}$$

We note that the denominator of this last equation can itself be simplified:

$$\begin{aligned}
&\gamma^2\left(\beta_L/\beta_1^\star + \cos\theta^\star\right)\left(\beta_L/\beta_2^\star - \cos\theta^\star\right) - \sin^2\theta^\star \\
=\ &\gamma^2\left(\frac{\beta_L^2}{\beta_1^\star\beta_2^\star} + \beta_L\left(\frac{1}{\beta_2^\star} - \frac{1}{\beta_1^\star}\right)\cos\theta^\star - \cos^2\theta^\star - \frac{1}{\gamma^2} + \frac{\cos^2\theta^\star}{\gamma^2}\right) \\
=\ &\gamma^2\left(\frac{\beta_L^2}{\beta_1^\star\beta_2^\star} - 1 + \beta_L^2 + \beta_L\left(\frac{1}{\beta_2^\star} - \frac{1}{\beta_1^\star}\right)\cos\theta^\star - \cos^2\theta^\star\underbrace{\left(1 - \frac{1}{\gamma^2}\right)}_{=\beta_L^2}\right)
\end{aligned} \qquad (5.50)$$

Finally, we can write:

$$\tan\alpha = \frac{\beta_L \frac{\beta_1^\star + \beta_2^\star}{\beta_1^\star\beta_1^\star}\sqrt{1 - \cos^2\theta^\star}}{\gamma D} \qquad (5.51)$$

where we have defined D in the denominator as:

$$D(\cos\theta^\star) = -\beta_L^2\cos^2\theta^\star + \beta_L\frac{\beta_1^\star - \beta_2^\star}{\beta_1^\star\beta_2^\star}\cos\theta^\star + \left(\frac{\beta_L^2}{\beta_1^\star\beta_2^\star} - 1 + \beta_L^2\right) \qquad (5.52)$$

which is a second-degree polynomial with negative concavity. The opening angle α in the laboratory frame as a function of the cosine of the decay angle in the center-of-mass frame is plotted in Figure 5.1. A PYTHON code to compute this angle is available on the GITHUB[1]. The dashed curve represents the case $\beta_L = 0.8$, $\beta_1^\star = 1$, and $\beta_2^\star = 1$, which is equivalent to massless final-state particles, travelling at the speed of light. We recover the situation observed in part (a) of the exercise where the minimum opening angle occurs for $\cos\theta^\star = 0$.

We note that the expression Eq. (5.52) is, as expected, symmetric under the interchange $1 \leftrightarrow 2$ and $\cos\theta^\star \leftrightarrow -\cos\theta^\star$. Then, without loss of generality we can just relabel the final state particles and assume that $\beta_1^\star \leq \beta_2^\star$. We should now consider the following three cases (see Figure 5.1):

– $\beta_L \leq \beta_1^\star \leq \beta_2^\star$ (blue curve): the backward emission in the center-of-mass frame of a particle with a velocity $\beta_i^\star$ larger than β_L yields a backward-propagating particle in the laboratory frame. Hence, it is possible for the particles to be back-to-back in the laboratory frame and accordingly we find $\alpha_{max} = 180°$ (when $\cos\theta^\star = \pm 1$). The minimum opening angle can be found by solving:

$$\frac{\mathrm{d}\tan\alpha}{\mathrm{d}\cos\theta^\star} = -\frac{\beta_L(\beta_1^\star + \beta_2^\star)\left(\beta_L^2\left(\beta_1^\star\beta_2^\star\cos^3\theta^\star - \beta_1^\star\beta_2^\star\cos\theta^\star + \cos\theta^\star\right) + \beta_L(\beta_1^\star - \beta_2^\star) - \beta_1^\star\beta_2^\star\cos\theta^\star\right)}{\gamma\sqrt{1 - \cos^2\theta^\star}\left(\beta_L^2\left(\beta_1^\star\beta_2^\star\left(\cos^2\theta^\star - 1\right) - 1\right) + \beta_L\cos\theta^\star(\beta_2^\star - \beta_1^\star) + \beta_1^\star\beta_2^\star\right)^2} = 0 \qquad (5.53)$$

or

$$\beta_L^2\cos\theta^\star\left(\beta_1^\star\beta_2^\star\cos^2\theta^\star - \beta_1^\star\beta_2^\star + 1\right) + \beta_L(\beta_1^\star - \beta_2^\star) - \beta_1^\star\beta_2^\star\cos\theta^\star = 0 \qquad (5.54)$$

The analytical solution for $\cos\theta^\star$ is quite complex. However, one can easily see that the solution does not correspond, in general, to $\cos\theta^\star = 0$. One can find the solution numerically or read it

1 `https://github.com/CambridgeUniversityPress/Phenomenology-Particle-Physics`

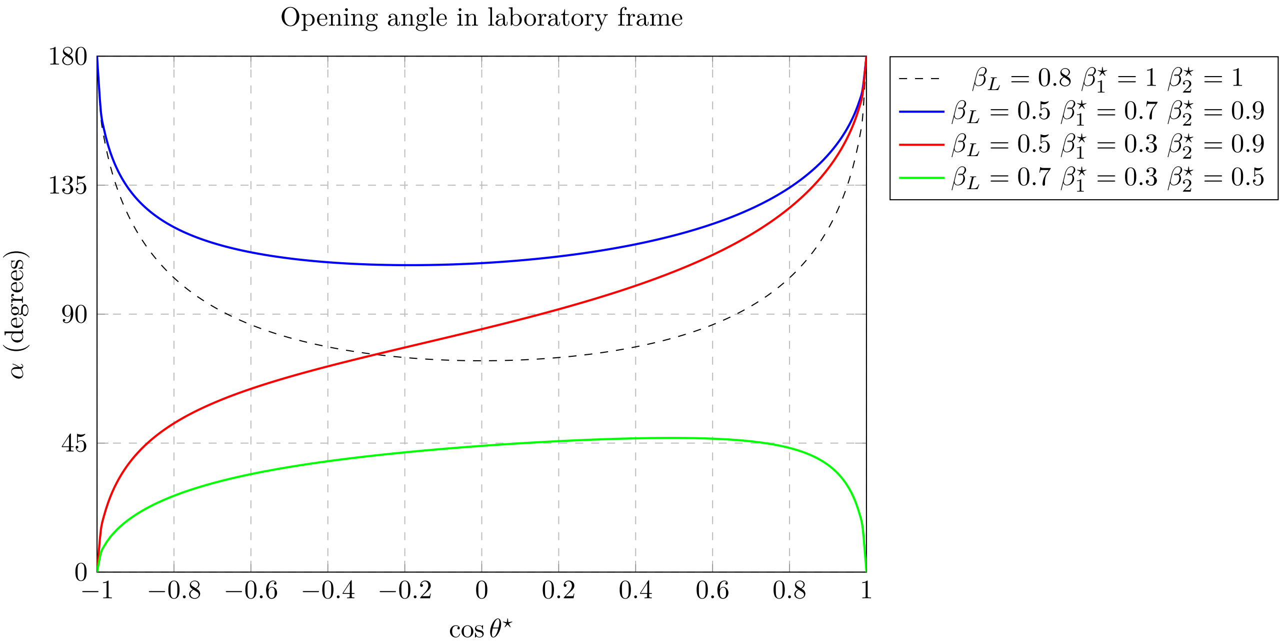

Figure 5.1 Opening angle in the laboratory frame as a function of the cosine of the decay angle in the center-of-mass frame.

off Figure 5.1 for the given set of parameters, for example for $\beta_L = 0.5$, $\beta_1^\star = 0.7$, and $\beta_2^\star = 0.9$ the value turns out to be $\cos\theta^\star \approx -0.19$. One can also verify that for $\beta_1^\star = \beta_2^\star = 1$, the solution simplifies to:

$$\beta_L^2 \cos\theta^\star \left(\cos^2\theta^\star\right) - \cos\theta^\star = 0 \quad \Longrightarrow \quad \cos\theta^\star = 0, \pm\frac{1}{\beta_L^2} \tag{5.55}$$

from which only the solution $\cos\theta^\star = 0$ is physical. This solution is, as expected, equivalent to what we found in part (a) of the exercise for massless final-state particles.

– $\beta_1^\star \leq \beta_L \leq \beta_2^\star$ (red curve): investigation of Eq. (5.54) reveals that in this case there is no solution with $|\cos\theta^\star| < 1$. Hence, there is no extremum in the physical range. One should focus on the two limiting values $\cos\theta^\star = \pm 1$. We note that the numerator of Eq. (5.51) tends to zero for $\cos\theta^\star \to \pm 1$. Hence, $\tan\alpha \to 0$ so $\alpha = 0$ or $180°$. We should look at the signs in order to separate the two solutions. The denominator is in this case equal to:

$$\begin{aligned} D(\pm 1) &= -\beta_L^2 \pm \beta_L \frac{\beta_1^\star - \beta_2^\star}{\beta_1^\star\beta_2^\star} + \left(\frac{\beta_L^2}{\beta_1^\star\beta_2^\star} - 1 + \beta_L^2\right) \\ &= \pm\beta_L \frac{\beta_1^\star - \beta_2^\star}{\beta_1^\star\beta_2^\star} + \frac{\beta_L^2}{\beta_1^\star\beta_2^\star} - 1 = \frac{1}{\beta_1^\star\beta_2^\star}\beta_L^2 \pm \frac{\beta_1^\star - \beta_2^\star}{\beta_1^\star\beta_2^\star}\beta_L - 1 \end{aligned} \tag{5.56}$$

These expressions are convex parabolas in β_L. The determinant Δ is given by:

$$\begin{aligned} \Delta &\equiv \frac{(\beta_1^\star - \beta_2^\star)^2}{(\beta_1^\star\beta_2^\star)^2} + \frac{4}{\beta_1^\star\beta_2^\star} \\ &= \frac{1}{(\beta_1^\star\beta_2^\star)^2}\left((\beta_1^\star - \beta_2^\star)^2 + 4\beta_1^\star\beta_2^\star\right) = \frac{(\beta_1^\star + \beta_2^\star)^2}{(\beta_1^\star\beta_2^\star)^2} \end{aligned} \tag{5.57}$$

Hence, the roots are (the signs must be considered separately!):

$$\beta_L^{\pm} = \frac{\beta_1^\star \beta_2^\star}{2}\left[\mp\frac{\beta_1^\star - \beta_2^\star}{\beta_1^\star\beta_2^\star} \pm \frac{(\beta_1^\star + \beta_2^\star)}{\beta_1^\star\beta_2^\star}\right] = \frac{1}{2}\left[\mp(\beta_1^\star - \beta_2^\star) \pm (\beta_1^\star + \beta_2^\star)\right] \tag{5.58}$$

Consequently, the solutions are:

$$\begin{aligned} D(+1): &\quad \beta_L^{\pm} = \frac{1}{2}\left[-(\beta_1^\star - \beta_2^\star) \pm (\beta_1^\star + \beta_2^\star)\right] = \{-\beta_1^\star, \beta_2^\star\} \\ D(-1): &\quad \beta_L^{\pm} = \frac{1}{2}\left[+(\beta_1^\star - \beta_2^\star) \pm (\beta_1^\star + \beta_2^\star)\right] = \{-\beta_2^\star, \beta_1^\star\} \end{aligned} \tag{5.59}$$

Since the parabolas are convex, we must have $D(+1) < 0$ and $D(-1) > 0$. Consequently, $\alpha(\cos\theta^\star = +1) = 180^\circ$ and $\alpha(\cos\theta^\star = -1) = 0$. For $\alpha(\cos\theta^\star = +1) = 180^\circ$, the two particles are emitted back-to-back. The slower particle 1 moves forward, while the faster particle 2 moves backward in the laboratory frame. For $\alpha(\cos\theta^\star = -1) = 0$, the two particles are collinear and forward moving. The slower particle 1, although emitted in the backward direction in the center-of-mass frame, is boosted forward in the laboratory frame, while the faster particle 2 moves forward in both frames.

– $\beta_1^\star \leq \beta_2^\star \leq \beta_L$ (green curve): in this case, we find that $D(+1) > 0$ and $D(-1) > 0$. Hence, the opening angle α vanishes for both cases of fully forward and backward emission ($\cos\theta^\star = \pm 1$), since both particles are boosted forward in the laboratory frame. One can use Eq. (5.54) to compute the maximum opening angle. For the chosen parameters $\beta_L = 0.7$, $\beta_1^\star = 0.3$, and $\beta_2^\star = 0.5$, this latter turns out to be at $\cos\theta^\star \approx 0.49$ with $\alpha_{max} \approx 46.8^\circ$.

(c) The opening angle is $10^\circ\pi/180^\circ = 0.175$ rad. The requirement to reconstruct the π^0 as two separate photons is equivalent to asking that the minimum opening angle exceeds the resolution of the detector. We use the approximation Eq. (5.40) to set:

$$\alpha_m \approx \frac{2}{\gamma}\breve{a} = \frac{2m_{\pi^0}}{E_{\pi^0}} > 0.175 \quad \Longrightarrow \quad E_{\pi^0} < \frac{2m_{\pi^0}}{0.175} \approx 1.54 \text{ GeV} \tag{5.60}$$

where we used $m_{\pi^0} = 135$ MeV.

6 The Lagrangian Formalism

6.1 An oscillating string using Lagrange formalism

As an application on a classical, non-relativistic, continuous system, we derive the equation of motion of an oscillating string (standing wave) from the Euler–Lagrange equation. A string is tensioned with a force F and its mass density (mass per length) is ρ [g/cm]. At $x = 0$ and $x = l$ the string is fixed, and the (small) transverse excitation at the position x and at the time t is $\phi(x,t)$. In classical mechanics the Lagrangian is given by:

$$L = \int_0^l \mathcal{L} \mathrm{d}x = T - V,$$

where $\mathcal{L}$ is the Lagrangian density, T is the kinetic energy of the system, and V its potential energy.

(a) *Derive the Lagrangian density $\mathcal{L}$ of the oscillating string. Hint: The length of the oscillating string is:*

$$\int_0^l \mathrm{d}s = \int_0^l \left(\mathrm{d}x^2 + \mathrm{d}\phi^2\right)^{\frac{1}{2}} = \int_0^l \left[1 + \left(\frac{\partial\phi}{\partial x}\right)^2\right]^{\frac{1}{2}} \mathrm{d}x$$

Thus, the expansion (in length) of the string for a small excitation ($\partial\phi/\partial x \ll 1$) from the zero position is $\int_0^l \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 \mathrm{d}x$ and, hence, the work necessary to expand the string by this amount against the force F is $\int_0^l \frac{1}{2} F (\frac{\partial\phi}{\partial x})^2 \mathrm{d}x$.

(b) *According to the principle of least action, the motion of the string between the times t_1 and t_2 is such that the action $S = \int_{t_1}^{t_2} L\mathrm{d}t$ is an extremum ($\delta S = 0$). Derive the Euler–Lagrange equation for the oscillating string. The boundary conditions are $\delta\phi(x,t_1) = \delta\phi(x,t_2) = 0$ for all x and $\delta\phi(0,t) = \delta\phi(l,t) = 0$ for all t.*

(c) *Derive with the Euler–Lagrange equation the wave equation for the oscillating string (standing wave).*

(d) *The Hamiltonian density is given by $\mathcal{H} = \Pi \cdot \frac{\partial\phi}{\partial t} - \mathcal{L}$, where $\Pi = \frac{\partial\mathcal{L}}{\left(\frac{\partial\phi}{\mathrm{d}t}\right)}$ is the canonical momentum density. Show with the Euler–Lagrange equation that the $E = \int_0^l \mathcal{H}\mathrm{d}x$ of the oscillating string is constant.*

Solution:

a) Let us consider an element of the elongated string along the x axis. The (small) transverse movement of the string at the position x is given by $\phi(x)$. Using the Pythagorean theorem, we can write the length of this elongated piece of string as:

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}\phi^2} = \sqrt{\mathrm{d}x^2 + \left(\frac{\partial\phi}{\partial x}\mathrm{d}x\right)^2} = \mathrm{d}x\left[1 + \left(\frac{\partial\phi}{\partial x}\right)^2\right]^{\frac{1}{2}} \approx 1 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 \tag{6.1}$$

where in the last term we have assumed very small deviations of the string ($\frac{\partial\phi}{\partial x}\mathrm{d}x \ll 1$). The full length of the excited string is:

$$\int_0^l \mathrm{d}s \approx \int_0^l \mathrm{d}x \left[1 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2\right] = \int_0^l \mathrm{d}x + \frac{1}{2}\int_0^l \left(\frac{\partial\phi}{\partial x}\right)^2 \mathrm{d}x = \ell + \delta\ell \tag{6.2}$$

The potential energy of the system is given by the work to be carried out to stretch the string:

$$V = F\delta\ell = \frac{1}{2}F\int_0^l \left(\frac{\partial\phi}{\partial x}\right)^2 \mathrm{d}x \tag{6.3}$$

On the other hand, the velocity of a given piece of string is given by $\partial\phi/\partial t$, so its kinetic energy is $(1/2)\rho\mathrm{d}x(\partial\phi/\partial t)^2$. Hence, the total kinetic energy of the string is given by:

$$T = \int \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 \rho\mathrm{d}x = \frac{\rho}{2}\int_0^l \left(\frac{\partial\phi}{\partial t}\right)^2 \mathrm{d}x \tag{6.4}$$

Regrouping the kinetic and the potential energy, the Lagrangian of the system can be expressed as:

$$L = \int_0^l \mathcal{L}\mathrm{d}x = T - V = \int_0^l \left[\frac{\rho}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{F}{2}\left(\frac{\partial\phi}{\partial x}\right)^2\right] \mathrm{d}x \tag{6.5}$$

Consequently, the Lagrangian density is just a function of the two variables $\partial\phi/\partial t$ and $\partial\phi/\partial x$:

$$\mathcal{L}\left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}\right) = \frac{\rho}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{F}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 \tag{6.6}$$

b) We first derive the Euler-Lagrange equations. The action S is:

$$S = \int_{t_1}^{t_2} L\mathrm{d}t = \int_{t_1}^{t_2} \mathrm{d}t \int_0^l \mathcal{L}\mathrm{d}x \quad\Longrightarrow\quad \delta S = \int_{t_1}^{t_2} \mathrm{d}t \int_0^l \mathrm{d}x \left[\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\delta\left(\frac{\partial\phi}{\partial t}\right) + \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\delta\left(\frac{\partial\phi}{\partial x}\right)\right] \tag{6.7}$$

Since ϕ is a continuous differentiable function, we can actually swap the order of the differentiations:

$$\delta\left(\frac{\partial\phi}{\partial t}\right) = \frac{\partial}{\partial t}(\delta\phi) \qquad \text{and} \qquad \delta\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}(\delta\phi) \tag{6.8}$$

which yields:

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \int_0^l \mathrm{d}x \left[\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\frac{\partial}{\partial t}(\delta\phi) + \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\frac{\partial}{\partial x}(\delta\phi)\right] \tag{6.9}$$

By partial integration of the time term, we find:

$$\int_{t_1}^{t_2} \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\frac{\partial}{\partial t}(\delta\phi)\,\mathrm{d}t = \underbrace{\left.\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\delta\phi\right|_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2} \mathrm{d}t\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\right)\delta\phi \tag{6.10}$$

where we used the boundary condition that $\delta\phi = 0$ for $t = t_1, t_2$. Similarly, we find for the position:

$$\int_0^l \mathrm{d}x \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)} \frac{\partial}{\partial x}(\delta\phi) = \underbrace{\left.\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\delta\phi\right|_0^l}_{=0} - \int_0^l \mathrm{d}x \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\right)\delta\phi \tag{6.11}$$

where we used $\delta\phi = 0$ at $x = 0, \ell$. Hence, we can rewrite the action:

$$\delta S = -\int_{t_1}^{t_2} \mathrm{d}t \int_0^l \mathrm{d}x \left[\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\right)\right]\delta\phi \tag{6.12}$$

Now according to the principle of least action, the motion of the string between the times t_1 and t_2 is such that the action $S = \int_{t_1}^{t_2} L\mathrm{d}t$ is an extremum ($\delta S = 0$). Since this should be true for any $\delta\phi$, we find the Euler-Lagrange equations:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\right) = 0 \tag{6.13}$$

c) For the case of the string, where the Lagrangian density is given by Eq. 6.6, we find:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\right) = \frac{\partial}{\partial t}\left(\rho\frac{\partial\phi}{\partial t}\right) = \rho\frac{\partial^2\phi}{\partial t^2} \qquad \text{and} \qquad \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\right) = \frac{\partial}{\partial x}\left(-F\frac{\partial\phi}{\partial x}\right) = -F\frac{\partial^2\phi}{\partial x^2} \tag{6.14}$$

The Euler-Lagrange equation gives:

$$\rho\frac{\partial^2\phi}{\partial t^2} - F\frac{\partial^2\phi}{\partial x^2} = 0 \qquad \Longrightarrow \qquad \frac{\partial^2\phi}{\partial t^2} - \frac{F}{\rho}\frac{\partial^2\phi}{\partial x^2} = 0 \tag{6.15}$$

which is the well-known wave equation. The speed of propagation of the wave is given by $v = \sqrt{F/\rho}$.

d) The Hamiltonian density is given by

$$\mathcal{H} = \Pi\cdot\frac{\partial\phi}{\partial t} - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\frac{\partial\phi}{\partial t} - \mathcal{L} \qquad \Longrightarrow \qquad E = \int_0^l \mathcal{H}\mathrm{d}x = \int_0^l \mathrm{d}x\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\frac{\partial\phi}{\partial t} - \mathcal{L}\right] \tag{6.16}$$

Hence, the time derivative is:

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \int_0^l \mathrm{d}x\left[\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\right)\frac{\partial\phi}{\partial t} + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial\mathcal{L}}{\partial t}\right] \\
&= \int_0^l \mathrm{d}x\left[\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\right)\frac{\partial\phi}{\partial t} + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t}\right) - \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial x}\right)\right] \\
&= \int_0^l \mathrm{d}x\left[\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\right)\frac{\partial\phi}{\partial t} + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\frac{\partial^2\phi}{\partial t\partial x}\right] \\
&= \int_0^l \mathrm{d}x\left[\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\right)\frac{\partial\phi}{\partial t} - \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\frac{\partial^2\phi}{\partial t\partial x}\right]
\end{aligned} \tag{6.17}$$

We apply again partial integration:

$$\int_0^l \mathrm{d}x \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)} \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right) = \underbrace{\left.\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)} \frac{\partial\phi}{\partial t}\right|_0^l}_{=0} - \int_0^l \mathrm{d}x \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\right)\frac{\partial\phi}{\partial t} \tag{6.18}$$

where we used the boundary $\phi(0,t) = \phi(l,t)$ for all t. Consequently, using the Euler-Lagrange equation, we find that for a solution of the motion of the string the energy is independent of time, hence is conserved:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_0^l \mathrm{d}x \underbrace{\left[\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\right)\right]}_{=0 \text{ (Euler-Lagrange equation)}} \frac{\partial\phi}{\partial t} \qquad \Longrightarrow \qquad \frac{\mathrm{d}E}{\mathrm{d}t} = 0 \tag{6.19}$$

6.2 Conserved charge in a Lorentz boost

Show that the conserved charge associated with a Lorentz boost is given by:

$$L^{\mu\nu} = -x^\nu p^\mu + x^\mu p^\nu \tag{6.20}$$

Hint*: Consider a Lagrangian of the form $\mathcal{L} = \mathcal{L}(x(\tau), \dot{x}(\tau), \tau)$.*

Solution:

From the Lagrangian of the form $\mathcal{L} = \mathcal{L}(x(\tau), \dot{x}(\tau), \tau)$, we can derive the action:

$$S[x(t)] = \int \mathrm{d}^4 x ă\mathcal{L}(x(\tau), \dot{x}(\tau), \tau) \tag{6.21}$$

Its variation can be expressed as:

$$\begin{aligned}
\delta S &= \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}\delta \dot{x}^\mu\right] = \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}\delta\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right)\right] \\
&= \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}\frac{\mathrm{d}}{\mathrm{d}\tau}\delta x^\mu\right] = \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}\delta x^\mu\right) - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}\right)\delta x^\mu\right] \\
&= \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}\right)\right]\delta x^\mu + \int \mathrm{d}^4x \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}\delta x^\mu\right)
\end{aligned} \tag{6.22}$$

The first term vanishes because the term in brackets corresponds to the Euler-Lagrange equation. Hence:

$$\delta S = \int \mathrm{d}^4x \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}\delta x^\mu\right) \tag{6.23}$$

An infinitesimal proper Lorentz transformation can be written as (see Eq. (B.88) of Appendix B of the book):

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \qquad \Longrightarrow \qquad \delta x^\mu = x'^\mu - x^\mu = \omega^{\mu\nu}x_\nu \tag{6.24}$$

where the $\omega^\mu{}_\nu$ is an anti-symmetric tensor. The variation of the action under such a transformation then becomes:

$$\delta S = \int \mathrm{d}^4x \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}\omega^{\mu\nu}x_\nu\right) = \int \mathrm{d}^4x \frac{1}{2}\omega^{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}x_\nu - \frac{\partial\mathcal{L}}{\partial \dot{x}^\nu}x_\mu\right) \equiv \int \mathrm{d}^4x \frac{1}{2}\omega^{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}\tau}L_{\mu\nu} \tag{6.25}$$

where we have used the anti-symmetry of $\omega^{\mu\nu}$. The conserved quantity for a true solution for which $\delta S = 0$ is the rotational tensor:

$$L^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} x^\nu - \frac{\partial \mathcal{L}}{\partial \dot{x}_\nu} x^\mu \equiv -x^\nu p^\mu + x^\mu p^\nu \tag{6.26}$$

where we have defined the canonical momentum:

$$p^\mu \equiv -\frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} \tag{6.27}$$

We note that the quantities L^{ij} $(i = 1, 2, 3)$ coincide with the conserved angular momenta. We consider L^{0i} $(i = 1, 2, 3)$:

$$L^{0i} = x^0 p^i - x^i p^0 = tp^i - x^i E \qquad \Longrightarrow \qquad L^{01} = tp_x - xE = t\gamma m\beta - x\gamma m = \gamma m(t\beta - x) \tag{6.28}$$

which is the quantity which we already encountered in **Ex. 1.2**, from which we infer that the mass is the conserved charge.

6.3 Relativistic motion of a charged particle in a uniform magnetic field

We consider a charged particle with momentum $\vec{p}$ and charge e moving in a uniform magnetic field $\vec{B}$.

(a) Show that the relativistic extension of the equation of motion is identical to the classical expression provided that the relativistic definition of the momentum is used:

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = e\vec{E} + e\vec{v} \times \vec{B} \tag{6.29}$$

(b) Show that the trajectory of such a particle is a helix.

(c) Find the relation between the radius of curvature of the trajectory and the momentum of the particle.

(d) Show that the transverse momentum p_T expressed in GeV/c of a particle of charge $q = ze$ moving inside a uniform magnetic field $\vec{B}$ whose strength is expressed in Tesla is:

$$p_T = 0.3z|\vec{B}|R \tag{6.30}$$

where R is the radius of curvature expressed in meters.

Solution:

(a) The proper time τ has been defined in Eq. (5.31) of the book. The velocity four-vector was also defined in Eqs. (5.33) and (5.34) of the book (we keep the constant c for clarity):

$$\eta^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \gamma(c, \vec{v}) \tag{6.31}$$

The four-momentum is then given by (see Eq. (5.35) of the book):

$$p^\mu = m\eta^\mu = (\gamma mc, \gamma m\vec{v}) \tag{6.32}$$

The relativistic version of the Lorentz equation involves the 4-momentum, the velocity four-vector, and the electromagnetic tensor $F^{\mu\nu}$. We have:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{6.33}$$

The electromagnetic force acting on a point particle of charge e can then be written in a covariant form as:

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \frac{e}{c}F^{\mu\nu}\eta_\nu \qquad \text{(relativistic formulation of the Lorentz force)} \tag{6.34}$$

Intuitively it should be clear how this is going to work. A moving particle experiences the magnetic force. But if we boost to its rest frame, there is no magnetic force. Instead, the magnetic field transforms into an electric field and we find the same force, now interpreted as an electric force.

Let us consider the spatial components of the equation $\mu = 1, 2, 3$. We find that (remember that $\eta_\mu = \gamma(c, -\vec{v})$):

$$\begin{aligned}
\mu = 1: \quad & \frac{\mathrm{d}p_x}{\mathrm{d}\tau} = \frac{e}{c}\gamma\left(cE_x + v_yB_z - v_zB_y\right) \\
\mu = 2: \quad & \frac{\mathrm{d}p_y}{\mathrm{d}\tau} = \frac{e}{c}\gamma\left(cE_y - v_xB_z + v_zB_x\right) \\
\mu = 3: \quad & \frac{\mathrm{d}p_z}{\mathrm{d}\tau} = \frac{e}{c}\gamma\left(cE_z + v_xB_y - v_yB_x\right)
\end{aligned} \tag{6.35}$$

or

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}\tau} = \frac{e}{c}\gamma\left(c\vec{E} + \vec{v}\times\vec{B}\right) = \frac{e}{\gamma}\left(\vec{E} + \vec{v}/c\times\vec{B}\right) \tag{6.36}$$

We can now observe that $\mathrm{d}t/\mathrm{d}\tau = ă\gamma$ hence $\mathrm{d}t = \gamma\mathrm{d}\tau$, so we find that the relativistic extension of the time-evolution for a charge e is identical to the classical expression provided that the relativistic definition of the momentum $\vec{p} = \gamma m\vec{v}$ is used:

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = e\vec{E} + e\vec{v}/c\times\vec{B} \tag{6.37}$$

The relativistic formulation also contains an extra equation coming from $\mu = 0$. This reads:

$$\mu = 0: \quad \frac{\mathrm{d}p^0}{\mathrm{d}\tau} = \frac{e}{c}\gamma c\vec{E}\cdot\vec{v} \quad \Longrightarrow \quad \frac{\mathrm{d}p^0}{\mathrm{d}t} = e\vec{E}\cdot\vec{v} = e\vec{E}\cdot\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \tag{6.38}$$

This equation simply tells us, as expected, that the (relativistic) energy of the particle increases by the work done by the electric field, and that the magnetic field does not exert any work.

(b) Let us begin by assuming $\vec{E} = 0$ and a uniform magnetic field, with direction along the z-axis, $\vec{B} = B_0\hat{z}$. The time evolution of the momentum can be written in two components, one parallel and one perpendicular to the field:

$$\frac{\mathrm{d}p_z}{\mathrm{d}t} = 0 \quad \text{and} \quad \frac{\mathrm{d}\vec{p}_\perp}{\mathrm{d}t} = e(\vec{v}_\perp\times\vec{B}) = \frac{eB_0}{\gamma m}(\vec{p}_\perp\times\hat{z}) \tag{6.39}$$

One immediately deduces that p_z is a constant of motion and that at each moment of time the change of $\vec{p}_\perp$ is proportional to $\vec{p}_\perp\times\hat{z}$ hence perpendicular to $\vec{p}_\perp$. Thus the transverse momentum is constant in amplitude, but changing in direction. Consequently, the momentum momentum $p = \sqrt{p_\perp^2 + p_z^2}$ and the energy $E = \sqrt{p^2 + m^2}$ are also constant (the magnetic force does not exert any work on the particle). This implies a uniform circular motion in the plane perpendicular to the magnetic field. In general, the motion is therefore the superposition of a circular orbit and a uniform drift normal to the circle. It represents an helix.

(c) We examine the helical motion in more details. We recognize the relativistic **cyclotron frequency** defined in Eqs. (3.2) and (14.26) of the book:

$$\omega_c = \frac{eB_0}{\gamma m} \tag{6.40}$$

Hence, the equations governing the transverse momentum can be expressed as:

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = \omega_c p_y \quad \text{and} \quad \frac{\mathrm{d}p_y}{\mathrm{d}t} = -\omega_c p_x \tag{6.41}$$

By taking the time derivatives and combining the results with the original expressions, we find the simple harmonic oscillator equations:

$$\frac{\mathrm{d}^2 p_x}{\mathrm{d}t^2} + \omega_c^2 p_x = 0 \quad \text{and} \quad \frac{\mathrm{d}^2 p_y}{\mathrm{d}t^2} + \omega_c^2 p_y = 0 \tag{6.42}$$

The solutions of this system of equations can be taken as harmonic oscillations in p_x and p_y which are 90° apart, having the general solution

$$p_x = -p_\perp \sin(\omega_c t + \phi) \quad \text{and} \quad p_y = p_\perp \cos(\omega_c t + \phi) \tag{6.43}$$

or using $p_x = \gamma m \dot{x}$ and $p_y = \gamma m \dot{y}$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{p_\perp}{\gamma m} \sin(\omega_c t + \phi) \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{p_\perp}{\gamma m} \cos(\omega_c t + \phi) \tag{6.44}$$

By further integration, one finds the transverse motion in its most general form:

$$x = R\cos(\omega_c t + \phi) + x_0 \quad \text{and} \quad y = R\sin(\omega_c t + \phi) + y_0 \tag{6.45}$$

where the radius of the circle centered at (x_0, y_0) is given by:

$$R = \frac{p_\perp}{\gamma m \omega_c} = \frac{p_\perp}{eB_0} \tag{6.46}$$

Hence,

$$p_\perp = eB_0 R \tag{6.47}$$

(d) We first verify the SI units of the equation to cross-check that it indeed yields a quantity in units of momentum. We have:

$$p_T = ze[\mathrm{C}] \times B[\mathrm{T}] \times R[\mathrm{m}] \quad \Longrightarrow \quad [\mathrm{C\,T\,m}] = [\mathrm{C\,Vs/m^2\,m}] = [\mathrm{J/(m/s)}] \tag{6.48}$$

where we used $[\mathrm{T}] = [\mathrm{Vs/m^2}]$ and $[\mathrm{CV}] = [\mathrm{J}]$. This gives:

$$[\mathrm{J/(m/s)}] = [\mathrm{kg\,(m^2/s^2)/(m/s)}] = [\mathrm{kg\,m/s}] \quad \square \tag{6.49}$$

Hence, our original formula gives the momentum in J/(m/s), which is indeed what we expect. We want to compute p_T in GeV/c. We use the fact that 1 GeV $= e \times 10^9$ J. We divide our expression by 1 GeV/c:

$$\frac{p_T}{1\ \mathrm{GeV/c}} = \frac{zecB[\mathrm{T}]R[\mathrm{m}]}{1\ \mathrm{GeV}} = \frac{zecB[\mathrm{T}]R[\mathrm{m}]}{e \times 10^9\ \mathrm{J}} \approx \frac{z(3 \times 10^8\ \mathrm{m/s})B[\mathrm{T}]R[\mathrm{m}]}{10^9\ \mathrm{J}} = 0.3zB[\mathrm{T}]R[\mathrm{m}] \tag{6.50}$$

where we approximated the speed of light to 3×10^8 m/s, valid to better than the permil level.

7 Free Boson Fields

7.1 Transformation of scalar field Lagrangian density

Consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\sum_{i=1,2}\left[(\partial_\mu\phi_i)(\partial^\mu\phi_i) - m^2\phi_i^2\right]$$

where ϕ_1, ϕ_2 are real scalar fields.

(a) *Consider the field transformation given by:*

$$\phi_1 \to \phi_1' = \phi_1\cos\alpha - \phi_2\sin\alpha \quad \text{and} \quad \phi_2 \to \phi_2' = \phi_1\sin\alpha + \phi_2\cos\alpha \tag{7.1}$$

You may recognize this as a rotation by angle α in the two-dimensional space with ϕ_1 along the horizontal axis and ϕ_2 along the vertical axis.

Show that the Lagrangian density is invariant under this transformation.

(b) *Find the infinitesimal form of the transformation given by Eq. (7.1).*

(c) *Show that*

$$j^\mu = \phi_1\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_1$$

Solution:

a) We consider the change of the Lagrangian under the given transformation:

$$\mathcal{L} \quad \to \quad \mathcal{L}' = \frac{1}{2}\left[\underbrace{(\partial_\mu\phi_1')(\partial^\mu\phi_1') + (\partial_\mu\phi_2')(\partial^\mu\phi_2')}_{\text{of the form } f_1'g_1'+f_2'g_2'}\right] - \frac{1}{2}m^2\left[\underbrace{(\phi_1')^2 + (\phi_2')^2}_{\text{of the form } f_1'g_1'+f_2'g_2'}\right] \tag{7.2}$$

where $f_i = \partial_\mu\phi_i$ and $g_i = \partial^\mu\phi_i$ in the first bracket (there is implicitly also a sum over μ, but we can consider Lorentz indices μ one by one), and $f_i = g_i = \phi_i$ in the second bracket. We now note that the field transformation just acts as:

$$f_1' = f_1\cos\alpha - f_2\sin\alpha \quad \text{and} \quad f_2' = f_1\sin\alpha + f_2\cos\alpha \tag{7.3}$$

and the same for the g functions. Consequently, we find

$$\begin{aligned} f_1'g_1' + f_2'g_2' &= (f_1\cos\alpha - f_2\sin\alpha)(g_1\cos\alpha - g_2\sin\alpha) + (f_1\sin\alpha + f_2\cos\alpha)(g_1\sin\alpha + g_2\cos\alpha) \\ &= (f_1g_1 + f_2g_2)(\cos^2\alpha + \sin^2\alpha) + (f_1g_2 + f_2g_1 - f_1g_2 - f_2g_1)\cos\alpha\sin\alpha \\ &= f_1g_1 + f_2g_2 \end{aligned} \tag{7.4}$$

from which it follows that $\mathcal{L}' = \mathcal{L}$.

b) The change of the fields under an infinitesimal transformation $\delta\alpha$ is given by:

$$\delta\phi_1 = \phi_1' - \phi_1 = \phi_1 \cos\delta\alpha - \phi_2 \sin\delta\alpha - \phi_1 \approx -\phi_2\delta\alpha \tag{7.5}$$

and

$$\delta\phi_2 = \phi_1 \sin\delta\alpha + \phi_2 \cos\delta\alpha - \phi_2 \approx \phi_1\delta\alpha \tag{7.6}$$

c) We seek the corresponding conserved current. We need to consider ϕ_1 and ϕ_2 as two independent fields. The general form of the conserved current is given by Eq. (6.65) of the book. We then have:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\Phi_1 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\Phi_2 \tag{7.7}$$

where Φ_i are defined such that $\delta\phi_i = \Phi_i\delta\alpha$ (see Eq. (6.57) of the book). Therefore in our case we have $\Phi_1 = -\phi_2$ and $\Phi_2 = \phi_1$. Moreover, we note that

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} = \partial^\mu\phi_i \tag{7.8}$$

Hence, putting all the terms together gives the desired result:

$$j^\mu = \phi_1\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_1 \tag{7.9}$$

7.2 Discrete transformation of the free scalar field

*How does a complex scalar field transform under a discrete P, C, and T transformation? (*Hint: *consider what happens on the annihilation–creation operators.) What about CP and CPT? What happens in the case of the real scalar field?*

Solution:

Let us consider a complex scalar field as discussed in Section 7.9 of the book. From Eq. (7.80) of the book, we have

$$\varphi(x) \equiv \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \left(a(\vec{p})e^{-ipx} + b^\dagger(\vec{p})e^{+ipx}\right)$$

$$\varphi^\dagger(x) \equiv \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \left(a^\dagger(\vec{p})e^{+ipx} + b(\vec{p})e^{-ipx}\right)$$

with $p^0 = E_p = +\sqrt{\vec{p}^2 + m^2}$. This field has specific transformations under discrete P, C, T operations.

- P **transformation**: A scalar field should transform under parity as follows:

$$P\varphi(t,\vec{x})P^{-1} = \eta_P\varphi(t,-\vec{x}) \qquad \text{and} \qquad P\varphi^\dagger(t,\vec{x})P^{-1} = \eta_P^*\varphi^\dagger(t,-\vec{x}) \tag{7.10}$$

where the constant η_P is the intrinsic parity of the field ϕ. We want that two parity transformations equal the identity $P^2 = \mathbb{1}$, therefore the parity can be either positive or negative, $\eta_P = \pm 1$. We now compute the action of P on $\varphi(x)$:

$$\begin{aligned} P\varphi(x)P^{-1} &= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \left(Pa(\vec{p})P^{-1}e^{-i(E_pt-\vec{p}\cdot\vec{x})} + Pb^\dagger(\vec{p})P^{-1}e^{+i(E_pt-\vec{p}\cdot\vec{x})}\right) \\ &= \int \frac{\mathrm{d}^3\tilde{\vec{p}}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_{\tilde{p}}}} \left(Pa(-\tilde{\vec{p}})P^{-1}e^{-i(E_{\tilde{p}}t-\tilde{\vec{p}}\cdot(-\vec{x}))} + Pb^\dagger(-\tilde{\vec{p}})P^{-1}e^{+i(E_{\tilde{p}}t-\tilde{\vec{p}}\cdot(-\vec{x}))}\right) \end{aligned} \tag{7.11}$$

where we made the change of integration variable $\tilde{\vec{p}} = -\vec{p}$, without changing the result, and similarly for $\phi^\dagger$.

We notice that this gives us the expected answer for the transformed field provided that the creation and annihilation operators follow the rules:

$$Pa(\vec{p})P^{-1} = \eta_P a(-\vec{p}) \qquad \text{and} \qquad Pa^\dagger(\vec{p})P^{-1} = \eta_P a^\dagger(-\vec{p}) \tag{7.12}$$

In addition:

$$Pb(\vec{p})P^{-1} = \eta_P b(-\vec{p}) \qquad \text{and} \qquad Pb^\dagger(\vec{p})P^{-1} = \eta_P b^\dagger(-\vec{p}) \tag{7.13}$$

Indeed we find:

$$\begin{aligned} P\varphi(x)P^{-1} &= \eta_P \int \frac{\mathrm{d}^3\tilde{\vec{p}}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_{\tilde{p}}}} \left(a(\tilde{\vec{p}})e^{-i(E_{\tilde{p}}t - \tilde{\vec{p}}\cdot(-\vec{x}))} + b^\dagger(\tilde{\vec{p}})e^{+i(E_{\tilde{p}}t - \tilde{\vec{p}}\cdot(-\vec{x}))} \right) \\ &= \eta_P \phi(t, -\vec{x}) \end{aligned} \tag{7.14}$$

Thus we have the parity transformation of $\varphi(x)$ that we wanted. A similar derivation yields the results for $\varphi^\dagger$.

- T **transformation**: Time reversal is a rather special transformation due to the distinguished role of time in quantum mechanics. As discussed in Section 8.21 of the book for the case of the Dirac fields, according to Wigner, time reversal (sometimes called motion reversal) is defined by an anti-unitary operator which also conjugates plain complex numbers.

$$T\varphi(t, \vec{x})T^{-1} = \eta_T \varphi^*(-t, \vec{x}) \tag{7.15}$$

We compute the action of T on $\varphi(x)$:

$$T\varphi(x)T^{-1} = \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \left(Ta(\vec{p})T^{-1}e^{-ipx} + Tb^\dagger(\vec{p})T^{-1}e^{+ipx} \right) \tag{7.16}$$

As for the case of the parity transformation, we then impose that the creation and annihilation operators follow the rule:

$$\begin{aligned} Ta(\vec{p})T^{-1} = \eta_T a(-\vec{p}) \qquad &\text{and} \qquad Ta^\dagger(\vec{p})T^{-1} = \eta_T^* a^\dagger(-\vec{p}), \\ Tb(\vec{p})T^{-1} = \eta_T^* b(-\vec{p}) \qquad &\text{and} \qquad Tb^\dagger(\vec{p})T^{-1} = \eta_T b^\dagger(-\vec{p}) \end{aligned} \tag{7.17}$$

with $|\eta_T| = 1$. We then have:

$$\begin{aligned} T\varphi(x)T^{-1} &= \eta_T \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \left(a(-\vec{p})e^{-ipx} + b^\dagger(-\vec{p})e^{+ipx} \right) \\ &= \eta_T \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \left(a(-\vec{p})e^{-i(E_p t - \vec{p}\cdot\vec{x})} + b^\dagger(-\vec{p})e^{+i(E_p t - \vec{p}\cdot\vec{x})} \right) \\ &= \eta_T \int \frac{d^3\tilde{\vec{p}}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_{\tilde{p}}}} \left(a(\tilde{\vec{p}})e^{-i(E_{\tilde{p}}t - \tilde{\vec{p}}\cdot(-\vec{x}))} + b^\dagger(\tilde{\vec{p}})e^{+i(E_{\tilde{p}}t - \tilde{\vec{p}}\cdot(-\vec{x}))} \right) \\ &= \eta_T \int \frac{d^3\tilde{\vec{p}}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_{\tilde{p}}}} \left(a(\tilde{\vec{p}})e^{+i(E_{\tilde{p}}(-t) - \tilde{\vec{p}}\cdot\vec{x})} + b^\dagger(\tilde{\vec{p}})e^{-i(E_{\tilde{p}}(-t) - \tilde{\vec{p}}\cdot\vec{x})} \right) \\ &= \eta_T \varphi^*(-t, \vec{x}) \end{aligned} \tag{7.18}$$

Similarly for $\varphi^\dagger$.

- C **transformation**: Charge conjugation is defined to transform a particle into its antiparticle, and thus maps between the fields φ and $\varphi^\dagger$. A choice for the transformation of the annihilation operators is:

$$Ca(\vec{p})C^{-1} = \eta_C b(\vec{p}) \qquad \text{and} \qquad Cb(\vec{p})C^{-1} = \eta_C^* a(\vec{p}) \tag{7.19}$$

  One finds that indeed:

$$C\varphi(x)C^{-1} = \eta_C \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \left(b(\vec{p})e^{-ipx} + a^\dagger(\vec{p})e^{+ipx}\right) = \eta_C \varphi^\dagger(x) \tag{7.20}$$

- CP **transformation:** As a consequence of the rules find above, CP transforms $\varphi(t,\vec{x})$ to $\varphi^\dagger(t,-\vec{x})$.
- CPT **transformation:** CPT transforms $\varphi(x)$ to $\varphi^\dagger(-x)$.

For the case of a real scalar field, we have, from Eq. (7.34) of the book:

$$\phi(x) \equiv \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \left(a(\vec{p})e^{-ipx} + a^\dagger(\vec{p})e^{+ipx}\right) \tag{7.21}$$

Similarly, we find that:

$$P\phi(t,\vec{x})P^{-1} = \phi(t,-\vec{x}) \qquad \text{and} \qquad T\phi(t,\vec{x})T^{-1} = \phi(-t,\vec{x}) \tag{7.22}$$

while the scalar field is invariant under C.

8 Free Fermion Dirac Fields

8.1 Gamma matrices

The 4×4 γ-matrices are given (in the Pauli–Dirac representation) by

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Prove the following relations:

(a) $(\gamma^0)^2 = \mathbb{1}, \qquad (\gamma^k)^2 = -\mathbb{1} \qquad k = 1,\ 2,\ 3.$

(b) $\gamma^{0\dagger} = \gamma^0, \qquad (\gamma^k)^\dagger = -\gamma^k.$

(c) $\{\gamma^\mu,\ \gamma^\nu\} = 2\, g^{\mu\nu}\ \mathbb{1}.$

Solution:

Although we explicitly work in the Pauli-Dirac representation, we derive fundamental relations which are valid in any representation.

a) The γ^μ matrices are 4×4 matrices which are subdivided into 2×2 submatrices. Accordingly, in the following we will be using block matrix convention where each entry is itself a 2×2 matrix. We have:

$$\left(\gamma^0\right)^2 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \mathbb{1} \tag{8.1}$$

Similarly, the Pauli matrices σ_i are 2×2 matrices. We have:

$$\left(\gamma^k\right)^2 = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} = \begin{pmatrix} -\left(\sigma^k\right)^2 & 0 \\ 0 & -\left(\sigma^k\right)^2 \end{pmatrix} = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = -\mathbb{1} \tag{8.2}$$

where we used that

$$\left(\sigma^k\right)^2 = \mathbb{1} \tag{8.3}$$

b) We trivially have that $\gamma^{0\dagger} = \gamma^0$. In addition:

$$\left(\gamma^k\right)^\dagger = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & -\sigma^{k\dagger} \\ \sigma^{k\dagger} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix} = -\gamma^k, \tag{8.4}$$

where we used the fact that Pauli matrices are Hermitian.

c) We consider the anticommutator of a pair of γ matrices. We have:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \{\gamma^\nu, \gamma^\mu\} \tag{8.5}$$

We now consider each case one-by-one:

$$\mu = \nu = 0: \quad 2\left(\gamma^0\right)^2 = 2 \cdot \mathbb{1} = 2g^{00}\mathbb{1} \tag{8.6}$$

$$\mu = \nu = k: \quad 2\left(\gamma^k\right)^2 = -2 \cdot \mathbb{1} = 2g^{kk}\mathbb{1},\ k = 1,2,3 \tag{8.7}$$

$$\mu = 0, \nu = k: \quad \gamma^0\gamma^k = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} \tag{8.8}$$

$$\gamma^k\gamma^0 = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^k \\ -\sigma^k & 0 \end{pmatrix} = -\begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} = -\gamma^0\gamma^k \tag{8.9}$$

$$\Longrightarrow \{\gamma^0, \gamma^k\} = \{\gamma^k, \gamma^0\} = 0 = 2g^{0k}\mathbb{1} \quad (g^{0k} = 0) \tag{8.10}$$

$$\mu = i, \nu = k: \quad \gamma^i\gamma^k = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^i\sigma^k & 0 \\ 0 & -\sigma^i\sigma^k \end{pmatrix} \tag{8.11}$$

$$i \neq k \quad \gamma^k\gamma^i = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \begin{pmatrix} \sigma^i\sigma^k & 0 \\ 0 & \sigma^i\sigma^k \end{pmatrix} \quad (\sigma^i\sigma^k = -\sigma^k\sigma^i) \tag{8.12}$$

$$\Longrightarrow \{\gamma^i, \gamma^k\} = 0 = 2g^{ik}\mathbb{1} \quad (g^{ik} = 0 \quad \text{for} \quad i \neq k) \tag{8.13}$$

8.2 Bi-linear covariants

The Lorentz transformation of spinors $\psi(x)$ between two inertial frames can be written as:

$$\psi'(x') = S(\Lambda)\psi(x); \quad x' = \Lambda\, x$$

where S is a 4×4 matrix, which depends on the Lorentz transformation between the two inertial frames.

(a) For a boost in the x direction with Lorentz factor $\gamma = (1-\beta^2)^{-1/2}$, S is given by:

$$S = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} \quad \textit{with} \quad a_\pm = \pm\sqrt{\frac{1}{2}(\gamma \pm 1)}$$

Show that: $S^\dagger S = \gamma \begin{pmatrix} 1 & -\beta\sigma_1 \\ -\beta\sigma_1 & 1 \end{pmatrix}$, $S^\dagger\gamma^0 S = \gamma^0$.

(b) Show that $\overline{\psi}\gamma^5\psi$ is invariant under the Lorentz transformation S for a boost in the x direction and that it changes sign under the parity transformation $\psi' = \gamma^0\psi$, i.e., it is a pseudoscalar.

(c) Show that $v^\mu \equiv \overline{\psi}\gamma^\mu\psi$ in the Dirac representation is a polar vector, i.e., under a Lorentz boost in the x direction the components transform like a four-vector, and under the parity transformation as:

$$(v^0)' = v^0 \qquad \text{and} \qquad \vec{v}\,' = -\vec{v}$$

Solution:

(a) We start from Eq. (8.167) of the book:

$$\Psi'(x) = S(\Lambda)\Psi(\Lambda^{-1}x) \quad \text{where} \quad S(\Lambda) = \exp\left(\frac{i}{2}\Omega_{\mu\nu}S^{\mu\nu}\right) \tag{8.14}$$

The expression for $S(\Lambda)$ in the Pauli-Dirac representation (without loss of generality) is given in Eq. (8.197) of the book:

$$S(\Lambda)_{PD} = \begin{pmatrix} \cosh(\phi/2)\mathbb{1} & \sinh(\phi/2)(\vec{\sigma}\cdot\hat{n}) \\ \sinh(\phi/2)(\vec{\sigma}\cdot\hat{n}) & \cosh(\phi/2)\mathbb{1} \end{pmatrix} \tag{8.15}$$

where

$$\cosh\left(\frac{\phi}{2}\right) = \sqrt{\frac{\gamma+1}{2}} \quad \text{and} \quad \sinh\left(\frac{\phi}{2}\right) = \pm\sqrt{\frac{\gamma-1}{2}} \tag{8.16}$$

For a boost in the x direction, we have $\hat{n} = (1,0,0)$. Hence:

$$S(\beta_x)_{PD} = \begin{pmatrix} \sqrt{\frac{\gamma+1}{2}}\mathbb{1} & \pm\sqrt{\frac{\gamma-1}{2}}\sigma_1 \\ \pm\sqrt{\frac{\gamma-1}{2}}\sigma_1 & \sqrt{\frac{\gamma+1}{2}}\mathbb{1} \end{pmatrix} \equiv \begin{pmatrix} a_+\mathbb{1} & a_-\sigma_1 \\ a_-\sigma_1 & a_+\mathbb{1} \end{pmatrix} \tag{8.17}$$

where (we chose the negative values for the off diagonal terms):

$$a_\pm = \pm\sqrt{\frac{1}{2}(\gamma\pm 1)} \tag{8.18}$$

We then directly get:

$$S^\dagger = \begin{pmatrix} {a_+}^*\mathbb{1} & {a_-}^*{\sigma_1}^\dagger \\ {a_-}^*{\sigma_1}^\dagger & a_+\mathbb{1}^* \end{pmatrix} = \begin{pmatrix} a_+\mathbb{1} & a_-\sigma_1 \\ a_-\sigma_1 & a_+\mathbb{1} \end{pmatrix} = S \tag{8.19}$$

since

$${\sigma_1}^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \tag{8.20}$$

and $a_\pm$ is real. Therefore:

$$S^\dagger S = S^2 = \begin{pmatrix} \left({a_+}^2 + {a_-}^2\right)\mathbb{1} & 2a_+a_-\sigma_1 \\ 2a_+a_-\sigma_1 & \left({a_+}^2+{a_-}^2\right)\mathbb{1} \end{pmatrix} = \gamma\begin{pmatrix} \mathbb{1} & -\beta\sigma_1 \\ -\beta\sigma_1 & \mathbb{1} \end{pmatrix} \tag{8.21}$$

since

$${a_+}^2 + {a_-}^2 = \frac{1}{2}\gamma + \frac{1}{2} + \frac{1}{2}\gamma - \frac{1}{2} = \gamma$$

and

$$2a_+a_- = -2\sqrt{\frac{1}{2}(\gamma+1)}\sqrt{\frac{1}{2}(\gamma-1)} = -\sqrt{\gamma^2-1} = -\sqrt{\frac{1}{1-\beta^2}-1} = -\beta\sqrt{\frac{1}{1-\beta^2}} = -\beta\gamma\ .$$

We also find (again in the Pauli-Dirac representation):

$$S^\dagger\gamma^0 = S\gamma^0 = \begin{pmatrix} a_+\mathbb{1} & a_-\sigma_1 \\ a_-\sigma_1 & a_+\mathbb{1} \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \begin{pmatrix} a_+\mathbb{1} & -a_-\sigma_1 \\ a_-\sigma_1 & -a_+ \end{pmatrix} \tag{8.22}$$

and therefore

$$S^\dagger\gamma^0 S = S\gamma^0 S = \begin{pmatrix} \left({a_+}^2 - {a_-}^2\right)\mathbb{1} & 0 \\ 0 & -\left({a_+}^2-{a_-}^2\right)\mathbb{1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \gamma^0 \tag{8.23}$$

since ${a_+}^2 - {a_-}^2 = \frac{1}{2}\gamma + \frac{1}{2} - \frac{1}{2}\gamma + \frac{1}{2} = 1$.

(b) We compute the transformation for the specific case of the boost in the x-direction. The transformation is

$$\psi \overset{boost}{\longrightarrow} S\psi \qquad \text{and} \qquad \overline{\psi} = \psi^\dagger\gamma^0 \overset{boost}{\longrightarrow} \psi^\dagger S^\dagger\gamma^0 \tag{8.24}$$

Hence:

$$\overline{\psi}\gamma^5\psi \overset{boost}{\longrightarrow} \psi^\dagger S^\dagger\gamma^0\gamma^5 S\psi = \psi^\dagger \underbrace{S^\dagger\gamma^0 S}_{\gamma^0}\gamma^5\psi = \overline{\psi}\gamma^5\psi \qquad \square \tag{8.25}$$

where we used the fact that:

$$\begin{aligned}\gamma^5 S &= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} a_+\mathbb{1} & a_-\sigma_1 \\ a_-\sigma_1 & a_+\mathbb{1} \end{pmatrix} = \begin{pmatrix} a_-\sigma_1 & a_+\mathbb{1} \\ a_+\mathbb{1} & a_-\sigma_1 \end{pmatrix} = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \\ &= S\gamma^5 \end{aligned} \tag{8.26}$$

Consequently, the quantity is Lorentz invariant. We now consider its transformation under parity. The parity transformation has been given in Eq. (8.204) of the book. We find:

$$\psi \overset{parity}{\longrightarrow} \gamma^0\psi \qquad \text{and} \qquad \overline{\psi} = \psi^\dagger\gamma^0 \overset{parity}{\longrightarrow} (\gamma^0\psi)^\dagger\gamma^0 = \psi^\dagger\gamma^{0\dagger}\gamma^0 = \psi^\dagger(\gamma^0)^2 = \psi^\dagger \tag{8.27}$$

Consequently:

$$\overline{\psi}\gamma^5\psi \overset{parity}{\longrightarrow} \psi^\dagger\gamma^5\gamma^0\psi = -\psi^\dagger\gamma^0\gamma^5\psi = -\overline{\psi}\gamma^5\psi \qquad \square \tag{8.28}$$

where we used $\gamma^5\gamma^0 = -\gamma^0\gamma^5$. The quantity $\overline{\psi}\gamma^5\psi$ is therefore a pseudo-scalar, since it is invariant under LT and changes sign under Parity.

(c) We derive the transformations for each component of the 4-vector $\mathrm{v}^\mu \equiv \overline{\psi}\gamma^\mu\psi$ with $\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$ and $\psi^\dagger = \left(\psi_A^\dagger, \psi_B^\dagger\right)$. The timelike component is:

$$\mathrm{v}^0 = \overline{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi = (\psi_A^\dagger\psi_A + \psi_B^\dagger\psi_B) \tag{8.29}$$

and for the three spatial components v^k, $k = 1, 2, 3$, we find:

$$\mathrm{v}^k = \overline{\psi}\gamma^k\psi = \psi^\dagger\gamma^0\gamma^k\psi = \left(\psi_A^\dagger, \psi_B^\dagger\right)\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \quad \longrightarrow \quad \mathrm{v}^k = \psi_A^\dagger\sigma_k\psi_B + \psi_B^\dagger\sigma_k\psi_A \tag{8.30}$$

We now consider the transformation of this quantity under a LT:

- For $k = 0$:

$$\begin{aligned}\mathrm{v}^0 = \overline{\psi}\gamma^0\psi \quad &\overset{boost}{\longrightarrow} \quad (\psi^\dagger S^\dagger\gamma^0)\gamma^0(S\psi) = \psi^\dagger S^\dagger S\psi = \left(\psi_A^\dagger, \psi_B^\dagger\right)\gamma\begin{pmatrix} 1 & -\beta\sigma^1 \\ -\beta\sigma^1 & 1 \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \\ &= \quad \gamma\left(\psi_A^\dagger, \psi_B^\dagger\right)\begin{pmatrix} \psi_A & -\beta\sigma^1\psi_B \\ \psi_B & -\beta\sigma^1\psi_A \end{pmatrix} = \gamma\left[\underbrace{\psi_A^\dagger\psi_A + \psi_B^\dagger\psi_B}_{\mathrm{v}^0} - \beta(\underbrace{\psi_A^\dagger\sigma^1\psi_B + \psi_B^\dagger\sigma^1\psi_A}_{\mathrm{v}^1})\right] \\ &= \quad \gamma(\mathrm{v}^0 - \beta\mathrm{v}^1) \end{aligned} \tag{8.31}$$

- For the other components $k = 1, 2, 3$ we get:

$$\mathrm{v}^k = \overline{\psi}\gamma^k\psi \quad \overset{boost}{\longrightarrow} \quad (\psi^\dagger S^\dagger\gamma^0)\gamma^k S\psi = \psi^\dagger(S^\dagger\gamma^0\gamma^k S)\psi \tag{8.32}$$

We note that:

$$\begin{aligned} S^\dagger\gamma^0\gamma^k S &= S\gamma^0\gamma^0\gamma^k S = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}\begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} \\ &= \begin{pmatrix} a_+a_-\sigma_k\sigma_1 + a_+a_-\sigma_1\sigma_k & a_+^2\sigma_k + a_-^2\sigma_1\sigma_k\sigma_1 \\ a_-^2\sigma_1\sigma_k\sigma_1 + a_+^2\sigma_k & a_+a_-\sigma_1\sigma_k + a_+a_-\sigma_k\sigma_1 \end{pmatrix} \end{aligned} \tag{8.33}$$

where we used $(\sigma_i)^2 = 1$, $\sigma_i\sigma_j = -\sigma_j\sigma_i$ $(i \neq j)$ and $a_+^2 - a_-^2 = 1$. We now distinguish the case $k = 1$ from $k = 2, 3$:

(a) For $k = 1$:

$$S^\dagger\gamma^0\gamma_1 S = \gamma\begin{pmatrix} -\beta & \sigma_1 \\ \sigma_1 & -\beta \end{pmatrix} \tag{8.34}$$

$$\begin{aligned} \mathrm{v}^1 &\overset{boost}{\longrightarrow} \gamma\left(\psi_A^\dagger, \psi_B^\dagger\right)\begin{pmatrix} -\beta & \sigma_1 \\ \sigma_1 & -\beta \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \gamma\left[-\beta\underbrace{\psi_A^\dagger\psi_A + \psi_B^\dagger\psi_B}_{\mathrm{v}^0} + \underbrace{\psi_A^\dagger\sigma_1\psi_B + \psi_B^\dagger\sigma_1\psi_A}_{\mathrm{v}^1}\right] \\ &= \gamma(\mathrm{v}^1 - \beta\mathrm{v}^0) \end{aligned} \tag{8.35}$$

(b) For $k = 2, 3$:

$$S^\dagger\gamma^0\gamma^k S = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} = \gamma^0\gamma^k \tag{8.36}$$

$$\mathrm{v}^k \overset{boost}{\longrightarrow} \psi^\dagger S^\dagger\gamma^0\gamma^k S\psi = \psi^\dagger\gamma^0\gamma^k\psi = \overline{\psi}\gamma^k\psi = \mathrm{v}^k \tag{8.37}$$

Therefore, v transforms for a boost in the x-direction according to:

$$\mathrm{v}^{0\prime} = \gamma(\mathrm{v}^0 - \beta\mathrm{v}^1) \quad \mathrm{v}^{1\prime} = \gamma(\mathrm{v}^1 - \beta\mathrm{v}^0) \quad \mathrm{v}^{2\prime} = \mathrm{v}^2 \quad \mathrm{v}^{3\prime} = \mathrm{v}^3 \tag{8.38}$$

which corresponds to the Lorentz transformation of a four-vector.

We now consider the transformation under parity.

- For $k = 0$:

$$(\mathrm{v}^0) = \overline{\psi}\gamma^0\psi \overset{parity}{\longrightarrow} (\psi^\dagger\underbrace{\gamma^{0\dagger}}_{\gamma^0}\gamma^0)\gamma^0(\gamma^0\psi) = \psi^\dagger\gamma^0\underbrace{(\gamma^0)^2}_{\mathbb{1}}\gamma^0\psi = \overline{\psi}\gamma^0\psi = \mathrm{v}^0 \tag{8.39}$$

- For $k = 1, 2, 3$:

$$(\mathrm{v}^k) = \overline{\psi}\gamma^k\psi \overset{parity}{\longrightarrow} \psi^\dagger\underbrace{\gamma^{0\dagger}\gamma^0}_{\mathbb{1}}\underbrace{\gamma^k\gamma^0}_{-\gamma^0\gamma^k}\psi = -\psi^\dagger\gamma^0\gamma^k\psi = -\overline{\psi}\gamma^k\psi = -\mathrm{v}^k \tag{8.40}$$

Hence the Parity transformation gives:

$$(\mathrm{v}^0) \overset{parity}{\longrightarrow} \mathrm{v}^0 \qquad \text{and} \qquad (\mathrm{v}^k) \overset{parity}{\longrightarrow} -\mathrm{v}^k \quad (k = 1, 2, 3) \tag{8.41}$$

We have therefore shown that v is Lorentz invariant under a boost in the x-direction and transforms under Parity as above. $\square$

8.3 Algebra of the Lorentz group for the spinors

Given the anticommutation relation of the γ matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$$

prove that the matrices

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

satisfy the generator algebra of the Lorentz group:

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\,(g^{\mu\sigma}S^{\nu\rho} + g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho})$$

Solution:

The $S^{\mu\nu}$ were defined in Eq. (8.169) of the book as the generators of the Lorentz group for the spinors. To prove that they satisfy the commutation relations of the Lorentz group, we preliminary note that:

$$S^{\mu\nu} = \frac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{i}{4}(2\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{i}{2}(\gamma^\mu\gamma^\nu - \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}) = \frac{i}{2}(\gamma^\mu\gamma^\nu - g^{\mu\nu}) \tag{8.42}$$

We then calculate their commutation relations:

$$[S^{\mu\nu}, S^{\rho\sigma}] = \left[\frac{i}{2}(\gamma^\mu\gamma^\nu - g^{\mu\nu}), \frac{i}{2}(\gamma^\rho\gamma^\sigma - g^{\rho\sigma})\right] \tag{8.43}$$

We note that the terms proportional to the metric tensor, do not give contributions to the commutator. Therefore:

$$[S^{\mu\nu}, S^{\rho\sigma}] = -\frac{1}{4}[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma] = -\frac{1}{4}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu) \tag{8.44}$$

We exploit the relation $\gamma^\nu\gamma^\rho = 2g^{\nu\rho} - \gamma^\rho\gamma^\nu$ several times with convenient permutations of the indices to make appear terms proportional to the metric tensors $g^{\mu\sigma}$, $g^{\nu\rho}$, $g^{\mu\rho}$, $g^{\nu\sigma}$:

$$\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] &= -\frac{1}{4}(\gamma^\mu(2g^{\nu\rho} - \gamma^\rho\gamma^\nu)\gamma^\sigma - \gamma^\rho(2g^{\sigma\mu} - \gamma^\mu\gamma^\rho)\gamma^\nu) \\
&= -\frac{1}{4}(2g^{\nu\rho}\gamma^\mu\gamma^\sigma - \gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma - 2g^{\sigma\mu}\gamma^\rho\gamma^\nu + \gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu) \\
&= -\frac{1}{4}(2g^{\nu\rho}\gamma^\mu\gamma^\sigma - 2g^{\sigma\mu}\gamma^\rho\gamma^\nu - (2g^{\mu\rho} - \gamma^\rho\gamma^\mu)\gamma^\nu\gamma^\sigma + \gamma^\rho\gamma^\mu(2g^{\sigma\nu} - \gamma^\nu\gamma^\sigma)) \\
&= -\frac{1}{4}(2g^{\nu\rho}\gamma^\mu\gamma^\sigma - 2g^{\sigma\mu}\gamma^\rho\gamma^\nu - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + 2g^{\sigma\nu}\gamma^\rho\gamma^\mu)
\end{aligned} \tag{8.45}$$

By expanding each term as:

$$2g^{\nu\rho}\gamma^\mu\gamma^\sigma = g^{\nu\rho}(\gamma^\mu\gamma^\sigma + \gamma^\mu\gamma^\sigma) = g^{\nu\rho}(\gamma^\mu\gamma^\sigma + 2g^{\mu\sigma} - \gamma^\sigma\gamma^\mu) \tag{8.46}$$

and by using the fact that the metric tensor is symmetric, we obtain:

$$\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] = -\frac{1}{4}(g^{\nu\rho}(\gamma^\mu\gamma^\sigma - \gamma^\sigma\gamma^\mu) - g^{\sigma\mu}(\gamma^\rho\gamma^\nu - \gamma^\nu\gamma^\rho) + \\
- g^{\mu\rho}(\gamma^\nu\gamma^\sigma - \gamma^\sigma\gamma^\nu) + g^{\sigma\nu}(\gamma^\rho\gamma^\mu - \gamma^\mu\gamma^\rho))
\end{aligned} \tag{8.47}$$

Finally:

$$\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] &= i(g^{\nu\rho}S^{\mu\sigma} - g^{\sigma\mu}S^{\rho\nu} - g^{\mu\rho}S^{\nu\sigma} + g^{\sigma\nu}S^{\rho\mu}) \\
&= i(g^{\nu\rho}S^{\mu\sigma} + g^{\mu\sigma}S^{\nu\rho} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho})
\end{aligned} \tag{8.48}$$

where we used that $S^{\mu\nu} = -S^{\nu\mu}$ and $g^{\mu\nu} = g^{\nu\mu}$. $\square$

8.4 Chirality and helicity

(a) Show that the chirality is not a good quantum number for a massive fermion by checking the commutator $[H, \gamma_5]$.

(b) Show that helicity is conserved although it depends on the choice of the coordinate system.

Solution:

(a) The chirality of defined by the γ^5 matrix as discussed in Section 8.19 of the book. We first consider the case of a massless fermion. The Dirac Hamiltonian is given by Eq. (8.2) of the book. For $m = 0$ we find:

$$H_{Dirac} = \vec{\alpha} \cdot \vec{p} = \alpha^i p^i = \gamma^0 \gamma^i p_i \tag{8.49}$$

(the index i implies the summing convention from 0 to 3). We now consider the commutation

$$\begin{aligned}
[H_{Dirac}, \gamma_5] &= [\gamma^0 \gamma^i p^i, \gamma_5] = p^i \left\{ \gamma^0 \left[\gamma^i, \gamma_5\right] + \left[\gamma^0, \gamma_5\right] \gamma^i \right\} \\
&= p^i \left\{ \gamma^0 \left(\gamma^i \gamma_5 - \gamma_5 \gamma^i\right) + \left(\gamma^0 \gamma_5 - \gamma_5 \gamma^0\right) \gamma^i \right\} \\
&= p^i \left\{ \gamma^0 \left(\gamma^i \gamma_5 + \gamma_5 \gamma^i\right) - \left(\gamma^0 \gamma_5 + \gamma_5 \gamma^0\right) \gamma^i \right\} \\
&= 0
\end{aligned} \tag{8.50}$$

where we used the anti-commutation relation $\{\gamma^5, \gamma^\mu\} = 0$. Now we can easily the mass term and consider again the commutator:

$$[H_{Dirac}, \gamma_5] = [\vec{\alpha} \cdot \vec{p}, \gamma_5] = m \left(\gamma^0 \gamma^5 - \gamma^5 \gamma^0\right) = 2m \gamma^0 \gamma^5 \neq 0 \tag{8.51}$$

Hence, the **chirality is not a good quantum number for a massive fermion**. It is not always conserved as the particle evolves in time (unless the particle is massless). In addition, the generators of the Lorentz transformation have been introduced in Eq. (8.168) of the book:

$$S^{\mu\nu} \equiv \frac{i}{4} \left[\gamma^\mu, \gamma^\nu\right] \tag{8.52}$$

Consequently, we find that:

$$\begin{aligned}
[S^{\mu\nu}, \gamma^5] &= \frac{i}{4} \left[\left[\gamma^\mu, \gamma^\nu\right], \gamma^5\right] = \frac{i}{4} \left[\left[\gamma^\mu, \gamma^\nu\right] \gamma^5 - \gamma^5 \left[\gamma^\mu, \gamma^\nu\right]\right] \\
&= \frac{i}{4} \left[\gamma^\mu \gamma^\nu \gamma^5 - \gamma^\nu \gamma^\mu \gamma^5 - \gamma^5 \gamma^\mu \gamma^\nu + \gamma^5 \gamma^\nu \gamma^\mu\right] \\
&= \frac{i}{4} \left[\gamma^\mu \gamma^\nu \gamma^5 - \gamma^\nu \gamma^\mu \gamma^5 - \gamma^\mu \gamma^\nu \gamma^5 + \gamma^\nu \gamma^\mu \gamma^5\right] = 0
\end{aligned} \tag{8.53}$$

where we used $\gamma^5 \gamma^\mu = -\gamma^\mu \gamma^5$ (see Appendix E of the book). Hence, **chirality is a Lorentz invariant**.

(b) The helicity operator is defined in Eq. (8.138) of the book. It represents the normalized projection of the spin along the flight direction of the particle:

$$h \equiv \frac{\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|} = \begin{pmatrix} \dfrac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} & 0 \\ 0 & \dfrac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} \end{pmatrix} = \begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix} \tag{8.54}$$

where we introduced the unit vector in the direction of the momentum of the particle $\hat{p} \equiv \vec{p}/|\vec{p}|$. It was already shown in Eq. (8.139) of the book that

$$[\vec{\Sigma} \cdot \vec{p}, H_{Dirac}] = \left([\vec{\Sigma}, H_{Dirac}] \cdot \vec{p}\right) = (-2i(\vec{\alpha} \times \vec{p}) \cdot \vec{p}) = 0 \tag{8.55}$$

Hence, **helicity is a good quantum number**: it is conserved always because it commutes with the Hamiltonian. That is, its value does not change with time within a given reference frame. However, **it is not Lorentz invariant** since it is a product of an axial vector with a 3-vector.

9 Interacting Fields and Propagator Theory

9.1 Toy model with interacting scalar fields

In Section 9.8 of the book, we have considered a simple toy model with real scalar and complex scalar fields. We have computed the scattering amplitude for the process $\pi^+\pi^- \to \pi^+\pi^-$.

(a) In a similar fashion, compute the amplitude for the $\pi^0\pi^0 \to \pi^0\pi^0$ process.

(b) Assuming an additional coupling between the neutral and charged fields given by g', compute the amplitudes for $\pi^+\pi^- \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$.

Solution:
In the toy model, we assume the existence of a triplet of real scalar fields $\phi_i(x)$ $(i = 1, 2, 3)$, which are rewritten in terms of one real scalar field $\phi(x)$ and two conjugated complex scalar fields φ and $\varphi^\dagger$. The $\phi(x)$ describes the π^0 bosons, while the φ, $\varphi^\dagger$ correspond to the π^+, π^- bosons. In addition, we assume the existence of a second real scalar field called $\sigma(x)$, which describes the σ boson of rest mass M. So finally, we have the following fields (see Eq. (9.13) of the book):

$$\varphi \equiv \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right), \quad \varphi^\dagger = \frac{1}{\sqrt{2}}\left(\phi_1 - i\phi_2\right), \quad \phi \equiv \phi_3, \quad \text{and} \quad \sigma \tag{9.1}$$

According to Eqs. (9.20) and (9.42) of the book, we introduce in the common way the following sets of creation–annihilation operators associated to each fields:

$$\begin{cases} \phi(x) = \displaystyle\int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \left(c(\vec{p})e^{-ip\cdot x} + c^\dagger(\vec{p})e^{+ip\cdot x}\right) \\ \varphi(x) \equiv \displaystyle\int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \left(a(\vec{p})e^{-ip\cdot x} + b^\dagger(\vec{p})e^{+ip\cdot x}\right) \\ \varphi^\dagger(x) \equiv \displaystyle\int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \left(a^\dagger(\vec{p})e^{+ip\cdot x} + b(\vec{p})e^{-ip\cdot x}\right) \\ \sigma(x) = \displaystyle\int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \left(d(\vec{p})e^{-ip\cdot x} + d^\dagger(\vec{p})e^{+ip\cdot x}\right) \end{cases} \tag{9.2}$$

with the suitable set of commutations rules for bosons (see e.g. Eq. (9.21) of the book for an example).

To define the Lagrangian, we start with its basic form with the triplet ϕ_i fields. In addition to the self-interaction term of type quartic type ϕ^4, we also include the direct σ-ϕ local interaction term. We then rewrite it as a function of the "physical" fields ϕ, φ, $\varphi^\dagger$ and σ. Mathematically, we have for the interaction part of the Lagrangian (compare with Eq. (9.77) of the book):

$$\begin{aligned} \mathcal{L}_{int} &= -g(\phi_1^2 + \phi_2^2 + \phi_3^2)^2 - \lambda\sigma(\phi_1^2 + \phi_2^2 + \phi_3^2) + \cdots \\ &= -g(\phi_1^2 + \phi_2^2)^2 + 2g(\phi_1^2 + \phi_2^2)\phi_3^2 - g\phi_3^4 - \lambda\sigma(\phi_1^2 + \phi_2^2) - \lambda\sigma\phi_3^2 + \ldots \\ &= -g\phi^4 - \lambda\sigma\phi^2 - 4g(\varphi^\dagger\varphi)^2 - 4g\varphi^\dagger\varphi\phi^2 - 2\lambda\sigma(\varphi^\dagger\varphi) + \cdots \end{aligned} \tag{9.3}$$

where we used:

$$(\varphi^\dagger \varphi) = \frac{1}{2}\left(\phi_1 - i\phi_2\right)\left(\phi_1 + i\phi_2\right) = \frac{1}{2}\left(\phi_1^2 + \phi_2^2\right) \tag{9.4}$$

The Lagrangian describes four-point interactions among the π bosons, as well as interactions between the σ and the π's. The Lagrangian density has the units of energy to the fourth power $[\mathcal{L}_{int}] = 4$ (see Eq. (6.22) of the book). Since the field has dimension $[\varphi] = 1$, then g must be dimensionless, while $[\lambda] = 1$.

(a) $\pi^0(q_1)\pi^0(q_2) \to \pi^0(q_3)\pi^0(q_4)$ scattering amplitude: We first consider the **four-point interaction**. The matrix element at first order is given by (see Section 9.13 of the book):

$$\langle q_3 q_4| S_1 |q_1 q_2\rangle = (-i)\langle q_3 q_4| T\left(\int \mathrm{d}^4 x g\phi^4(x) |q_1 q_2\rangle\right) = (-ig)\langle q_3 q_4| T\left(\int \mathrm{d}^4 x \phi\phi\phi\phi\right)|q_1 q_2\rangle \tag{9.5}$$

By applying Wick's theorem, the shorthand notation of all possible contractions is given by (compare with Eq. (9.155) of the book):

$$T(\phi\phi\phi\phi) = \phi\phi\phi\phi + \overbracket{\phi\phi}\phi\phi + \overbracket{\phi\phi}\,\overbracket{\phi\phi} \tag{9.6}$$

We have seen in Section 9.13 of the book that only the first term is relevant to the scattering amplitude we want to calculate. Hence, we just need to consider (compare with Eq. (9.79) of the book):

$$\langle q_3 q_4| S_1 |q_1 q_2\rangle = (-ig)\langle q_3 q_4| \int \mathrm{d}^4 x \phi\phi\phi\phi\, |q_1 q_2\rangle \tag{9.7}$$

We can write the initial state as:

$$|q_1 q_2\rangle = \sqrt{(2\pi)^3 2E_{q1}(2\pi)^3 2E_{q2}}\; c^\dagger(\vec{q}_1)c^\dagger(\vec{q}_2)|0\rangle \equiv \mathcal{N}_i c^\dagger(\vec{q}_1)c^\dagger(\vec{q}_2)|0\rangle \tag{9.8}$$

and the final state:

$$\langle q_3 q_4| = \sqrt{(2\pi)^3 2E_{q3}(2\pi)^3 2E_{q4}}\,\langle 0| c(\vec{q}_3)c(\vec{q}_4) \equiv \mathcal{N}_f \langle 0| c(\vec{q}_3)c(\vec{q}_4) \tag{9.9}$$

where $\mathcal{N}_i$ and $\mathcal{N}_f$ are the normalisation factors (see Eqs. (9.24)–(9.26) of the book). Introducing the short hand notation $c_i \equiv c(\vec{p}_i)e^{-ip_i\cdot x}$, we find that the relevant term is given by (compare with Eq. (9.81) of the book):

$$\begin{aligned}\langle q_3 q_4| S_1 |q_1 q_2\rangle =& (-ig)\mathcal{N}_i\mathcal{N}_f \langle 0| c(\vec{q}_3)c(\vec{q}_4) \\ &\times \int \mathrm{d}^4 x \int \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^{3/2}\sqrt{2E_{p1}}}\int \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^{3/2}\sqrt{2E_{p2}}}\int \frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^{3/2}\sqrt{2E_{p3}}}\int \frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^{3/2}\sqrt{2E_{p4}}} \\ &(c_1 + c_1^\dagger)\times(c_2 + c_2^\dagger)\times(c_3 + c_3^\dagger)\times(c_4 + c_4^\dagger)\; c^\dagger(\vec{q}_1)c^\dagger(\vec{q}_2)|0\rangle\end{aligned}$$

From the total of 16 terms obtained by multiplying the terms in brackets with creation–annihilation operators, we need to only consider those who contain exactly two creation and two annihilation operators in order to match the initial and final states. Therefore the following six combinations $\mathcal{C}_i$ $(i = 1, \ldots, 6)$ remain:

$$c_1 c_2 c_3^\dagger c_4^\dagger\ ,\ c_1 c_2^\dagger c_3 c_4^\dagger\ ,\ c_1 c_2^\dagger c_3^\dagger c_4\ ,\ c_1^\dagger c_2 c_3 c_4^\dagger\ ,\ c_1^\dagger c_2 c_3^\dagger c_4\ ,\ \ c_1^\dagger c_2^\dagger c_3 c_4 \tag{9.10}$$

The detailed calculation is the same as in the case of the pure ϕ^4 theory, as performed in **Ex. 9.2**. Hence we directly write the result Eq. (9.76) in **Ex. 9.2**:

$$\langle q_3 q_4| S_1 |q_1 q_2\rangle = (-24ig)(2\pi)^4\delta^4(q_3 + q_4 - q_1 - q_2) \tag{9.11}$$

where we recognize the result of Eq. (9.162) of the book. The corresponding Feynman diagram is shown in Figure 9.1.

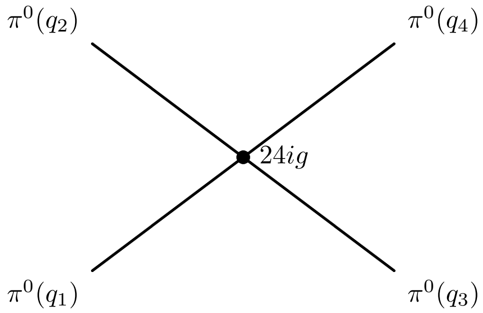

Figure 9.1 Four-point vertex for pion scattering $\pi^0(q_1) + \pi^0(q_2) \to \pi^0(q_3) + \pi^0(q_4)$.

We now focus on the **scattering diagrams** with exchange of a σ boson. We consider the $-\lambda\sigma\phi^2$ term in the interaction Lagrangian. Including the initial and final states and integrating over two space-time points x_1 and x_2, the amplitude is (compare with Eq. (9.89) of the book):

$$\begin{aligned}\langle q_3 q_4 | S_2 | q_1 q_2 \rangle &= \frac{(-\lambda i)^2}{2!} \langle q_3 q_4 | T \left[\int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 \sigma(x_1)\phi^2(x_1)\sigma(x_2)\phi^2(x_2) \right] | q_1 q_2 \rangle \\ &= \frac{(-\lambda i)^2}{2!} \langle q_3 q_4 | \int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 T \left[\sigma(x_1)\sigma(x_2)\phi(x_1)\phi(x_1)\phi(x_2)\phi(x_2) \right] | q_1 q_2 \rangle \end{aligned} \quad (9.12)$$

We are going to drop the factor 2! **because there is a second, identical term that comes from interchanging** x_1 **and** x_2**!** As the σ fields have no external legs, we consider the following term:

$$\langle q_3 q_4 | \phi(x_1)\phi(x_1)\phi(x_2)\phi(x_2) | q_1 q_2 \rangle \quad (9.13)$$

The initial and final states are as before. Introducing the short hand notation $c_{ij} \equiv c(\vec{p}_i) e^{-ip_i \cdot x_j}$, we can rewrite the considered term as (we omit the normalizations for clarity):

$$\langle 0 | c(\vec{q}_3) c(\vec{q}_4) \int \mathrm{d}^4 x \int \mathrm{d}^3 \vec{p}_1 \int \mathrm{d}^3 \vec{p}_2 \int \mathrm{d}^3 \vec{p}_3 \int \mathrm{d}^3 \vec{p}_4$$
$$\times (c_{11} + c_{11}^\dagger) \times (c_{21} + c_{21}^\dagger) \times (c_{32} + c_{32}^\dagger) \times (c_{42} + c_{42}^\dagger)\ c^\dagger(\vec{q}_1) c^\dagger(\vec{q}_2) | 0 \rangle$$

From the total of 16 terms obtained by multiplying the terms in brackets with creation–annihilation operators, we need to only consider those who contain exactly two creation and two annihilation operators in order to match the initial and final states. Therefore the following six combinations $\mathcal{C}_i$ $(i = 1, \ldots, 6)$ remain:

$$c_{11}c_{21}c_{32}^\dagger c_{42}^\dagger \ , \ c_{11}c_{21}^\dagger c_{32}c_{42}^\dagger \ , \ c_{11}c_{21}^\dagger c_{32}^\dagger c_{42} \ , \ c_{11}^\dagger c_{21}c_{32}c_{42}^\dagger \ , \ c_{11}^\dagger c_{21}c_{32}^\dagger c_{42} \ , \ c_{11}^\dagger c_{21}^\dagger c_{32}c_{42} \quad (9.14)$$

Wick's theorem lets us replace the time-ordered product with the normal ordered product. **Normal ordering means that all annihilation operators are moved to the right of the expression, while the creation operators are moved to the left**. Hence, we must consider the **normal ordered** product of our operators:

$$c_{32}^\dagger c_{42}^\dagger c_{11}c_{21} \ , \ c_{21}^\dagger c_{42}^\dagger c_{11}c_{32} \ , \ c_{21}^\dagger c_{32}^\dagger c_{11}c_{42} \ , \ c_{11}^\dagger c_{42}^\dagger c_{21}c_{32} \ , \ c_{11}^\dagger c_{32}^\dagger c_{21}c_{42} \ , \ c_{11}^\dagger c_{21}^\dagger c_{32}c_{42} \quad (9.15)$$

Let us for instance compute the last normal product of the list:

$$\langle 0 | : \mathcal{C}_6 : | 0 \rangle = \underbrace{\langle 0 | c(\vec{q}_3) c(\vec{q}_4) c_{11}^\dagger c_{21}^\dagger}_{\mathcal{A}} \underbrace{c_{32} c_{42} c^\dagger(\vec{q}_1) c^\dagger(\vec{q}_2) | 0 \rangle}_{\mathcal{B}} \quad (9.16)$$

We will use the commutation rule $\left[c(\vec{p}), c^\dagger(\vec{p}\,')\right] = \delta^3(\vec{p}-\vec{p}\,')$ (see Eq. (7.83) of the book). Let us first consider $\mathcal{B}$:

$$\begin{aligned}
\mathcal{B} &= c_{32}c_{42}c^\dagger(\vec{q}_1)c^\dagger(\vec{q}_2)\left|0\right\rangle = c_{32}\left[\delta^3(\vec{p}_4-\vec{q}_1)e^{-ip_4\cdot x_2} + c^\dagger(\vec{q}_1)c_{42}\right]c^\dagger(\vec{q}_2)\left|0\right\rangle \\
&= c_{32}\left\{\delta^3(\vec{p}_4-\vec{q}_1)e^{-ip_4\cdot x_2}c^\dagger(\vec{q}_2) + c^\dagger(\vec{q}_1)\left[\delta^3(\vec{p}_4-\vec{q}_2)e^{-ip_4\cdot x} + \underbrace{\cancel{c^\dagger(\vec{q}_2)c_{42}}}_{c_{42}|0\rangle=0}\right]\right\}\left|0\right\rangle \\
&= \left[\delta^3(\vec{p}_4-\vec{q}_1)c_{32}c^\dagger(\vec{q}_2) + \delta^3(\vec{p}_4-\vec{q}_2)c_{32}c^\dagger(\vec{q}_1)\right]e^{-ip_4\cdot x_2}\left|0\right\rangle \\
&= \left[\delta^3(\vec{p}_4-\vec{q}_1)\left[\delta^3(\vec{p}_3-\vec{q}_2)e^{-ip_3\cdot x_2} + \cancel{c^\dagger(\vec{q}_2)c_{32}}\right] + \delta^3(\vec{p}_4-\vec{q}_2)\left[\delta^3(\vec{p}_3-\vec{q}_1)e^{-ip_3\cdot x_2} + \cancel{c^\dagger(\vec{q}_1)c_{32}}\right]\right]e^{-ip_4\cdot x_2}\left|0\right\rangle \\
&= \left[\delta^3(\vec{p}_4-\vec{q}_1)\delta^3(\vec{p}_3-\vec{q}_2) + \delta^3(\vec{p}_4-\vec{q}_2)\delta^3(\vec{p}_3-\vec{q}_1)\right]e^{-ip_3\cdot x_2}e^{-ip_4\cdot x_2}\left|0\right\rangle
\end{aligned} \tag{9.17}$$

Similarly:

$$\mathcal{A} = \left\langle 0\right|\left[\delta^3(\vec{p}_1-\vec{q}_4)\delta^3(\vec{p}_2-\vec{q}_3) + \delta^3(\vec{p}_1-\vec{q}_3)\delta^3(\vec{p}_2-\vec{q}_4)\right]e^{+ip_1\cdot x_1}e^{+ip_2\cdot x_1} \tag{9.18}$$

Consequently,

$$\begin{aligned}
\left\langle 0\right| : \mathcal{C}_6 : \left|0\right\rangle =& [\delta^3(\vec{p}_4-\vec{q}_1)\delta^3(\vec{p}_3-\vec{q}_2) + \delta^3(\vec{p}_4-\vec{q}_2)\delta^3(\vec{p}_3-\vec{q}_1)] \\
&\times [\delta^3(\vec{p}_1-\vec{q}_4)\delta^3(\vec{p}_2-\vec{q}_3) + \delta^3(\vec{p}_1-\vec{q}_3)\delta^3(\vec{p}_2-\vec{q}_4)]e^{+ip_1\cdot x_1}e^{+ip_2\cdot x_1}e^{-ip_3\cdot x_2}e^{-ip_4\cdot x_2}
\end{aligned} \tag{9.19}$$

Integration over the $\vec{p}_i$'s will yield the following four exponential terms:

$$\begin{aligned}
\int d^3\vec{p}_1\int d^3\vec{p}_2\int d^3\vec{p}_3\int d^3\vec{p}_4\left\langle 0\right|\mathcal{C}_6\left|0\right\rangle : \quad & e^{+iq_4\cdot x_1}e^{+iq_3\cdot x_1}e^{-iq_2\cdot x_2}e^{-iq_1\cdot x_2} + e^{+iq_3\cdot x_1}e^{+iq_4\cdot x_1}e^{-iq_2\cdot x_2}e^{-iq_1\cdot x_2} \\
& + e^{+iq_4\cdot x_1}e^{+iq_3\cdot x_1}e^{-iq_1\cdot x_2}e^{-iq_2\cdot x_2} + e^{+iq_3\cdot x_1}e^{+iq_4\cdot x_1}e^{-iq_1\cdot x_2}e^{-iq_2\cdot x_2} \\
& = 4e^{-i(q_2+q_1)\cdot x_2}e^{+i(q_4+q_3)\cdot x_1}
\end{aligned} \tag{9.20}$$

Now let us move to the other terms. The same calculation as above should be repeated each time. Let us generalize the expression and solve the following product, where a, b, c, d and i, j, k, l are generic indices:

$$\left\langle 0\right| : \mathcal{C}_m : \left|0\right\rangle = \underbrace{\left\langle 0\right| c(\vec{q}_3)c(\vec{q}_4)c^\dagger_{ab}c^\dagger_{cd}}_{\mathcal{A}}\underbrace{c_{ij}c_{kl}c^\dagger(\vec{q}_1)c^\dagger(\vec{q}_2)\left|0\right\rangle}_{\mathcal{B}} \tag{9.21}$$

As before, let us first consider $\mathcal{B}$:

$$\begin{aligned}
\mathcal{B} &= c_{ij}c_{kl}c^\dagger(\vec{q}_1)c^\dagger(\vec{q}_2)\left|0\right\rangle = c_{ij}\left[\delta^3(\vec{p}_k-\vec{q}_1)e^{-ip_k\cdot x_l} + c^\dagger(\vec{q}_1)c_{kl}\right]c^\dagger(\vec{q}_2)\left|0\right\rangle \\
&= c_{ij}\left\{\delta^3(\vec{p}_k-\vec{q}_1)e^{-ip_k\cdot x_l}c^\dagger(\vec{q}_2) + c^\dagger(\vec{q}_1)\left[\delta^3(\vec{p}_k-\vec{q}_2)e^{-ip_k\cdot x_l} + \underbrace{\cancel{c^\dagger(\vec{q}_2)c_{kl}}}_{c_{kl}|0\rangle=0}\right]\right\}\left|0\right\rangle \\
&= \left[\delta^3(\vec{p}_k-\vec{q}_1)c_{ij}c^\dagger(\vec{q}_2) + \delta^3(\vec{p}_k-\vec{q}_2)c_{ij}c^\dagger(\vec{q}_1)\right]e^{-ip_k\cdot x_l}\left|0\right\rangle \\
&= \left[\delta^3(\vec{p}_k-\vec{q}_1)\left[\delta^3(\vec{p}_i-\vec{q}_2)e^{-ip_i\cdot x_j} + \cancel{c^\dagger(\vec{q}_2)c_{ij}}\right] + \delta^3(\vec{p}_k-\vec{q}_2)\left[\delta^3(\vec{p}_i-\vec{q}_1)e^{-ip_i\cdot x_j} + \cancel{c^\dagger(\vec{q}_1)c_{ij}}\right]\right]e^{-ip_k\cdot x_l}\left|0\right\rangle \\
&= \left[\delta^3(\vec{p}_k-\vec{q}_1)\delta^3(\vec{p}_i-\vec{q}_2) + \delta^3(\vec{p}_k-\vec{q}_2)\delta^3(\vec{p}_i-\vec{q}_1)\right]e^{-ip_i\cdot x_j}e^{-ip_k\cdot x_l}\left|0\right\rangle
\end{aligned} \tag{9.22}$$

Similarly,

$$\begin{aligned}
\mathcal{A} &= \langle 0|\, c(\vec{q}_3)c(\vec{q}_4)c^\dagger_{ab}c^\dagger_{cd} = \langle 0|\, c(\vec{q}_3)\left[\delta^3(\vec{p}_a-\vec{q}_4)e^{+ip_a\cdot x_b} + c^\dagger_{ab}c(\vec{q}_4)\right]c^\dagger_{cd} \\
&= \langle 0|\, c(\vec{q}_3)\left\{\delta^3(\vec{p}_a-\vec{q}_4)e^{+ip_a\cdot x_b}c^\dagger_{cd} + c^\dagger_{ab}\left[\delta^3(\vec{p}_a-\vec{q}_4)e^{+ip_a\cdot x_b} + \underbrace{\cancel{c^\dagger_{cd}c(\vec{q}_4)}}_{=0}\right]\right\} \\
&= \langle 0|\left[\delta^3(\vec{p}_a-\vec{q}_4)e^{+ip_a\cdot x_b}c(\vec{q}_3)c^\dagger_{cd} + \delta^3(\vec{p}_a-\vec{q}_4)e^{+ip_a\cdot x_b}c(\vec{q}_3)c^\dagger_{ab}\right] \\
&= \langle 0|\left[\delta^3(\vec{p}_a-\vec{q}_4)e^{+ip_a\cdot x_b}\delta^3(\vec{p}_c-\vec{q}_3)e^{+ip_c\cdot x_d} + \delta^3(\vec{p}_a-\vec{q}_4)e^{+ip_a\cdot x_b}\delta^3(\vec{p}_c-\vec{q}_3)e^{+ip_c\cdot x_d}\right] \\
&= \langle 0|\left[\delta^3(\vec{p}_a-\vec{q}_4)\delta^3(\vec{p}_c-\vec{q}_3) + \delta^3(\vec{p}_c-\vec{q}_4)\delta^3(\vec{p}_a-\vec{q}_3)\right]e^{+ip_a\cdot x_b}e^{+ip_c\cdot x_d}
\end{aligned} \tag{9.23}$$

Consequently:

$$\begin{aligned}
\langle 0|:\mathcal{C}_m:|0\rangle = &\left[\delta^3(\vec{p}_k-\vec{q}_1)\delta^3(\vec{p}_i-\vec{q}_2) + \delta^3(\vec{p}_k-\vec{q}_2)\delta^3(\vec{p}_i-\vec{q}_1)\right] \\
&\times\left[\delta^3(\vec{p}_a-\vec{q}_4)\delta^3(\vec{p}_c-\vec{q}_3) + \delta^3(\vec{p}_c-\vec{q}_4)\delta^3(\vec{p}_a-\vec{q}_3)\right]e^{-ip_i\cdot x_j}e^{-ip_k\cdot x_l}e^{+ip_a\cdot x_b}e^{+ip_c\cdot x_d}
\end{aligned} \tag{9.24}$$

Integration over the $\vec{p}_m$'s ($m = i,k,a,c$) will yield the following four exponential terms, which only depend on the b,d and j,l indices:

$$\begin{aligned}
\int d^3\vec{p}_1\int d^3\vec{p}_2\int d^3\vec{p}_3\int d^3\vec{p}_4 : \quad & e^{-iq_2\cdot x_j}e^{-iq_1\cdot x_l}e^{+iq_4\cdot x_b}e^{+iq_3\cdot x_d} + e^{-iq_2\cdot x_j}e^{-iq_1\cdot x_l}e^{+iq_3\cdot x_b}e^{+iq_4\cdot x_d} \\
& + e^{-iq_1\cdot x_j}e^{-iq_2\cdot x_l}e^{+iq_4\cdot x_b}e^{+iq_3\cdot x_d} + e^{-iq_1\cdot x_j}e^{-iq_2\cdot x_l}e^{+iq_3\cdot x_b}e^{+iq_4\cdot x_d}
\end{aligned} \tag{9.25}$$

We now apply this generic result to compute all the $\vec{p}_m$ integrated $\mathcal{C}_i$ terms, which we call $\mathcal{D}_i$:

$$\begin{aligned}
\mathcal{D}_1(b=2,d=2,j=1,l=1): \quad & e^{-iq_2\cdot x_1}e^{-iq_1\cdot x_1}e^{+iq_4\cdot x_2}e^{+iq_3\cdot x_2} + e^{-iq_2\cdot x_1}e^{-iq_1\cdot x_1}e^{+iq_3\cdot x_2}e^{+iq_4\cdot x_2} \\
& + e^{-iq_1\cdot x_1}e^{-iq_2\cdot x_1}e^{+iq_4\cdot x_2}e^{+iq_3\cdot x_2} + e^{-iq_1\cdot x_1}e^{-iq_2\cdot x_1}e^{+iq_3\cdot x_2}e^{+iq_4\cdot x_2} \\
& = 4e^{-i(q_1+q_2)\cdot x_1}e^{+i(q_3+q_4)\cdot x_2} \\
\mathcal{D}_2(b=1,d=2,j=1,l=2): \quad & e^{-iq_2\cdot x_1}e^{-iq_1\cdot x_2}e^{+iq_4\cdot x_1}e^{+iq_3\cdot x_2} + e^{-iq_2\cdot x_1}e^{-iq_1\cdot x_2}e^{+iq_3\cdot x_1}e^{+iq_4\cdot x_2} \\
& + e^{-iq_1\cdot x_1}e^{-iq_2\cdot x_2}e^{+iq_4\cdot x_1}e^{+iq_3\cdot x_2} + e^{-iq_1\cdot x_1}e^{-iq_2\cdot x_2}e^{+iq_3\cdot x_1}e^{+iq_4\cdot x_2} \\
& = e^{-i(q_2-q_4)\cdot x_1}e^{+i(q_3-q_1)\cdot x_2} + e^{-i(q_2-q_3)\cdot x_1}e^{+i(q_4-q_1)\cdot x_2} \\
& + e^{-i(q_1-q_4)\cdot x_1}e^{+i(q_3-q_2)\cdot x_2} + e^{-i(q_1-q_3)\cdot x_1}e^{+i(q_4-q_2)\cdot x_2} \\
\mathcal{D}_3(b=1,d=2,j=1,l=2): \quad & \text{same as } \mathcal{D}_2 \\
\mathcal{D}_4(b=1,d=2,j=1,l=2): \quad & \text{same as } \mathcal{D}_2 \\
\mathcal{D}_5(b=1,d=2,j=1,l=2): \quad & \text{same as } \mathcal{D}_2 \\
\mathcal{D}_6(b=1,d=1,j=2,l=2): \quad & e^{-iq_2\cdot x_2}e^{-iq_1\cdot x_2}e^{+iq_4\cdot x_1}e^{+iq_3\cdot x_1} + e^{-iq_2\cdot x_2}e^{-iq_1\cdot x_2}e^{+iq_3\cdot x_1}e^{+iq_4\cdot x_1} \\
& + e^{-iq_1\cdot x_2}e^{-iq_2\cdot x_2}e^{+iq_4\cdot x_1}e^{+iq_3\cdot x_1} + e^{-iq_1\cdot x_2}e^{-iq_2\cdot x_2}e^{+iq_3\cdot x_1}e^{+iq_4\cdot x_1} \\
& = 4e^{-i(q_1+q_2)\cdot x_2}e^{+i(q_3+q_4)\cdot x_1} \quad \text{(see Eq. (9.20))} \quad \square
\end{aligned} \tag{9.26}$$

The term with the field σ does not act on external four-momenta of the initial or final states. It leads to an internal line corresponding to the Feynman propagator of virtual particles. Recalling that the σ particle has a rest mass M, we have:

$$\langle 0|\, T\left[\sigma(x_1)\sigma(x_2)\right]|0\rangle = D_F(x_1-x_2) = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x_1-x_2)}}{(p^2-M^2+i\epsilon)} \tag{9.27}$$

Putting all the bits and pieces together, we obtain:

$$\langle q_3 q_4 | \, S_2 \, | q_1 q_2 \rangle = i(-\lambda i)^2 \int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{e^{-ip\cdot(x_1-x_2)}}{(p^2 - M^2 + i\epsilon)} \left[\sum_{i=1}^{6} \mathcal{D}_i \right] \tag{9.28}$$

Let us first treat the term containing $\mathcal{D}_6$ in details:

$$\begin{aligned}
\langle q_3 q_4 | \, S_{2,6} \, | q_1 q_2 \rangle &= i(-\lambda i)^2 \int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{e^{-ip\cdot(x_1-x_2)}}{(p^2 - M^2 + i\epsilon)} \left[4 e^{-i(q_1+q_2)\cdot x_2} e^{+i(q_3+q_4)\cdot x_1} \right] \\
&= 4(-\lambda i)^2 \int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{(p^2 - M^2 + i\epsilon)} \left[e^{-i(q_1+q_2-p)\cdot x_2} e^{+i(q_3+q_4-p)\cdot x_1} \right] \\
&= 4(-\lambda i)^2 (2\pi)^4 \int \mathrm{d}^4 p \frac{i}{(p^2 - M^2 + i\epsilon)} \left[\delta^4(q_1 + q_2 - p) \delta^4(q_3 + q_4 - p) \right] \\
&= 4(-\lambda i)^2 (2\pi)^4 \delta^4(q_1 + q_2 - q_3 - q_4) \frac{i}{((q_1 + q_2)^2 - M^2 + i\epsilon)}
\end{aligned} \tag{9.29}$$

We note that the $\mathcal{D}_1$ terms leads to the same result. Hence:

$$\langle q_3 q_4 | \, S_{2,(1,6)} \, | q_1 q_2 \rangle = (-2\lambda i)^2 (2\pi)^4 \delta^4(q_1 + q_2 - q_3 - q_4) \frac{i}{((q_1 + q_2)^2 - M^2 + i\epsilon)} \tag{9.30}$$

We can now focus on the rest of the combinations. We find identical four terms, hence we can write:

$$\begin{aligned}
\langle q_3 q_4 | \, S_{2,(2,3,4,5)} \, | q_1 q_2 \rangle &= 4 \times i(-\lambda i)^2 \int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{e^{-ip\cdot(x_1-x_2)}}{(p^2 - M^2 + i\epsilon)} \\
&\quad \times \Big[e^{-i(q_2-q_4)\cdot x_1} e^{+i(q_3-q_1)\cdot x_2} + e^{-i(q_2-q_3)\cdot x_1} e^{+i(q_4-q_1)\cdot x_2} \\
&\quad + e^{-i(q_1-q_4)\cdot x_1} e^{+i(q_3-q_2)\cdot x_2} + e^{-i(q_1-q_3)\cdot x_1} e^{+i(q_4-q_2)\cdot x_2} \Big] \\
&= 4(-\lambda i)^2 \int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{(p^2 - M^2 + i\epsilon)} \\
&\quad \times \Big[e^{-i(q_2-q_4+p)\cdot x_1} e^{+i(q_3-q_1+p)\cdot x_2} + e^{-i(q_2-q_3+p)\cdot x_1} e^{+i(q_4-q_1+p)\cdot x_2} \\
&\quad + e^{-i(q_1-q_4+p)\cdot x_1} e^{+i(q_3-q_2+p)\cdot x_2} + e^{-i(q_1-q_3+p)\cdot x_1} e^{+i(q_4-q_2+p)\cdot x_2} \Big] \\
&= 4(-\lambda i)^2 (2\pi)^4 \delta^4(q_1 + q_2 - q_3 - q_4) \int \mathrm{d}^4 p \frac{i}{(p^2 - M^2 + i\epsilon)} \\
&\quad \times \Big[\delta^4(q_2 - q_4 + p) \delta^4(q_3 - q_1 + p) + \delta^4(q_2 - q_3 + p) \delta^4(q_4 - q_1 + p) \\
&\quad + \delta^4(q_1 - q_4 + p) \delta^4(q_3 - q_2 + p) + \delta^4(q_1 - q_3 + p) \delta^4(q_4 - q_2 + p) \Big] \\
&= 4(-\lambda i)^2 (2\pi)^4 \delta^4(q_1 + q_2 - q_3 - q_4) \\
&\quad \times \left[\frac{2i}{((q_1 - q_3)^2 - M^2 + i\epsilon)} + \frac{2i}{((q_1 - q_4)^2 - M^2 + i\epsilon)} \right] \\
&= (-2\lambda i)^2 (2\pi)^4 \delta^4(q_1 + q_2 - q_3 - q_4) \\
&\quad \times \left[\frac{i}{((q_1 - q_3)^2 - M^2 + i\epsilon)} + \frac{i}{((q_1 - q_4)^2 - M^2 + i\epsilon)} \right]
\end{aligned} \tag{9.31}$$

Let us analyse the last two results for $S_{2,(1,6)}$ and $S_{2,(2,3,4,5)}$. The final result is the sum of the two contributions. The terms $(2\pi)^4 \delta^4(q_3 + q_4 - q_1 - q_2)$ correspond to the overall energy–momentum conservation between initial and final states. We also recognize Feynman propagators for the σ particle in the momentum space with well-defined kinematical configurations given by $q_1 + q_2$ for the first calculation, and $q_1 - q_3$ and $q_1 - q_4$ for the second term.

We can write the total amplitude as a function of the Feynman propagator in momentum space (see Eq. (9.63) of the book):

$$\tilde{D}_F(p) \equiv \frac{i}{(p^2 - M^2 + i\epsilon)} \tag{9.32}$$

We finally find:

$$\langle q_3 q_4 | S_2 | q_1 q_2 \rangle = (-2\lambda i)^2 (2\pi)^4 \delta^4(q_1 + q_2 - q_3 - q_4) \left[\tilde{D}_F(q_1 + q_2) + \tilde{D}_F(q_1 - q_3) + \tilde{D}_F(q_1 - q_4)\right] \tag{9.33}$$

The corresponding Feynman diagrams are shown in Figure 9.2. The first diagram (on the left) corresponds to the σ-annihilation process. In the second and third diagrams (middle and right), the σ is exchanged between the two π^0 bosons. Since the π^0 are indistinguishable, there are two possible diagrams, the first one where e.g. $\pi^0(q_4)$ emanates from the vertex at the top, and the second one where $\pi^0(q_4)$ emanates from the vertex at the bottom.

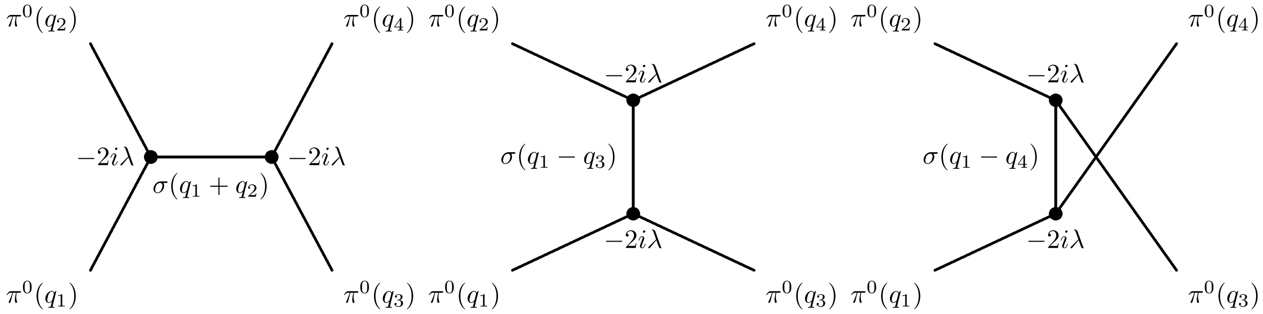

Figure 9.2 The diagrams for the $\pi^0\pi^0$-scattering process in our toy model: (left) σ-annihilation diagram; (middle and right) σ-exchange diagrams.

(b) $\pi^+\pi^- \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$ scattering amplitudes: By T invariance, the two amplitudes for $\pi^+\pi^- \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$ are identical. We first consider the **four-point interaction** for the $\pi^+\pi^- \to \pi^0\pi^0$ case. The matrix element for $\pi^+(q_1)\pi^-(q_2) \to \pi^0(q_3)\pi^0(q_4)$ at first order is given by:

$$\langle q_3 q_4 | S_1 | q_1 q_2 \rangle = (-2ig') \langle q_3 q_4 | T \left(\int d^4x \varphi^\dagger \varphi \phi^2 \right) | q_1 q_2 \rangle \tag{9.34}$$

We can write the initial state as:

$$|q_1 q_2\rangle = \sqrt{(2\pi)^3 2E_{q1} (2\pi)^3 2E_{q2}}\, b^\dagger(\vec{q}_1) a^\dagger(\vec{q}_2) |0\rangle \equiv \mathcal{N}_i b^\dagger(\vec{q}_1) a^\dagger(\vec{q}_2) |0\rangle \tag{9.35}$$

and the final state:

$$\langle q_3 q_4| = \sqrt{(2\pi)^3 2E_{q3} (2\pi)^3 2E_{q4}}\, \langle 0| c(\vec{q}_3) c(\vec{q}_4) \equiv \mathcal{N}_f \langle 0| c(\vec{q}_3) c(\vec{q}_4) \tag{9.36}$$

where $\mathcal{N}_i$ and $\mathcal{N}_f$ are the normalisation factors of the initial and final states. Introducing the short hand notations as before, we can write the matrix element as:

$$\begin{aligned}\langle q_3 q_4 | S_1 | q_1 q_2 \rangle =& (-i2g') \mathcal{N}_i \mathcal{N}_f \langle 0| c(\vec{q}_3) c(\vec{q}_4) \\ & \times \int d^4x \int \frac{d^3\vec{p}_1}{(2\pi)^{3/2}\sqrt{2E_{p1}}} \int \frac{d^3\vec{p}_2}{(2\pi)^{3/2}\sqrt{2E_{p2}}} \int \frac{d^3\vec{p}_3}{(2\pi)^{3/2}\sqrt{2E_{p3}}} \int \frac{d^3\vec{p}_4}{(2\pi)^{3/2}\sqrt{2E_{p4}}} \\ & (b_1 + a_1^\dagger) \times (a_2 + b_2^\dagger) \times (c_3 + c_3^\dagger) \times (c_4 + c_4^\dagger)\, b^\dagger(\vec{q}_1) a^\dagger(\vec{q}_2) |0\rangle\end{aligned}$$

The detailed calculation is the same as in the case of the pure ϕ^4 theory, as performed in **Ex. 9.2**. Hence we directly write the result Eq. (9.89) in **Ex. 9.2**:

$$\langle q_3 q_4| S_1 |q_1 q_2\rangle = (-4ig')(2\pi)^4 \delta^4(q_3 + q_4 - q_1 - q_2) \tag{9.37}$$

The corresponding Feynman diagram is shown in Figure 9.3(left). We now move to the **scattering diagrams** with exchange of a σ boson. As shown in Figure 9.3(right), the process occurs via the $-\lambda\sigma\phi^2$ which couples the σ to a pair of π^0 bosons, and in addition the $-2\lambda\sigma\varphi^\dagger\varphi$ term which yields a direct coupling between the σ and the $\pi^+\pi^-$ pair. The amplitude is given by:

$$\begin{aligned}\langle q_3 q_4| S_2 |q_1 q_2\rangle &= \frac{2(-\lambda i)^2}{2!} \langle q_3 q_4| T \left[\int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 \sigma(x_1)\varphi^\dagger(x_1)\varphi(x_1)\sigma(x_2)\phi^2(x_2)\right] |q_1 q_2\rangle \\ &= \frac{2(-\lambda i)^2}{2!} \langle q_3 q_4| \int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 T\left[\sigma(x_1)\sigma(x_2)\varphi^\dagger(x_1)\varphi(x_1)\phi(x_2)\phi(x_2)\right] |q_1 q_2\rangle \end{aligned} \tag{9.38}$$

We are going to drop the factor 2! because there is a second, identical term that comes from interchanging x_1 and x_2! The σ fields have no external legs and will lead to the Feynman propagator of the σ being exchanged. We hence consider the following term:

$$\langle q_3 q_4| \varphi^\dagger(x_1)\varphi(x_1)\phi(x_2)\phi(x_2) |q_1 q_2\rangle \tag{9.39}$$

The initial and final states are as before. Introducing the short hand notation as before for the operators, we can rewrite the considered term as (we omit the normalizations for clarity):

$$\begin{aligned}&\langle 0| c(\vec{q}_3)c(\vec{q}_4) \int \mathrm{d}^4 x \int \mathrm{d}^3\vec{p}_1 \int \mathrm{d}^3\vec{p}_2 \int \mathrm{d}^3\vec{p}_3 \int \mathrm{d}^3\vec{p}_4 \\ &\qquad (b_{11} + a_{11}^\dagger) \times (a_{21} + b_{21}^\dagger) \times (c_{32} + c_{32}^\dagger) \times (c_{42} + c_{42}^\dagger)\; b^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2) |0\rangle\end{aligned}$$

From the total of 16 terms obtained by multiplying the terms in brackets with creation–annihilation operators, we need to only consider those who contain exactly two creation and two annihilation operators in order to match the initial and final states. Therefore only one combination $\mathcal{C}_1$ remains:

$$b_{11} a_{21} c_{32}^\dagger c_{42}^\dagger \tag{9.40}$$

Since the a, b and c operators commute, the normal ordering does not change anything. We find:

$$\langle 0| : \mathcal{C}_1 : |0\rangle = \underbrace{\langle 0| c(\vec{q}_3)c(\vec{q}_4)c_{32}^\dagger c_{42}^\dagger}_{\mathcal{A}} \underbrace{b_{11} a_{21} b^\dagger(\vec{q}_1) a^\dagger(\vec{q}_2) |0\rangle}_{\mathcal{B}} \tag{9.41}$$

Using the commutation rules of the annihilation and creation operators, we find for $\mathcal{B}$:

$$\begin{aligned}\mathcal{B} &= b_{11} b^\dagger(\vec{q}_1) a_{21} a^\dagger(\vec{q}_2) |0\rangle = \left[\delta^3(\vec{p}_1 - \vec{q}_1)e^{-ip_1\cdot x_1} + \underbrace{\cancel{b^\dagger(\vec{q}_1) b_{11}}}_{b_{11}|0\rangle = 0}\right] \left[\delta^3(\vec{p}_2 - \vec{q}_2)e^{-ip_2\cdot x_1} + \underbrace{\cancel{a^\dagger(\vec{q}_2) a_{21}}}_{a_{21}|0\rangle = 0}\right] |0\rangle \\ &= \left[\delta^3(\vec{p}_1 - \vec{q}_1)\delta^3(\vec{p}_2 - \vec{q}_2)\right] e^{-ip_1\cdot x_1} e^{-ip_2\cdot x_1} |0\rangle \end{aligned} \tag{9.42}$$

Similarly:

$$\begin{aligned}\mathcal{A} &= \langle 0| c(\vec{q}_3)c(\vec{q}_4)c_{32}^\dagger c_{42}^\dagger = \langle 0| c(\vec{q}_3) \left[\delta^3(\vec{p}_3 - \vec{q}_4)e^{+ip_3\cdot x_2} + c_{32}^\dagger c(\vec{q}_4)\right] c_{42}^\dagger \\ &= \langle 0| c(\vec{q}_3) \left\{ \delta^3(\vec{p}_3 - \vec{q}_4)e^{+ip_3\cdot x_2} c_{42}^\dagger + c_{32}^\dagger \left[\delta^3(\vec{p}_4 - \vec{q}_4)e^{+ip_4\cdot x_2} + \underbrace{\cancel{c_{42}^\dagger c(\vec{q}_4)}}_{=0}\right]\right\} \\ &= \langle 0| \left[\delta^3(\vec{p}_3 - \vec{q}_4)e^{+ip_3\cdot x_2} c(\vec{q}_3) c_{42}^\dagger + \delta^3(\vec{p}_4 - \vec{q}_4)e^{+ip_4\cdot x_2} c(\vec{q}_3) c_{32}^\dagger\right] \\ &= \langle 0| \left[\delta^3(\vec{p}_3 - \vec{q}_4)e^{+ip_3\cdot x_2}\delta^3(\vec{p}_4 - \vec{q}_3)e^{+ip_4\cdot x_2} + \delta^3(\vec{p}_4 - \vec{q}_4)e^{+ip_4\cdot x_2}\delta^3(\vec{p}_3 - \vec{q}_3)e^{+ip_3\cdot x_2}\right] \\ &= \langle 0| \left[\delta^3(\vec{p}_3 - \vec{q}_4)\delta^3(\vec{p}_4 - \vec{q}_3) + \delta^3(\vec{p}_4 - \vec{q}_4)\delta^3(\vec{p}_3 - \vec{q}_3)\right] e^{+ip_3\cdot x_2} e^{+ip_4\cdot x_2} \end{aligned} \tag{9.43}$$

Consequently:

$$\begin{aligned}\langle 0|:\mathcal{C}_1:|0\rangle &= \left[\delta^3(\vec{p}_3-\vec{q}_4)\delta^3(\vec{p}_4-\vec{q}_3)+\delta^3(\vec{p}_4-\vec{q}_4)\delta^3(\vec{p}_3-\vec{q}_3)\right] \\ &\quad\times\left[\delta^3(\vec{p}_1-\vec{q}_1)\delta^3(\vec{p}_2-\vec{q}_2)\right] e^{-ip_1\cdot x_1}e^{-ip_2\cdot x_1}e^{+ip_3\cdot x_2}e^{+ip_4\cdot x_2}\end{aligned} \tag{9.44}$$

Integration over the $\vec{p}_m$'s ($m=1,\dots,4$) yields the following four exponential terms:

$$\begin{aligned}\int \mathrm{d}^3\vec{p}_1\int \mathrm{d}^3\vec{p}_2\int \mathrm{d}^3\vec{p}_3\int \mathrm{d}^3\vec{p}_4 :\ & e^{-iq_1\cdot x_1}e^{-iq_2\cdot x_1}e^{+iq_4\cdot x_2}e^{+iq_3\cdot x_2}+e^{-iq_1\cdot x_1}e^{-iq_2\cdot x_1}e^{+iq_3\cdot x_2}e^{+iq_4\cdot x_2} \\ & = 2e^{-i(q_1+q_2)\cdot x_1}e^{+i(q_3+q_4)\cdot x_2}\end{aligned} \tag{9.45}$$

Putting all the bits and pieces together, we obtain:

$$\begin{aligned}\langle q_3q_4|\,S_2\,|q_1q_2\rangle &= i2(-\lambda i)^2\int \mathrm{d}^4x_1\int \mathrm{d}^4x_2\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x_1-x_2)}}{(p^2-M^2+i\epsilon)}\left[2e^{-i(q_1+q_2)\cdot x_1}e^{+i(q_3+q_4)\cdot x_2}\right] \\ &= (-2\lambda i)^2(2\pi)^4\delta^4(q_1+q_2-q_3-q_4)\frac{i}{((q_1+q_2)^2-M^2+i\epsilon)}\end{aligned} \tag{9.46}$$

The corresponding Feynman diagrams are shown in Figure 9.3. The first diagram (on the left) corresponds to the four point interaction. The second diagram (on the right) yields the σ-annihilation process.

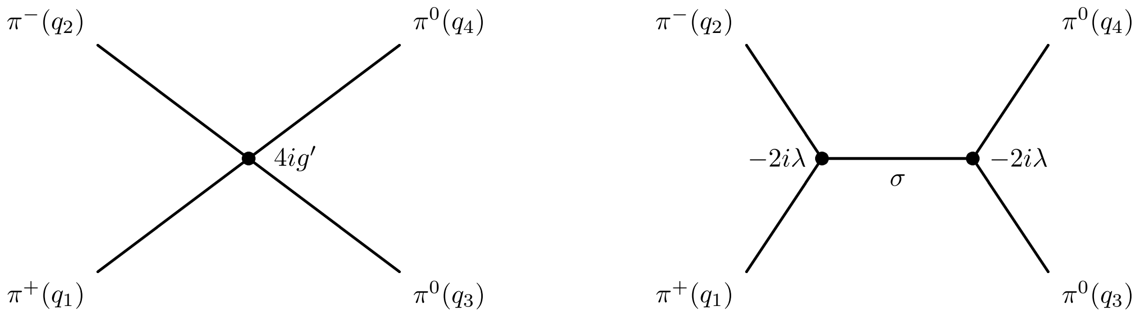

Figure 9.3 Lagrangian terms and diagrams contributing to the $\pi^+\pi^-\to\pi^0\pi^0$ scattering amplitudes in our toy model. (left) four point interaction (right) σ-exchange. Similar diagrams can be drawn for the $\pi^0\pi^0\to\pi^+\pi^-$ case.

9.2 Scattering in the ϕ^4 theory

Compute the amplitudes and the differential cross-sections for the following scattering processes within the ϕ^4 theory:

(a) $\pi^+\pi^-\to\pi^+\pi^-$

(b) $\pi^0\pi^0\to\pi^0\pi^0$

(c) $\pi^+\pi^-\to\pi^0\pi^0$ *and* $\pi^0\pi^0\to\pi^+\pi^-$

Solution:

To describe the above interactions in the ϕ^4 theory, we assume the existence of a triplet of real scalar fields $\phi_i(x)$ $(i = 1, 2, 3)$, which are rewritten in terms of one physical real scalar field $\phi(x)$ and two physical conjugated complex scalar fields φ and $\varphi^\dagger$, as defined in Eq. (9.2) of **Ex. 9.1**. The interaction part of the Lagrangian is written as:

$$\mathcal{L}_{int} = -g(\phi_1^2 + \phi_2^2 + \phi_3^2)^2 = -g\phi^4 - 4g(\varphi^\dagger\varphi)^2 - 4g\varphi^\dagger\varphi\phi^2 \tag{9.47}$$

where we used Eq. (9.4) of **Ex. 9.1**. The Lagrangian density has the units of energy to the fourth power $[\mathcal{L}_{int}] = 4$ (see Eq. (6.22) of the book). Since the field has dimension $[\varphi] = 1$, then g must be dimensionless. For clarity, we have omitted the conventional extra factor 4! in the denominator (i.e. the **symmetry factor**) of the ϕ^4 term found in Eq. (9.141) of the book. We will add it at the end of the calculations.

(a) $\pi^+(q_1)\pi^-(q_2) \to \pi^+(q_3)\pi^-(q_4)$ scattering amplitude: At first order, we should calculate:

$$\langle q_3q_4| S_1 |q_1q_2\rangle = (-4i)\langle q_3q_4| T\left(\int \mathrm{d}^4x g(\varphi^\dagger\varphi)^2\right)|q_1q_2\rangle = (-4ig)\langle q_3q_4| T\left(\int \mathrm{d}^4x \varphi^\dagger\varphi\varphi^\dagger\varphi\right)|q_1q_2\rangle \tag{9.48}$$

Following the derivation in Section 9.13 of the book, we now use Wick's theorem to transform the time-ordered product into a normal-ordered one. Using the shorthand notation of all possible contractions (see Section 9.13 of the book), we have:

$$T(\varphi^\dagger\varphi\varphi^\dagger\varphi) = \varphi^\dagger\varphi\varphi^\dagger\varphi + \overbracket{\varphi^\dagger\varphi}\varphi^\dagger\varphi + \overbracket{\varphi^\dagger\varphi}\,\overbracket{\varphi^\dagger\varphi} \tag{9.49}$$

We have seen in the book that only the first term is relevant to the scattering amplitude we want to calculate. We can write the initial state as:

$$|q_1q_2\rangle = \sqrt{(2\pi)^3 2E_{q1}(2\pi)^3 2E_{q2}}\; b^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)|0\rangle \equiv \mathcal{N}_i b^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)|0\rangle \tag{9.50}$$

and the final state:

$$\langle q_3q_4| = \sqrt{(2\pi)^3 2E_{q3}(2\pi)^3 2E_{q4}}\,\langle 0| b(\vec{q}_3)a(\vec{q}_4) \equiv \mathcal{N}_f \langle 0| b(\vec{q}_3)a(\vec{q}_4) \tag{9.51}$$

where $\mathcal{N}_i$ and $\mathcal{N}_f$ are the normalisation factors of the initial and final states. Introducing the short hand notation $a_i \equiv a(\vec{p}_i)e^{-ip_i\cdot x}$ and $b_i \equiv b(\vec{p}_i)e^{-ip_i\cdot x}$, we find that the relevant term is given by (compare with Eq. (9.81) of the book):

$$\begin{aligned}\langle q_3q_4| S_1 |q_1q_2\rangle =&(-4ig)\mathcal{N}_i\mathcal{N}_f \langle 0| b(\vec{q}_3)a(\vec{q}_4)\\ &\times \int \mathrm{d}^4x \int \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^{3/2}\sqrt{2E_{p1}}}\int \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^{3/2}\sqrt{2E_{p2}}}\int \frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^{3/2}\sqrt{2E_{p3}}}\int \frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^{3/2}\sqrt{2E_{p4}}}\\ &(b_1 + a_1^\dagger)\times(a_2 + b_2^\dagger)\times(b_3 + a_3^\dagger)\times(a_4 + b_4^\dagger)\; b^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)|0\rangle\end{aligned}$$

From the total of 16 terms obtained by multiplying the terms in brackets with creation–annihilation operators, we need to only consider those who contain exactly two creation and two annihilation operators in order to match the initial and final states. Therefore the following four combinations $\mathcal{C}_i$ $(i = 1, \ldots, 4)$ remain:

$$b_1a_2a_3^\dagger b_4^\dagger\ ,\ b_1b_2^\dagger a_3^\dagger a_4\ ,\ a_1^\dagger a_2b_3b_4^\dagger\ ,\ a_1^\dagger b_2^\dagger b_3a_4 \tag{9.52}$$

Wick's theorem lets us replace the time-ordered product with the normal ordered product. **Normal ordering means that all annihilation operators are moved to the right of the expression, while the**

creation operators are moved to the left. Hence, we must consider the **normal ordered** product of our operators:

$$a_3^\dagger b_4^\dagger b_1 a_2 \ , \ b_2^\dagger a_3^\dagger b_1 a_4 \ , \ a_1^\dagger b_4^\dagger a_2 b_3 \ , \ a_1^\dagger b_2^\dagger b_3 a_4 \tag{9.53}$$

Note that the a and b operators commute, so we can regroup them as we wish, as long as we keep the relative order within the a's and b's. Hence, we can express them for convenience as $b_i^\dagger a_j^\dagger b_k a_l$ series:

$$\underbrace{b_4^\dagger a_3^\dagger b_1 a_2}_{i,j,k,l=4,3,1,2} \ , \ \underbrace{b_2^\dagger a_3^\dagger b_1 a_4}_{2,3,1,4} \ , \ \underbrace{b_4^\dagger a_1^\dagger b_3 a_2}_{4,1,3,2} \ , \ \underbrace{b_2^\dagger a_1^\dagger b_3 a_4}_{2,1,3,4} \tag{9.54}$$

Let us for instance compute the braket using the last term of the list:

$$\langle 0| : \mathcal{C}_4 : |0\rangle = \underbrace{\langle 0|\, b(\vec{q}_3) b_2^\dagger a(\vec{q}_4) a_1^\dagger}_{\mathcal{A}} \underbrace{b_3 b^\dagger(\vec{q}_1) a_4 a^\dagger(\vec{q}_2)\,|0\rangle}_{\mathcal{B}} \tag{9.55}$$

We will use the commutation rule $\left[a(\vec{p}), a^\dagger(\vec{p}\,')\right] = \delta^3(\vec{p}-\vec{p}\,')$ and $\left[b(\vec{p}), b^\dagger(\vec{p}\,')\right] = \delta^3(\vec{p}-\vec{p}\,')$ (see Eq. (7.83) of the book). Hence:

$$a(\vec{p})a^\dagger(\vec{p}\,') = \delta^3(\vec{p}-\vec{p}\,') + a^\dagger(\vec{p}\,')a(\vec{p}) \tag{9.56}$$

and similarly for the b operator. Let us first consider $\mathcal{B}$:

$$\begin{aligned} \mathcal{B} &= b_3 b^\dagger(\vec{q}_1) a_4 a^\dagger(\vec{q}_2)\,|0\rangle = \left[\delta^3(\vec{p}_3-\vec{q}_1)e^{-ip_3\cdot x} + \underbrace{\cancel{b^\dagger(\vec{q}_1) b_3}}_{b_3|0\rangle=0}\right]\left[\delta^3(\vec{p}_4-\vec{q}_2)e^{-ip_4\cdot x} + \underbrace{\cancel{a^\dagger(\vec{q}_2) a_4}}_{a_4|0\rangle=0}\right]|0\rangle \\ &= \left[\delta^3(\vec{p}_3-\vec{q}_1)\delta^3(\vec{p}_4-\vec{q}_2)\right] e^{-ip_3\cdot x} e^{-ip_4\cdot x}\,|0\rangle \end{aligned} \tag{9.57}$$

Similarly:

$$\mathcal{A} = \langle 0|\left[\delta^3(\vec{p}_2-\vec{q}_3)\delta^3(\vec{p}_1-\vec{q}_4)\right] e^{+ip_1\cdot x} e^{+ip_2\cdot x} \tag{9.58}$$

Consequently,

$$\langle 0| : \mathcal{C}_4 : |0\rangle = \left[\delta^3(\vec{p}_3-\vec{q}_1)\delta^3(\vec{p}_4-\vec{q}_2)\delta^3(\vec{p}_2-\vec{q}_3)\delta^3(\vec{p}_1-\vec{q}_4)\right] e^{+ip_1\cdot x} e^{+ip_2\cdot x} e^{-ip_3\cdot x} e^{-ip_4\cdot x} \tag{9.59}$$

We now reintroduce the integrations (for clarity, we drop the normalisations in the integrations which will eventually cancel the $\mathcal{N}_i\mathcal{N}_f$ factor) and calculate the contribution of this combination to the amplitude:

$$\begin{aligned} \langle q_3 q_4|\, S_{1,4}\, |q_1 q_2\rangle &= (-4ig)\int \mathrm{d}^4x \int \mathrm{d}^3\vec{p}_1 \int \mathrm{d}^3\vec{p}_2 \int \mathrm{d}^3\vec{p}_3 \int \mathrm{d}^3\vec{p}_4 \\ &\quad \times \left[\delta^3(\vec{p}_3-\vec{q}_1)\delta^3(\vec{p}_4-\vec{q}_2)\delta^3(\vec{p}_2-\vec{q}_3)\delta^3(\vec{p}_1-\vec{q}_4)\right] e^{i(p_1+p_2-p_3-p_4)\cdot x} \\ &= (-4ig)\int \mathrm{d}^4x e^{i(q_4+q_3-q_1-q_2)\cdot x} = (-4ig)(2\pi)^4\delta^4(q_3+q_4-q_1-q_2) \end{aligned} \tag{9.60}$$

The three other normal ordered combinations $:\mathcal{C}_1:$, $:\mathcal{C}_2:$ and $:\mathcal{C}_3:$ can be found by swapping appropriately the i,j,k,l indices, according to Eq. (9.54). They each contribute to the amplitude in the same way since each individual one does not depend on the i,j,k,l indices. Hence we can write:

$$i\mathcal{M}(\pi^+\pi^- \to \pi^+\pi^-) = 4\times(-4ig) = -16ig \tag{9.61}$$

where we recover the result of Eq. (9.162) of the book. Hence, if we now reintroduce the **symmetry factor** 4! in the definition of the Lagrangian, the amplitude of the process becomes:

$$i\mathcal{M}(\pi^+\pi^- \to \pi^+\pi^-) = -\frac{16}{4!}ig = -\frac{2}{3}ig \tag{9.62}$$

In order to compute the differential cross-section, we use Eq. (5.145) of the book, which yields in the center-of-mass system:

$$\left(\frac{\mathrm{d}\sigma(\pi^+\pi^- \to \pi^+\pi^-)}{\mathrm{d}\Omega}\right)_{CMS} = \left(\frac{S}{64\pi^2 s}\right)|\mathcal{M}|^2 = \frac{4}{9}\left(\frac{g^2}{64\pi^2 s}\right) = \frac{g^2}{144\pi^2 s} \tag{9.63}$$

The differential cross-section is isotropic in the CMS system, as there are no spins involved.

(b) $\pi^0(q_1)\pi^0(q_2) \to \pi^0(q_3)\pi^0(q_4)$ scattering amplitude: This case has been treated in Section 9.13 of the book, and here we follow the same procedure as above. The matrix element at first order (compare with Eq. (9.79) of the book) is given by:

$$\langle q_3 q_4|\, S_1 \,|q_1 q_2\rangle = (-i)\,\langle q_3 q_4|\, T\left(\int \mathrm{d}^4 x g\phi^4(x)\,|q_1 q_2\rangle\right) = (-ig)\,\langle q_3 q_4|\, T\left(\int \mathrm{d}^4 x \phi\phi\phi\phi\right)|q_1 q_2\rangle \tag{9.64}$$

By applying Wick's theorem, the shorthand notation of all possible contractions (see Section 9.13 of the book) is given by:

$$T(\phi\phi\phi\phi) = \phi\phi\phi\phi + \phi\phi\phi\phi + \phi\phi\phi\phi \tag{9.65}$$

We have seen in the book that only the first term is relevant to the scattering amplitude we want to calculate. We can write the initial state as:

$$|q_1 q_2\rangle = \sqrt{(2\pi)^3 2E_{q1}(2\pi)^3 2E_{q2}}\; c^\dagger(\vec{q}_1)c^\dagger(\vec{q}_2)\,|0\rangle \equiv \mathcal{N}_i c^\dagger(\vec{q}_1)c^\dagger(\vec{q}_2)\,|0\rangle \tag{9.66}$$

and the final state:

$$\langle q_3 q_4| = \sqrt{(2\pi)^3 2E_{q3}(2\pi)^3 2E_{q4}}\,\langle 0|\, c(\vec{q}_3)c(\vec{q}_4) \equiv \mathcal{N}_f\,\langle 0|\, c(\vec{q}_3)c(\vec{q}_4) \tag{9.67}$$

where $\mathcal{N}_i$ and $\mathcal{N}_f$ are the normalisation factors of the initial and final states. Introducing the short hand notation $c_i \equiv c(\vec{p}_i)e^{-ip_i\cdot x}$, we find that the relevant term is given by (compare with Eq. (9.81) of the book):

$$\begin{aligned}\langle q_3 q_4|\, S_1\,|q_1 q_2\rangle =& (-ig)\mathcal{N}_i\mathcal{N}_f\,\langle 0|\, c(\vec{q}_3)c(\vec{q}_4)\\ &\times \int \mathrm{d}^4 x \int \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^{3/2}\sqrt{2E_{p1}}}\int \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^{3/2}\sqrt{2E_{p2}}}\int \frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^{3/2}\sqrt{2E_{p3}}}\int \frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^{3/2}\sqrt{2E_{p4}}}\\ &(c_1+c_1^\dagger)\times(c_2+c_2^\dagger)\times(c_3+c_3^\dagger)\times(c_4+c_4^\dagger)\; c^\dagger(\vec{q}_1)c^\dagger(\vec{q}_2)\,|0\rangle\end{aligned}$$

From the total of 16 terms obtained by multiplying the terms in brackets with creation–annihilation operators, we need to only consider those who contain exactly two creation and two annihilation operators in order to match the initial and final states. Therefore the following six combinations $\mathcal{C}_i$ $(i = 1, \ldots, 6)$ remain:

$$c_1 c_2 c_3^\dagger c_4^\dagger\;,\;\; c_1 c_2^\dagger c_3 c_4^\dagger\;,\;\; c_1 c_2^\dagger c_3^\dagger c_4\;,\;\; c_1^\dagger c_2 c_3 c_4^\dagger\;,\;\; c_1^\dagger c_2 c_3^\dagger c_4\;,\;\; c_1^\dagger c_2^\dagger c_3 c_4 \tag{9.68}$$

Wick's theorem lets us replace the time-ordered product with the normal ordered product. **Normal ordering means that all annihilation operators are moved to the right of the expression, while the creation operators are moved to the left**. Hence, we must consider the **normal ordered** product of our operators:

$$c_3^\dagger c_4^\dagger c_1 c_2\;,\;\; c_2^\dagger c_4^\dagger c_1 c_3\;,\;\; c_2^\dagger c_3^\dagger c_1 c_4\;,\;\; c_1^\dagger c_4^\dagger c_2 c_3\;,\;\; c_1^\dagger c_3^\dagger c_2 c_4\;,\;\; c_1^\dagger c_2^\dagger c_3 c_4 \tag{9.69}$$

Let us for instance compute the braket using the last term of the list:

$$\langle 0|:\mathcal{C}_6:|0\rangle = \underbrace{\langle 0|\, c(\vec{q}_3)c(\vec{q}_4)c_1^\dagger c_2^\dagger}_{\mathcal{A}}\underbrace{c_3 c_4 c^\dagger(\vec{q}_1)c^\dagger(\vec{q}_2)\,|0\rangle}_{\mathcal{B}} \tag{9.70}$$

We will use the commutation rule $\left[c(\vec{p}), c^\dagger(\vec{p}\,')\right] = \delta^3(\vec{p} - \vec{p}\,')$ (see Eq. (7.83) of the book). Let us first consider $\mathcal{B}$:

$$\begin{aligned}
\mathcal{B} &= c_3 c_4 c^\dagger(\vec{q}_1) c^\dagger(\vec{q}_2)\,|0\rangle = c_3\left[\delta^3(\vec{p}_4 - \vec{q}_1)e^{-ip_4\cdot x} + c^\dagger(\vec{q}_1)c_4\right] c^\dagger(\vec{q}_2)\,|0\rangle \\
&= c_3\left\{\delta^3(\vec{p}_4 - \vec{q}_1)e^{-ip_4\cdot x}c^\dagger(\vec{q}_2) + c^\dagger(\vec{q}_1)\left[\delta^3(\vec{p}_4 - \vec{q}_2)e^{-ip_4\cdot x} + \underbrace{\cancel{c^\dagger(\vec{q}_2)c_4}}_{c_4|0\rangle=0}\right]\right\}|0\rangle \\
&= \left[\delta^3(\vec{p}_4 - \vec{q}_1)c_3 c^\dagger(\vec{q}_2) + \delta^3(\vec{p}_4 - \vec{q}_2)c_3 c^\dagger(\vec{q}_1)\right] e^{-ip_4\cdot x}\,|0\rangle \\
&= \left[\delta^3(\vec{p}_4 - \vec{q}_1)\left[\delta^3(\vec{p}_3 - \vec{q}_2)e^{-ip_3\cdot x} + \cancel{c^\dagger(\vec{q}_2)c_3}\right] + \delta^3(\vec{p}_4 - \vec{q}_2)\left[\delta^3(\vec{p}_3 - \vec{q}_1)e^{-ip_3\cdot x} + \cancel{c^\dagger(\vec{q}_1)c_3}\right]\right] e^{-ip_4\cdot x}\,|0\rangle \\
&= \left[\delta^3(\vec{p}_4 - \vec{q}_1)\delta^3(\vec{p}_3 - \vec{q}_2) + \delta^3(\vec{p}_4 - \vec{q}_2)\delta^3(\vec{p}_3 - \vec{q}_1)\right] e^{-ip_3\cdot x}e^{-ip_4\cdot x}\,|0\rangle
\end{aligned} \tag{9.71}$$

Similarly:

$$\mathcal{A} = \langle 0|\left[\delta^3(\vec{p}_1 - \vec{q}_4)\delta^3(\vec{p}_2 - \vec{q}_3) + \delta^3(\vec{p}_1 - \vec{q}_3)\delta^3(\vec{p}_2 - \vec{q}_4)\right] e^{+ip_1\cdot x}e^{+ip_2\cdot x} \tag{9.72}$$

Consequently,

$$\begin{aligned}
\langle 0| : \mathcal{C}_6 : |0\rangle =& [\delta^3(\vec{p}_4 - \vec{q}_1)\delta^3(\vec{p}_3 - \vec{q}_2) + \delta^3(\vec{p}_4 - \vec{q}_2)\delta^3(\vec{p}_3 - \vec{q}_1)] \\
&\times [\delta^3(\vec{p}_1 - \vec{q}_4)\delta^3(\vec{p}_2 - \vec{q}_3) + \delta^3(\vec{p}_1 - \vec{q}_3)\delta^3(\vec{p}_2 - \vec{q}_4)]e^{+ip_1\cdot x}e^{+ip_2\cdot x}e^{-ip_3\cdot x}e^{-ip_4\cdot x} \\
&\equiv [\mathcal{C}_{61} + \mathcal{C}_{62} + \mathcal{C}_{63} + \mathcal{C}_{64}]\, e^{i(p_1+p_2-p_3-p_4)\cdot x}
\end{aligned} \tag{9.73}$$

where

$$\begin{aligned}
\mathcal{C}_{61} &\equiv \delta^3(\vec{p}_4 - \vec{q}_1)\delta^3(\vec{p}_3 - \vec{q}_2)\delta^3(\vec{p}_1 - \vec{q}_4)\delta^3(\vec{p}_2 - \vec{q}_3) \\
\mathcal{C}_{62} &\equiv \delta^3(\vec{p}_4 - \vec{q}_1)\delta^3(\vec{p}_3 - \vec{q}_2)\delta^3(\vec{p}_1 - \vec{q}_3)\delta^3(\vec{p}_2 - \vec{q}_4) \\
\mathcal{C}_{63} &\equiv \delta^3(\vec{p}_4 - \vec{q}_2)\delta^3(\vec{p}_3 - \vec{q}_1)\delta^3(\vec{p}_1 - \vec{q}_4)\delta^3(\vec{p}_2 - \vec{q}_3) \\
\mathcal{C}_{64} &\equiv \delta^3(\vec{p}_4 - \vec{q}_2)\delta^3(\vec{p}_3 - \vec{q}_1)\delta^3(\vec{p}_1 - \vec{q}_3)\delta^3(\vec{p}_2 - \vec{q}_4)
\end{aligned} \tag{9.74}$$

We now reintroduce the integrations (for clarity, we drop the normalisations in the integrations which cancel the $\mathcal{N}_i\mathcal{N}_f$ factor) and calculate the contribution of this combination to the amplitude:

$$\begin{aligned}
\langle q_3 q_4|\, S_{1,6}\,|q_1 q_2\rangle &= (-ig)\int \mathrm{d}^4x \int \mathrm{d}^3\vec{p}_1 \int \mathrm{d}^3\vec{p}_2 \int \mathrm{d}^3\vec{p}_3 \int \mathrm{d}^3\vec{p}_4\,(\mathcal{C}_{61} + \mathcal{C}_{62} + \mathcal{C}_{63} + \mathcal{C}_{64})\, e^{i(p_1+p_2-p_3-p_4)\cdot x} \\
&= (-ig)\int \mathrm{d}^4x \left[e^{i(q_4+q_3-q_2-q_1)} + (q_3 \leftrightarrow q_4) + e^{i(q_3+q_4-q_2-q_1)} + (q_1 \leftrightarrow q_2)\right] \\
&= 4(-ig)\int \mathrm{d}^4x e^{i(q_3+q_4-q_1-q_2)\cdot x} \\
&= (-4ig)(2\pi)^4\delta^4(q_3 + q_4 - q_1 - q_2)
\end{aligned} \tag{9.75}$$

The other five normal ordered combinations $: \mathcal{C}_i :$ $(i = 1, 5)$ contribute in the same way to the amplitude, hence we can write:

$$i\mathcal{M}(\pi^0\pi^0 \to \pi^0\pi^0) = 6 \times (-4ig) = -24ig \tag{9.76}$$

where we recover the result of Eq. (9.162) of the book. Hence, if we now reintroduce the **symmetry factor** 4! in the definition of the Lagrangian, the amplitude of the process becomes:

$$i\mathcal{M}(\pi^0\pi^0 \to \pi^0\pi^0) = -\frac{24}{4!}ig = -ig \tag{9.77}$$

which is Eq. (9.158) of the book.

In order to compute the differential cross-section, we use Eq. (5.145) of the book, which yields in the center-of-mass system:

$$\left(\frac{\mathrm{d}\sigma(\pi^0\pi^0 \to \pi^0\pi^0)}{\mathrm{d}\Omega}\right)_{CMS} = \left(\frac{S}{64\pi^2 s}\right)|\mathcal{M}|^2 = \frac{1}{2}\left(\frac{g^2}{64\pi^2 s}\right) = \frac{g^2}{128\pi^2 s} \tag{9.78}$$

where we used a symmetry factor $S = 1/2$ since the two final state particles are indistinguishable. The differential cross-section is isotropic in the CMS system, as there are no spins involved.

(c) $\pi^+\pi^- \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$ scattering amplitudes: As before, we start with a triplet of fields ϕ_i, which are combined into physical neutral and charged fields. The quartic coupling between these neutral and charged fields is generically given by:

$$\mathcal{L}'_{int} = -g'(\phi_1^2 + \phi_2^2)\phi_3^2 = -2g'\left(\varphi^\dagger\varphi\right)\phi^2 \tag{9.79}$$

It is already included in the Lagrangian when we start from the $g(\sum\phi_i^2)^2$ term, as shown in Eq. (9.47). By T invariance, the two amplitudes for $\pi^+\pi^- \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$ are identical. The matrix element for $\pi^+(q_1)\pi^-(q_2) \to \pi^0(q_3)\pi^0(q_4)$ at first order is given by:

$$\langle q_3q_4|\, S_1\, |q_1q_2\rangle = (-2ig')\,\langle q_3q_4|\, T\left(\int \mathrm{d}^4x\varphi^\dagger\varphi\phi^2\right)|q_1q_2\rangle \tag{9.80}$$

We apply as before Wick's theorem and retain only the first term. We can write the initial state as:

$$|q_1q_2\rangle = \sqrt{(2\pi)^3 2E_{q1}(2\pi)^3 2E_{q2}}\; b^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)\,|0\rangle \equiv \mathcal{N}_i b^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)\,|0\rangle \tag{9.81}$$

and the final state:

$$\langle q_3q_4| = \sqrt{(2\pi)^3 2E_{q3}(2\pi)^3 2E_{q4}}\,\langle 0|\, c(\vec{q}_3)c(\vec{q}_4) \equiv \mathcal{N}_f\,\langle 0|\, c(\vec{q}_3)c(\vec{q}_4) \tag{9.82}$$

where $\mathcal{N}_i$ and $\mathcal{N}_f$ are the normalisation factors of the initial and final states. Introducing the short hand notations as before, we find that the relevant term is given by

$$\begin{aligned}\langle q_3q_4|\, S_1\, |q_1q_2\rangle =& (-i2g')\mathcal{N}_i\mathcal{N}_f\,\langle 0|\, c(\vec{q}_3)c(\vec{q}_4) \\ &\times \int \mathrm{d}^4x \int \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^{3/2}\sqrt{2E_{p1}}}\int \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^{3/2}\sqrt{2E_{p2}}}\int \frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^{3/2}\sqrt{2E_{p3}}}\int \frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^{3/2}\sqrt{2E_{p4}}} \\ &(b_1 + a_1^\dagger)\times(a_2 + b_2^\dagger)\times(c_3 + c_3^\dagger)\times(c_4 + c_4^\dagger)\; b^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)\,|0\rangle\end{aligned}$$

From the total of 16 terms obtained by multiplying the terms in brackets with creation–annihilation operators, we need to only consider those who contain exactly two creation and two annihilation operators in order to match the initial and final states. Therefore only one combination $\mathcal{C}_1$ remains:

$$b_1a_2c_3^\dagger c_4^\dagger \tag{9.83}$$

Since the a, b and c operators commute, the normal ordering does not change anything. We find:

$$\langle 0|: \mathcal{C}_1 :|0\rangle = \underbrace{\langle 0|\, c(\vec{q}_3)c(\vec{q}_4)c_3^\dagger c_4^\dagger}_{\mathcal{A}}\underbrace{b_1a_2b^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)\,|0\rangle}_{\mathcal{B}} \tag{9.84}$$

Using the commutation rules of the annihilation and creation operators, we find for $\mathcal{B}$:

$$\begin{aligned}\mathcal{B} &= b_1b^\dagger(\vec{q}_1)a_2a^\dagger(\vec{q}_2)\,|0\rangle = \left[\delta^3(\vec{p}_1-\vec{q}_1)e^{-ip_1\cdot x} + \underbrace{\cancel{b^\dagger(\vec{q}_1)b_1}}_{b_1|0\rangle=0}\right]\left[\delta^3(\vec{p}_2-\vec{q}_2)e^{-ip_2\cdot x} + \underbrace{\cancel{a^\dagger(\vec{q}_2)a_2}}_{a_2|0\rangle=0}\right]|0\rangle \\ &= \left[\delta^3(\vec{p}_1-\vec{q}_1)\delta^3(\vec{p}_2-\vec{q}_2)\right]e^{-ip_1\cdot x}e^{-ip_2\cdot x}\,|0\rangle\end{aligned} \tag{9.85}$$

Similarly:

$$\begin{aligned}
\mathcal{A} &= \langle 0| c(\vec{q}_3)c(\vec{q}_4)c_3^\dagger c_4^\dagger = \langle 0| c(\vec{q}_3)\left[\delta^3(\vec{p}_3-\vec{q}_4)e^{+ip_3\cdot x} + c_3^\dagger c(\vec{q}_4)\right] c_4^\dagger \\
&= \langle 0| c(\vec{q}_3)\left\{\delta^3(\vec{p}_3-\vec{q}_4)e^{+ip_3\cdot x}c_4^\dagger + c_3^\dagger\left[\delta^3(\vec{p}_4-\vec{q}_4)e^{+ip_4\cdot x} + \underbrace{\cancel{c_4^\dagger c(\vec{q}_4)}}_{=0}\right]\right\} \\
&= \langle 0| \left[\delta^3(\vec{p}_3-\vec{q}_4)e^{+ip_3\cdot x}c(\vec{q}_3)c_4^\dagger + \delta^3(\vec{p}_4-\vec{q}_4)e^{+ip_4\cdot x}c(\vec{q}_3)c_3^\dagger\right] \\
&= \langle 0| \left[\delta^3(\vec{p}_3-\vec{q}_4)e^{+ip_3\cdot x}\delta^3(\vec{p}_4-\vec{q}_3)e^{+ip_4\cdot x} + \delta^3(\vec{p}_4-\vec{q}_4)e^{+ip_4\cdot x}\delta^3(\vec{p}_3-\vec{q}_3)e^{+ip_3\cdot x}\right] \\
&= \langle 0| \left[\delta^3(\vec{p}_3-\vec{q}_4)\delta^3(\vec{p}_4-\vec{q}_3) + \delta^3(\vec{p}_4-\vec{q}_4)\delta^3(\vec{p}_3-\vec{q}_3)\right] e^{+ip_3\cdot x}e^{+ip_4\cdot x} \qquad (9.86)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\langle 0| :\mathcal{C}_1: |0\rangle &= \delta^3(\vec{p}_1-\vec{q}_1)\delta^3(\vec{p}_2-\vec{q}_2)\left[\delta^3(\vec{p}_3-\vec{q}_4)\delta^3(\vec{p}_4-\vec{q}_3) + \delta^3(\vec{p}_4-\vec{q}_4)\delta^3(\vec{p}_3-\vec{q}_3)\right] \\
&\quad \times e^{-ip_1\cdot x}e^{-ip_2\cdot x}e^{+ip_3\cdot x}e^{+ip_4\cdot x} \\
&\equiv \delta^3(\vec{p}_1-\vec{q}_1)\delta^3(\vec{p}_2-\vec{q}_2)\left[\mathcal{C}_{11}+\mathcal{C}_{12}\right]e^{i(p_3+p_4-p_1-p_2)\cdot x} \qquad (9.87)
\end{aligned}$$

We now reintroduce the integrations (for clarity, we drop the normalisations in the integrations which cancel the $\mathcal{N}_i\mathcal{N}_f$ factor) and calculate the contribution of this combination to the amplitude:

$$\begin{aligned}
\langle q_3q_4| S_{1,6} |q_1q_2\rangle &= (-2ig')\int \mathrm{d}^4x \int \mathrm{d}^3\vec{p}_1 \int \mathrm{d}^3\vec{p}_2 \int \mathrm{d}^3\vec{p}_3 \int \mathrm{d}^3\vec{p}_4\, (\mathcal{C}_{11}+\mathcal{C}_{12})\, e^{i(p_3+p_4-p_1-p_2)\cdot x} \\
&= (-2ig')\int \mathrm{d}^4x \left[e^{i(q_4+q_3-q_1-q_2)} + (q_3 \leftrightarrow q_4)\right] \\
&= 2(-2ig')\int \mathrm{d}^4x e^{i(q_3+q_4-q_1-q_2)\cdot x} \\
&= (-4ig')(2\pi)^4\delta^4(q_3+q_4-q_1-q_2) \qquad (9.88)
\end{aligned}$$

Hence:

$$\langle q_3q_4| S_1 |q_1q_2\rangle = (-4ig')(2\pi)^4\delta^4(q_3+q_4-q_1-q_2) \qquad (9.89)$$

where we recover the result of Eq. (9.163) of the book. Hence, if we now reintroduce the **symmetry factor** 4! in the definition of the Lagrangian, the amplitude of the process becomes:

$$i\mathcal{M}(\pi^+\pi^- \to \pi^0\pi^0) = i\mathcal{M}(\pi^0\pi^0 \to \pi^+\pi^-) = -\frac{4}{4!}ig' = -\frac{ig'}{6} \qquad (9.90)$$

In order to compute the differential cross-section, we use Eq. (5.145) of the book, which yields in the center-of-mass system:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CMS} = \left(\frac{S}{64\pi^2 s}\right)|\mathcal{M}|^2 = \frac{S}{36}\left(\frac{g'^2}{64\pi^2 s}\right) = \left\{\begin{array}{l} \left(\frac{g'^2}{4608\pi^2 s}\right) \text{for } \breve{a}\pi^+\pi^- \to \pi^0\pi^0 \\ \left(\frac{g'^2}{2304\pi^2 s}\right) \text{for } \breve{a}\pi^0\pi^0 \to \pi^+\pi^- \end{array}\right. \qquad (9.91)$$

where we used a symmetry factor $S = 1/2$ for the first reaction since the two final state particles are indistinguishable. The differential cross-section is isotropic in the CMS system, as there are no spins involved.

9.3 Yukawa potential

The full Lagrangian of the Yukawa theory was introduced to be given by:

$$\mathcal{L}_{Yukawa} = \mathcal{L}_{KG} + \mathcal{L}_{Dirac} + \mathcal{L}_{int} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_\phi^2 \phi^2 + i\overline{\Psi}\left(\gamma^\mu \partial_\mu - m\right)\Psi + g\phi(\overline{\Psi}\Psi) \tag{9.92}$$

where m_ϕ is the rest mass of the neutral scalar particle, m is the rest mass of the Dirac fermion f, and g is the coupling constant of the Yukawa theory. Consider the scattering of distinguishable fermions in the non-relativistic limit.

(a) *Sketch the corresponding Feynman diagram at tree level.*

(b) *In the non-relativistic limit, one can write the four-vectors of the incoming and outgoing particles as $p = (m, \vec{p})$, $p' = (m, \vec{p}')$, $k = (m, \vec{k})$, and $k' = (m, \vec{k}')$. Show that*

$$(p' - p)^2 \simeq -|\vec{p}' - \vec{p}|^2 \quad \text{and} \quad u^{(s)}(p) \simeq \sqrt{m}\begin{pmatrix} u_A^s \\ u_A^s \end{pmatrix} \tag{9.93}$$

where u_A^s is an orthonormal spin-dependent bi-spinor.

(c) *Show that*

$$\bar{u}^{(s')}(p')u^{(s)}(p) = 2m\delta^{s,s'} \tag{9.94}$$

and similarly for $\bar{u}^{(r')}(k')u^{(r)}(k)$.

(d) *Show that the scattering amplitude then becomes:*

$$i\mathcal{M} = \frac{ig^2}{|\vec{q}|^2 + m_\phi^2} 2m\delta^{s,s'} 2m\delta^{r,r'} \tag{9.95}$$

where $\vec{q} \equiv \vec{p}' - \vec{p}$.

(e) *What is the consequence for the spins?*

(f) *Convince yourself that the amplitude shows that the interaction can be expressed as due to a potential, whose Fourier transform is given by:*

$$\tilde{V}(\vec{q}) = \frac{-g^2}{|\vec{q}|^2 + m_\phi^2} \tag{9.96}$$

(g) *Invert the Fourier transform to find the Yukawa potential:*

$$V(r) = -\frac{g^2}{4\pi}\frac{1}{r}e^{-m_\phi r} \tag{9.97}$$

This result shows that the Yukawa interaction between two scalar fermions is attractive, with a limited range given by the mass of the exchanged particle ϕ.

(h) *Now consider the interaction between a fermion and an antifermion, and show that in this case also, the Yukawa interaction is attractive.*

Solution:

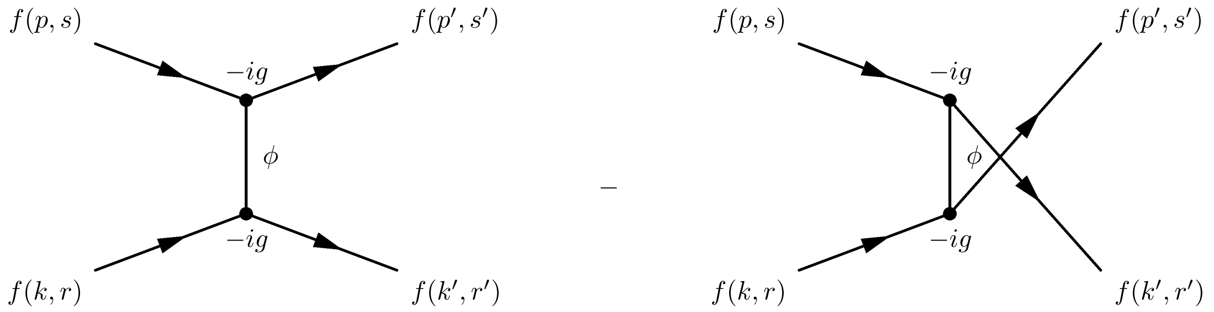

Figure 9.4 The diagrams for the scattering process of two fermions in the Yukawa theory. The amplitudes of the two diagrams must be subtracted from one another. Since for simplicity we assume that the fermions are distinguishable, we only need to consider the diagram on the left (see text).

a) The Feynman diagrams in the Yukawa theory describing the scattering between two identical fermions are those depicted in Figure 9.4. The kinematics is defined by the energy–momentum 4-vectors p, k, p', k' and the corresponding spin states of the particles by r, s, r', s'. These are the diagrams of Figure 9.15 of the book, having renamed the momenta $q_2 \to p$, $q_4 \to p'$, $q_1 \to k$, $q_3 \to k'$. In order to simplify the problem, we assume that the two fermions are somehow distinguishable, hence we only need to consider the diagram on the left.

b) We consider the kinematics of the process and assign the four-momenta in the reaction $f(p^\mu) + f(k^\mu) \to p'^\mu) + f(k'^\mu)$. These four momenta are also reported on the Feynman diagram discussed in part a) above. We place ourselves in the center-of-mass system of the interaction, where all incoming and outgoing momenta have the same magnitude, and we write the 4-momenta of the incoming and outgoing particles as:

$$p^\mu \equiv (E, \vec{p}) = (\gamma_L m, \vec{p}) = (\gamma_L m, \gamma m \vec{\beta}_p) \tag{9.98}$$

and similarly $p'^\mu \simeq (\gamma_L m, \vec{p}\,')$, $k^\mu \simeq (\gamma_L m, \vec{k})$, $k'^\mu \simeq (\gamma_L m, \vec{k}')$, where m is the rest mass of the fermions, and γ_L represents the Lorentz factor of the particles and the $\vec{\beta}_p, \vec{\beta}_k, \ldots$ are the velocity vectors (since the all particles have the same mass and same magnitude of momentum due to energy-momentum conservation, the Lorentz factor is a unique factor).

It follows immediately that:

$$(p' - p)^2 = (\gamma_L m - \gamma_L m,\, \vec{p'} - \vec{p})^2 = -\left(\vec{p'} - \vec{p}\right)^2 = -|\vec{p'} - \vec{p}|^2 \tag{9.99}$$

To derive the form of the spinor, we start from the Dirac equation for a plane wave solution. For $p^\mu = (E, \vec{p}) = (\gamma_L m, \vec{p}) = (\gamma_L m, \gamma m \vec{\beta}_p)$, we find (see Eq. (8.66) of the book):

$$(\gamma^\mu p_\mu - m\mathbb{1})\Psi^{(s)} = (\gamma^0 p_0 - \vec{\gamma} \cdot \vec{p} - m\mathbb{1})\Psi^{(s)} = m(\gamma_L \gamma^0 - \gamma_L \vec{\gamma} \cdot \vec{\beta}_p - \mathbb{1})\Psi^{(s)} = 0 \tag{9.100}$$

with $s = 1, 2$ (we keep only two solutions to take into account the spin of the fermions). We introduce the 2×2 compact notation in the Weyl representation (compare with Eqs. (8.16) and (8.22) of the book):

$$\gamma^0 = \beta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^k = \beta\alpha^k = \begin{pmatrix} 0 & +\sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad \mathbb{1} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{9.101}$$

where β and $\vec{\alpha}$ are the 4×4 Dirac matrices. Hence:

$$m\left[\gamma_L \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} - \gamma_L \begin{pmatrix} 0 & +\vec{\sigma} \cdot \vec{\beta} \\ -\vec{\sigma} \cdot \vec{\beta} & 0 \end{pmatrix} - \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\right]\Psi^{(s)} = 0 \tag{9.102}$$

or

$$m\left[\left(\begin{array}{cc} -\mathbb{1} & \gamma_L\left(\mathbb{1}-\vec{\sigma}\cdot\vec{\beta}\right) \\ \gamma_L\left(\mathbb{1}+\vec{\sigma}\cdot\vec{\beta}\right) & -\mathbb{1} \end{array}\right)\right]\Psi^{(s)} = 0 \tag{9.103}$$

In the non-relativistic limit where $|\vec{\beta}| \ll 1$ and $\gamma_L \to 1$, we simply have:

$$m\left(\begin{array}{cc} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{array}\right)\Psi^{(s)} \simeq 0 \tag{9.104}$$

We write the spinor solutions $\Psi^{(s)}$ as:

$$\Psi^{(s)} = u^{(s)}e^{-ip\cdot x} \qquad \text{where} \quad u^{(s)} = \left(\begin{array}{c} u_A^s \\ u_B^s \end{array}\right) \tag{9.105}$$

where u_A^s and u_B^s are bi-spinors normalised such that $u_A^{s\dagger}u_A^{s'} = \delta^{ss'}$ and similarly for u_B. Consequently,

$$m\left(\begin{array}{cc} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{array}\right)\left(\begin{array}{c} u_A^s \\ u_B^s \end{array}\right) \simeq 0 \quad \Longrightarrow \quad \left\{\begin{array}{l} -u_A^s + u_B^s \simeq 0 \\ +u_A^s - u_B^s \simeq 0 \end{array}\right. \quad \Longrightarrow \quad u_A^s \simeq u_B^s \tag{9.106}$$

Hence, we can write:

$$u^{(s)}(p) \simeq \mathcal{N}\left(\begin{array}{c} u_A^s \\ u_A^s \end{array}\right) = \sqrt{m}\left(\begin{array}{c} u_A^s \\ u_A^s \end{array}\right) \tag{9.107}$$

where $\mathcal{N} = \sqrt{m}$ is the conventional normalisation.

c) From the form of the bi-spinors derived above, we immediately see that:

$$\begin{aligned} \bar{u}^{(s')}(p')u^{(s)}(p) &= u^{(s')\dagger}(p')\gamma^0 u^{(s)}(p) = m\,(u_A^{s'\dagger}, u_A^{s'\dagger})\left(\begin{array}{cc} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{array}\right)\left(\begin{array}{c} u_A^s \\ u_A^s \end{array}\right) \\ &= 2m u_A^{s'\dagger}u_A^s = 2m\delta^{ss'} \end{aligned} \tag{9.108}$$

and similarly for $\bar{u}^{(r')}(k')u^{(r)}(k)$. Compare with Eq. (8.136) of the book.

d) We can readily obtain the amplitude from the Feynman diagram for the process in Figure 9.4(left) using the rules **[YUK]** defined in Section 9.18 of the book (compare with Eq. (9.199) of the book):

$$i\mathcal{M} = \underbrace{(-ig)\bar{u}^{s'}(p')u^s(p)}_{-ig\text{ vertex + external legs}}\underbrace{\left(\frac{i}{(p-p')^2 - m_\phi^2 + i\epsilon}\right)}_{\phi\text{ propagator}}\underbrace{(-ig)\bar{u}^{r'}(k')u^r(k)}_{-ig\text{ vertex + external legs}} \tag{9.109}$$

Hence:

$$\begin{aligned} i\mathcal{M} &= (-ig)^2\bar{u}^{s'}(p')u^s(p)\frac{i}{(p-p')^2 - m_\phi^2 + i\epsilon}\bar{u}^{r'}(k')u^r(k) \\ &\simeq -ig^2\frac{1}{-|\vec{p}' - \vec{p}|^2 - m_\phi^2}2m\delta^{ss'}2m\delta^{rr'} \\ &= \frac{ig^2}{|\vec{q}|^2 + m_\phi^2}2m\delta^{ss'}2m\delta^{rr'} \end{aligned} \tag{9.110}$$

where in the last equality we have used the definition of momentum transfer $\vec{q} \equiv \vec{p}' - \vec{p}$.

e) As a consequence of Eq. 9.110, the spin of each particle has to be conserved separately.

f) In non-relativistic quantum mechanics, the Born approximation of the scattering amplitude is expressed in terms of the Fourier transform of the potential causing the perturbation (see Eq. (4.200) of the book):

$$T_{fi} \simeq -i\tilde{V}(\vec{q})\, 2\pi\, \delta(E_{p'} - E_p) \tag{9.111}$$

By equating it to the scattering amplitude of Eq. 9.110 and assuming according to the answer (e) above that the spin of each particle is conserved, we find that:

$$\tilde{V}(\vec{q}) \simeq -g^2 \frac{1}{|\vec{q}|^2 + m_\phi^2} \tag{9.112}$$

where we have omitted the normalisation factors $2m2m$ given that in non-relativistic quantum mechanics, wave functions are normalized to unity.

g) By inverting the Fourier transform $\tilde{V}(\vec{q})$, we get (we denote $q = |\vec{q}|$ and $r = |\vec{x}|$ in the following):

$$\begin{aligned}
V(\vec{x}) &= \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \tilde{V}(\vec{q}) e^{i\vec{q}\cdot\vec{x}} = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{-g^2}{q^2 + m_\phi^2} e^{i\vec{q}\cdot\vec{x}} \\
&= -\frac{g^2}{(2\pi)^3} \int q^2 dq \int \mathrm{d}\phi \int \mathrm{d}\cos\theta \frac{e^{iqr\cos\theta}}{q^2 + m_\phi^2} = -\frac{g^2}{4\pi^2} \int_0^{+\infty} dq\, q^2 \frac{e^{iqr} - e^{-iqr}}{iqr(q^2 + m_\phi^2)} \\
&= -\frac{g^2}{4\pi^2 ir} \left(\int_0^{+\infty} dq\, q \frac{e^{iqr}}{q^2 + m_\phi^2} - \int_0^{+\infty} dq\, q \frac{e^{-iqr}}{q^2 + m_\phi^2} \right) \\
&= -\frac{g^2}{4\pi^2 ir} \int_{-\infty}^{+\infty} dq \frac{q e^{iqr}}{q^2 + m_\phi^2}
\end{aligned} \tag{9.113}$$

In order to perform the integral we use Cauchy's residue theorem (see Appendix A.4 of the book). We note that the argument has a simple pole at $q = +im_\phi$. The result is then:

$$V(r) = -\frac{g^2}{4\pi^2 ir}(2\pi i)\frac{im_\phi e^{-m_\phi r}}{2im_\phi} = -\frac{g^2}{4\pi}\frac{e^{-m_\phi r}}{r} \tag{9.114}$$

This result indeed shows that the Yukawa interaction between two fermions is attractive, with a limited range given by the mass of the exchanged scalar particle m_ϕ.

h) Now we consider the case of the interaction between a fermion and an antifermion. The corresponding diagrams are shown in Figure 9.5. As before, we consider for simplicity only the σ-exchange diagram (left diagram on the figure). Its amplitude is given by (compare with Eq. (9.203) of the book):

$$i\mathcal{M}(f\bar{f} \to f\bar{f}) = -ig^2 \left[\bar{u}^{(r')}(k') u^{(r)}(k) \frac{1}{(p - p')^2 - m_\phi^2 + i\epsilon} \bar{v}^{(s)}(p) v^{(s')}(p') \right] \tag{9.115}$$

The amplitude can be derived from the fermion–fermion interaction amplitude by noting that an incoming antifermion is represented by the spinor $\bar{v}$ and an outgoing antifermion is described by v. Let us derive the $\bar{v}^{(r')}(p)v^{(r)}(p')$ factor. We first recall that the Dirac equation for the v spinors is equivalent to that for a spinor u up to the sign (see Eq. (8.135) of the book):

$$(\not{p} + m)v = 0 \quad \Longrightarrow \quad v^{(s)}(p) \simeq \mathcal{N} \begin{pmatrix} v_A^s \\ -v_A^s \end{pmatrix} = \sqrt{m} \begin{pmatrix} v_A^s \\ -v_A^s \end{pmatrix} \tag{9.116}$$

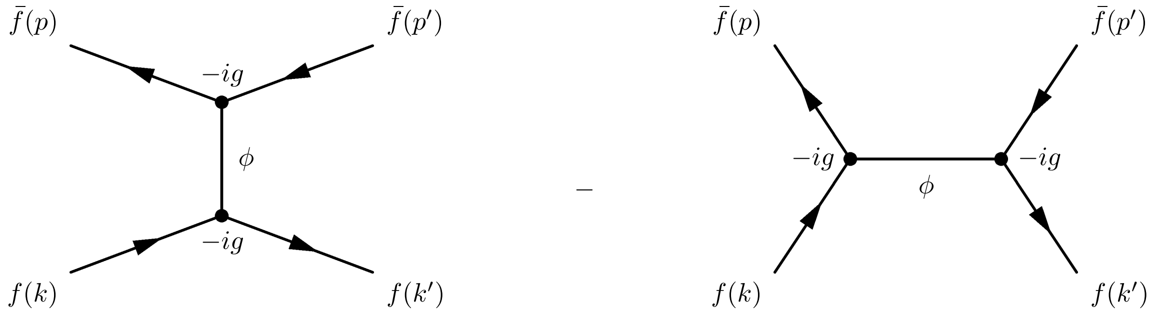

Figure 9.5 The diagrams for the scattering process of a fermion–antifermion pair in the Yukawa theory. The amplitudes of the two diagrams must be subtracted from one another. For simplicity, we only need to consider the diagram on the left (see text).

This immediately yields:

$$\begin{aligned}\bar{v}^{(s)}(p)v^{(s')}(p') &= v^{(s)\dagger}(p)\gamma^0 v^{(s')}(p') = m\,(v_A^{s\dagger}, -v_A^{s\dagger})\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} v_A^{s'} \\ -v_A^{s'} \end{pmatrix} \\ &= -2m v_A^{s\dagger} v_A^{s'} = -2m\delta^{ss'} \end{aligned} \tag{9.117}$$

Compare with Eq. (8.136) of the book. So naively we would expect an extra minus sign and therefore an opposite sign in the potential for fermion–antifermion interaction as compared to that for fermion–fermion, given by Eq. (9.112). **However, we must pay attention here!** We have already noted in the captions of the Figures 9.4ăand 9.5ăthat the two diagrams (left and right) must be **subtracted**ăfrom one another due to a minus sign coming from the Fermi-Dirac statistics of our fermions. In our present case, we should compare **the relative sign between the first diagrams** (on the left) of our two cases $ff \to ff$ and $f\bar{f} \to f\bar{f}$. We note that for the fermion–fermion case, if we were to derive the amplitude of the diagram from the basic theory rather than the rules, we would have compute (compare with Eqs. (9.192), (9.193) and (9.194) of the book:

$$ff \to ff: \quad \langle 0|a_{r'}(k')a_{s'}(p')\overline{\Psi}(x_1)\Psi(x_1)\overline{\Psi}(x_2)\Psi(x_2)a_r^\dagger(k)a_s^\dagger(p)|0\rangle \tag{9.118}$$

while for the fermion–antifermion case, we have:

$$f\bar{f} \to f\bar{f}: \quad \langle 0|a_{r'}(k')b_{s'}(p')\overline{\Psi}(x_1)\Psi(x_1)\overline{\Psi}(x_2)\Psi(x_2)a_r^\dagger(k)b_s^\dagger(p)|0\rangle \tag{9.119}$$

since $\Psi \propto a + b^\dagger$ and $\overline{\Psi} \propto a^\dagger + b$ (see Eq. (9.194) of the book). Untangling the fermion–fermion expression, led to a minus sign in front of the equation (see Eq. (9.195) of the book):

$$\begin{aligned} & a_{k'}a_{p'}a_{x_1}^\dagger a_{x_1}a_{x_2}^\dagger a_{x_2}a_k^\dagger a_p^\dagger \quad \rightsquigarrow \quad -a_{k'}a_{x_1}^\dagger a_{p'}a_{x_1}a_{x_2}^\dagger a_{x_2}a_k^\dagger a_p^\dagger \\ & \rightsquigarrow +a_{k'}a_{x_1}^\dagger a_{p'}a_{x_2}^\dagger a_{x_1}a_{x_2}a_k^\dagger a_p^\dagger \quad \rightsquigarrow \quad -a_{k'}a_{x_1}^\dagger a_{p'}a_{x_2}^\dagger a_{x_1}a_k^\dagger a_{x_2}a_p^\dagger \end{aligned} \tag{9.120}$$

For the fermion–antifermion expression, we find the following untangling of the contractions:

$$\begin{aligned} & a_{k'} b_{p'} a^\dagger_{x_1} a_{x_1} b_{x_2} b^\dagger_{x_2} a^\dagger_k b^\dagger_p \quad \rightsquigarrow \quad -a_{k'} a^\dagger_{x_1} b_{p'} a_{x_1} b_{x_2} b^\dagger_{x_2} a^\dagger_k b^\dagger_p \\ & \rightsquigarrow -a_{k'} a^\dagger_{x_1} a_{x_1} b_{x_2} b_{p'} b^\dagger_{x_2} a^\dagger_k b^\dagger_p \quad \rightsquigarrow \quad +a_{k'} a^\dagger_{x_1} b_{x_2} b_{p'} b^\dagger_{x_2} a_{x_1} a^\dagger_k b^\dagger_p \end{aligned} \tag{9.121}$$

Hence, the amplitude for the fermion–antifermion scattering accordingly has the opposite sign as the one in the fermion–fermion scattering case. Consequently, the change of sign in Eq. (9.117) is cancelled, and the **Yukawa interaction is universally attractive for both fermion–fermion and fermion–antifermion pairs**!

10 Quantum Electrodynamics

10.1 Classical Coulomb potential

Derive Coulomb's law using classical field theory. Start from the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu \tag{10.1}$$

and represent a single charge e at the origin as:

$$J_\mu = \left(e\delta^3(\vec{x}), \vec{0}\right) \tag{10.2}$$

In the Lorenz gauge, the solution is $\Box^2 A_\nu(x) = J_\nu(x)$. Since $\delta^3(\vec{x})$ is time-independent, show that the solution to find is:

$$\Box^2 A_0(x) = e\delta^3(\vec{x}) \tag{10.3}$$

Derive the known result:

$$A_0(x) = \frac{e}{4\pi}\frac{1}{r} \tag{10.4}$$

where $r = |\vec{x}|$.

Solution:

We want to derive Coulomb's law using the classical field theory. As discussed in Section 10.2 of the book, the entire classical Maxwell theory of electromagnetism can be summarized into the single Lorentz-invariant Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu \tag{10.5}$$

where the sources are given by the electric charge–current density J_μ and the electromagnetic field tensor $F^{\mu\nu}$ is given by:

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \tag{10.6}$$

The wave equation is found with the help of the Euler–Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial A^\nu} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\right) = 0 \tag{10.7}$$

We immediately find that (compare with Eq. (10.13) of the book):

$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} = \frac{\partial}{\partial(\partial^\mu A^\nu)}\left[-\frac{1}{4}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha)\right] = -(\partial_\mu A_\nu - \partial_\nu A_\mu) = -F_{\mu\nu} \tag{10.8}$$

and

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = -J_\nu \tag{10.9}$$

Hence:

$$\frac{\partial \mathcal{L}}{\partial A^\nu} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} \right) = 0 \quad \Longrightarrow \quad \partial^\mu F_{\mu\nu} = J_\nu \tag{10.10}$$

which is Eq. (10.17) of the book. In the Lorenz gauge $\partial_\mu A^\mu \equiv 0$, we find that:

$$\partial^\mu F_{\mu\nu} = \partial^\mu (\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial^\mu \partial_\mu A_\nu - \partial^\mu \partial_\nu A_\mu = \partial^\mu \partial_\mu A_\nu \quad \Longrightarrow \quad \Box^2 A_\nu(x) = J_\nu(x) \tag{10.11}$$

Since the charge distribution is time-independent, we are dealing with an electrostatic problem. We recall that by definition $A^\mu \equiv (A^0, \vec{A}) = (\phi, \vec{A})$ where the electric field $\vec{E}$ is given by:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} = -\nabla A^0 - \frac{\partial \vec{A}}{\partial t} \tag{10.12}$$

It is then natural to assume that only the time component of the electromagnetic four-vector is relevant and that it is time independent. We then focus on A^0. It is defined by:

$$\Box^2 A_0(x) = J_0(x) \quad \Longrightarrow \quad \underbrace{\partial_0 \partial^0 A_0(x)}_{=0} - \partial_k \partial^k A_0(x) = e\delta^3(\vec{x}) \tag{10.13}$$

Let us verify that the known result satisfies this equation:

$$-\partial_k \partial^k \left(\frac{e}{4\pi} \frac{1}{r} \right) \stackrel{?}{=} e\delta^3(\vec{x}) \quad \Longrightarrow \quad -\frac{e}{4\pi} \partial_k \partial^k \left(\frac{1}{r} \right) = -\frac{e}{4\pi} \left(-4\pi\delta^3\vec{x} \right) = e\delta^3\vec{x} \stackrel{?}{=} e\delta^3(\vec{x}) \qquad \Box \tag{10.14}$$

where we used Eq. (A.55) of the book.

10.2 The Coulomb force in QED

Refer to Exercise 9.3 and compute the non-relativistic limit of the interaction between two fermions in QED at tree level via the exchange of a photon.

(a) *Sketch the corresponding Feynman diagram at tree level.*

(b) *Comparing to the Yukawa case, show that the QED potential is a repulsive potential with an infinite range:*

$$V(r) = \frac{e^2}{4\pi r} \tag{10.15}$$

(c) *Show that, on the other hand, for fermion–antifermion scattering, the potential is attractive. Hence, the exchange of a vector boson makes the interaction repulsive for like fermions (i.e., equal charge) and attractive for fermion–antifermion (i.e., opposite charges), as expected from the classical Coulomb force!*

Solution:

(a) We define the kinematics of the reaction as follows:

$$f(p,s) + f(k,r) \to f(p',s') + f(k',r') \tag{10.16}$$

where the energy–momentum 4-vectors are given by p, k, p', k' and the corresponding spin states of the particles by r, s, r', s'. We draw the Feynman diagram for the scattering of two distinguishable fermions via the exchange of a photon to lowest order and use the QED rules **[R]** defined in Section 10.16 of the book to calculate the corresponding amplitude. The Feynman diagram is shown in Figure 10.1(left). The corresponding amplitude is given by:

$$i\mathcal{M} = \underbrace{(-ie)\bar{u}^{s'}(p')\gamma^\mu u^s(p)}_{e\gamma^\mu\text{vertex + external legs}} \underbrace{\left(\frac{-ig_{\mu\nu}}{q^2+i\epsilon}\right)}_{\text{photon propagator}} \underbrace{(-ie)\bar{u}^{r'}(k')\gamma^\nu u^r(k)}_{e\gamma^\nu\text{vertex + external legs}} \tag{10.17}$$

where we have defined the momentum exchange $q \equiv p' - p$.

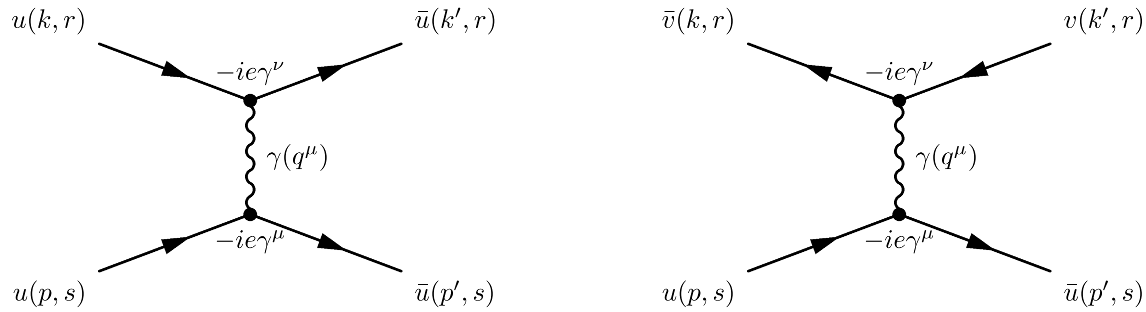

Figure 10.1 Feynman diagram for the scattering of two distinguishable fermions (left) and fermion–antifermion (right) via the exchange of a photon to lowest order.

We compare the QED amplitude above with that obtained for the Yukawa interaction between fermions, as given by Eq. (9.109) in **Ex. 9.3**.

$$i\mathcal{M} = \underbrace{(-ig)\bar{u}^{s'}(p')u^s(p)}_{-ig\text{ vertex + external legs}} \underbrace{\left(\frac{i}{(p-p')^2 - m_\phi^2 + i\epsilon}\right)}_{\phi\text{ propagator}} \underbrace{(-ig)\bar{u}^{r'}(k')u^r(k)}_{-ig\text{ vertex + external legs}} \tag{10.18}$$

We note that the QED amplitude can be obtained from the Yukawa amplitude by the following replacements:

- vertex factors: $(-ig) \longrightarrow (-ie\gamma^\mu)$, $(-ig) \longrightarrow (-ie\gamma^\nu)$
- propagator $i/(q^2+i\epsilon) \longrightarrow -ig^{\mu\nu}/(q^2+i\epsilon)$ (note the **minus** sign!)
- mass $m_\phi \longrightarrow m_\gamma = 0$.

Basically, the Yukawa interaction is scalar since the exchanged boson ϕ is spinless, while the QED interaction has a vector structure with the corresponding introduction of the μ and ν Lorentz indices, due to the spin-1 nature of the exchanged photon.

(b) In order to recover the Coulomb interaction, we consider the non-relativistic limit. In **Ex. 9.3**, we have shown that the spinor u in Weyl representation can in this case be written as:

$$u^s(p) \simeq \sqrt{m}\begin{pmatrix} u_A^s \\ u_A^s \end{pmatrix} \tag{10.19}$$

We introduce the 2×2 compact notation in the Weyl representation (compare with Eqs. (8.16) and (8.22) of the book):

$$\gamma^0 = \beta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^k = \beta\alpha^k = \begin{pmatrix} 0 & +\sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad \mathbb{1} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{10.20}$$

where β and $\vec{\alpha}$ are the 4×4 Dirac matrices.

We compute the vector-like vertex term in the non-relativistic limit:

$$\bar{u}^{s'}(p')\gamma^\mu u^s(p) = \begin{cases} m\begin{pmatrix} u_A^{s'\dagger} & u_A^{s'\dagger}\end{pmatrix}\underbrace{\gamma^0\gamma^0}_{=\mathbb{1}}\begin{pmatrix} u_A^s \\ u_A^s\end{pmatrix} = 2mu_A^{s'\dagger}u_A^s = 2m\delta^{ss'} & (\mu=0) \\ m\begin{pmatrix} u_A^{s'\dagger} & u_A^{s'\dagger}\end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix} 0 & +\sigma^k \\ -\sigma^k & 0\end{pmatrix}\begin{pmatrix} u_A^s \\ u_A^s\end{pmatrix} = 0 & (\mu=k=1,2,3)\end{cases} \tag{10.21}$$

Thus we have:

$$\begin{aligned} i\mathcal{M} &= (-ie)^2\left[\underbrace{\bar{u}^{s'}(p')\gamma^0u^s(p)}_{2m\delta^{ss'}}\left(\frac{-ig_{00}}{q^2+i\epsilon}\right)\underbrace{\bar{u}^{r'}(k')\gamma^0u^r(k)}_{2m\delta^{rr'}}+\underbrace{\bar{u}^{s'}(p')\gamma^ku^s(p)\left(\frac{-ig_{kk}}{q^2+i\epsilon}\right)\bar{u}^{r'}(k')\gamma^ku^r(k)}_{=0}\right] \\ &= (+ie^2)\frac{1}{q^2+i\epsilon}2m\delta^{ss'}2m\delta^{rr'} \end{aligned} \tag{10.22}$$

where we used that $g_{00}=1$. Consequently,

$$\text{QED:}\quad i\mathcal{M} = (+ie^2)\frac{1}{-|\vec{p}'-\vec{p}|^2+i\epsilon}2m\delta^{ss'}2m\delta^{rr'} = \frac{-ie^2}{|\vec{q}|^2-i\epsilon}2m\delta^{ss'}2m\delta^{rr'} \tag{10.23}$$

where we used the result $q^2=-|\vec{p}'-\vec{p}|^2=-|\vec{q}|^2$ derived for the Yukawa case in Eq. (9.99) of **Ex. 9.3**. We compare the QED result with the amplitude that we derived in the Yukawa case Eq. (9.110):

$$\text{Yukawa:}\quad i\mathcal{M} = \frac{ig^2}{|\vec{q}|^2+m_\phi^2}2m\delta^{ss'}2m\delta^{rr'} \tag{10.24}$$

We note the extra factor (-1) in the QED case, coming from the $-g_{00}=-1$ term in the photon propagator. Following the same considerations as for the Yukawa case, we write:

$$T_{fi}\simeq -i\tilde{V}(\vec{q})\,2\pi\,\delta(E_{p'}-E_p) \tag{10.25}$$

and the corresponding potential in momentum space for the QED interaction between two fermions is therefore:

$$\tilde{V}_{QED}(\vec{q}) = +\frac{e^2}{q^2-i\epsilon} \tag{10.26}$$

By inverting the Fourier transform $\tilde{V}(\vec{q})$, we get (we denote $q=|\vec{q}|$ and $r=|\vec{x}|$ in the following and the mass of the photon equal to m_γ):

$$\begin{aligned} V(\vec{x}) &= \int\frac{\mathrm{d}^3q}{(2\pi)^3}\tilde{V}(\vec{q})e^{i\vec{q}\cdot\vec{x}} = \int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{+e^2}{q^2+m_\gamma^2}e^{i\vec{q}\cdot\vec{x}} \\ &= +\frac{e^2}{(2\pi)^3}\int q^2dq\int\mathrm{d}\phi\int\mathrm{d}\cos\theta\frac{e^{iqr\cos\theta}}{q^2+m_\phi^2} = -\frac{g^2}{4\pi^2}\int_0^{+\infty}dq\,q^2\frac{e^{iqr}-e^{-iqr}}{iqr(q^2+m_\gamma^2)} \\ &= +\frac{e^2}{4\pi^2ir}\left(\int_0^{+\infty}dq\,q\frac{e^{iqr}}{q^2+m_\gamma^2}-\int_0^{+\infty}dq\,q\frac{e^{-iqr}}{q^2+m_\gamma^2}\right) \\ &= +\frac{e^2}{4\pi^2ir}\int_{-\infty}^{+\infty}dq\frac{qe^{iqr}}{q^2+m_\gamma^2} \end{aligned} \tag{10.27}$$

In order to perform the integral we use Cauchy's residue theorem (see Appendix A.4 of the book). We note that the argument has a simple pole at $q = +im_\gamma$. The result is then (compare with Eq. (9.114) of **Ex. 9.3**):

$$V(r) = \lim_{m_\gamma \to 0} \frac{e^2}{4\pi^2 ir}(2\pi i)\frac{im_\gamma e^{-m_\gamma r}}{2im_\gamma} = +\frac{e^2}{4\pi r} \tag{10.28}$$

where we recovered the Coulomb potential with an infinite range (while the Yukawa potential was limited in range due to m_ϕ). The **Coulomb potential is repulsive for fermion-fermion scattering**, as expected from classical electromagnetism.

(c) For fermion-antifermion scattering, we consider the Feynman diagram shown in Figure 10.1(right). Its amplitude is given by:

$$i\mathcal{M} = \underbrace{(-ie)\bar{u}^{s'}(p')\gamma^\mu u^s(p)}_{e\gamma^\mu \text{vertex + external legs}} \underbrace{\left(\frac{-ig_{\mu\nu}}{q^2+i\epsilon}\right)}_{\text{photon propagator}} \underbrace{(-ie)\bar{v}^r(k)\gamma^\nu v^{r'}(k')}_{e\gamma^\nu \text{vertex + external legs}} \tag{10.29}$$

We use the solution for the spinor v which we derived in Eq. (9.116) of **Ex. 9.3**):

$$(\not{p} + m)v = 0 \quad \Longrightarrow \quad v^{(s)}(p) \simeq \mathcal{N}\begin{pmatrix} v_A^s \\ -v_A^s \end{pmatrix} = \sqrt{m}\begin{pmatrix} v_A^s \\ -v_A^s \end{pmatrix} \tag{10.30}$$

As before, we compute the vector-like vertex term in the non-relativistic limit for antispinors:

$$\bar{v}^r(k)\gamma^\mu v^{r'}(k') = \begin{cases} m\begin{pmatrix} v_A^{r\dagger} & v_A^{r\dagger} \end{pmatrix}\underbrace{\gamma^0\gamma^0}_{=\mathbb{1}}\begin{pmatrix} v_A^{r'} \\ v_A^{r'} \end{pmatrix} = 2mv_A^{r\dagger}v_A^{r'} = 2m\delta^{rr'} & (\mu = 0) \\ m\begin{pmatrix} v_A^{r'\dagger} & v_A^{r'\dagger} \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & +\sigma^k \\ -\sigma^k & 0 \end{pmatrix}\begin{pmatrix} v_A^r \\ v_A^r \end{pmatrix} = 0 & (\mu = k = 1,2,3) \end{cases} \tag{10.31}$$

This is the opposite of what we obtained in the Yukawa case (compare with Eq. (9.117) of **Ex. 9.3**), where the sign changed for the v spinors compared to the u case:

$$\begin{aligned} \text{Yukawa:} \quad \bar{v}^{(r)}(k)v^{(r)}(k) &= v^{(r')\dagger}(k')\gamma^0 v^{(r)}(k) = m\,(v_A^{r'\dagger}, -v_A^{r'\dagger})\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} v_A^s \\ -v_A^s \end{pmatrix} \\ &= -2mv_A^{r'\dagger}v_A^r = -2m\delta^{rr'} \end{aligned} \tag{10.32}$$

Thus we have:

$$\begin{aligned} i\mathcal{M} &= (-ie)^2\left[\underbrace{\bar{u}^{s'}(p')\gamma^0 u^s(p)}_{2m\delta^{ss'}}\left(\frac{-ig_{00}}{q^2+i\epsilon}\right)\underbrace{\bar{v}^r(k')\gamma^0 v^{r'}(k)}_{2m\delta^{rr'}}\right] \\ &= (+ie^2)\frac{1}{q^2+i\epsilon}2m\delta^{ss'}2m\delta^{rr'} \end{aligned} \tag{10.33}$$

As a result, the **QED potential between a fermion–antifermion pair is attractive**. And one similarly sees that the potential for antifermion–antifermion interaction is repulsive. So, we see that the fact that equal charges repel and unlike charges attract is a consequence of the vector structure of the interaction, with the exchange of the photon with the $-g^{00}$ term at its propagator!

10.3 Scattering of two charged scalar particles via photon exchange

Consider the scattering of two charged pions $\pi^\pm + \pi^\pm \to \pi^\pm + \pi^\pm$ via the electromagnetic exchange of a photon. Compute the differential cross-section in the center-of-mass system.

Solution:

We are considering the electrodynamics of scalar fields, as discussed in Section 10.18 of the book. Since the theory is manifestly C-invariant, the $\pi^+ + \pi^+ \to \pi^+ + \pi^+$ and $\pi^- + \pi^- \to \pi^- + \pi^-$ amplitudes are identical. We define the kinematics of the reaction as follows:

$$\pi^\pm(p) + \pi^\pm(k) \to \pi^\pm(p') + \pi^\pm(k') \tag{10.34}$$

where the energy–momentum 4-vectors are given by p, k, p', k'. We draw the Feynman diagrams at tree level and use the rules **[S]** defined in Section 10.18 of the book to calculate the corresponding amplitudes. The Feynman diagrams are shown in Figure 10.2 (compare with Figure 10.5 of the book). The corresponding amplitudes are given by:

$$i\mathcal{M}_1 = \underbrace{(-ie)(p+p')^\mu}_{\text{vertex + external legs}} \underbrace{\left(\frac{-ig_{\mu\nu}}{(p'-p)^2 + i\epsilon}\right)}_{\text{photon propagator}} \underbrace{(-ie)(k+k')^\nu}_{\text{vertex + external legs}} \tag{10.35}$$

and

$$i\mathcal{M}_2 = \underbrace{(-ie)(p+k')^\mu}_{\text{vertex + external legs}} \underbrace{\left(\frac{-ig_{\mu\nu}}{(k'-p)^2 + i\epsilon}\right)}_{\text{photon propagator}} \underbrace{(-ie)(k+p')^\nu}_{\text{vertex + external legs}} \tag{10.36}$$

We see that the two diagrams can be obtained from one another by the swap $p' \leftrightarrow k'$. The two diagrams must be added, since for bosons the result must be symmetric under the interchange of two particles.

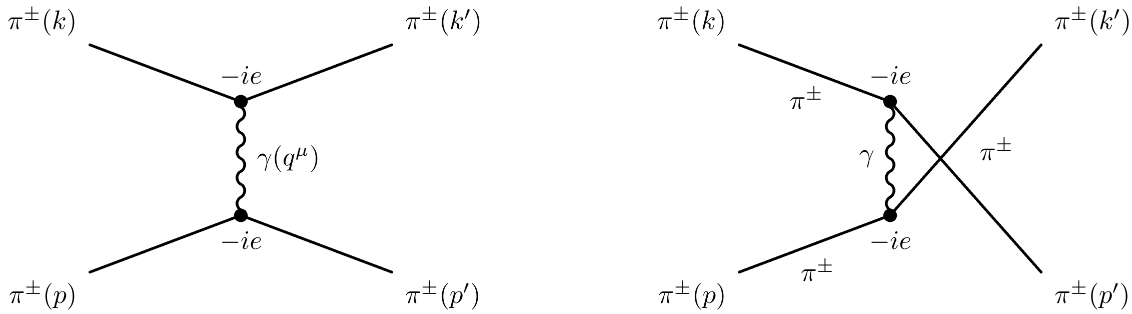

Figure 10.2 Electrodynamics of scalar fields: Feynman diagram for the scattering of two charged scalar particles via photon exchange to lowest order.

The amplitude $\mathcal{M}_1$ can be simplified to:

$$i\mathcal{M}_1 = (-i)(-ie)^2 \left(\frac{(p+p')^\mu (k+k')_\mu}{(p'-p)^2 + i\epsilon}\right) = +ie^2 \left(\frac{p\cdot k + p'\cdot k + p\cdot k' + p'\cdot k'}{(p'-p)^2 + i\epsilon}\right) \tag{10.37}$$

The amplitude is manifestly Lorentz invariant since it is composed scalar products of four-vectors, and such scalar products are Lorentz invariant! We introduce the Mandelstam variables s, t and u, as it is customary to express the amplitude of $2 \to 2$ scattering processes (see Section 11.8 of the book for the detailed definition of the Mandelstam variables). We have in our choice of kinematic variables:

$$s = (k+p)^2 = (k'+p')^2, \qquad t = (p-p')^2 = (k'-k)^2, \qquad u = (p-k')^2 = (p'-k)^2 \tag{10.38}$$

where we used that $k^\mu + p^\mu = k'^\mu + p'^\mu$ by energy–momentum conservation. Consequently, we can derive that:

$$s = k^2 + p^2 + 2k\cdot p = 2m^2 + 2k\cdot p \quad \longrightarrow \quad k\cdot p = k'\cdot p' = \frac{1}{2}(s-2m^2) \tag{10.39}$$

$$t = p^2 + p'^2 - 2p\cdot p' = 2m^2 - 2p\cdot p' \quad \longrightarrow \quad p\cdot p' = k\cdot k' = -\frac{1}{2}(t-2m^2) \tag{10.40}$$

$$u = p^2 + k'^2 - 2p\cdot k' = 2m^2 - 2p\cdot k' \quad \longrightarrow \quad p\cdot k' = p'\cdot k = -\frac{1}{2}(u-2m^2) \tag{10.41}$$

where m is the rest mass of the charged pions. Consequently, the amplitude is then simply:

$$i\mathcal{M}_1 = +ie^2\frac{1}{2}\left(\frac{(s-2m^2)-(u-2m^2)-(u-2m^2)+(s-2m^2)}{t+i\epsilon}\right) = (+ie^2)\frac{s-u}{t} \tag{10.42}$$

This expression is very simple and manifestly Lorentz invariant (since the Mandelstam variables are also LT invariant). The amplitude $\mathcal{M}_2$ can be immediately found by noting that the $p' \leftrightarrow k'$ corresponds to the $t \leftrightarrow u$ swap. Hence:

$$i\mathcal{M}_2 = +ie^2\frac{1}{2}\left(\frac{(s-2m^2)-(t-2m^2)-(t-2m^2)+(s-2m^2)}{u+i\epsilon}\right) = (+ie^2)\frac{s-t}{u} \tag{10.43}$$

We now make a choice of reference system. We express the differential cross-section in the center-of-mass system of the reaction. We use Eq. (5.145) which is valid the masses of the particles are all identical (or in the ultrarelativistic case where the differences of masses can be neglected):

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CMS} = \left(\frac{1}{64\pi^2 s}\right) S|\mathcal{M}_1+\mathcal{M}_2|^2 = \left(\frac{1}{128\pi^2 s}\right)\left(\mathcal{M}_1^2+\mathcal{M}_2^2+2\mathcal{M}_1\mathcal{M}_2^*\right) \tag{10.44}$$

where we note that squaring the total amplitude $\mathcal{M}_1+\mathcal{M}_2$ yields the amplitude squared of the two diagrams $\mathcal{M}_1^2$ and $\mathcal{M}_2^2$ plus an interference term $2\mathcal{M}_1\mathcal{M}_2^*$. We also introduced a statistic factor $S = 1/2$ since the two final state particles are indistinguishable. We consequently find:

$$\begin{aligned}
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CMS} &= \left(\frac{e^4}{128\pi^2 s}\right)\left[\frac{(s-u)^2}{t^2}+\frac{(s-t)^2}{u^2}+2\frac{(s-u)(s-t)}{ut}\right] \\
&= \left(\frac{e^4}{128\pi^2 s}\right)\left[\frac{(s-u)^2u^2+(s-t)^2t^2+2ut(s-u)(s-t)}{u^2t^2}\right] \\
&= \left(\frac{e^4}{128\pi^2 s}\right)\left[\frac{(s^2+u^2-2us)u^2+(s^2+t^2-2ts)t^2+2ut(s^2-us-ts+t^2)}{u^2t^2}\right] \\
&= \left(\frac{e^4}{128\pi^2 s}\right)\left[\frac{u^4+u^2s^2-2u^3s+s^2t^2+t^4-2t^3s+2uts^2-2u^2ts-2ut^2s+2u^2t^2}{u^2t^2}\right] \\
&= \left(\frac{e^4}{128\pi^2 s}\right)\left[\frac{u^4+t^4+2u^2t^2-2u^2s(t+u)-2t^2s(t+u)+s^2(t^2+2ut+u^2)}{u^2t^2}\right] \\
&= \left(\frac{e^4}{128\pi^2 s}\right)\left[\frac{(u^2+t^2)^2-2s(t+u)(u^2+t^2)-s^2(t+u)^2}{u^2t^2}\right] \\
&= \left(\frac{e^4}{128\pi^2 s}\right)\left[\frac{(u^2-s(t+u)+t^2)^2}{u^2t^2}\right]
\end{aligned} \tag{10.45}$$

We analyse the kinematics in the center-of-mass system. All particles possess identical energies in the CMS system and their momenta are back-to-back in the initial and final states. Since all particles have the same rest mass, the magnitudes of the initial and final state momenta are identical. Hence, we can write:

$$p^\mu = (E, \vec{p}), \quad k^\mu = (E, -\vec{p}), \quad p'^\mu = (E, \vec{p}'), \quad k'^\mu = (E, -\vec{p}') \tag{10.46}$$

where $|\vec{p}| = |\vec{p}'|$. It then follows that:

$$s = (k+p)^2 = (2E)^2 = 4E^2 \tag{10.47}$$

$$t = (p-p')^2 = (E-E)^2 - (\vec{p}-\vec{p}')^2 = -\left(\vec{p}^2 + \vec{p}'^2 - 2\vec{p}\cdot\vec{p}'\right) = -2p^2(1-\cos\theta) \tag{10.48}$$

$$u = (p-k')^2 = (E-E)^2 - (\vec{p}+\vec{p}')^2 = -\left(\vec{p}^2 + \vec{p}'^2 + 2\vec{p}\cdot\vec{p}'\right) = -2p^2(1+\cos\theta) \tag{10.49}$$

where $p \equiv |\vec{p}| = |\vec{p}'|$, and θ is the scattering angle, defined as the angle between the incoming $\vec{p}$ and the outgoing $\vec{p}'$ momenta. We note that:

$$u^2 + t^2 = 4p^4(1+\cos\theta)^2 + 4p^4(1-\cos\theta)^2 = 4p^4\left((1-\cos\theta)^2 + (1+\cos\theta)^2\right) = 8p^4\left(1+\cos^2\theta\right) \tag{10.50}$$

$$s(t+u) = 4E^2\left(2p^2(1-\cos\theta) + 2p^2(1+\cos\theta)\right) = -4E^2 4p^2 \tag{10.51}$$

$$u^2t^2 = 4p^4(1+\cos\theta)^2 4p^4(1-\cos\theta)^2 = 16p^8\left((1+\cos\theta)(1-\cos\theta)\right)^2 = 16p^8\left(1-\cos^2\theta\right)^2 \tag{10.52}$$

We can plug these expressions into our differential cross-section to find:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CMS} = \left(\frac{e^4}{128\pi^2 4E^2}\right)\left[\frac{\left(8p^4\left(1+\cos^2\theta\right) + 4E^2 4p^2\right)^2}{16p^8\left(1-\cos^2\theta\right)^2}\right] = \left(\frac{e^4}{128\pi^2 E^2}\right)\left[\frac{\left(p^2\left(1+\cos^2\theta\right) + 2E^2\right)^2}{p^4\left(1-\cos^2\theta\right)^2}\right] \tag{10.53}$$

We can introduce the velocity of the particles $\beta \equiv p/E$ to find:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CMS} = \left(\frac{e^4}{128\pi^2 E^2}\right)\left[\frac{\left(p^2\left(1+\cos^2\theta\right) + 2E^2\right)^2}{p^4\left(1-\cos^2\theta\right)^2}\right] = \left(\frac{e^4}{128\pi^2 E^2}\right)\left[\frac{\left(\beta^2\left(1+\cos^2\theta\right) + 2\right)^2}{\beta^4\left(1-\cos^2\theta\right)^2}\right] \tag{10.54}$$

Hence, we obtain that for the ultra-relativistic case $\beta \to 1$, the differential cross-section is simply:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CMS} = \left(\frac{e^4}{128\pi^2 E^2}\right)\left[\frac{\left(\cos^2\theta + 3\right)^2}{\left(1-\cos^2\theta\right)^2}\right] \tag{10.55}$$

The behavior of the differential cross-section is plotted in Figure 10.3.

10.4 Annihilation of two charged scalar particles into two other charged scalar particles via photon exchange

Consider the following hypothetical reaction within electrodynamics (see Section 10.18) of scalar particles $\tilde{e}^+\tilde{e}^- \to \gamma^* \to \tilde{\mu}^+\tilde{\mu}^-$, where $\tilde{e}^\pm$ and $\tilde{\mu}^\pm$ are point-like spinless scalar particles with rest masses m and M.

(a) Write down the amplitude at tree level assuming the coupling given by the electromagnetism of scalar particles.

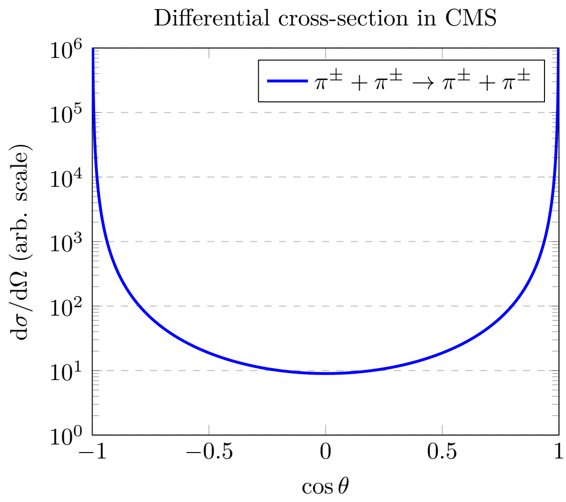

Figure 10.3 Differential cross-section for $\pi^\pm + \pi^\pm \to \pi^\pm + \pi^\pm$ in the center-of-mass system as a function of $\cos\theta$.

(b) Show that the amplitude can be written as:

$$\mathcal{M} = e^2 \frac{u-t}{s} \tag{10.56}$$

where s, t, u are the Mandelstam variables.

(c) Use the general phase space result derived in Exercise 5.4 to show that this leads to the following differential cross-section in the center-of-mass system:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{\alpha^2}{4s}\sqrt{\frac{s-4M^2}{s-4m^2}}\left(1-\frac{4m^2}{s}\right)\left(1-\frac{4M^2}{s}\right)\cos^2\theta \tag{10.57}$$

(d) Compute the total cross-section and show that it is given by:

$$\sigma = \frac{\pi\alpha^2}{3s}\sqrt{\frac{s-4M^2}{s-4m^2}}\left\{1-\frac{4\left(m^2+M^2\right)}{s}+\frac{16m^2M^2}{s^2}\right\} \tag{10.58}$$

(e) Take the high-energy limit $s \gg m^2, M^2$ and show that the results become:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{\alpha^2}{4s}\cos^2\theta \quad \text{and} \quad \sigma = \frac{\pi\alpha^2}{3s} \tag{10.59}$$

where α is the fine-structure constant.

(f) Can you motivate these results in terms of the spin of the photon?

(g) Compare the results obtained to that for $e^+e^- \to \mu^+\mu^-$ derived in Chapter 11.

Solution:

a) We are considering the electrodynamics of scalar fields, as discussed in Section 10.18 of the book. We define the kinematics of the reaction as follows:

$$\tilde{e}^-(p) + \tilde{e}^+(k) \to \tilde{\mu}^-(p') + \tilde{\mu}^+(k') \tag{10.60}$$

where the energy–momentum 4-vectors are given by p, k, p', k'. The reaction $\tilde{e}^+\tilde{e}^- \to \gamma^* \to \tilde{\mu}^+\tilde{\mu}^-$ necessarily proceeds via annihilation of the initial charged scalars and creation of the two other charged scalars. We draw the Feynman diagrams at tree level and use the rules **[S]** defined in Section 10.18 of the book to calculate the corresponding amplitudes. The Feynman diagram is shown in Figure 10.4. We note that $\tilde{e}^+$ and $\tilde{e}^-$ as well as $\tilde{\mu}^+$ and $\tilde{\mu}^-$ are assumed to form particle–antiparticle pairs, hence we need to adapt our rules **[S3]** to take that into account. The corresponding amplitude is then given by:

$$i\mathcal{M} = \underbrace{(-ie)(p-k)^\mu}_{\text{vertex + external legs}} \underbrace{\left(\frac{-ig_{\mu\nu}}{(p+k)^2+i\epsilon}\right)}_{\text{photon propagator}} \underbrace{(-ie)(p'-k')^\nu}_{\text{vertex + external legs}} \tag{10.61}$$

where in the vertex factors we have followed the convention of the arrows shown in the Feynman diagram in order to define the minus sign between the two four-vectors.

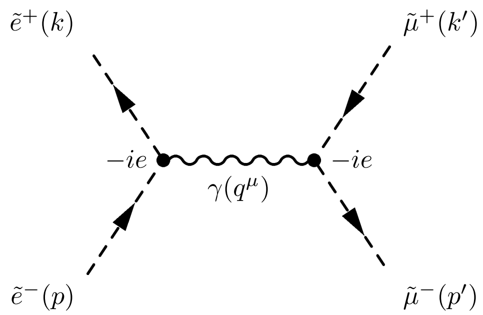

Figure 10.4 Electrodynamics of scalar fields: Feynman diagram for the annihilation of two oppositely charged scalar particles into two other scalar particles at lowest order.

The amplitude can then be simplified to:

$$i\mathcal{M} = -i(-ie)^2\frac{(p-k)^\mu(p'-k')_\mu}{(p+k)^2+i\epsilon} = +ie^2\frac{p\cdot p' - k\cdot p' - p\cdot k' + k\cdot k'}{(p+k)^2+i\epsilon} \tag{10.62}$$

b) We introduce the Mandelstam variables s, t and u, as it is customary to express the amplitude of $2\to 2$ scattering processes (see Section 11.8 of the book for the detailed definition of the Mandelstam variables). We have in our choice of kinematic variables:

$$s = (k+p)^2 = (k'+p')^2, \qquad t = (p-p')^2 = (k'-k)^2, \qquad u = (p-k')^2 = (p'-k)^2 \tag{10.63}$$

where we used that $k^\mu + p^\mu = k'^\mu + p'^\mu$ by energy–momentum conservation. Consequently, we can derive that:

$$t = p^2 + p'^2 - 2p\cdot p' = m^2 + M^2 - 2p\cdot p' \quad\longrightarrow\quad p\cdot p' = k\cdot k' = -\frac{1}{2}(t - m^2 - M^2) \tag{10.64}$$

$$u = p^2 + k'^2 - 2p\cdot k' = m^2 + M^2 - 2p\cdot k' \quad\longrightarrow\quad p\cdot k' = p'\cdot k = -\frac{1}{2}(u - m^2 - M^2) \tag{10.65}$$

where m and M are resp. the rest masses of the initial and final states particles. We insert these four-products into our amplitude and immediately obtain the desired result:

$$\begin{aligned} i\mathcal{M} &= +ie^2 \frac{-\frac{1}{2}(t-m^2-M^2)+\frac{1}{2}(u-m^2-M^2)+\frac{1}{2}(u-m^2-M^2)-\frac{1}{2}(t-m^2-M^2)}{s} \\ &= +ie^2 \frac{-(t-m^2-M^2)+(u-m^2-M^2)}{s} = +ie^2 \frac{u-t}{s} \end{aligned} \tag{10.66}$$

or

$$\mathcal{M} = e^2 \frac{(u-t)}{s} \qquad \square \tag{10.67}$$

c) We use the result derived in **Ex. 5.4** for the differential cross-section in the center-of-mass system of the reaction involving particles with masses m in the initial state and M in the final state:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CMS}} = \left(\frac{1}{64\pi^2 s}\right) \sqrt{\frac{\lambda(s,M^2,M^2)}{\lambda(s,m^2,m^2)}} |\mathcal{M}|^2 \tag{10.68}$$

where $\lambda(x,y,z) \equiv x^2+y^2+z^2-2xy-2yz-2zx$ is the Källén or triangle function is given by Eq. (5.132) of the book. We can expand:

$$\begin{aligned} \lambda(s,M^2,M^2) &= s^2+M^2+M^2-2sM^2-2M^4-2sM^2 = s^2+2M^4-4sM^2-2M^4 \\ &= s(s-4M^2) \end{aligned} \tag{10.69}$$

and similarly replacing M by m:

$$\lambda(s,m^2,m^2) = s(s-4m^2) \tag{10.70}$$

We can now insert these expressions and the amplitude squared into the differential cross-section to find:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CMS}} = \frac{e^4}{64\pi^2 s} \sqrt{\frac{s-4M^2}{s-4m^2}} \frac{(u-t)^2}{s^2} \tag{10.71}$$

We analyse the kinematics in the center-of-mass system. All particles possess identical energies in the CMS system and their momenta are back-to-back in the initial and final states, however the momenta in the initial and final states have different magnitudes due to the different rest masses. Hence, we can write:

$$p^\mu = (E,\vec{p}), \quad k^\mu = (E,-\vec{p}), \quad p'^\mu = (E,\vec{p}\,'), \quad k'^\mu = (E,-\vec{p}\,') \tag{10.72}$$

where

$$|\vec{p}| = \sqrt{E^2-m^2} \quad \text{and} \quad |\vec{p}\,'| = \sqrt{E^2-M^2} \tag{10.73}$$

It then follows that:

$$s = (k+p)^2 = (2E)^2 = 4E^2 \tag{10.74}$$

$$\begin{aligned} t &= (p-p')^2 = (E-E)^2 - (\vec{p}-\vec{p}\,')^2 = -\left(\vec{p}^{\,2}+\vec{p}\,'^2-2\vec{p}\cdot\vec{p}\,'\right) \\ &= -\left(2E^2-m^2-M^2-2\sqrt{E^2-m^2}\sqrt{E^2-M^2}\cos\theta\right) \end{aligned} \tag{10.75}$$

$$\begin{aligned} u &= (p-k')^2 = (E-E)^2 - (\vec{p}+\vec{p}\,')^2 = -\left(\vec{p}^{\,2}+\vec{p}\,'^2+2\vec{p}\cdot\vec{p}\,'\right) \\ &= -\left(2E^2-m^2-M^2+2\sqrt{E^2-m^2}\sqrt{E^2-M^2}\cos\theta\right) \end{aligned} \tag{10.76}$$

Hence:

$$u-t = -4\sqrt{E^2-m^2}\sqrt{E^2-M^2}\cos\theta = -4\sqrt{\frac{s}{4}-m^2}\sqrt{\frac{s}{4}-M^2}\cos\theta \tag{10.77}$$

Consequently:

$$\frac{(u-t)^2}{s^2} = \frac{16\left(\frac{s}{4}-m^2\right)\left(\frac{s}{4}-M^2\right)\cos^2\theta}{s^2} = \left(1-\frac{4m^2}{s}\right)\left(1-\frac{4M^2}{s}\right)\cos^2\theta \tag{10.78}$$

We replace the above expression in Eq. (10.71) above and finally obtain:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CMS}} = \frac{\alpha^2}{4s}\sqrt{\frac{s-4M^2}{s-4m^2}}\left(1-\frac{4m^2}{s}\right)\left(1-\frac{4M^2}{s}\right)\cos^2\theta \tag{10.79}$$

where we have used the fine structure constant $\alpha = e^2/4\pi$.

d) The differential cross-section can be expressed as:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CMS}} = \sigma_0(s,m,M)\cos^2\theta \tag{10.80}$$

where

$$\sigma_0(s,m,M) \equiv \frac{\alpha^2}{4s}\sqrt{\frac{s-4M^2}{s-4m^2}}\left(1-\frac{4m^2}{s}\right)\left(1-\frac{4M^2}{s}\right) \tag{10.81}$$

The total cross section is obtained by integration over the solid angle in momentum space:

$$\sigma = \int \mathrm{d}\Omega\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \sigma_0(s,m,M)\int(\mathrm{d}\cos\theta)\cos^2\theta\int\mathrm{d}\phi = \sigma_0(s,m,M)\left(\frac{2}{3}\right)(2\pi) \tag{10.82}$$

Also we can express $\sigma_0(s,m,M)$ as follows:

$$\sigma_0(s,m,M) = \frac{\alpha^2}{4s}\sqrt{\frac{s-4M^2}{s-4m^2}}\left(1-\frac{4(m^2+M^2)}{s}+\frac{16m^2M^2}{s^2}\right) \tag{10.83}$$

Hence:

$$\begin{aligned}\sigma(s) &= (2\pi)\left(\frac{2}{3}\right)\frac{\alpha^2}{4s}\sqrt{\frac{s-4M^2}{s-4m^2}}\left(1-\frac{4(m^2+M^2)}{s}+\frac{16m^2M^2}{s^2}\right)\\ &= \frac{\pi\alpha^2}{3s}\sqrt{\frac{s-4M^2}{s-4m^2}}\left(1-\frac{4(m^2+M^2)}{s}+\frac{16m^2M^2}{s^2}\right)\end{aligned} \tag{10.84}$$

We note that by construction $s \geq 4m^2$. On the other hand, we see that the reaction has a kinematical threshold set by

$$s = 4E^2 \geq 4M^2 \quad\Longrightarrow\quad E \geq M \tag{10.85}$$

which implies that the initial state particles must have enough energy to produce the final state $\tilde{\mu}^+\tilde{\mu}^-$ particles.

e) We first study the case where the rest mass of the initial particles can be neglected, i.e. $s \gg m^2$ and $M^2 \gg m^2$. Since

$$\sqrt{\frac{s-4M^2}{s-4m^2}} \approx \sqrt{\frac{s-4M^2}{s}} = \sqrt{1-\frac{4M^2}{s}} \quad\text{and}\quad \left(1-\frac{4m^2}{s}\right)\approx 1 \tag{10.86}$$

We then obtain:

$$\sigma(s) = \frac{\pi\alpha^2}{3s}\sqrt{1-\frac{4M^2}{s}}\left(1-\frac{4M^2}{s}\right) = \frac{\pi\alpha^2}{3s}\left(1-\frac{4M^2}{s}\right)^{3/2} \tag{10.87}$$

where the term in brackets in the kinematical threshold factor. As before, the energy of the initial state particles must satisfy $E > M$ in order to produce the final state particles. As a matter of fact, it is worthwhile noting that we can rewrite the kinematical threshold factor as a function of the velocity β of the final state particles. Indeed:

$$\left(1-\frac{4M^2}{s}\right)^{3/2}=\left(\frac{s-4M^2}{s}\right)^{3/2}=\left(\frac{4E^2-4M^2}{4E^2}\right)^{3/2}=\left(\frac{p^2}{E^2}\right)^{3/2}=\left(\beta^2\right)^{3/2}=\beta^3 \tag{10.88}$$

Hence the total cross-section as a function of the velocity of the final state particle is simply:

$$\sigma(s)=\frac{\pi\alpha^2}{3s}\beta^3 \tag{10.89}$$

where β is the velocity of the final state particles in the center-of-mass of the reaction. The total cross-section as a function of the energy of the initial state particles is shown in Figure 10.5 for various rest masses M of the final state particles. The threshold effect is clearly visible. **Hence, the measurement of the cross-section as a function of the energy near the threshold can be used to precisely measure the rest mass of the final state particles.**

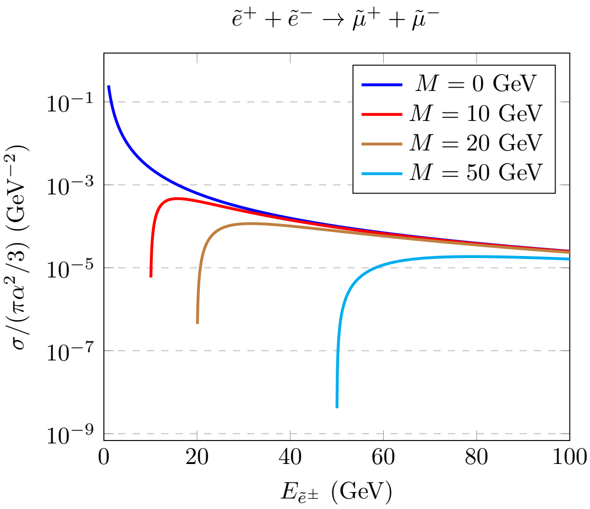

Figure 10.5 Total cross-section for $\tilde{e}^+ + \tilde{e}^- \to \tilde{\mu}^+ + \tilde{\mu}^-$ as a function of initial state particle energies in the center-of-mass system.

We now move to the ultra-relativistic case where $s \gg m^2, M^2$. One can immediately see that the expressions for the cross-sections simplify greatly since:

$$\sqrt{\frac{s-4M^2}{s-4m^2}}\approx 1 \quad \text{and} \quad \left(1-\frac{4m^2}{s}\right)\left(1-\frac{4M^2}{s}\right)\approx 1 \tag{10.90}$$

Consequently in the ultra-relativistic case, we have:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{CMS}}=\frac{\alpha^2}{4s}\cos^2\theta \qquad \text{and} \qquad \sigma=\frac{\pi\alpha^2}{3s} \tag{10.91}$$

f) The photon is a spin-1 vector boson, hence it has three independent spin eigenstates with respect to a certain z axis:

$$L=1: \qquad |L,L_z\rangle=|1,+1\rangle\,,|1,0\rangle\,,|1,-1\rangle \tag{10.92}$$

We must consider the photon spin eigenstate relative to the initial and final state axes, which we label z and z'. See Figure 10.6. The amplitude will be proportional to the overlap between the wave-functions, which are related by the rotational covariance of angular momentum eigenstates. The change of quantization frame is obtained with the help of the Wigner matrices $D^J_{L_z L'_z}$:

$$|L, L'_z\rangle = \sum_{L_z} D^J_{L_z L'_z} |L, L_z\rangle \tag{10.93}$$

For $L = 1$, the Wigner matrices are given by:

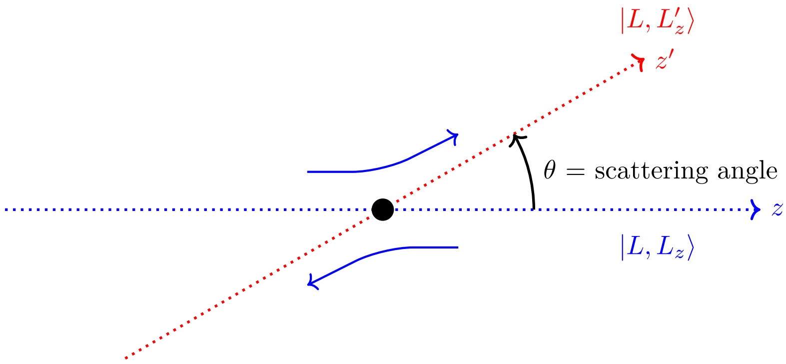

Figure 10.6 Rotational covariance of angular momentum eigenstates. The change of quantization frame is obtained with the help of the Wigner matrices $D^J_{L_z L'_z}$.

$$D^1_{1,1} = \frac{1+\cos\theta}{2}, \quad D^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}, \quad D^1_{1,-1} = \frac{1-\cos\theta}{2}, \quad D^1_{0,0} = \cos\theta \tag{10.94}$$

In the case of the scattering of two scalar particles, hence spinless, we have $L = 1$ and $L_z = L_{z'} = 0$. Hence, we expect the amplitude and the cross-section to be proportional to:

$$\mathcal{M} \propto D^1_{0,0} = \cos\theta \quad \Longrightarrow \quad \frac{d\sigma}{d\Omega} \propto |\mathcal{M}|^2 = \cos^2\theta \tag{10.95}$$

This explains why:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CMS}} = \frac{\alpha^2}{4s}\cos^2\theta \quad \Longrightarrow \quad \sigma = \int d\Omega \frac{d\sigma}{d\Omega} = (2\pi)\left(\frac{2}{3}\right)\frac{\alpha^2}{4s} = \frac{\pi\alpha^2}{3s} \tag{10.96}$$

We note that the cross-section vanishes for $\theta = 90°$, which is a clear signal for the scattering of scalar particles.

g) In the case of the $e^+e^- \to \gamma^* \to \mu^+\mu^-$ scattering, the initial and final state particles are fermions with spin-1/2. As discussed in Section 10.17 of the book, the chiral structure of QED is such that the chirality is conserved at the vertices. In the ultra-relativistic case, this implies that the helicity of the particles is conserved at the vertex. It also implies that in $e^+e^- \to \mu^+\mu^-$ scattering, the electron and the positron in the initial state must have opposite helicities, and so must the muon and antimuon in the final state. See Section 11.14 of the book for a detailed derivation. Hence we have:

$$L = 1, \qquad L_z = L_{z'} = \pm 1 \tag{10.97}$$

The state $|1,0\rangle$ is prohibited. We must incoherently add the two cross-sections with $L_z = +1$ and $L_{z'} = -1$ to find:

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e^+e^- \to \gamma^* \to \mu^+\mu^-) &\propto \frac{1}{2}\left(\left(D^1_{1,1}\right)^2 + \left(D^1_{1,-1}\right)^2\right) \\
&= \frac{1}{2}\left((1+\cos\theta)^2 + (1-\cos\theta)^2\right) \\
&= \frac{1}{2}\left(1 + 2\cos\theta + \cos^2\theta + 1 - 2\cos\theta + \cos^2\theta\right) \\
&= 1 + \cos^2\theta \qquad (10.98)
\end{aligned}$$

This is clearly different than the case of the scattering of scalar particles where we obtained:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\tilde{e}^+\tilde{e}^- \to \gamma^* \to \tilde{\mu}^+\tilde{\mu}^-) \propto \cos^2\theta \qquad (10.99)$$

The angular distribution gives us information on the spins of the interacting particles.

11 Computations in QED

11.1 Electron–muon scattering amplitudes with helicity spinors

We consider the following QED process $e^- + \mu^- \to e^- + \mu^-$.

(a) *Derive the amplitude at first order choosing p_1 and p_2 as the initial four-momenta for the e^- and the μ^-, and p_3, p_4 the corresponding final momenta.*

(b) *Show that in the center-of-mass frame with incoming and outgoing electron given by $p_1^\mu = (E_1, 0, 0, p)$ and $p_3^\mu = (E_1, p\sin\theta, 0, p\cos\theta)$, one has:*

$$
\begin{aligned}
(\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1)) &= 2\left(E_1\cos\frac{\theta}{2}, p\sin\frac{\theta}{2}, -ip\sin\frac{\theta}{2}, p\cos\frac{\theta}{2}\right) \\
(\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1)) &= 2\left(m_e\sin\frac{\theta}{2}, 0, 0, 0\right) \\
(\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1)) &= 2\left(E_1\cos\frac{\theta}{2}, p\sin\frac{\theta}{2}, ip\sin\frac{\theta}{2}, p\cos\frac{\theta}{2}\right) \\
(\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1)) &= -2\left(m_e\sin\frac{\theta}{2}, 0, 0, 0\right)
\end{aligned} \tag{11.1}
$$

where the arrows correspond to each helicity state of the spinor.

(c) *Write down the incoming and outgoing muon four-momenta p_2 and p_4, and the helicity eigenstate spinors $u_\downarrow(p_2)$, $u_\downarrow(p_4)$, $u_\uparrow(p_2)$, $u_\uparrow(p_4)$ (take the muon mass as M and the muon energy to be E_2). By comparing the forms of the muon and electron spinors, explain how the muon currents can be written down without any further calculation.*

$$
\begin{aligned}
(\bar{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2)) &= 2\left(E_2\cos\frac{\theta}{2}, -p\sin\frac{\theta}{2}, -ip\sin\frac{\theta}{2}, -p\cos\frac{\theta}{2}\right) \\
(\bar{u}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_2)) &= 2\left(M\sin\frac{\theta}{2}, 0, 0, 0\right) \\
(\bar{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2)) &= 2\left(E_2\cos\frac{\theta}{2}, -p\sin\frac{\theta}{2}, ip\sin\frac{\theta}{2}, -p\cos\frac{\theta}{2}\right) \\
(\bar{u}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_2)) &= -2\left(M\sin\frac{\theta}{2}, 0, 0, 0\right)
\end{aligned} \tag{11.2}
$$

(d) *Explain why some of the above currents vanish in the relativistic limit where the electron and muon mass can be neglected. Sketch the spin configurations which are allowed in this limit.*

(e) *Show that in the relativistic limit, the matrix element squared $|\mathcal{M}_{LL}|^2$ for the case where the incoming electron and incoming muon are both left-handed is given by:*

$$|\mathcal{M}_{LL}|^2 = \frac{4e^4 s^2}{(p_1 - p_3)^4} \tag{11.3}$$

where $s = (p_1 + p_2)^2$. Why is the result independent of θ?

(f) *Find a similar expression for the matrix element $|\mathcal{M}_{RL}|^2$ for a right-handed incoming electron and a left-handed incoming muon, and explain why it vanishes when $\theta = \pi$. Write down the corresponding results for $|\mathcal{M}_{LL}|^2$ and $|\mathcal{M}_{LR}|^2$.*

(g) *Show that, in the relativistic limit, the differential cross-section for unpolarized electron–muon scattering in the center-of-mass frame is:*

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\frac{4 + (1+\cos\theta)^2}{(1-\cos\theta)^2} \tag{11.4}$$

(h) *Show that the spin-averaged matrix element squared for unpolarized electron–muon scattering can be written as:*

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\frac{e^4}{(p_1 - p_3)^4} L^{\mu\nu} W_{\mu\nu} \tag{11.5}$$

where the spin-summed electron and muon tensors are given by:

$$\begin{aligned} L^{\mu\nu} &= \sum_{spins} \left(\bar{u}(p_3)\gamma^\mu u(p_1)\right)\left(\bar{u}(p_3)\gamma^\nu u(p_1)\right)^* \\ W_{\mu\nu} &= \sum_{spins} \left(\bar{u}(p_4)\gamma_\mu u(p_2)\right)\left(\bar{u}(p_4)\gamma_\nu u(p_2)\right)^* \end{aligned} \tag{11.6}$$

(i) *Using the currents from part (b), show that the components of electron tensor $L^{\mu\nu}$ are:*

$$\begin{Bmatrix} L_{00} & L_{01} & L_{02} & L_{03} \\ L_{10} & L_{11} & L_{12} & L_{13} \\ L_{20} & L_{21} & L_{22} & L_{23} \\ L_{30} & L_{31} & L_{32} & L_{33} \end{Bmatrix} = 8 \begin{Bmatrix} E_1^2 c^2 + m_e^2 s^2 & E_1 psc & 0 & E_1 pc^2 \\ E_1 psc & p^2 s^2 & 0 & p^2 sc \\ 0 & 0 & p^2 s^2 & 0 \\ E_1 pc^2 & p^2 sc & 0 & p^2 c^2 \end{Bmatrix} \text{ where } s = \sin\tfrac{\theta}{2} \text{ and } c = \cos\tfrac{\theta}{2}.$$

(j) *Verify that $L^{\mu\nu} = 4\left(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}\left(m_e^2 - p_1 \cdot p_3\right)\right)$.*

Solution:

a) We define the kinematics of the reaction as follows:

$$e^-(p_1^\mu, s) + \mu^-(p_2^\mu, r) \to e^-(p_3^\mu, s') + \mu^-(p_4^\mu, r') \tag{11.7}$$

where the energy–momentum 4-vectors are given by p_1^μ, p_2^μ, p_3^μ, p_4^μ and the corresponding spin states of the particles by s, r, s', r'. The QED tree-level Feynman diagram is shown in Figure 11.1.

By using the Feynman rules, we can directly construct the matrix element to find:

$$i\mathcal{M}_{e\mu\to e\mu} = \underbrace{(-ie)\bar{u}^{s'}(p_3)\gamma^\mu u^s(p_1)}_{e\gamma^\mu\text{vertex + external legs}} \underbrace{\left(\frac{-ig_{\mu\nu}}{q^2 + i\epsilon}\right)}_{\text{photon propagator}} \underbrace{(-ie)\bar{u}^{r'}(p_4)\gamma^\nu u^r(p_2)}_{e\gamma^\nu\text{vertex + external legs}} \tag{11.8}$$

where we have defined the momentum exchange $q \equiv p_1 - p_3$. We can contract the Lorentz indices to give:

$$i\mathcal{M}_{e\mu\to e\mu} = (ie^2)\frac{\left(\bar{u}^{s'}(p_3)\gamma^\mu u^s(p_1)\right)\left(\bar{u}^{r'}(p_4)\gamma_\mu u^r(p_2)\right)}{(p_1 - p_3)^2 + i\epsilon} \tag{11.9}$$

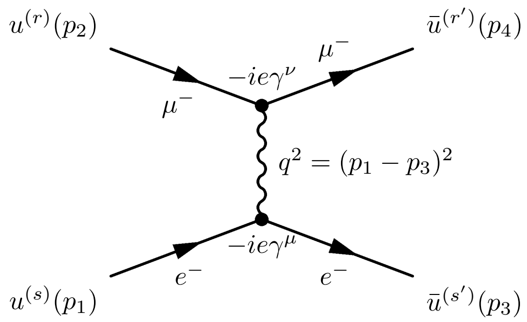

Figure 11.1 Electron-muon scattering between an electron and a muon: QED tree-level Feynman diagram.

b) A four momentum is defined by energy and 3-momentum, e.g. $p^\mu = (E, \vec{p})$. In the center-of-mass system (CMS), the 3-momenta of the incoming particles are equal in magnitude but opposed in direction:

$$\text{CMS:} \quad \vec{p}_e = -\vec{p}_\mu \quad \Longrightarrow \quad p \equiv |\vec{p}_e| = |\vec{p}_\mu| \tag{11.10}$$

Since the electron and muon have different rest masses, they have different energies in the CMS. We can define the z-axis along the direction of incoming particles. In the final state, we have the same situation where the 3-momenta of the outgoing particles are equal in magnitude but opposed in direction. The back-to-back pair is however rotated with respect to the initial state momenta by the scattering angle θ. By energy-momentum conservation, the energies of the initial and final state electrons are identical. The same for the muons. However, the electrons and the muons have different energies. We consider first the electron. We can describe this situation defining in cartesian coordinates ($p^\mu = (E, p_x, p_y, p_z)$):

$$p_1^\mu = (E_1, 0, 0, p) \quad \text{and} \quad p_3^\mu = (E_1, p\sin\theta, 0, p\cos\theta) \tag{11.11}$$

where $E_1 = \sqrt{p^2 + m_e^2}$ is the energy of the electrons. We have used the azimuthal symmetry to set the y projection of the 3-momenta to zero.

We now focus on the amplitude $i\mathcal{M}_{e\mu\to e\mu}$ and **analyse it as a function of the helicity state of the incoming and outgoing fermions**. The amplitude is proportional to e^2, as expected since we have two vertices in the Feynman diagram. The second part of the amplitude is the ratio of product of two vector current, i.e. of the form $\bar{u}\gamma^\mu u$ divided by the photon propagator. Each fermion can possess two helicity states, which we denote with the $\uparrow$ and $\downarrow$ signs. The helicity eigenstates of spinors for particles were defined Eq. (8.148) of the book (in spherical coordinates within the Pauli-Dirac representation):

$$u_\uparrow(E, p, \theta, \phi) = N \begin{pmatrix} c \\ se^{i\phi} \\ \alpha c \\ \alpha s e^{i\phi} \end{pmatrix} \quad \text{and} \quad u_\downarrow(E, p, \theta, \phi) = N \begin{pmatrix} -s \\ ce^{i\phi} \\ \alpha s \\ -\alpha c e^{i\phi} \end{pmatrix} \tag{11.12}$$

where

$$N = \sqrt{E+m}, \quad c = \cos(\theta/2), \quad s = \sin(\theta/2) \quad \text{and} \quad \alpha = \frac{p}{E+m} \tag{11.13}$$

For the initial state electron ($\theta = 0$, $\phi = 0$), we find:

$$u_\uparrow(p_1) = u_\uparrow(E_1, p, 0, 0) = N \begin{pmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{pmatrix} \quad \text{and} \quad u_\downarrow(p_1) = u_\downarrow(E_1, p, 0, 0) = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\alpha \end{pmatrix} \tag{11.14}$$

and similarly for the final state electron ($\theta = \theta$, $\phi = 0$):

$$u_\uparrow(p_3) = u_\uparrow(E_1, p, \theta, 0) = N \begin{pmatrix} c \\ s \\ \alpha c \\ \alpha s \end{pmatrix} \quad \text{and} \quad u_\downarrow(p_3) = u_\downarrow(E_1, p, \theta, 0) = N \begin{pmatrix} -s \\ c \\ \alpha s \\ -\alpha c \end{pmatrix} \tag{11.15}$$

We can simplify the spinors by introducing the following 2×2 block notation:

$$u_\uparrow(p_i) = N \begin{pmatrix} u_{i\uparrow} \\ \alpha u_{i\uparrow} \end{pmatrix} \quad \text{and} \quad u_\downarrow(p_i) = N \begin{pmatrix} u_{i\downarrow} \\ -\alpha u_{i\downarrow} \end{pmatrix} \tag{11.16}$$

where

$$u_{1\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_{1\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_{3\uparrow} = \begin{pmatrix} c \\ s \end{pmatrix}, \quad u_{3\downarrow} = \begin{pmatrix} -s \\ c \end{pmatrix} \tag{11.17}$$

Using the same 2×2 block notation, we can express the Dirac matrices in the Pauli-Dirac representation as (see Eq. (8.31) of the book):

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \text{and} \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \tag{11.18}$$

where σ_k are the 2×2 Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11.19}$$

We note that

$$\gamma^0\gamma^0 = (\gamma^0)^2 = \mathbb{1}, \quad \text{and} \quad \gamma^0\gamma^k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \tag{11.20}$$

We now want to explicitly calculate the vector current for the four helicity combinations $(\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1))$, $(\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1))$, $(\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1))$ and $(\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1))$. Hence, we need to calculate:

$$\bar{u}_a(p_3)\gamma^\mu u_b(p_1) = u_a^\dagger(p_3)\gamma^0\gamma^\mu u_b(p_1) = \left(\underbrace{u_a^\dagger(p_3)u_b(p_1)}_{\mu=0}, \underbrace{u_a^\dagger(p_3)\gamma^0\gamma^k u_b(p_1)}_{\mu=k} \right) \tag{11.21}$$

where $a =\uparrow,\downarrow$ and $b =\uparrow,\downarrow$ and where we used $\gamma^0\gamma^0 = \mathbb{1}$ and $k = 1, 2, 3$. Introducing the 2×2 notation, we find:

$$\begin{aligned} \bar{u}_a(p_3)\gamma^\mu u_b(p_1) &= N^2 \left(\begin{pmatrix} u_{3a}^\dagger & \pm\alpha u_{3a}^\dagger \end{pmatrix} \begin{pmatrix} u_{1b} \\ \pm\alpha u_{1b} \end{pmatrix}, \begin{pmatrix} u_{3a}^\dagger & \pm\alpha u_{3a}^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \begin{pmatrix} u_{1b} \\ \pm\alpha u_{1b} \end{pmatrix} \right) \\ &= N^2 \left(\begin{pmatrix} u_{3a}^\dagger & \pm\alpha u_{3a}^\dagger \end{pmatrix} \begin{pmatrix} u_{1b} \\ \pm\alpha u_{1b} \end{pmatrix}, \begin{pmatrix} u_{3a}^\dagger & \pm\alpha u_{3a}^\dagger \end{pmatrix} \begin{pmatrix} \pm\alpha\sigma_k u_{1b} \\ \sigma_k u_{1b} \end{pmatrix} \right) \end{aligned} \tag{11.22}$$

where the $+$ sign is for $\uparrow$ and $-$ sign for $\downarrow$ of the corresponding index a or b.

Now we explicitly compute:

$$\begin{aligned} u_{3\uparrow}^\dagger u_{1\uparrow} &= \begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c, \qquad u_{3\uparrow}^\dagger u_{1\downarrow} = \begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = s, \\ u_{3\downarrow}^\dagger u_{1\uparrow} &= \begin{pmatrix} -s & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -s, \quad u_{3\downarrow}^\dagger u_{1\downarrow} = \begin{pmatrix} -s & c \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c \end{aligned} \tag{11.23}$$

and

$$u_{3\uparrow}^{\dagger}\sigma_k u_{1\uparrow} = \begin{pmatrix} c & s \end{pmatrix} \sigma_k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = s \quad (k=1) \\ \begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} 0 \\ i \end{pmatrix} = is \ (k=2) \\ \begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c \quad (k=3) \end{cases} \tag{11.24}$$

$$u_{3\uparrow}^{\dagger}\sigma_k u_{1\downarrow} = \begin{pmatrix} c & s \end{pmatrix} \sigma_k \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{cases} \begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c \\ \begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} -i \\ \text{ă}0 \end{pmatrix} = -ic \\ \begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} 0 \\ \text{ă}-1 \end{pmatrix} = -s \end{cases}$$

$$u_{3\downarrow}^{\dagger}\sigma_k u_{1\uparrow} = \begin{pmatrix} -s & c \end{pmatrix} \sigma_k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} -s & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -s & c \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c \\ \begin{pmatrix} -s & c \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -s & c \end{pmatrix} \begin{pmatrix} 0 \\ i \end{pmatrix} = ic \\ \begin{pmatrix} -s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -s & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -s \end{cases}$$

$$u_{3\downarrow}^{\dagger}\sigma_k u_{1\downarrow} = \begin{pmatrix} -s & c \end{pmatrix} \sigma_k \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{cases} \begin{pmatrix} -s & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -s & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -s \\ \begin{pmatrix} -s & c \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -s & c \end{pmatrix} \begin{pmatrix} -i \\ \text{ă}0 \end{pmatrix} = is \\ \begin{pmatrix} -s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -s & c \end{pmatrix} \begin{pmatrix} 0 \\ \text{ă}-1 \end{pmatrix} = -c \end{cases}$$

Hence, we have for instance:

$$\begin{aligned} \bar{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) &= N^2 \left(\begin{pmatrix} u_{3\uparrow}^{\dagger} & +\alpha u_{3\uparrow}^{\dagger} \end{pmatrix} \begin{pmatrix} u_{1\uparrow} \\ +\alpha u_{1\uparrow} \end{pmatrix}, \begin{pmatrix} u_{3\uparrow}^{\dagger} & +\alpha u_{3\uparrow}^{\dagger} \end{pmatrix} \begin{pmatrix} +\alpha\sigma_k u_{1\uparrow} \\ \sigma_k u_{1\uparrow} \end{pmatrix} \right) \\ &= N^2 \left(u_{3\uparrow}^{\dagger}u_{1\uparrow} + (+\alpha)u_{3\uparrow}^{\dagger}(+\alpha)u_{1\uparrow}, u_{3\uparrow}^{\dagger}\sigma_k(+\alpha)u_{1\uparrow} + (+\alpha)u_{3\uparrow}^{\dagger}\sigma_k u_{1\uparrow} \right) \end{aligned} \tag{11.25}$$

where we have written the $(+\alpha)$ terms close to the corresponding spinors to keep track of the signs. We obtain:

$$\bar{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = N^2 \left((1+\alpha^2)c, 2\alpha s, 2i\alpha s, 2\alpha c \right) \tag{11.26}$$

We note that:

$$(1 \pm \alpha^2) = 1 \pm \frac{p^2}{(E_1+m_e)^2} = \frac{(E_1+m_e)^2 \pm p^2}{(E_1+m_e)^2} = \frac{E_1^2 + 2m_eE_1 + E_1^2 \pm p^2}{(E_1+m_e)^2} = \frac{E_1^2 + 2m_eE_1 + m_e^2 \pm p^2}{(E_1+m_e)^2} \tag{11.27}$$

Hence:

$$(1+\alpha^2) = \frac{2E_1^2 + 2m_e E_1}{(E_1+m_e)^2} = \frac{2E_1(E_1+m_e)}{(E_1+m_e)^2} = \frac{2E_1}{(E_1+m_e)} \tag{11.28}$$

and similarly:

$$(1-\alpha^2) = \frac{2m_e^2 + 2m_e E_1}{(E_1+m_e)^2} = \frac{2m_e(E_1+m_e)}{(E_1+m_e)^2} = \frac{2m_e}{(E_1+m_e)} \tag{11.29}$$

Consequently, we arrive at the desired result:

$$\begin{aligned}\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) &= (E_1+m_e)\left(\frac{2E_1}{E_1+m_e}c, \frac{2p}{E_1+m_e}s, \frac{2p}{E_1+m_e}is, \frac{2p}{E_1+m_e}c\right) \\ &= 2\,(E_1 c, ps, ips, pc)\end{aligned} \tag{11.30}$$

Similarly, we obtain for the other combinations:

$$\begin{aligned}\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) &= N^2\left(u_{3\uparrow}^\dagger u_{1\uparrow} + (+\alpha)u_{3\uparrow}^\dagger(-\alpha)u_{1\downarrow}, u_{3\uparrow}^\dagger\sigma_k(-\alpha)u_{1\downarrow} + (+\alpha)u_{3\uparrow}^\dagger\sigma_k u_{1\downarrow}\right) \\ &= (E_1+m_e)\left((1-\alpha^2)u_{3\uparrow}^\dagger u_{1\downarrow}, 0, 0, 0\right) \\ &= 2\,(m_e s, 0, 0, 0)\end{aligned} \tag{11.31}$$

and

$$\begin{aligned}\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) &= N^2\left(u_{3\downarrow}^\dagger u_{1\uparrow} + (-\alpha)u_{3\downarrow}^\dagger(+\alpha)u_{1\uparrow}, u_{3\downarrow}^\dagger\sigma_k(+\alpha)u_{1\uparrow} + (-\alpha)u_{3\downarrow}^\dagger\sigma_k u_{1\uparrow}\right) \\ &= (E_1+m_e)\left((1-\alpha^2)u_{3\downarrow}^\dagger u_{1\uparrow}, 0, 0, 0\right) \\ &= 2\,(-m_e s, 0, 0, 0)\end{aligned} \tag{11.32}$$

and

$$\begin{aligned}\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) &= N^2\left(u_{3\downarrow}^\dagger u_{1\downarrow} + (-\alpha)u_{3\downarrow}^\dagger(-\alpha)u_{1\downarrow}, u_{3\downarrow}^\dagger\sigma_k(-\alpha)u_{1\downarrow} + (-\alpha)u_{3\downarrow}^\dagger\sigma_k u_{1\downarrow}\right) \\ &= (E_1+m_e)\left((1+\alpha^2)u_{3\downarrow}^\dagger u_{1\downarrow}, -2\alpha u_{3\downarrow}^\dagger\sigma_k u_{1\downarrow}\right) \\ &= 2\,(E_1 c, ps, -ips, pc)\end{aligned} \tag{11.33}$$

c) As before, we analyse the kinematics in the center-of-mass system. The 3-momenta of the electron and muons are equal in magnitude but opposed in direction. Hence, we can compute the currents for the muons using the results derived for the electrons, but with the replacement:

$$p^\mu(E_1, \vec{p}) \quad \Longrightarrow \quad p^\mu(E_2, -\vec{p}) \tag{11.34}$$

where $E_2 = \sqrt{p^2 + M^2}$ is the energy of the muons. We can then describe this situation defining starting from $p_1^\mu = (E_1, 0, 0, p)$ and $p_3^\mu = (E_1, p\sin\theta, 0, p\cos\theta)$ as:

$$p_2^\mu = (E_2, 0, 0, -p) \quad \text{and} \quad p_4^\mu = (E_2, -p\sin\theta, 0, -p\cos\theta) \tag{11.35}$$

In order to define the helicity eigenstates, we note that in spherical coordinates ($\vec{p} = (p, \theta, \phi)$), we have:

$$p = |\vec{p}|, \quad \theta = \arccos\frac{p_z}{p} \quad \phi = \operatorname{atan2}(p_y, p_x) \tag{11.36}$$

where the inverse tangent must be suitably defined, taking into account the correct quadrant of (x, y):

$$\operatorname{atan} 2(y,x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases} \tag{11.37}$$

We have used the azimuthal symmetry to set the y projection of the 3-momenta to zero, hence $\vec{p} = (p_x, 0, p_z)$. In cartesian coordinates, the transformation is $\vec{p}(p_x, 0, p_z) \longrightarrow -\vec{p} = (-p_x, 0, -p_z)$. In spherical coordinates, the transformation is $\vec{p}(p, \theta, 0) \longrightarrow -\vec{p}(p, \theta, \phi)$, which is then equivalent to:

$$p \longrightarrow p, \quad \theta \longrightarrow \arccos\frac{-p_z}{p}, \quad \phi \longrightarrow \operatorname{atan} 2(0, -p_x) \tag{11.38}$$

Consequently:

$$p \longrightarrow p, \quad \theta \longrightarrow \pi - \theta, \quad \phi = 0 \longrightarrow \phi = \pi \tag{11.39}$$

We substitute these results in the helicity eigenstates to find for the initial state muon $\vec{p} = (p, \pi, \pi)$. We have $c = \cos(\pi/2) = 0$, $s = \sin(\pi/2) = 1$ and $e^{i\pi} = -1$. Hence:

$$u_\uparrow(p_2) = N\begin{pmatrix} c \\ se^{i\phi} \\ \alpha c \\ \alpha s e^{i\phi} \end{pmatrix} = N\begin{pmatrix} 0 \\ -1 \\ 0 \\ -\alpha \end{pmatrix} \quad \text{and} \quad u_\downarrow(p_2) = N\begin{pmatrix} -s \\ ce^{i\phi} \\ \alpha s \\ -\alpha c e^{i\phi} \end{pmatrix} = N\begin{pmatrix} -1 \\ 0 \\ \alpha \\ 0 \end{pmatrix} \tag{11.40}$$

where $N = \sqrt{E_2 + M}$ and $\alpha = p/(E_2 + M)$. For the final state muon $\vec{p} = (p, \pi - \theta, \pi)$ we have:

$$u_\uparrow(p_4) = N\begin{pmatrix} \cos\left(\frac{\pi-\theta}{2}\right) \\ -\sin\left(\frac{\pi-\theta}{2}\right) \\ \alpha\cos\left(\frac{\pi-\theta}{2}\right) \\ -\alpha\sin\left(\frac{\pi-\theta}{2}\right) \end{pmatrix} = N\begin{pmatrix} s \\ -c \\ \alpha s \\ -\alpha c \end{pmatrix} \quad \text{and} \quad u_\downarrow(p_4) = N\begin{pmatrix} -\sin\left(\frac{\pi-\theta}{2}\right) \\ -\cos\left(\frac{\pi-\theta}{2}\right) \\ \alpha\sin\left(\frac{\pi-\theta}{2}\right) \\ \alpha\cos\left(\frac{\pi-\theta}{2}\right) \end{pmatrix} = N\begin{pmatrix} -c \\ -s \\ \alpha c \\ \alpha s \end{pmatrix} \tag{11.41}$$

We can compare these spinors to the results obtained for the electrons. We (perhaps not surprisingly) find that:

$$u_\uparrow(p_2) \sim -u_\downarrow(p_1, -\alpha) = N\begin{pmatrix} 0 \\ -1 \\ 0 \\ -\alpha \end{pmatrix}, \quad u_\downarrow(p_2) \sim -u_\uparrow(p_1, -\alpha) = N\begin{pmatrix} -1 \\ 0 \\ \alpha \\ 0 \end{pmatrix} \tag{11.42}$$

and

$$u_\uparrow(p_4) \sim -u_\downarrow(p_3, -\alpha) = N\begin{pmatrix} s \\ -c \\ \alpha s \\ -\alpha c \end{pmatrix}, \quad u_\downarrow(p_4) \sim -u_\uparrow(p_3, -\alpha) = N\begin{pmatrix} -c \\ -s \\ \alpha c \\ \alpha s \end{pmatrix} \tag{11.43}$$

In other words, **we can pass from electrons spinors to muon spinors by the replacements $\alpha \to -\alpha$, $m_e \to M$, and swapping the $\uparrow\to\downarrow$ and $\downarrow\to\uparrow$.** This result can be directly applied to write down the muon currents without any further calculations, by noting that changing the sign of α is equivalent to change the sign of p and that E_1 should be replaced by E_2. Hence:

$$\begin{aligned} (\bar{u}_\uparrow(p_4)\gamma^\nu u_\uparrow(p_2)) &\sim (\bar{u}_\downarrow(p_3, -p)\gamma^\nu u_\downarrow(p_1, -p)) = 2\,(E_2 c, (-p)s, -i(-p)s, (-p)c) \\ &= 2\,(E_2 c, -ps, ips, -pc) \end{aligned} \tag{11.44}$$

and

$$\begin{aligned}(\bar{u}_\downarrow(p_4)\gamma^\nu u_\downarrow(p_2)) &\sim (\bar{u}_\uparrow(p_3,-p)\gamma^\nu u_\uparrow(p_1,-p)) = 2\,(E_2c,(-p)s,i(-p)s,(-p)c)\\ &= 2\,(E_2c,-ps,-ips,-pc) \end{aligned}\tag{11.45}$$

and similarly

$$\begin{aligned}(\bar{u}_\uparrow(p_4)\gamma^\nu u_\downarrow(p_2)) &= 2\,(Ms,0,0,0)\\ (\bar{u}_\downarrow(p_4)\gamma^\nu u_\uparrow(p_2)) &= -2\,(Ms,0,0,0)\end{aligned}\tag{11.46}$$

d) In the ultra-relativistic limit, the electron and muon masses can be neglected and $E_1 = E_2 = p$. For the electron currents, we directly obtain:

$$(\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1)) = 2E\,(c,s,is,c)\,,\quad (\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1)) = 2E\,(c,s,-is,c)\tag{11.47}$$

and

$$(\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1)) = 0,\quad (\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1)) = 0\tag{11.48}$$

Similarly for the muon currents:

$$(\bar{u}_\uparrow(p_4)\gamma^\nu u_\uparrow(p_2)) = 2E\,(c,-s,is,-c)\,,\quad (\bar{u}_\downarrow(p_4)\gamma^\nu u_\downarrow(p_2)) = 2E\,(c,-s,-is,-c)\tag{11.49}$$

and

$$(\bar{u}_\uparrow(p_4)\gamma^\nu u_\downarrow(p_2)) = 0,\quad (\bar{u}_\downarrow(p_4)\gamma^\nu u_\uparrow(p_2)) = 0\tag{11.50}$$

The currents where the helicity flips are vanishing. This is just the case of the general result that helicity is conserved at the vertices in the ultra-relativistic limit. This result is rooted in the chiral structure of QED, which is discussed in Section 10.17 of the book. Indeed, the vector form of the currents can be written as (see Eq. (10.165) of the book):

$$j^\mu = \bar{u}\gamma^\mu u = \overline{u_R}\gamma^\mu u_R + \overline{u_L}\gamma^\mu u_L\tag{11.51}$$

where u_R and u_L are the right-handed and left-handed chiral projections of the spinor u. Since in the ultra-relativistic limit, the chirality state coincide with the helicity states, this results translates directly into the conservation of helicities at very high energies.

e) We want to compute the matrix element squared $|\mathcal{M}_{LL}|^2$ for the case where the incoming electron and incoming muon are both left-handed. In general, we have:

$$\mathcal{M}_{s,r} \propto \sum_{s,r'=L,R}\left(\bar{u}^{s'}(p_3)\gamma^\mu u^s(p_1)\right)\left(\bar{u}^{r'}(p_4)\gamma_\mu u^r(p_2)\right)\tag{11.52}$$

However, we have recalled under part d) that the chiral structure of QED only allows for same chiral states at the vertex (i.e. $j^\mu = \overline{u_R}\gamma^\mu u_R + \overline{u_L}\gamma^\mu u_L$). Therefore, the initial state handednesses determine the final state ones, and the only non-vanishing amplitudes are:

$$\underbrace{\mathcal{M}_{LL\to LL}}_{=\mathcal{M}_{LL}},\quad \underbrace{\mathcal{M}_{LR\to LR}}_{=\mathcal{M}_{LR}},\quad \underbrace{\mathcal{M}_{RL\to RL}}_{=\mathcal{M}_{RL}},\quad \underbrace{\mathcal{M}_{RR\to RR}}_{=\mathcal{M}_{RR}}\tag{11.53}$$

In the ultra-relativistic case, there is a direct equivalence between chiral and helicity states (see Section 8.24 of the book). For particles, we have (see Eq. (8.250) of the book):

$$u_L \leftrightarrow u_\downarrow \quad\text{and}\quad u_R \leftrightarrow u_\uparrow\tag{11.54}$$

Therefore, we can write:

$$\begin{aligned}\mathcal{M}_{LL} \propto (\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1))\, g_{\mu\nu}\, (\bar{u}_\downarrow(p_4)\gamma^\nu u_\downarrow(p_2)) &= 2E\,(c,s,-is,c)^\mu\, g_{\mu\nu} 2E\,(c,-s,-is,-c)^\nu \\ &= 4E^2(c^2+s^2+s^2+c^2) = 8E^2 \qquad (11.55)\end{aligned}$$

Hence:

$$|\mathcal{M}_{LL}|^2 = \left|(ie^2)\frac{8E^2}{(p_1-p_3)^2}\right|^2 = \frac{64e^4E^4}{(p_1-p_3)^4} \qquad (11.56)$$

We now introduce the Mandelstam variable s (note that unfortunately we as well used s for $s = \sin\frac{\theta}{2}$ but not in the following!):

$$s = (p_1+p_2)^2 = ((E,\vec{p}) + (E,-\vec{p}))^2 = (2E)^2 = 4E^2 \qquad (11.57)$$

Hence:

$$|\mathcal{M}_{LL}|^2 = \frac{4e^4(4E^2)^2}{(p_1-p_3)^4} = \frac{4e^4s^2}{(p_1-p_3)^4} \qquad (11.58)$$

The result is independent of the scattering angle θ, since its dependence canceled in the product of the vector currents. The reason is that the electron and the muon have the same helicity, and since their 3-momenta are opposite to each other, then the spins of the electron and the muon are opposite. The total angular momentum therefore vanishes. In this configuration, the vector current yields an interaction in the $L=0$ state, which therefore is isotropic in the center-of-mass system.

f) We want to compute the matrix element squared $|\mathcal{M}_{RL}|^2$ for a right-handed incoming electron and a left-handed incoming muon. For the same reasons mentioned above, we only need to consider the process $RL \to RL$. Hence:

$$\begin{aligned}\mathcal{M}_{RL} \propto (\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1))\, g_{\mu\nu}\, (\bar{u}_\downarrow(p_4)\gamma^\nu u_\downarrow(p_2)) &= 2E\,(c,s,is,c)^\mu\, g_{\mu\nu} 2E\,(c,-s,-is,-c)^\nu \\ &= 4E^2(c^2+s^2-s^2+c^2) = 8E^2c^2 \\ &= 4E^2(1+\cos\theta) \qquad (11.59)\end{aligned}$$

Consequently, we obtain:

$$|\mathcal{M}_{RL}|^2 = \frac{e^4(4E^2(1+\cos\theta))^2}{(p_1-p_3)^4} = \frac{e^4s^2(1+\cos\theta)^2}{(p_1-p_3)^4} \qquad (11.60)$$

We note that the electron and the muon have the opposite helicities, and since their 3-momenta are opposite to each other, then the spins of the electron and the muon are pointing on the same direction. In this configuration, the vector current yields an interaction in the $L=1$ state. We must consider the photon spin eigenstate relative to the initial and final state axes, which we label z and z'. See Figure 11.2. The amplitude will be proportional to the overlap between the wave-functions, which are related by the rotational covariance of angular momentum eigenstates. The change of quantization frame is obtained with the help of the Wigner matrices $D^J_{L_zL'_z}$:

$$|L, L'_z\rangle = \sum_{L_z} D^J_{L_zL'_z}\, |L, L_z\rangle \qquad (11.61)$$

For $L=1$, the Wigner matrices are given by:

$$D^1_{1,1} = \frac{1+\cos\theta}{2}, \quad D^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}, \quad D^1_{1,-1} = \frac{1-\cos\theta}{2}, \quad D^1_{0,0} = \cos\theta \qquad (11.62)$$

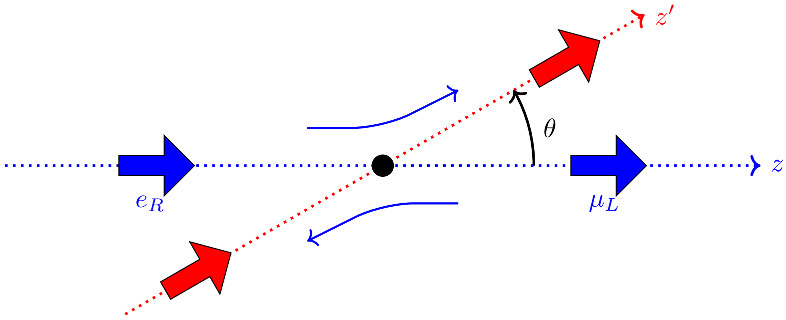

Figure 11.2 Spin configurations in the $\mathcal{M}_{RL}$ amplitude.

Hence, we expect the amplitude $\mathcal{M}_{RL}$ and its squared to be proportional to:

$$\mathcal{M}_{RL} \propto D^1_{1,1} = 1 + \cos\theta \quad \Longrightarrow \quad \frac{d\sigma}{d\Omega} \propto |\mathcal{M}_{RL}|^2 = (1 + \cos\theta)^2 \tag{11.63}$$

The amplitude vanishes for $\theta = \pi$ due to angular momentum conservation (compare with Figure 11.13 of the book).

We now compute the remaining two configurations. We can readily find that:

$$\begin{aligned}\mathcal{M}_{RR} \propto (\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1))\, g_{\mu\nu}\, (\bar{u}_\uparrow(p_4)\gamma^\nu u_\uparrow(p_2)) &= 2E\,(c, s, is, c)^\mu\, g_{\mu\nu} 2E\,(c, -s, is, -c)^\nu \\ &= 4E^2(c^2 + s^2 + s^2 + c^2) = 8E^2 \end{aligned} \tag{11.64}$$

Hence:

$$|\mathcal{M}_{LL}|^2 = |\mathcal{M}_{RR}|^2 = \frac{4e^4 s^2}{(p_1 - p_3)^4} \tag{11.65}$$

For $\mathcal{M}_{LR}$ we note that the reaction $LR \to LR$ and $RL \to LR$ are related by the parity transformation. Since QED conserves parity, the two amplitudes must be identical. Consequently, we obtain:

$$|\mathcal{M}_{RL}|^2 = |\mathcal{M}_{LR}|^2 = \frac{e^4 s^2 (1 + \cos\theta)^2}{(p_1 - p_3)^4} \tag{11.66}$$

(the same argument also holds for $|\mathcal{M}_{LL}|^2$ and $|\mathcal{M}_{RR}|^2$).

g) We want to compute the differential cross-section for unpolarised electron muon scattering in the centre of mass frame. In this case, we need to average over the spin configurations of the initial state and add over the ones of the final state (as discussed in Section 11.1 of the book). Hence the unpolarised spin-averaged matrix element is equal to:

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \sum_{s,r,s',r'} |\mathcal{M}_{s,r,s',r'}|^2 = \frac{1}{4}\left(|\mathcal{M}_{LL}|^2 + |\mathcal{M}_{RL}|^2 + |\mathcal{M}_{LR}|^2 + |\mathcal{M}_{RR}|^2\right) \\ &= \frac{e^4 s^2}{4\,(p_1 - p_3)^4}\left(8 + 2(1 + \cos\theta)^2\right) = \frac{e^4 s^2}{2\,(p_1 - p_3)^4}\left(4 + (1 + \cos\theta)^2\right) \end{aligned} \tag{11.67}$$

Furthermore, we recall that in the relativistic limit, we have $p_1^\mu = (E, 0, 0, E)$ and $p_3^\mu = (E, E\sin\theta, 0, E\cos\theta)$, hence:

$$\begin{aligned}(p_1 - p_3)^2 &= (0, -E\sin\theta, 0, E(1 - \cos\theta))^2 = E^2 \sin^2\theta + E^2(1 - \cos\theta)^2 \\ &= E^2\left(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\right) = 2E^2(1 - \cos\theta) = \frac{s}{2}(1 - \cos\theta) \end{aligned} \tag{11.68}$$

Hence:

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4 s^2}{2\left(\frac{s}{2}(1-\cos\theta)\right)^2}\left(4+(1+\cos\theta)^2\right) = 2e^4\left(\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}\right) \tag{11.69}$$

We include the phase space factor using Eq. (5.145) of the book and arrive at the differential cross-section:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CMS} = \frac{1}{64\pi^2 s}\langle|\mathcal{M}|^2\rangle = \frac{e^4}{32\pi^2 s}\left(\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}\right) = \frac{\alpha^2}{2s}\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2} \tag{11.70}$$

where we introduced the fine structure constant $\alpha = e^2/4\pi$. This result is equivalent to Eq. (11.152) of the book. However, it was calculated using explicit helicity spinors, while in the book we used the Casimir's trick and the trace theorems to reach the same result.

h) We follow the discussion in Section 11.10 of the book. We start from the amplitude for the process, where for clarify we remove the spin state indices:

$$\mathcal{M} = \frac{e^2}{(p_1-p_3)^2}\left(\bar{u}(p_3)\gamma^\mu u(p_1)\right)\left(\bar{u}(p_4)\gamma_\mu u(p_2)\right) \tag{11.71}$$

We note that the complex conjugate of the amplitude is simply equal to (we change the Lorentz index to ν) since the product $\bar{u}\gamma^\mu u$ for a given index ν is just a c-number:

$$\mathcal{M}^* = \frac{e^2}{(p_1-p_3)^2}\left(\bar{u}(p_3)\gamma^\nu u(p_1)\right)^*\left(\bar{u}(p_4)\gamma_\nu u(p_2)\right)^* \tag{11.72}$$

Hence, the spin-averaged matrix element squared for unpolarised electron muon scattering can be written as:

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{e^4}{(p_1-p_3)^4}\left(\frac{1}{2}\right)\sum_s\left(\frac{1}{2}\right)\sum_r\sum_{s',r'}\mathcal{M}\mathcal{M}^* \\
&= \frac{1}{4}\frac{e^4}{(p_1-p_3)^4}\sum_{s,r,s',r'}\left(\bar{u}(p_3)\gamma^\mu u(p_1)\right)\left(\bar{u}(p_3)\gamma^\nu u(p_1)\right)^*\left(\bar{u}(p_4)\gamma_\mu u(p_2)\right)\left(\bar{u}(p_4)\gamma_\nu u(p_2)\right)^* \\
&= \frac{1}{4}\frac{e^4}{(p_1-p_3)^4}L^{\mu\nu}W_{\mu\nu}
\end{aligned} \tag{11.73}$$

where

$$\begin{aligned}
L^{\mu\nu} &\equiv \sum_{s,s'}\left(\bar{u}(p_3)\gamma^\mu u(p_1)\right)\left(\bar{u}(p_3)\gamma^\nu u(p_1)\right)^* \\
W_{\mu\nu} &\equiv \sum_{r,r'}\left(\bar{u}(p_4)\gamma_\mu u(p_2)\right)\left(\bar{u}(p_4)\gamma_\nu u(p_2)\right)^*
\end{aligned} \tag{11.74}$$

i) We first note that the tensor $L^{\mu\nu}$ is symmetric by construction, i.e. $L^{\mu\nu} = L^{\nu\mu}$. Hence, we just need to compute the 10 terms L^{00}, L^{01}, L^{02}, L^{03}, L^{11}, L^{12}, L^{13}, L^{22}, L^{23} and L^{33}. We work in the helicity basis and recall that:

$$\begin{aligned}
&\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2\left(E_1 c, ps, ips, pc\right) \quad &&\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2\left(E_1 c, ps, -ips, pc\right) \\
&\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2\left(m_e s, 0, 0, 0\right) &&\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2\left(-m_e s, 0, 0, 0\right)
\end{aligned}$$

Hence, we have:

$$L_{00} = \underbrace{(2E_1c)(2E_1c)^*}_{\uparrow\uparrow} + \underbrace{(2m_es)(2m_es)^*}_{\uparrow\downarrow} + \underbrace{(-2m_es)(-2m_es)^*}_{\downarrow\uparrow} + \underbrace{(2E_1c)(2E_1c)^*}_{\downarrow\downarrow} = 8\left(E_1^2c^2 + m_e^2s^2\right) \tag{11.75}$$

and similarly

$$\begin{aligned}
L_{01} &= (2E_1c)(2ps)^* + (2E_1c)(2ps)^* = 8E_1psc \\
L_{02} &= (2E_1c)(2ips)^* + (2E_1c)(-2ips)^* = 0 \\
L_{03} &= (2E_1c)(pc)^* + (2E_1c)(pc)^* = 8E_1pc^2 \\
L_{11} &= (2ps)(2ps)^* + (2ps)(2ps)^* = 8p^2s^2 \\
L_{12} &= (2ps)(2ips)^* + (2ps)(-2ips)^* = 0 \\
L_{13} &= (2ps)(pc)^* + (2ps)(pc)^* = 8p^2sc \\
L_{22} &= (2ips)(2ips)^* + (-2ips)(-2ips)^* = 8p^2s^2 \\
L_{23} &= (2ips)(pc)^* + (-2ips)(pc)^* = 0 \\
L_{33} &= (2pc)(2pc)^* + (2pc)(2pc)^* = 8p^2c^2
\end{aligned} \tag{11.76}$$

So finally:

$$\begin{Bmatrix} L_{00} & L_{01} & L_{02} & L_{03} \\ L_{10} & L_{11} & L_{12} & L_{13} \\ L_{20} & L_{21} & L_{22} & L_{23} \\ L_{30} & L_{31} & L_{32} & L_{33} \end{Bmatrix} = 8 \begin{Bmatrix} E_1^2c^2 + m_e^2s^2 & E_1psc & 0 & E_1pc^2 \\ E_1psc & p^2s^2 & 0 & p^2sc \\ 0 & 0 & p^2s^2 & 0 \\ E_1pc^2 & p^2sc & 0 & p^2c^2 \end{Bmatrix} \tag{11.77}$$

j) We recall the kinematics of the electrons $p_1^\mu = (E_1, 0, 0, p)$ and $p_3^\mu = (E_1, p\sin\theta, 0, p\cos\theta)$. Starting from the result:

$$L^{\mu\nu} = 4\left(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}\left(m_e^2 - p_1 \cdot p_3\right)\right) \tag{11.78}$$

we note that in our case:

$$m_e^2 - p_1 \cdot p_3 = m_e^2 - (E_1)^2 + p^2\cos\theta = -p^2(1-\cos\theta) = -2p^2s^2 \tag{11.79}$$

Using $L^{\mu\nu} = 4\left(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}\left(m_e^2 - p_1 \cdot p_3\right)\right)$, we therefore have:

$$\begin{aligned}
L^{00} &= 4\left(p_1^0p_3^0 + p_3^0p_1^0 + g^{00}\left(-2p^2s^2\right)\right) = 4\left(2E_1^2 - 2p^2s^2\right) = 8\left(E_1^2 - (E_1^2 - m_e^2)s^2\right) \\
&= 8\left(E_1^2\left(1-s^2\right) - (-m_e^2)s^2\right) = 8\left(E_1^2c^2 + m_e^2s^2\right)
\end{aligned} \tag{11.80}$$

Similarly, noting that $\cos\theta = 1 - 2s^2$ and that $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-(1-2s^2)^2} = \sqrt{4s^2(1-s^2)} = 2sc$:

$$\begin{aligned}
L^{01} &= 4\left(p_1^0p_3^1 + p_3^0p_1^1\right) = 4\left(E_1p\sin\theta + 0\right) = 8E_1psc \\
L^{02} &= 4\left(p_1^0p_3^2 + p_3^0p_1^2\right) = 0 \\
L^{03} &= 4\left(p_1^0p_3^3 + p_3^0p_1^3\right) = 4\left(E_1p\cos\theta + E_1p\right) = 4E_1p(\cos\theta+1) = 8E_1pc^2 \\
L^{11} &= 4g^{11}\left(-2p^2s^2\right) = 8p^2s^2 \\
L^{12} &= 4\left(p_1^1p_3^2 + p_3^1p_1^2\right) = 0 \\
L^{13} &= 4\left(p_1^1p_3^3 + p_3^1p_1^3\right) = 4\left(0 + p^2\sin\theta\right) = 8p^2sc \\
L^{22} &= 4g^{22}\left(-2p^2s^2\right) = 8p^2s^2 \\
L^{23} &= 4\left(p_1^2p_3^3 + p_3^2p_1^3\right) = 0 \\
L^{33} &= 4\left(2p_1^3p_3^3 + g^{33}\left(-2p^2s^2\right)\right) = 8p^2\cos\theta + 8p^2s^2 = 8p^2\left(1-2s^2\right) + 8p^2s^2 = 8p^2c^2
\end{aligned} \tag{11.81}$$

where we used $g^{00} = 1$, $g^{kk} = -1$ ($k = 1, 2, 3$), otherwise $g^{\mu\nu} = 0$ ($\mu \neq \nu$).

So we have verified that the spin-summed 4×4 electron current tensor $L^{\mu\nu}$ which we somewhat painfully found with the help of the explicit spinors calculations, can be in fact written in a very compact form involving on the kinematical four-vectors. The spin dependence and the spinors have totally disappeared. Similarly, we can write the spin-summed muon current tensor by replacing $p_1^\mu \to p_2^\mu$, $p_3^\mu \to p_4^\mu$, and $m_e \to M$:

$$W^{\mu\nu} = 4\left(p_2^\mu p_4^\nu + p_4^\mu p_2^\nu + g^{\mu\nu}\left(M^2 - p_2\cdot p_4\right)\right) \tag{11.82}$$

Consequently if neglect m_e relative to M, we find:

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\frac{e^4}{(p_1-p_3)^4}4\left(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu - g^{\mu\nu}(p_1\cdot p_3)\right)4\left(p_{\mu2}p_{\nu4} + p_{\mu4}p_{\nu2} + g_{\mu\nu}\left(M^2 - p_2\cdot p_4\right)\right)\\
&= \frac{4e^4}{(p_1-p_3)^4}\left(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu - g^{\mu\nu}(p_1\cdot p_3)\right)\left(p_{\mu2}p_{\nu4} + p_{\mu4}p_{\nu2} + g_{\mu\nu}\left(M^2 - p_2\cdot p_4\right)\right)\\
&= \frac{4e^4}{(p_1-p_3)^4}\Big[\left(p_1^\mu p_3^\nu\right)\left(p_{\mu2}p_{\nu4} + p_{\mu4}p_{\nu2} + g_{\mu\nu}\left(M^2 - p_2\cdot p_4\right)\right)\\
&\quad + \left(p_3^\mu p_1^\nu\right)\left(p_{\mu2}p_{\nu4} + p_{\mu4}p_{\nu2} + g_{\mu\nu}\left(M^2 - p_2\cdot p_4\right)\right)\\
&\quad - g^{\mu\nu}(p_1\cdot p_3)\left(p_{\mu2}p_{\nu4} + p_{\mu4}p_{\nu2} + g_{\mu\nu}\left(M^2 - p_2\cdot p_4\right)\right)\Big]\\
&= \frac{4e^4}{(p_1-p_3)^4}\Big[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_3\cdot p_2) + (p_1\cdot p_3)\left(M^2 - p_2\cdot p_4\right)\\
&\quad (p_3\cdot p_2)(p_1\cdot p_4) + (p_3\cdot p_4)(p_1\cdot p_2) + (p_3\cdot p_1)\left(M^2 - p_2\cdot p_4\right)\\
&\quad - (p_1\cdot p_3)\left((p_2\cdot p_4) + (p_4\cdot p_2) + 4\left(M^2 - p_2\cdot p_4\right)\right)\Big]\\
&= \frac{4e^4}{(p_1-p_3)^4}\Big[2(p_1\cdot p_2)(p_3\cdot p_4) + 2(p_1\cdot p_4)(p_3\cdot p_2) + 2(p_1\cdot p_3)\left(M^2 - p_2\cdot p_4\right)\\
&\quad - (p_1\cdot p_3)\left(2(p_2\cdot p_4) + 4\left(M^2 - p_2\cdot p_4\right)\right)\Big]\\
&= \frac{4e^4}{(p_1-p_3)^4}\Big[2(p_1\cdot p_2)(p_3\cdot p_4) + 2(p_1\cdot p_4)(p_3\cdot p_2) + 2(p_1\cdot p_3)\left(M^2 - p_2\cdot p_4\right)\\
&\quad - 2(p_1\cdot p_3)\left(2M^2 - p_2\cdot p_4\right)\Big]\\
&= \frac{8e^4}{(p_1-p_3)^4}\Big[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_3\cdot p_2) - M^2(p_1\cdot p_3)\Big]
\end{aligned} \tag{11.83}$$

where we recovered Eq. (11.126) of the book which was obtained in a totally different way using Casimir's trick and the trace theorems! □

11.2 Mott scattering in a different way

Show that the Mott scattering differential cross-section can be derived from the $e^- + \mu^- \to e^- + \mu^-$ process relative to the muon rest frame and letting the mass of the muon go to infinity.

Solution:

We define the kinematics of the reaction as follows:

$$e^-(p_1^\mu, s) + \mu^-(p_2^\mu, r) \to e^-(p_3^\mu, s') + \mu^-(p_4^\mu, r') \tag{11.84}$$

where the energy–momentum 4-vectors are given by p_1^μ, p_2^μ, p_3^μ, p_4^μ and the corresponding spin states of the particles by s, r, s', r'. The QED tree-level Feynman diagram is shown in Figure 11.1 of **Ex. 11.1**. The amplitude can be expressed as (see part a) of **Ex. 11.1**):

$$i\mathcal{M}_{e\mu\to e\mu} = (ie^2)\frac{\left(\bar{u}^{s'}(p_3)\gamma^\mu u^s(p_1)\right)\left(\bar{u}^{r'}(p_4)\gamma_\mu u^r(p_2)\right)}{(p_1-p_3)^2 + i\epsilon} \tag{11.85}$$

and we computed its matrix element squared in part j) of **Ex. 11.1**:

$$\langle|\mathcal{M}|^2\rangle = \frac{8e^4}{(p_1-p_3)^4}\Big[(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_3\cdot p_2)-M^2(p_1\cdot p_3)\Big] \tag{11.86}$$

where we neglected the mass of the electron relative to that of the muon. In addition, we have energy-momentum conservation:

$$p_1^\mu+p_2^\mu=p_3^\mu+p_4^\mu \quad\Longrightarrow\quad p_4^\mu=p_1^\mu+p_2^\mu-p_3^\mu \tag{11.87}$$

The four-momentum of the exchanged photon is:

$$q^\mu=(p_1-p_3)^\mu \quad\Longrightarrow\quad q^2=(p_1-p_3)^2=p_1^2+p_3^2-2p_1\cdot p_3=2m_e^2-2p_1\cdot p_3=-2p_1\cdot p_3 \tag{11.88}$$

since we neglect the electron mass, and also:

$$p_4^2=(p_1-p_3+p_2)^2=(p_1-p_3)^2+2(p_1-p_3)\cdot p_2+p_2^2=q^2+2(p_1-p_3)\cdot p_2+M^2=M^2 \tag{11.89}$$

Consequently:

$$q^2+2(p_1-p_3)\cdot p_2=0 \quad\Longrightarrow q^2=-2(p_1-p_3)\cdot p_2 \tag{11.90}$$

The squared amplitude takes the following form:

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{8e^4}{q^4}\Big[(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_3\cdot p_2)+\frac{M^2q^2}{2}\Big]\\
&= \frac{8e^4}{q^4}\Big[(p_1\cdot p_2)(p_3\cdot(p_1+p_2-p_3))+(p_1\cdot(p_1+p_2-p_3))(p_3\cdot p_2)+\frac{M^2q^2}{2}\Big]\\
&= \frac{8e^4}{q^4}\Big[(p_1\cdot p_2)(p_3\cdot(p_1+p_2))+(p_1\cdot(p_2-p_3))(p_3\cdot p_2)+\frac{M^2q^2}{2}\Big]\\
&= \frac{8e^4}{q^4}\Big[-\frac{q^2}{2}(p_1\cdot p_2)+(p_1\cdot p_2)(p_3\cdot p_2)+(p_1\cdot p_2)(p_3\cdot p_2)+\frac{q^2}{2}(p_3\cdot p_2)+\frac{M^2q^2}{2}\Big]\\
&= \frac{8e^4}{q^4}\Big[2(p_1\cdot p_2)(p_3\cdot p_2)+\frac{M^2q^2}{2}-\frac{q^2}{2}(p_1\cdot p_2)+\frac{q^2}{2}(p_3\cdot p_2)\Big]\\
&= \frac{8e^4}{q^4}\Big[2(p_1\cdot p_2)(p_3\cdot p_2)+\frac{q^2}{2}\left(M^2-(p_1-p_3)\cdot p_2\right)\Big]\\
&= \frac{8e^4}{q^4}\Big[2(p_1\cdot p_2)(p_3\cdot p_2)+\frac{q^2}{2}\left(M^2+\frac{q^2}{2}\right)\Big]
\end{aligned} \tag{11.91}$$

We compute the differential cross-section in the rest frame of the initial state muon. Neglecting m_e, we can accordingly define the kinematics as:

$$p_1^\mu=(E,0,0,E),\quad p_2^\mu=(M,0,0,0),\quad p_3^\mu=(E',0,0,E'\cos\theta) \tag{11.92}$$

where E and E' are the energies of the incident and the outgoing electron. We find that:

$$q^2=-2(p_1-p_3)\cdot p_2=-2M(E-E') \tag{11.93}$$

and as well that:

$$q^2=-2p_1\cdot p_3=-2EE'(1-\cos\theta)=-4EE'\sin^2\left(\frac{\theta}{2}\right) \tag{11.94}$$

Consequently, although it might appear that E' and θ are *two independent*ăparameters, they are uniquely related by:

$$2M(E-E')=2EE'(1-\cos\theta) \quad\Longrightarrow\quad \frac{ME}{E'}=M+E(1-\cos\theta) \tag{11.95}$$

In addition:

$$(p_1 \cdot p_2) = ME, \quad (p_3 \cdot p_2) = ME' \tag{11.96}$$

Using these relation, we now can rewrite the squared amplitude in the following form:

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{8e^4}{q^4}\left[2MEME' + \frac{q^2}{2}\left(M^2 - \frac{4EE'\sin^2\left(\frac{\theta}{2}\right)}{2}\right)\right] = \frac{8e^4}{q^4}2M^2EE'\left[1 + \frac{q^2}{2}\left(\frac{1}{2EE'} - \frac{\sin^2\left(\frac{\theta}{2}\right)}{M^2}\right)\right] \\
&= \frac{8e^4}{q^4}2M^2EE'\left[1 + \frac{q^2}{4EE'} - \frac{q^2\sin^2\left(\frac{\theta}{2}\right)}{2M^2}\right] = \frac{8e^4}{q^4}2M^2EE'\left[1 - \sin^2\left(\frac{\theta}{2}\right) - \frac{q^2\sin^2\left(\frac{\theta}{2}\right)}{2M^2}\right] \\
&= \frac{8e^4}{q^4}2M^2EE'\left[\cos^2\left(\frac{\theta}{2}\right) - \frac{q^2}{2M^2}\sin^2\left(\frac{\theta}{2}\right)\right]
\end{aligned} \tag{11.97}$$

We now introduce the phase-space factor Eq. (5.151) of Section 5.15 of the book, derived for a fixed target frame, noting that $p = E$ for the electrons, since we neglected their rest mass:

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{lab} &= \frac{1}{64\pi^2 M(E + M - E\cos\theta)}\frac{E'}{E}\langle|\mathcal{M}|^2\rangle = \frac{1}{64\pi^2 M\frac{ME}{E'}}\frac{E'}{E}\langle|\mathcal{M}|^2\rangle \\
&= \frac{1}{64\pi^2 M^2}\left(\frac{E'}{E}\right)^2\langle|\mathcal{M}|^2\rangle
\end{aligned} \tag{11.98}$$

where we made use of Eq. (11.95). Finally, the differential cross-section in the initial state muon rest frame can be expressed as:

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{lab} &= \frac{1}{64\pi^2 M^2}\left(\frac{E'}{E}\right)^2\frac{8e^4}{q^4}2M^2EE'\left[\cos^2\left(\frac{\theta}{2}\right) - \frac{q^2}{2M^2}\sin^2\left(\frac{\theta}{2}\right)\right] \\
&= \frac{16e^4}{64\pi^2}\left(\frac{E'}{E}\right)^2\frac{1}{\left(4EE'\sin^2\left(\frac{\theta}{2}\right)\right)^2}EE'\left[\cos^2\left(\frac{\theta}{2}\right) - \frac{q^2}{2M^2}\sin^2\left(\frac{\theta}{2}\right)\right] \\
&= \frac{e^4}{64\pi^2}\frac{1}{E^2\sin^4\left(\frac{\theta}{2}\right)}\left(\frac{E'}{E}\right)\left[\cos^2\left(\frac{\theta}{2}\right) - \frac{q^2}{2M^2}\sin^2\left(\frac{\theta}{2}\right)\right] \\
&= \frac{\alpha^2}{4E^2\sin^4\left(\frac{\theta}{2}\right)}\left(\frac{E'}{E}\right)\left[\cos^2\left(\frac{\theta}{2}\right) - \frac{q^2}{2M^2}\sin^2\left(\frac{\theta}{2}\right)\right]
\end{aligned} \tag{11.99}$$

where we introduced the fine structure constant $\alpha = e^2/4\pi$. **This equation represents a fundamental result, valid for the scattering of relativistic electrons off a point-like Dirac particle of rest mass M initially at rest!** As is discussed in Section 16.1 of the book (see Eqs. (16.16) and (16.17) of the book), it can be understood as a correction to the Rutherford scattering formula:

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \underbrace{\frac{\alpha^2}{4E^2\sin^4\left(\frac{\theta}{2}\right)}}_{\text{Rutherford}}\underbrace{\left(\frac{E'}{E}\right)}_{\text{muon recoil}}\left[\underbrace{\cos^2\left(\frac{\theta}{2}\right)}_{\text{overlap spin 1/2}}\underbrace{-\frac{q^2}{2M^2}\sin^2\left(\frac{\theta}{2}\right)}_{\text{spin–spin magnetic}}\right] \tag{11.100}$$

The term (E'/E) is due to the muon recoil, the term $\propto \cos^2(\theta/2)$ takes into account the spins of the electron and the muon, and the term $\propto \sin^2(\theta/2)$ is a purely magnetic spin–spin interaction between the spin of the electron and the spin of the muon.

We can now take the limit $M \to \infty$, i.e. $|q^2| \ll M$. We note that Eq. (11.95) implies in this case:

$$\frac{E}{E'} = 1 + \frac{E}{M}(1 - \cos\theta) \quad \stackrel{M\to\infty}{\Longrightarrow} E = E' \tag{11.101}$$

which is expected since in this limit there is no muon recoil. Hence, we arrive at the Mott scattering cross-section:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{Mott} = \underbrace{\frac{\alpha^2}{4E^2 \sin^4\left(\frac{\theta}{2}\right)}}_{\text{Rutherford}} \left[\cos^2\left(\frac{\theta}{2}\right)\right] \tag{11.102}$$

which is Eq. (11.51) of the book (see also Figure 11.2 of the book). □

11.3 Pair creation

Compute the amplitude and the cross-section for the process $\gamma\gamma \to e^+e^-$.

Solution:

This process is related by crossing symmetry to the processes $e^+e^- \to \gamma\gamma$ and Compton $e^-\gamma \to e^-\gamma$ discussed in Sections 11.15 and 11.16 of the book. As a matter of fact, **the processes $\gamma\gamma \to e^+e^-$ and $e^+e^- \to \gamma\gamma$ are related by the T transformation, and under T invariance, we would expect both amplitudes to be identical**. Let us verify this! The kinematics of $\gamma\gamma \to e^+e^-$ is given by:

$$\gamma(k_1, \epsilon_1) + \gamma(k_2, \epsilon_2) \to e^-(p, s) + e^+(k, r) \tag{11.103}$$

with $k_1^\mu + k_2^\mu = p^\mu + k^\mu$. The corresponding Feynman diagrams at tree level are shown in Figure 11.3. These are examples of diagrams in which there is a fermion propagator between the two vertices between a fermion and a photon. The fermion propagators have four-momentum $q^\mu = p^\mu - k_1^\mu = k_2^\mu - k^\mu$ and $\tilde{q}^\mu = p^\mu - k_2^\mu = k_1^\mu - k^\mu$. Starting at the end of the positron and working backwards, the contribution from the first diagram using the

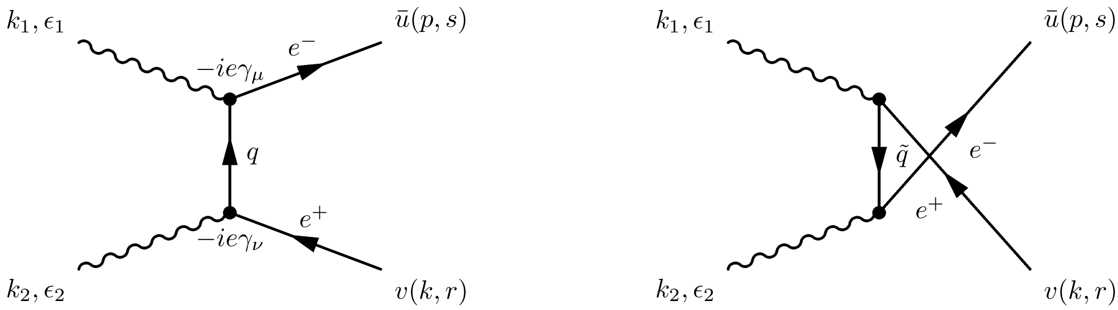

Figure 11.3 Feynman diagrams for $\gamma\gamma \to e^+e^-$ at tree level.

Feynman rules is:

$$i\mathcal{M}_1 = \epsilon_1^\mu(k_1)\epsilon_2^\nu(k_2)\bar{u}(p,s)(-ie\gamma_\mu)\frac{i(\not{q} + m_e)}{q^2 - m_e^2}(-ie\gamma_\nu)v(k,r) \tag{11.104}$$

with $q^\mu = p^\mu - k_1^\mu = k_2^\mu - k^\mu$. Likewise, for the crossed diagram we swap the two fermions on the vertices, and obtain:

$$i\mathcal{M}_2 = \epsilon_1^\mu(k_1)\epsilon_2^\nu(k_2)\bar{u}(p,s)(-ie\gamma_\nu)\frac{i(\tilde{\not{q}} + m_e)}{\tilde{q}^2 - m_e^2}(-ie\gamma_\mu)v(k,r) \tag{11.105}$$

with $\tilde{q}^\mu = p^\mu - k_2^\mu = k_1^\mu - k^\mu$.

From now on, we will neglect the rest masses of the electrons (the same conclusion will be obtained keeping the electron masses but the calculations will be longer). The net amplitude of the two diagrams can be expressed by factorizing the incoming photon polarization vectors:

$$i\mathcal{M} = \epsilon_{1\mu}(k_1)\epsilon_{2\nu}(k_2)i\mathcal{M}^{\mu\nu} = \epsilon_{1\mu}(k_1)\epsilon_{2\nu}(k_2)\left(i\mathcal{M}_1^{\mu\nu} + i\mathcal{M}_2^{\mu\nu}\right) \tag{11.106}$$

We use the Mandelstam variable $t = (p - k_1)^2$ and rewrite $i\mathcal{M}_1^{\mu\nu}$:

$$i\mathcal{M}_1^{\mu\nu} = \frac{(-ie^2)}{t}\bar{u}(p,s)\gamma^\mu(\not{p} - \not{k}_1)\gamma^\nu v(k,r) \tag{11.107}$$

and similarly with $u = (p - k_2)^2$, we find for $i\mathcal{M}_2^{\mu\nu}$:

$$i\mathcal{M}_2^{\mu\nu} = \frac{(-ie^2)}{u}\bar{u}(p,s)\gamma^\nu(\not{p} - \not{k}_2)\gamma^\mu v(k,r) \tag{11.108}$$

Combining the two diagrams, we will get the following matrix element squared:

$$|\mathcal{M}|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2\,\mathrm{Re}(\mathcal{M}_1\mathcal{M}_2^*) \tag{11.109}$$

Let us now focus first on the square of the matrix element of $\mathcal{M}_1$:

$$|\mathcal{M}_1|^2 = \frac{e^4}{t^2}\bar{u}(p,s)\not{\epsilon}_1(k_1)(\not{p} - \not{k}_1)\not{\epsilon}_2(k_2)v(k,r)\bar{v}(k,r)\not{\epsilon}_2^*(k_2)(\not{p} - \not{k}_1)\not{\epsilon}_1^*(k_1)u(p,s) \tag{11.110}$$

In order to get the unpolarized matrix element, we *sum* over the outgoing fermion helicities and find:

$$\sum_{s,r}|\mathcal{M}_1|^2 = \frac{e^4}{t^2}\,\mathrm{Tr}\left(\not{p}\not{\epsilon}_1^*(k_1)(\not{p} - \not{k}_1)\not{\epsilon}_2^*(k_2)\not{k}\not{\epsilon}_2(k_2)(\not{p} - \not{k}_1)\not{\epsilon}_1(k_1)\right) \tag{11.111}$$

and *average* over the *two* helicities of each of the initial-state photons to find (compare with Eqs. (11.203) and (11.204) of the book):

$$\langle|\mathcal{M}_1|^2\rangle = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\sum_{s,r,\lambda_1,\lambda_2}|\mathcal{M}_1|^2 = \frac{e^4}{4t^2}\,\mathrm{Tr}\left(\not{p}\gamma^\mu(\not{p} - \not{k}_1)\gamma^\nu\not{k}\gamma_\nu(\not{p} - \not{k}_1)\gamma_\mu\right) \tag{11.112}$$

We can use the property **[D2]** of the gamma matrices (see Appendix E.1 of the book) to simplify the expression, noting that $\gamma_\mu\not{p}\gamma^\mu = -2\not{p}$ and $\gamma_\nu\not{k}\gamma^\nu = -2\not{k}$:

$$\begin{aligned}\langle|\mathcal{M}_1|^2\rangle &= \frac{e^4}{4t^2}\,\mathrm{Tr}\left((-2\not{p})(\not{p} - \not{k}_1)(-2\not{k})(\not{p} - \not{k}_1)\right)\\ &= \frac{e^4}{t^2}\,\mathrm{Tr}\left((\not{p}\not{p} - \not{p}\not{k}_1)(\not{k}\not{p} - \not{k}\not{k}_1)\right)\end{aligned} \tag{11.113}$$

We note that $\not{p}\not{p} = p_\alpha p_\beta\gamma^\alpha\gamma^\beta = p_\alpha p_\beta\left(2g^{\mu\nu} - \gamma^\beta\gamma^\alpha\right) = 2p\cdot p - \not{p}\not{p}$. Hence $\not{p}\not{p} = p\cdot p = m_e^2 = 0$. Therefore:

$$\begin{aligned}\langle|\mathcal{M}_1|^2\rangle &= -\frac{e^4}{t^2}\,\mathrm{Tr}\left(\not{p}\not{k}_1(\not{k}\not{p} - \not{k}\not{k}_1)\right) = -\frac{e^4}{t^2}\left[\underbrace{\mathrm{Tr}\left(\not{p}\not{k}_1\not{k}\not{p}\right)}_{=0} - \mathrm{Tr}\left(\not{p}\not{k}_1\not{k}\not{k}_1\right)\right]\\ &= \frac{e^4}{t^2}\,\mathrm{Tr}\left(\not{p}\not{k}_1\not{k}\not{k}_1\right)\end{aligned} \tag{11.114}$$

We now use the trace property **[TS3]** in the Appendix E.3 of the book to replace the trace with a sum of products of four-vectors:

$$\begin{aligned}\langle|\mathcal{M}_1|^2\rangle &= \frac{e^4}{t^2}4\left((p\cdot k_1)(k\cdot k_1) - (p\cdot k)\underbrace{(k_1\cdot k_1)}_{=0} + (p\cdot k_1)(k_1\cdot k)\right)\\ &= \frac{8e^4}{t^2}(k\cdot k_1)(p\cdot k_1)\end{aligned} \tag{11.115}$$

We may further simplify this formula by expressing all the momenta products in terms of the Mandelstam's variables s, t, and u. We have:

$$s = (k_1 + k_2)^2 = \underbrace{k_1^2}_{=0} + \underbrace{k_2^2}_{=0} + 2(k_1 \cdot k_2) = 2(k_1 \cdot k_2) \quad \rightarrow \quad k_1 \cdot k_2 = \frac{s}{2} \tag{11.116}$$

and

$$t = (p - k_1)^2 = \underbrace{p^2}_{=m_e=0} + \underbrace{k_1^2}_{=0} - 2(p \cdot k_1) = -2(p \cdot k_1) \quad \rightarrow \quad p \cdot k_1 = -\frac{t}{2} \tag{11.117}$$

and similarly

$$u = (p - k_2)^2 \quad \rightarrow \quad p \cdot k_2 = -\frac{u}{2} \tag{11.118}$$

Consequently, the averaged matrix element squared can be expressed as:

$$\begin{aligned} \langle |\mathcal{M}_1|^2 \rangle &= \frac{8e^4}{t^2} (k \cdot k_1) \left(-\frac{t}{2}\right) = \frac{8e^4}{t^2} ((k_1 + k_2 - p) \cdot k_1) \left(-\frac{t}{2}\right) \\ &= \frac{8e^4}{t^2} \left(\underbrace{k_1^2}_{=0} + k_2 \cdot k_1 - p \cdot k_1\right) \left(-\frac{t}{2}\right) = \frac{8e^4}{t^2} \left(\underbrace{k_1^2}_{=0} + \frac{s}{2} + \frac{t}{2}\right) \left(-\frac{t}{2}\right) = \frac{2e^4}{t^2} (s+t)(-t) \\ &= \frac{2e^4}{t^2} ut \end{aligned} \tag{11.119}$$

where in the last line we used $s + t + u = 2m_e^2 = 0$ when neglecting the electron mass (see Eq. (11.88) of the book). The matrix element for the crossed diagram can be found by crossing symmetry, swapping p and k, and hence $t = (p - k_1)^2 \leftrightarrow u = (p - k_2)^2 = (k_1 - k)^2$. So we just need to swap $t \leftrightarrow u$ in our previous result:

$$\langle |\mathcal{M}_2|^2 \rangle = \frac{2e^4}{u^2} ut \tag{11.120}$$

A similar procedure must be applied to compute the interference term between the two graphs. For the interference term we have averaging as before on the initial-state helicities and summing over the spins of the final-state electrons:

$$\begin{aligned} \frac{1}{4} \sum_{s,r,\lambda_1,\lambda_2} 2\,\mathrm{Re}(\mathcal{M}_1 \mathcal{M}_2^*) &= \frac{1}{2} \sum_{s,r,\lambda_1,\lambda_2} \mathrm{Re}\left(\epsilon_1^\mu(k_1) \epsilon_2^\nu(k_2) \mathcal{M}_{1\mu\nu} \epsilon_1^{\rho*}(k_1) \epsilon_2^{\sigma*}(k_2) \mathcal{M}_{2\rho\sigma}^*\right) \\ &= \frac{1}{2} \sum_{s,r} \mathrm{Re}\, e \left(g^{\mu\rho} g^{\nu\sigma} \mathcal{M}_{1\mu\nu} \mathcal{M}_{2\rho\sigma}^*\right) = \frac{1}{2} \sum_{s,r} \mathrm{Re}\left(\mathcal{M}_{1\mu\nu} \mathcal{M}_2^{*\mu\nu}\right) \\ &= \frac{2e^4}{4ut} \sum_{s,r} \mathrm{Re}\left(\bar{u}(p) \gamma^\mu \not{q} \gamma^\nu v(k) \bar{v}(k) \gamma_\mu \not{\tilde{q}} \gamma_\nu u(p)\right) \end{aligned} \tag{11.121}$$

The sum over the electron spins gives us the following trace:

$$\mathrm{Tr}\left(\not{p} \gamma^\mu \not{q} \underbrace{\gamma^\nu \not{k} \gamma_\mu \not{\tilde{q}} \gamma_\nu}_{=-2\not{\tilde{q}}\gamma_\mu \not{k}}\right) = -2\,\mathrm{Tr}\left(\not{p} \gamma^\mu \underbrace{\not{q} \not{\tilde{q}} \gamma_\mu}_{=4(q\cdot\tilde{q})} \not{k}\right) = -8(q \cdot \tilde{q})\,\mathrm{Tr}\left(\not{p}\not{k}\right) = -32(q \cdot \tilde{q})(p \cdot k) \tag{11.122}$$

where we used $\gamma^\nu \gamma_\nu = 4\mathbb{1}$, $\gamma^\nu \not{a} \not{b} \gamma_\nu = 4(a \cdot b)\mathbb{1}$ and $\gamma^\nu \not{a} \gamma^\mu \not{c} \gamma_\nu = -2\not{c}\gamma^\mu \not{a}$ (see Eq. (11.225) of the book). Moving to the Mandelstam variables, we first noting that, neglecting the electron rest masses ($p^2 = k^2 = m_e^2 = 0$):

$$q \cdot \tilde{q} = (p - k_1) \cdot (p - k_2) = p^2 - p \cdot k_2 - p \cdot k_1 + k_1 \cdot k_2 = \frac{u}{2} + \frac{t}{2} + \frac{s}{2} = \frac{2m_e^2}{2} = 0 \tag{11.123}$$

So there is *no* interference term when we neglect the mass of the electrons! Finally, we find the total averaged matrix element squared:

$$\langle|\mathcal{M}|^2\rangle(\gamma\gamma\to e^+e^-) = 2e^4\left(\frac{ut}{t^2}+\frac{ut}{u^2}\right) = 2e^4\left(\frac{u}{t}+\frac{t}{u}\right) \qquad (m_e\longrightarrow 0) \tag{11.124}$$

As advertised at the beginning, this is exactly the same amplitude as for the T conjugated (time reversal) process (see Eq. (11.233) of the book):

$$\langle|\mathcal{M}|^2\rangle(e^+e^-\to\gamma\gamma) = 2e^4\left(\frac{u}{t}+\frac{t}{u}\right) \qquad (m_e\longrightarrow 0) \tag{11.125}$$

This shows that QED is indeed T-invariant!

We define the kinematics in the center-of-mass system. We can write (neglecting the electron rest masses):

$$k_1=(\omega,0,0,\omega),\quad k_2=(\omega,0,0,-\omega),\quad p=(E,0,0,E\cos\theta),\quad k=(E,0,0,-E\cos\theta) \tag{11.126}$$

where ω is the energy of the photons, E is the energy of the electron/positron, and θ is the scattering angle of the electron. We immediately find (when neglecting electron masses):

$$t=-2p\cdot k_1=2E(1-\cos\theta),\quad t=-2p\cdot k_2=2E(1+\cos\theta) \tag{11.127}$$

Using Eq. (5.145) of the book, we can introduce the phase-space factor for the differential cross-section in the center-of-mass system and write with $c=\cos\theta$:

$$\begin{aligned}\left(\frac{\mathrm{d}\sigma(\gamma\gamma\to e^+e^-)}{\mathrm{d}\Omega}\right)_{\mathrm{CMS}} &= \frac{1}{64\pi^2 s}2e^4\left(\frac{1+c}{1-c}+\frac{1-c}{1+c}\right) = \frac{2e^4}{64\pi^2 s}\left(\frac{(1+c)^2+(1-c)^2}{(1-c)(1+c)}\right)\\ &= \frac{e^4}{16\pi^2 s}\left(\frac{1+c^2}{1-c^2}\right)\\ &= \frac{\alpha^2}{s}\left(\frac{1+\cos^2\theta}{\sin^2\theta}\right)\end{aligned} \tag{11.128}$$

where $s=4\omega^2$ and we introduced the fine structure constant $\alpha=e^2/4\pi$. This expression is equivalent to the result obtained for $e^+e^-\to\gamma\gamma$ (see Eq. (11.230) of the book) up to a factor 2 because in our case the two final state particles are distinguishable. □

11.4 Electron–positron annihilation into two charged scalars

Consider the reaction $e^+(p)e^-(p')\to\gamma^*\to\tilde\mu^+(k_+)\tilde\mu^-(k_-)$, where $\tilde\mu^\pm$ are (spinless) scalar particles.

(a) *Show that the amplitude at tree level can be written as:*

$$i\mathcal{M} = ie^2\bar v(p')\gamma_\mu u(p)\frac{1}{q^2}(k_- - k_+)^\mu \tag{11.129}$$

where $q^\mu = k_- + k_+$.

(b) *Compute the differential cross-section in the center-of-mass system.*

(c) *Compute the total cross-section.*

(d) *What differences do you expect between this process $e^+(p)e^-(p')\to\gamma^*\to\tilde\mu^+(k_+)\tilde\mu^-(k_-)$ and the muon–antimuon pair production $e^+(p)e^-(p')\to\gamma^*\to\mu^+(k_+)\mu^-(k_-)$?*

Solution:

a) We define the kinematics of the reaction as follows:

$$e^+(p,s)e^-(p',r) \to \gamma^* \to \tilde{\mu}^+(k_+)\tilde{\mu}^-(k_-) \tag{11.130}$$

where the energy–momentum 4-vectors are given by p, p', k_+, k_-. The reaction necessarily proceeds via annihilation of the initial charged electron-positron pairs and creation of the two other charged scalars. We draw the Feynman diagrams at tree level and use the rules **[R]** defined in Section 10.16 and **[S]** defined in Section 10.18 of the book to calculate the corresponding amplitudes. The Feynman diagram is shown in Figure 11.4.

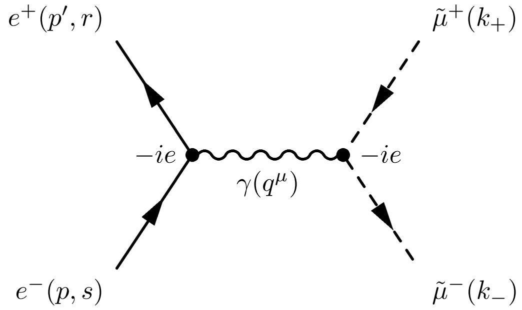

Figure 11.4 Feynman diagram for the annihilation of an electron-positron pair into two scalar particles at lowest order.

We note that the $\tilde{\mu}^+$ and $\tilde{\mu}^-$ are assumed to form particle–antiparticle pairs, hence we need to adapt our rules **[S3]** to take that into account. In the vertex factor for the final state scalars, we must follow the convention of the arrows shown in the Feynman diagram in order to define the minus sign between the two four-vectors. The corresponding amplitude is then given by:

$$\begin{aligned} i\mathcal{M} &= \underbrace{(-ie)\bar{v}(p')\gamma^\mu u(p)}_{\text{QED vertex + external fermions}} \underbrace{\left(\frac{-ig_{\mu\nu}}{(p+p')^2+i\epsilon}\right)}_{\text{photon propagator}} \underbrace{(-ie)(k_- - k_+)^\nu}_{\text{vertex + external legs}} \\ &= \frac{ie^2}{q^2}\bar{v}(p')\gamma_\mu(k_- - k_+)^\mu u(p) \end{aligned} \tag{11.131}$$

where by energy-momentum conservation, we have:

$$q^\mu = p^\mu + p'^\mu = k_-^\mu + k_+^\mu \tag{11.132}$$

b) In order to compute the differential cross-section, we must first compute the invariant matrix element squared:

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^* = \frac{e^4}{q^4}\bar{v}(p')\gamma_\mu(k_- - k_+)^\mu u(p)\bar{u}(p)\gamma_\nu(k_- - k_+)^\nu v(p') \tag{11.133}$$

We are interested in the unpolarized cross-section, hence, we average of the initial spins (no spins in the final state):

$$\langle|\mathcal{M}|^2\rangle = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\frac{e^4}{q^4}\sum_{s,r}\bar{v}(p')\gamma_\mu(k_- - k_+)^\mu u(p)\bar{u}(p)\gamma_\nu(k_- - k_+)^\nu v(p') \tag{11.134}$$

Using Casimir's trick (see Section 11.9 of the book) for the incoming fermions, we can write the expression as the following trace (we make use of the completeness relations Eq. (11.103) of the book):

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{e^4}{4q^4}\,\mathrm{Tr}\left((\not{p}' - m_e\mathbb{1})\,(\not{k}_- - \not{k}_+)\,(\not{p} + m_e\mathbb{1})\,(\not{k}_- - \not{k}_+)\right) \\ &= \frac{e^4}{4q^4}\left[\mathrm{Tr}\left(\not{p}'\,(\not{k}_- - \not{k}_+)\,\not{p}\,(\not{k}_- - \not{k}_+)\right) - m_e^2\,\mathrm{Tr}\left((\not{k}_- - \not{k}_+)\,(\not{k}_- - \not{k}_+)\right)\right] \qquad (11.135)\end{aligned}$$

where we used the property **[T3]** of Appendix A.2 of the book which states the trace of the product of an odd number of γ^μ matrices vanishes. The first trace leads to the following terms:

- $T_1 \equiv \mathrm{Tr}\,(\not{p}'\not{k}_-\not{p}\not{k}_-) = 4\left[2(p'\cdot k_-)(p\cdot k_-) - (p'\cdot p)(k_-\cdot k_-)\right]$
- $T_2 \equiv -\,\mathrm{Tr}\,(\not{p}'\not{k}_+\not{p}\not{k}_-) = -4\left[(p'\cdot k_+)(p\cdot k_-) - (p'\cdot p)(k_+\cdot k_-) + (p'\cdot k_-)(p\cdot k_+)\right]$
- $T_3 \equiv -\,\mathrm{Tr}\,(\not{p}'\not{k}_-\not{p}\not{k}_+) = -4\left[(p'\cdot k_-)(p\cdot k_+) - (p'\cdot p)(k_-\cdot k_+) + (p'\cdot k_+)(p\cdot k_-)\right]$
- $T_4 \equiv \mathrm{Tr}\,(\not{p}'\not{k}_+\not{p}\not{k}_+) = 4\left[2(p'\cdot k_+)(p\cdot k_+) - (p'\cdot p)(k_+\cdot k_+)\right]$

where we used the trace theorem **[TS2]** $\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)]$ of Appendix E.3 of the book. The second trace proportional to m_e^2 gives the following terms:

- $T_5 \equiv -m_e^2\,\mathrm{Tr}\,(\not{k}_-\not{k}_-) = -4m_e^2(k_-\cdot k_-) = -4m_e^2 k_-^2 = -4m_e^2 m_{\tilde\mu}^2$
- $T_6 \equiv m_e^2\,\mathrm{Tr}\,(\not{k}_-\not{k}_+) = 4m_e^2(k_-\cdot k_+)$
- $T_7 \equiv m_e^2\,\mathrm{Tr}\,(\not{k}_+\not{k}_-) = 4m_e^2(k_+\cdot k_-)$
- $T_8 \equiv -m_e^2\,\mathrm{Tr}\,(\not{k}_+\not{k}_+) = -4m_e^2(k_+\cdot k_+) = -4m_e^2 k_+^2 = -4m_e^2 m_{\tilde\mu}^2$

where we used the trace theorem **[TS1]** $\mathrm{Tr}\,(\not{a}\not{b}) = 4a\cdot b$ of Appendix E.3 of the book.

We should now introduce the Mandelstam variables. We find:

$$\begin{aligned}s &= (p+p')^2 = (k_- + k_+)^2 = p^2 + p'^2 + 2p\cdot p' = 2m_e^2 + 2p\cdot p' = 2m_{\tilde\mu}^2 + 2k_-\cdot k_+ \\ t &= (p-k_-)^2 = (k_+ - p')^2 = p^2 + k_-^2 - 2p\cdot k_- = m_e^2 + m_{\tilde\mu}^2 - 2p\cdot k_- = m_e^2 + m_{\tilde\mu}^2 - 2p'\cdot k_+ \\ u &= (p-k_+)^2 = (k_- - p')^2 = p^2 + k_+^2 - 2p\cdot k_+ = m_e^2 + m_{\tilde\mu}^2 - 2p\cdot k_+ = m_e^2 + m_{\tilde\mu}^2 - 2p'\cdot k_-\end{aligned} \qquad (11.136)$$

Accordingly:

$$\begin{aligned}p\cdot p' &= \frac{1}{2}\left(s - 2m_e^2\right) \equiv \frac{1}{2}\mathcal{S}_e, \qquad k_-\cdot k_+ = \frac{1}{2}\left(s - 2m_{\tilde\mu}^2\right) \equiv \frac{1}{2}\mathcal{S}_{\tilde\mu} \\ p\cdot k_- &= p'\cdot k_+ = -\frac{1}{2}\left(t - m_e^2 - m_{\tilde\mu}^2\right) \equiv -\frac{1}{2}\mathcal{T} \\ p\cdot k_+ &= p'\cdot k_- = -\frac{1}{2}\left(u - m_e^2 - m_{\tilde\mu}^2\right) \equiv -\frac{1}{2}\mathcal{U}\end{aligned} \qquad (11.137)$$

We understand the physical meaning of our variables by noting that in the ultra-relativistic limit where the rest masses can be neglected, $\mathcal{S}_e = \mathcal{S}_{\tilde\mu} = s$, $\mathcal{T} = t$ and $\mathcal{U} = u$. We now introduce these definitions in our traces T_i $(i = 1,\ldots,8)$:

- $T_1 = 4\left[2(-\frac{1}{2}\mathcal{U})(-\frac{1}{2}\mathcal{T}) - m_{\tilde\mu}^2(\frac{1}{2}\mathcal{S}_e)\right]$
- $T_2 = -4\left[(-\frac{1}{2}\mathcal{T})^2 - (\frac{1}{2}\mathcal{S}_e)(\frac{1}{2}\mathcal{S}_{\tilde\mu}) + (-\frac{1}{2}\mathcal{U})^2\right]$
- $T_3 = -4\left[(-\frac{1}{2}\mathcal{U})^2 - (\frac{1}{2}\mathcal{S}_e)(\frac{1}{2}\mathcal{S}_{\tilde\mu}) + (-\frac{1}{2}\mathcal{T})^2\right]$
- $T_4 = 4\left[2(-\frac{1}{2}\mathcal{T})(-\frac{1}{2}\mathcal{U}) - m_{\tilde\mu}^2(\frac{1}{2}\mathcal{S}_e)\right]$

- $T_5 = T_8 = -4m_e^2 m_{\tilde{\mu}}^2$
- $T_6 = T_7 = 4m_e^2 \frac{1}{2}\mathcal{S}_{\tilde{\mu}} = 2m_e^2 \mathcal{S}_{\tilde{\mu}}$

Consequently:

$$T_1 = T_4 = 2\left[\mathcal{UT} - m_{\tilde{\mu}}^2 \mathcal{S}_e\right] \tag{11.138}$$

and

$$T_2 = T_3 = -\left[\mathcal{U}^2 - \mathcal{S}_e \mathcal{S}_{\tilde{\mu}} + \mathcal{T}^2\right] \tag{11.139}$$

Hence, considering the first four terms:

$$\sum_{i=1}^{4} T_i = 4\left[\mathcal{UT} - m_{\tilde{\mu}}^2 \mathcal{S}_e\right] - 2\left[\mathcal{U}^2 - \mathcal{S}_e \mathcal{S}_{\tilde{\mu}} + \mathcal{T}^2\right] = -2\mathcal{U}^2 + 4\mathcal{UT} - 2\mathcal{T}^2 - 4m_{\tilde{\mu}}^2 \mathcal{S}_e + 2\mathcal{S}_e \mathcal{S}_{\tilde{\mu}} \tag{11.140}$$

We note that in the ultra-relativistic limit we have simply $s + t + u = 0$ (see Eq. (11.88) of the book), hence $s^2 = t^2 + u^2 + 2ut$, therefore:

$$\sum_{i=1}^{4} T_i = -2u^2 + 4ut - 2t^2 + 2s^2 = -2u^2 + 4ut - 2t^2 + 2(t^2 + u^2 + 2ut) = 8ut \tag{11.141}$$

So, we expect something similar for our general result, with just extra terms proportional to the rest masses to some power. Indeed, we find:

$$-2\mathcal{U}^2 + 4\mathcal{UT} - 2\mathcal{T}^2 = -2\left(\mathcal{U}^2 - 2\mathcal{UT}^2 + \mathcal{T}^2\right) = -2\left(\mathcal{U} - \mathcal{T}\right)^2 = -2(u-t)^2 \tag{11.142}$$

and

$$\begin{aligned}
2\mathcal{S}_e \mathcal{S}_{\tilde{\mu}} - 4m_{\tilde{\mu}}^2 \mathcal{S}_e &= 2\mathcal{S}_e\left(\mathcal{S}_{\tilde{\mu}} - 2m_{\tilde{\mu}}^2\right) = 2\left(s - 2m_e^2\right)\left(s - 4m_{\tilde{\mu}}^2\right) \\
&= 2s^2 - 2s\left(2m_e^2 + 4m_{\tilde{\mu}}^2\right) + 16m_e^2 m_{\tilde{\mu}}^2 \\
&= 2s\left(s - 2m_e^2 - 4m_{\tilde{\mu}}^2\right) + 16m_e^2 m_{\tilde{\mu}}^2 \\
&= 2s\left(-2m_{\tilde{\mu}}^2 - (t+u)\right) + 16m_e^2 m_{\tilde{\mu}}^2 \\
&= 2\left(2m_e^2 + 2m_{\tilde{\mu}}^2 - (t+u)\right)\left(-2m_{\tilde{\mu}}^2 - (t+u)\right) + 16m_e^2 m_{\tilde{\mu}}^2 \\
&= 4m_e^2\left(-2m_{\tilde{\mu}}^2 - (t+u)\right) + 2\left(2m_{\tilde{\mu}}^2 - (t+u)\right)\left(-2m_{\tilde{\mu}}^2 - (t+u)\right) + 16m_e^2 m_{\tilde{\mu}}^2 \\
&= -4m_e^2\left(t+u\right) - 2\left(2m_{\tilde{\mu}}^2 - (t+u)\right)\left(2m_{\tilde{\mu}}^2 + (t+u)\right) + 8m_e^2 m_{\tilde{\mu}}^2 \\
&= 8m_e^2 m_{\tilde{\mu}}^2 - 4m_e^2\left(t+u\right) - 8m_{\tilde{\mu}}^4 + 2(t+u)^2
\end{aligned} \tag{11.143}$$

where we used $s + t + u = 2m_e^2 + 2m_{\tilde{\mu}}^2$ (see Eq. (11.88) of the book). We can now collect all the 4 T_i ($i = 1, \ldots, 4$) terms and find:

$$\begin{aligned}
\sum_{i=1}^{4} T_i &= -2(u-t)^2 + 8m_e^2 m_{\tilde{\mu}}^2 - 4m_e^2\left(t+u\right) - 8m_{\tilde{\mu}}^4 + 2(t+u)^2 \\
&= -2(u^2 - 2ut + t^2) + 2(u^2 + 2ut + t^2) + 8m_e^2 m_{\tilde{\mu}}^2 - 4m_e^2\left(t+u\right) - 8m_{\tilde{\mu}}^4 \\
&= 8ut + 8m_e^2 m_{\tilde{\mu}}^2 - 4m_e^2\left(t+u\right) - 8m_{\tilde{\mu}}^4
\end{aligned} \tag{11.144}$$

We should now add the trace terms T_i ($i = 6, \ldots 8$) explicitly proportional to m_e^2. We can simplify the $T_6 = T_7$ terms, by noting:

$$\begin{aligned}
T_6 = T_7 &= 2m_e^2 \mathcal{S}_{\tilde{\mu}} = 2m_e^2(s - 2m_{\tilde{\mu}}^2) = 2m_e^2 s - 4m_e^2 m_{\tilde{\mu}}^2 \\
&= 2m_e^2\left(2m_e^2 + 2m_{\tilde{\mu}}^2 - (t+u)\right) - 4m_e^2 m_{\tilde{\mu}}^2 \\
&= 4m_e^4 - 2m_e^2(t+u)
\end{aligned} \tag{11.145}$$

Consequently, putting collecting all 8 terms, we finally find:

$$\begin{aligned}\sum_{i=1}^{8} T_i &= \underbrace{8ut + 8m_e^2 m_{\tilde{\mu}}^2 - 4m_e^2\,(t+u) - 8m_{\tilde{\mu}}^4}_{\sum_{i=1}^{4} T_i} \underbrace{-8m_e^2 m_{\tilde{\mu}}^2}_{=T_5+T_8} + \underbrace{8m_e^4 - 4m_e^2(t+u)}_{=T_6+T_7} \\ &= 8\left[ut - m_e^2(t+u) + m_e^4 - m_{\tilde{\mu}}^4\right] \end{aligned} \quad (11.146)$$

The spin averaged matrix element squared is therefore equal to:

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4q^4}\sum_{i=1}^{8} T_i = \frac{2e^4}{s^2}\left[ut - m_e^2(t+u) + m_e^4 - m_{\tilde{\mu}}^4\right] \quad (11.147)$$

where in the denominator we used that $q^2 = (p+p')^2 = s$.

We now define the kinematical variables in the center-of-mass system of the reaction. The initial state electrons and positron are moving with three-momenta in opposite direction but with same magnitudes $p \equiv |\vec{p}| = |-\vec{p}'|$. Similarly, the final state scalar particles are back-to-back with three-momenta with same magnitudes $k \equiv |\vec{k}_-| = |-\vec{k}_+|$. We define the four-momenta as:

$$p = (E_p, 0, 0, p), \quad p' = (E_p, 0, 0, -p), \quad k_- = (E_k, k\sin\theta, 0, k\cos\theta), \quad k_+ = (E_k, -k\sin\theta, 0, -k\cos\theta) \quad (11.148)$$

where $E_p = \sqrt{p + m_e^2}$, $E_k = \sqrt{k + m_{\tilde{\mu}}^2}$ and θ is the scattering angle. We express the Mandelstam variables as a function of the center-of-mass kinematical variables:

$$\begin{aligned} s &= (p+p')^2 = (2E_p)^2 = 4E_p^2 = (k_- + k_+)^2 = (2E_k)^2 = 4E_k^2 \quad \Longrightarrow \quad E \equiv E_p = E_k \\ t &= m_e^2 + m_{\tilde{\mu}}^2 - 2(p\cdot k_-) = m_e^2 + m_{\tilde{\mu}}^2 - 2(E^2 - pk\cos\theta) \\ u &= m_e^2 + m_{\tilde{\mu}}^2 - 2(p\cdot k_+) = m_e^2 + m_{\tilde{\mu}}^2 - 2(E^2 + pk\cos\theta) \end{aligned} \quad (11.149)$$

Hence:

$$\begin{aligned} ut &= \left(m_e^2 + m_{\tilde{\mu}}^2 - 2E^2 - 2pk\cos\theta\right)\left(m_e^2 + m_{\tilde{\mu}}^2 - 2E^2 + 2pk\cos\theta\right) \\ &= \left(m_e^2 + m_{\tilde{\mu}}^2 - 2E^2\right)^2 - 4p^2k^2\cos^2\theta \\ &= \left(m_e^2 + m_{\tilde{\mu}}^2\right)^2 - 4\left(m_e^2 + m_{\tilde{\mu}}^2\right)E^2 + 4E^4 - 4p^2k^2\cos^2\theta \end{aligned} \quad (11.150)$$

and

$$-m_e^2(t+u) = -2m_e^4 - 2m_e^2 m_{\tilde{\mu}}^2 + 4m_e^2 E^2 \quad (11.151)$$

Then:

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle &= \frac{2e^4}{s^2}\Big[\left(m_e^2 + m_{\tilde{\mu}}^2\right)^2 - 4\left(m_e^2 + m_{\tilde{\mu}}^2\right)E^2 + 4\left(E^4 - p^2k^2\cos^2\theta\right) \\ &\qquad -2m_e^4 - 2m_e^2 m_{\tilde{\mu}}^2 + 4m_e^2E^2 + m_e^4 - m_{\tilde{\mu}}^4\Big] \\ &= \frac{2e^4}{s^2}\Big[m_e^4 + 2m_e^2 m_{\tilde{\mu}}^2 + m_{\tilde{\mu}}^4 - 4\left(m_e^2 + m_{\tilde{\mu}}^2\right)E^2 + 4\left(E^4 - p^2k^2\cos^2\theta\right) \\ &\qquad -2m_e^4 - 2m_e^2 m_{\tilde{\mu}}^2 + 4m_e^2E^2 + m_e^4 - m_{\tilde{\mu}}^4\Big] \\ &= \frac{2e^4}{s^2}\Big[-4\left(m_e^2 + m_{\tilde{\mu}}^2\right)E^2 + 4\left(E^4 - p^2k^2\cos^2\theta\right) + 4m_e^2E^2\Big] \\ &= \frac{8e^4}{s^2}\Big[-m_{\tilde{\mu}}^2E^2 + E^4 - p^2k^2\cos^2\theta\Big] = \frac{e^4}{2}\left(\frac{-m_{\tilde{\mu}}^2E^2 + E^4 - p^2k^2\cos^2\theta}{E^4}\right) \end{aligned} \quad (11.152)$$

So, finally:

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{2}\left(-\frac{4m_{\tilde{\mu}}^2}{s} + 1 - \beta_p^2\beta_k^2\cos^2\theta\right) \tag{11.153}$$

where $\beta_p = p/E$ and $\beta_k = k/E$ are the velocities of the particles in the center-of-mass frame. In order to compute the differential cross-section, we use the result derived in **Ex. 5.4**:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CMS}} = \left(\frac{1}{64\pi^2 s}\right)\sqrt{\frac{\lambda(s,m_{\tilde{\mu}}^2,m_{\tilde{\mu}}^2)}{\lambda(s,m_e^2,m_e^2)}}\,\langle|\mathcal{M}|^2\rangle \tag{11.154}$$

where $\lambda(x,y,z) \equiv x^2+y^2+z^2-2xy-2yz-2zx$ is the Källén or triangle function is given by Eq. (5.132) of the book. We have shown in **Ex. 5.4** that:

$$\lambda(s,m_{\tilde{\mu}}^2,m_{\tilde{\mu}}^2) = s(s-4m_{\tilde{\mu}}^2) \quad \text{and} \quad \lambda(s,m_e^2,m_e^2) = s(s-4m_e^2) \tag{11.155}$$

Hence, the differential cross-section in the center-of-mass system can be written as:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CMS}} = \left(\frac{\alpha^2}{8s}\right)\sqrt{\frac{(s-4m_{\tilde{\mu}}^2)}{(s-4m_e^2)}}\left(-\frac{4m_{\tilde{\mu}}^2}{s} + 1 - \beta_p^2\beta_k^2\cos^2\theta\right) \tag{11.156}$$

where $s = 4E^2$ and where we introduced the fine structure constant $\alpha = e^2/4\pi$.

In the ultra-relativistic limit, we can neglect the rest masses, and the velocities will be equal to 1, hence the differential cross-section in the center-of-mass reduces to the simple expression:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CMS}} = \left(\frac{\alpha^2}{8s}\right)\left(1-\cos^2\theta\right) \qquad m_e \longrightarrow 0, m_{\tilde{\mu}} \longrightarrow 0 \tag{11.157}$$

c) Let us assume that $m_e \ll m_{\tilde{\mu}}$ and $s \gg m_e$. Then the velocities of the electron and positrons give $\beta_p \to 1$. We have:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CMS}} = \left(\frac{\alpha^2}{8s}\right)\sqrt{1-\frac{4m_{\tilde{\mu}}^2}{s}}\left(-\frac{4m_{\tilde{\mu}}^2}{s} + 1 - \beta_k^2\cos^2\theta\right) \tag{11.158}$$

The kinematical threshold factor under the square-root implies that $s \geq 4m_{\tilde{\mu}}^2$ in order for the final state scalar muons to be produced. For $s \gg m_{\tilde{\mu}}^2$, we have the ultra-relavistic case. The total cross-section is found by integration over the phase-space:

$$\sigma = \int \mathrm{d}\Omega\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\alpha^2}{8s}\right)\int(\mathrm{d}\cos\theta)(1-\cos^2\theta)\int\mathrm{d}\phi = \left(\frac{\alpha^2}{8s}\right)\left(\frac{4}{3}\right)(2\pi) = \left(\frac{\pi\alpha^2}{3s}\right) \tag{11.159}$$

d) We compare the differential cross-section for $e^+e^- \to \tilde{\mu}^+\tilde{\mu}^-$ with that of $e^+e^- \to \mu^+\mu^-$ and also with that of $\tilde{e}^+ + \tilde{e}^- \to \tilde{\mu}^+ + \tilde{\mu}^-$ which we computed in **Ex. 10.3**. Each of these reactions proceeds in the s-channel via a photon propagator. We must consider the photon spin eigenstate relative to the initial and final state axes, which we label z and z'. The amplitude will be proportional to the overlap between the wave-functions, which are related by the rotational covariance of angular momentum eigenstates. The change of quantization frame is obtained with the help of the Wigner matrices $D^J_{L_zL'_z}$:

$$|L,L'_z\rangle = \sum_{L_z} D^J_{L_zL'_z}\,|L,L_z\rangle \tag{11.160}$$

For $L=1$, the Wigner matrices are given by:

$$D^1_{1,1} = \frac{1+\cos\theta}{2}, \quad D^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}, \quad D^1_{1,-1} = \frac{1-\cos\theta}{2}, \quad D^1_{0,0} = \cos\theta \tag{11.161}$$

- For $e^+e^- \to \tilde{\mu}^+\tilde{\mu}^-$: the initial state particles are spin-1/2 fermions and the final state particles are scalar, hence final state is spinless. The unpolarized differential cross-section is then expected to be proportional to:
$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CMS}}(e^+e^- \to \tilde{\mu}^+\tilde{\mu}^-) \propto \left(D^1_{1,0}\right)^2 \propto \sin^2\theta \tag{11.162}$$
Indeed, we found:
$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CMS}}(e^+e^- \to \tilde{\mu}^+\tilde{\mu}^-) = \left(\frac{\alpha^2}{8s}\right)\left(1-\cos^2\theta\right) = \left(\frac{\alpha^2}{8s}\right)\sin^2\theta \tag{11.163}$$

- For $e^+e^- \to \mu^+\mu^-$: the initial state particles and the final state particles are spin-1/2 fermions. The unpolarized differential cross-section is then expected to be proportional to:
$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CMS}}(e^+e^- \to \mu^+\mu^-) \propto \left(D^1_{1,1}\right)^2+\left(D^1_{1,-1}\right)^2 \propto (1+\cos\theta)^2+(1-\cos\theta)^2 \propto 1+\cos^2\theta \tag{11.164}$$
Indeed, we found:
$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CMS}}(e^+e^- \to \mu^+\mu^-) = \left(\frac{\alpha^2}{4s}\right)\left(1+\cos^2\theta\right) \tag{11.165}$$

- For $\tilde{e}^+ + \tilde{e}^- \to \tilde{\mu}^+ + \tilde{\mu}^-$: the initial state particles and the final state particles are scalar particles. The unpolarized differential cross-section is then expected to be proportional to:
$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CMS}}(e^+e^- \to \mu^+\mu^-) \propto \left(D^1_{0,0}\right)^2 \propto \cos^2\theta \tag{11.166}$$
Indeed, we found:
$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CMS}}(\tilde{e}^+ + \tilde{e}^- \to \tilde{\mu}^+ + \tilde{\mu}^-) = \frac{\alpha^2}{4s}\cos^2\theta \tag{11.167}$$

These differential cross-sections are plotted in Figure 11.5.

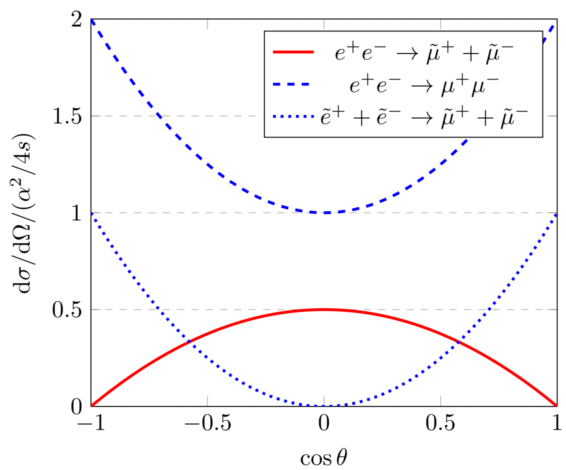

Figure 11.5 Differential cross-section in the center-of-mass system for $e^+e^- \to \tilde{\mu}^+ + \tilde{\mu}^-$, $e^+e^- \to \mu^+\mu^-$ and $\tilde{e}^+ + \tilde{e}^- \to \tilde{\mu}^+ + \tilde{\mu}^-$ in the ultra-relavistic limit.

12 QED Radiative Corrections

12.1 Running coupling

The coupling constant given in Eq. (12.57) is found to be scale-dependent.

(a) Prove that for any measurement, the parameter μ in this equation is arbitrary. To achieve this, assume two experiments using different values μ_a and μ_b and that Eq. (12.57) holds for one of them. Then show that it will necessarily hold for the other as well.

(b) Usually the running of the coupling is described using α in the low-energy limit (since the value is just given by the well-known fine-structure constant in this regime). Which value is usually chosen for μ^2 in this case? Explain where this choice originates from.

(c) The running coupling has a singularity at a certain energy, which is called the Landau pole. Determine the energy scale at which this happens. Does this result carry physical meaning? Is the scale at which the Landau pole appears accessible with current accelerator technology?

(d) Calculate the energy scales at which you expect the coupling to be 1%, 10%, and 100% stronger than $\alpha(0) \approx 1/137$, respectively. Are these scales accessible by current accelerator technology?

Solution:

a) We investigate Eq. (12.57) of the book:

$$\alpha\left(Q^2\right) = \frac{\alpha\left(\mu^2\right)}{1 - \dfrac{\alpha\left(\mu^2\right)}{3\pi}\ln\left(\dfrac{Q^2}{\mu^2}\right)} \quad \Longrightarrow \quad \alpha^{-1}\left(Q^2\right) = \alpha^{-1}\left(\mu^2\right) - \frac{1}{3\pi}\ln\left(\frac{Q^2}{\mu^2}\right) \tag{12.1}$$

We assume that the relation holds for Q^2 relative to the scale μ_A in experiment A:

$$\alpha^{-1}\left(Q^2\right) = \alpha^{-1}\left(\mu_A^2\right) - \frac{1}{3\pi}\ln\left(\frac{Q^2}{\mu_A^2}\right) \tag{12.2}$$

Since it holds for any value of Q^2, then it is also true for the specific case that $Q^2 = \mu_B^2$:

$$\alpha^{-1}\left(\mu_B^2\right) = \alpha^{-1}\left(\mu_A^2\right) - \frac{1}{3\pi}\ln\left(\frac{\mu_B^2}{\mu_A^2}\right) \tag{12.3}$$

which (incidentally) is just the scale relative to which experiment B is making its measurements. Hence:

$$\begin{aligned}
\alpha^{-1}\left(Q^2\right) &= \left(\alpha^{-1}\left(\mu_B^2\right) + \frac{1}{3\pi}\ln\left(\frac{\mu_B^2}{\mu_A^2}\right)\right) - \frac{1}{3\pi}\ln\left(\frac{Q^2}{\mu_A^2}\right) \\
&= \alpha^{-1}\left(\mu_B^2\right) - \frac{1}{3\pi}\ln\left(\frac{\mu_A^2}{\mu_B^2}\right) - \frac{1}{3\pi}\ln\left(\frac{Q^2}{\mu_A^2}\right) \\
&= \alpha^{-1}\left(\mu_B^2\right) - \frac{1}{3\pi}\ln\left(\frac{Q^2}{\cancel{\mu_A^2}}\frac{\cancel{\mu_A^2}}{\mu_B^2}\right) \\
&= \alpha^{-1}\left(\mu_B^2\right) - \frac{1}{3\pi}\ln\left(\frac{Q^2}{\mu_B^2}\right) \qquad \square \qquad (12.4)
\end{aligned}$$

The same equation holds for any scale μ ! It basically tells us about the "relative" shift of the coupling constant α between one scale and Q^2 and this shift does not depend on the absolute value of μ, but simply the ratio between Q^2 and μ, hence is independent of the absolute value of scale. For instance, relative to any scale, doubling Q^2 will decrease the inverse of the coupling constant by:

$$\begin{aligned}
\alpha^{-1}\left(2Q^2\right) &= \alpha^{-1}\left(\mu^2\right) - \frac{1}{3\pi}\ln\left(\frac{2Q^2}{\mu^2}\right) \\
&= \alpha^{-1}\left(\mu^2\right) - \frac{1}{3\pi}\left[\ln\left(\frac{Q^2}{\mu^2}\right) + \ln 2\right] \\
&= \alpha^{-1}\left(Q^2\right) - \frac{\ln 2}{3\pi} \qquad (12.5)
\end{aligned}$$

So the decrease of α^{-1} is independent of the absolute value of Q^2 (owing to the properties of the logarithm).

b) The fine structure constant is a very precisely measured fundamental constant of nature (see CODATA recommended values or `https://physics.nist.gov`):

$$\alpha^{-1} = 137.035999084(21) \qquad (12.6)$$

with a relative uncertainty of 1.5×10^{-10} ! Naively, one would assume that it corresponds to the value of the running coupling constant in the "static" case, i.e. as $Q^2 \to 0$. In other words, the fine structure constant should be equivalent to the running coupling constant at $Q^2 \simeq 0$:

$$\alpha^{-1} \simeq \alpha^{-1}(Q^2 = 0) \qquad (12.7)$$

However, we run into divergences due to the logarithm. A natural choice for the minimum scale of the running α^{-1} can be motivated by recalling the processes responsible for the renormalization of the electric charge. In this context, the main contributions to the self-energy of the photon comes from fermion–antifermion loops in its propagator (see Figure 12.4 of the book and read Chapter 12.5 of the book for detailed derivations). Looking at Eq. (12.54) of the book, we find that a reasonable choice of μ^2 for the low-energy limit to be rest mass of the lightest fermion in the loops, that is m_e. Hence, we can write:

$$\alpha^{-1} \simeq \alpha^{-1}(Q^2 \simeq m_e^2) \qquad (12.8)$$

c) We write:

$$\alpha^{-1}\left(Q^2\right) = \alpha^{-1}\left(m_e^2\right) - \frac{1}{3\pi}\ln\left(\frac{Q^2}{m_e^2}\right) \qquad (12.9)$$

To find the Landau-Pole, we need to find the condition for $\alpha^{-1}\left(Q^2\right) \to 0$. Consequently:

$$\alpha^{-1}\left(m_e^2\right) - \frac{1}{3\pi}\ln\left(\frac{Q^2_{pole}}{m_e^2}\right) = 0 \quad\Longrightarrow\quad \frac{\alpha\left(m_e^2\right)}{3\pi}\ln\left(\frac{Q^2_{pole}}{m_e^2}\right) = 1 \tag{12.10}$$

This leads to the following expression for the pole:

$$Q^2_{pole} = m_e^2 e^{3\pi/\alpha} \quad\Longrightarrow\quad Q_{pole} = m_e\, e^{3\pi/2\,\alpha} \tag{12.11}$$

where the rest mass of the electron is $m_e \approx 511\,\text{keV}$. Numerically,

$$Q_{pole} \approx (511\,\text{keV}) \times e^{1291} \approx 10^{286}\,\text{eV} = 10^{274}\,\text{TeV} \tag{12.12}$$

which we can compare to 14 TeV of the world's current highest energy collider, the CERN LHC. Clearly this energy scale cannot be reached with current accelerator technology! The pole is also much higher than the GUT scale ($\approx 10^{16}$ GeV), which is the energy scale at which the strong nuclear force, the weak nuclear force, and the electromagnetic force are believed to merge into a single unified force (see Chapter 1 and Section 31.5 of the book). In other words, it is the energy scale at which the three fundamental forces of nature, excluding gravity, are predicted to become indistinguishable and be governed by a new single set of rules. So, we do not expected the pole to have a physical meaning since QED should break at a lower scale.

d) We are looking for the energy scales at which the running coupling constant increases by a relative amount δ:

$$\alpha(Q^2) = \alpha(\mu^2) + \delta \cdot \alpha(\mu^2) = (1+\delta)\alpha(\mu^2) \quad\Longrightarrow\quad \delta = \frac{\alpha(Q^2)}{\alpha(\mu^2)} - 1 = \frac{\alpha^{-1}(\mu^2)}{\alpha^{-1}(Q^2)} - 1 \tag{12.13}$$

We have seen that the running of the coupling constant can be expressed at first loop order as:

$$\alpha(Q^2) = \frac{\alpha({m_e}^2)}{1 - \Delta\alpha_{ee}} \quad \text{where} \quad \Delta\alpha_{ee} = \frac{\alpha({m_e}^2)}{3\pi}\ln\left(\frac{Q^2}{{m_e}^2}\right) \tag{12.14}$$

Here, $\Delta\alpha_{ee}$ only accounts for the e^+e^- fermion loops in the photon propagator. As Q^2 increases, fermion loops with heavier fermion–antifermion pairs start to contribute, which are otherwise highly suppressed at low Q^2. For instance, we can have the same diagrams replacing the electron by the muon and the tau lepton. Hence, the running of the coupling constant can be expressed as:

$$\alpha(Q^2) \simeq \frac{\alpha(0)}{1 - \Delta\alpha_{ee} - \Delta\alpha_{\mu\mu} - \Delta\alpha_{\tau\tau}} \quad \text{where} \quad \Delta\alpha_{\ell\ell} = \frac{\alpha(0)}{3\pi}\ln\left(\frac{Q^2}{{m_\ell}^2}\right) \tag{12.15}$$

An accurate theoretical evaluation of $\alpha(s)$ within the Standard Model (so beyond QED) valid at high energies must take into account several effects. As discussed in Section 12.6 of the book, one can conveniently split the different contributions as:

$$\alpha_{SM}(s) \simeq \frac{\alpha(0)}{1 - \Delta\alpha_{leptons}(s) - \Delta\alpha_{hadrons}(s) - \Delta\alpha_{top}(s) - \Delta\alpha_{new}(s)} \tag{12.16}$$

where $\Delta\alpha_{leptons}$ is the correction introduced by the leptonic loops (electron, muon, and tau), $\Delta\alpha_{hadrons}$ is the correction introduced by hadronic corrections corresponding to loops of quarks, excluding the top quark, since it is much heavier than the other quarks. $\Delta\alpha_{top}$ is the contribution of the top quark loop. Finally, $\Delta\alpha_{new}(s)$ could contain the effect of "new" virtual particles in the higher-order diagrams. As discussed in the book, a lot of effort has gone into obtaining a precise the theoretical prediction at the scale $Q^2 = M_Z^2$. The result is:

$$\alpha_{SM}(M_Z^2) = \frac{1}{128.927 \pm 0.023} \quad \Longrightarrow \quad \delta \approx 6.3\% \tag{12.17}$$

This is significant and as been observed experimentally, as is discussed in the book. In particular, refer to Table 12.1 and Figure 12.7 of the book. Figure 12.1 shows the same plot as in the book, but with a highly expanded and logarithmic horizontal scale. The Q^2 range extends up to 10^{17} GeV, which is just beyond the predicted GUT scale. The solid thick line shows the theoretical expectation. The data points are from the measurements performed at the DORIS, PEP, PETRA, TRISTAN, and LEP e^+e^- colliders. The measurements agree with the theoretical expectations.

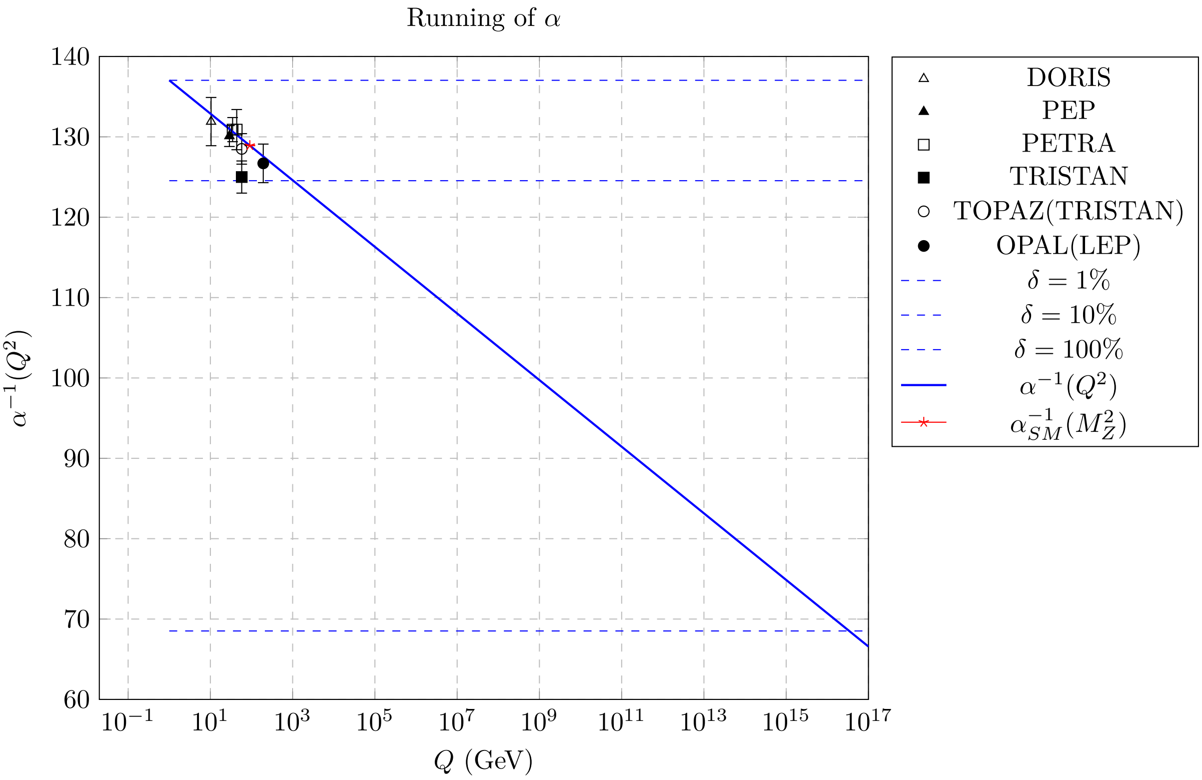

Figure 12.1 Measured values of the running coupling constant α at the DORIS, PEP, PETRA, TRISTAN, and LEP e^+e^- colliders. The solid line shows the theoretical expectation.

Here we want to estimate the scales for the three cases where δ =1%, 10%, and 100%.

- For $\delta = 1\%$: This is a relatively small variation and we can estimate the Q^2 by using the most simplistic formula with only the electron–positron loop contributing:

$$\alpha(Q^2) = (1+\delta)\alpha(m_e{}^2) \quad \Longrightarrow \quad \frac{\alpha(m_e{}^2)}{1-\Delta\alpha_{ee}} = (1+\delta)\alpha(m_e{}^2) \tag{12.18}$$

Hence:

$$\Delta\alpha_{ee} = \frac{\alpha(m_e{}^2)}{3\pi}\ln\left(\frac{Q^2}{m_e{}^2}\right) = \frac{\delta}{1+\delta} \quad \Longrightarrow \quad Q^2 = m_e^2 \cdot \exp\left[\frac{3\pi\delta}{\alpha(1+\delta)}\right] = m_e^2 \cdot \exp\left[\frac{3\pi\alpha^{-1}\delta}{(1+\delta)}\right] \tag{12.19}$$

Numerically, we find:

$$\sqrt{Q^2} \approx m_e \sqrt{e^{12.8}} \approx 600 m_e \approx 3 \times 10^8 \text{ eV} = 300\,\text{MeV} \tag{12.20}$$

This is clearly an accessible energy scale with an accelerator. Compare this to the largest electron-positron collider LEP at CERN, which reached 200 GeV.

- For $\delta = 10\%$ and $\delta = 100\%$: As mentioned above, the running fine structure constant increases by about 6.3% at a scale $Q^2 = M_Z^2$. Hence, the naive calculation with only the electron–positron loop will definitely yield an inaccurate result. One needs to include heavier particles and higher order loops to precisely estimate the behavior of α. We can read off the values from the theoretical curve shown in from Figure 12.1. We find:

$$\delta = 10\% \quad \Longrightarrow \quad Q \approx 10^3 \text{ GeV} = 1 \text{ TeV} \tag{12.21}$$

and

$$\delta = 100\% \quad \Longrightarrow \quad Q \approx 10^{16} \text{ GeV} \tag{12.22}$$

The TeV scale can be reached at the CERN LHC, as well as the future colliders under study, such as the ILC or the FCC. The 10^{16} scale is equivalent to the GUT scale, and is definitely not within reach of accelerators using known technology.

12.2 Light scattering by light

Compute the amplitude and cross-section for the process of light scattering by light.

Solution:

The scattering of light by light (or photon-photon scattering) does not occur classically in the linear regime of Maxwell's equation. It is a purely QED process that happens at higher orders, since, unlike the strong or the weak force, the electromagnetic force is mediated by the photon which is chargeless. In the Standard Model (SM), the reaction proceeds at one-loop level via virtual box diagrams involving electrically charged fermions (leptons and quarks) or $W^{\pm}$ bosons. In various extensions of the SM, extra contributions are possible, making the measurement sensitive to new physics.

We consider the simplest possible case within QED and limit ourselves to electrons in the loop. We define the kinematics and the photon polarizations as

$$\gamma(k_1, \epsilon_1) + \gamma(k_2, \epsilon_2) \to \gamma(k_3, \epsilon_3) + \gamma(k_4, \epsilon_4) \tag{12.23}$$

By energy-momentum conservation, we have:

$$k_1^\mu + k_2^\mu = k_3^\mu + k_4^\mu \tag{12.24}$$

The light-by-light occurs at lowest non-vanishing level via the production of a virtual electron–positron pair by the two initial-state photons, followed by the annihilation of the pair into the two final-state photons. It can be represented by six "square" diagrams with every possible attachment of the four external photons to the vertices. A first basic square diagram, labelled (A), is shown in Figure 12.2.

- **Amplitude of diagram (A).** The amplitude $\mathcal{M}_A$ can be explicitly written as

$$\begin{aligned}
\mathcal{M}_A &= (-1)\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \mathrm{Tr}\left[(-ie\gamma^\mu)\left[\frac{-i\,(\not{q} + m_e \mathbb{1})}{q^2 - m_e^2 + i\epsilon}\right](-ie\gamma^\sigma)\left[\frac{-i\,(\not{q} - \not{k}_3 + m_e\mathbb{1})}{(q-k_3)^2 - m_e^2 + i\epsilon}\right]\right. \\
&\quad \left.(-ie\gamma^\lambda)\left[\frac{-i\,(\not{q} - \not{k}_1 - \not{k}_2 + m_e\mathbb{1})}{(q-k_1-k_2)^2 - m_e^2 + i\epsilon}\right](-ie\gamma^\sigma)\left[\frac{-i\,(\not{q} - \not{k}_1 + m_e\mathbb{1})}{(q-k_1)^2 - m_e^2 + i\epsilon}\right] \times \epsilon_\mu(k_1)\epsilon^*_\sigma(k_3)\epsilon^*_\lambda(k_4)\epsilon_\nu(k_2)\right] \\
&= -(ie)^4 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu(\not{q} + m_e\mathbb{1})\gamma^\sigma(\not{q} - \not{k}_3 + m_e\mathbb{1})\gamma^\lambda(\not{q} - \not{k}_1 - \not{k}_2 + m_e\mathbb{1})\gamma^\nu(\not{q} - \not{k}_1 + m_e\mathbb{1})\right]}{[q^2 - m^2][(q-k_3)^2 - m_e^2][(q-k_1-k_2)^2 - m_e^2][(q-k_1)^2 - m_e^2]} \\
&\qquad \times \epsilon_\mu(k_1)\epsilon^*_\sigma(k_3)\epsilon^*_\lambda(k_4)\epsilon_\nu(k_2)
\end{aligned} \tag{12.25}$$

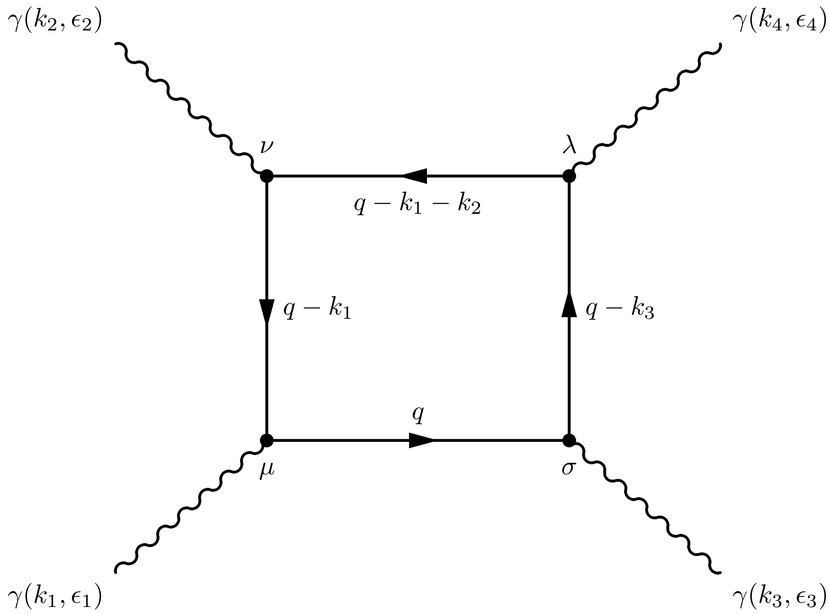

Figure 12.2 Basic square diagram (A) for light by light scattering. We note that $q-k_1-k_2=q-k_3-k_4$ by energy-momentum conservation.

where in the second line we have omitted for clarity the $+i\epsilon$ in the terms in the denominator. We define the **four-photon-photon-scattering tensor** $\mathcal{M}_A^{\mu\sigma\lambda\nu}$ such that

$$\mathcal{M}_A=(ie)^4\mathcal{M}_A^{\mu\sigma\lambda\nu}\epsilon_\mu(k_1)\epsilon^*_\sigma(k_3)\epsilon^*_\lambda(k_4)\epsilon_\nu(k_2) \tag{12.26}$$

where

$$\mathcal{M}_A^{\mu\sigma\lambda\nu}\equiv\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\mu(\not{q}+m_e\mathbb{1})\gamma^\sigma(\not{q}-\not{k}_3+m_e\mathbb{1})\gamma^\lambda(\not{q}-\not{k}_1-\not{k}_2+m_e\mathbb{1})\gamma^\nu(\not{q}-\not{k}_1+m_e\mathbb{1})\right]}{[q^2-m_e^2][(q-k_3)^2-m_e^2][(q-k_1-k_2)^2-m_e^2][(q-k_1)^2-m_e^2]} \tag{12.27}$$

• **Handling the denominator – Feynman integrals.** In order to compute higher-order loops in a Feynman diagram where the four-momentum in the loop must be integrated upon, there are clever tricks that were invented in order to solve them. The idea is to find ways to regroup the product of propagators in the denominator into a single term, however, at the cost of additional integrations. This the case of the Feynman integrals method which is discussed in Appendix F.4 of the book. We have (see Eq. (F.22) of the book):

$$\int_0^1\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\delta(x_1+x_2+x_3+x_4-1)\frac{3!}{(x_1a+x_2b+x_3c+x_4d)^4}=\frac{1}{abcd} \tag{12.28}$$

With this trick, we can write:

$$\begin{aligned}\mathcal{M}_A^{\mu\sigma\lambda\nu}\quad=\quad&3!\int\frac{\mathrm{d}^4q}{(2\pi)^4}\int_0^1\mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\int_0^{x_2}dx_3\\&\times\frac{\mathrm{Tr}\left[\cdots\right]}{\left[x_1(q^2-m_e^2)+x_2((q-k_3)^2-m_e^2)+x_3((q-k_1-k_2)^2-m_e^2)+x_4((q-k_1)^2-m_e^2)\right]^4}\end{aligned} \tag{12.29}$$

where $x_4 = 1 - x_1 - x_2 - x_3$. All of the terms in the denominator proportional to $x_i m_e^2$ $(i = 1, 2, 3, 4)$ cancel each other leaving only $-m_e^2$. Hence:

$$\begin{aligned} \mathcal{M}_A^{\mu\sigma\lambda\nu} &= 3! \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \int_0^{x_2} dx_3 \\ &\quad \times \frac{\mathrm{Tr}\Big[\cdots\Big]}{[x_1 q^2 + x_2(q-k_3)^2 + x_3(q-k_1-k_2)^2 + x_4(q-k_1)^2 - m_e^2]^4} \end{aligned} \tag{12.30}$$

We can now expand the squares and we note that the terms $x_1 q^2 + x_2 q^2 + x_3 q^2 + x_4 q^2 = q^2$. Also $k_1^2 = k_3^2 = 0$. Therefore:

$$\begin{aligned} \mathcal{M}_A^{\mu\sigma\lambda\nu} &= 3! \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \int_0^{x_2} dx_3 \\ &\quad \times \frac{\mathrm{Tr}\Big[\cdots\Big]}{[q^2 - 2x_2(q\cdot k_3) - 2x_3(q\cdot(k_1+k_2)) + x_3(k_1+k_2)^2 - 2x_4(q\cdot k_1) - m_e^2]^4} \end{aligned} \tag{12.31}$$

The term between the square brackets in the denominator can be rewritten as:

$$q^2 - 2q_\mu \underbrace{(x_2 k_3^\mu + x_3(k_1^\mu + k_2^\mu) + x_4 k_1^\mu)}_{\equiv A^\mu} + \underbrace{2x_3(k_1\cdot k_2) - m_e^2}_{\equiv B} \tag{12.32}$$

Moreover:

$$\begin{aligned} A^\mu &= x_2 k_3^\mu + x_3(k_1^\mu + k_2^\mu) + x_4 k_1^\mu = x_2 k_3^\mu + x_3 k_1^\mu + x_3 k_2^\mu + (1 - x_1 - x_2 - x_3) k_1^\mu \\ &= x_2(k_3^\mu - k_1^\mu) + x_3 k_2^\mu + (1 - x_1) k_1^\mu \end{aligned} \tag{12.33}$$

Finally, we have dealt with the denominator:

$$\mathcal{M}_A^{\mu\sigma\lambda\nu} = 3! \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \int_0^{x_2} dx_3 \frac{\mathrm{Tr}\Big[\cdots\Big]}{[q^2 - 2q\cdot A + B + i\epsilon]^4} \tag{12.34}$$

where we have reintroduced the $+i\epsilon$ prescription for the denominator. We can now change the integration variable by defining ℓ:

$$\ell^\mu \equiv q^\mu - A^\mu \quad \Longrightarrow \quad q^2 = (\ell + A)^2 = \ell^2 + 2A\cdot\ell + A^2 = \ell^2 + 2A\cdot(q - A) + A^2 = \ell^2 + 2A\cdot q - A^2 \tag{12.35}$$

Hence:

$$\mathcal{M}_A^{\mu\sigma\lambda\nu} = 3! \int \frac{\mathrm{d}^4 \ell}{(2\pi)^4} \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \int_0^{x_2} dx_3 \frac{\mathrm{Tr}\Big[\cdots\Big]}{[\ell^2 - A^2 + B + i\epsilon]^4} \tag{12.36}$$

The terms in the original denominator have been combined into a single term dependent on our integration variable ℓ^2 to the fourth power!

• **Handling the numerator of the amplitude.** We should now focus on the trace at the numerator, replacing q^μ by $\ell^\mu + A^\mu$. We find that:

$$\mathrm{Tr}\Big[\cdots\Big] = \mathrm{Tr}\big[\gamma^\mu(\not{\ell} + \not{A} + m_e)\gamma^\sigma(\not{\ell} + \not{A} - \not{k}_3 + m_e)\gamma^\lambda(\not{\ell} + \not{A} - \not{k}_1 - \not{k}_2 + m_e)\gamma^\nu(\not{\ell} + \not{A} - \not{k}_1 + m_e)\big] \tag{12.37}$$

where for clarity we removed the identity matrix after the m_e terms. We define for simplicity:

$$a \equiv A, \quad b \equiv A - k_3, \quad c \equiv A - k_1 - k_2, \quad d \equiv A - k_1 \tag{12.38}$$

then we have for the numerator:

$$\mathrm{Tr}\left[\cdots\right] = \mathrm{Tr}\left[\gamma^\mu(\not\ell + \not a + m_e)\gamma^\sigma(\not\ell + \not b + m_e)\gamma^\lambda(\not\ell + \not c + m_e)\gamma^\nu(\not\ell + \not d + m_e)\right] \tag{12.39}$$

• **Dimensional regularization.** Dimensional regularization is a very elegant technique to deal with the divergences of the space-time integrals encountered in higher-order calculations. It is briefly exposed in Appendix F.6 of the book. We start with the generic form of Eq. (F.34) of the book which considers the integration over a momentum $d^D\ell$ in D dimensions with the mth power of a typical propagator at the denominator:

$$I_m(\Delta) = \int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{1}{(\ell^2-\Delta)^m} = i\frac{(-1)^m}{2^D\pi^{D/2}}\frac{\Gamma(m-D/2)}{\Gamma(m)}\frac{1}{\Delta^{m-D/2}} \tag{12.40}$$

When the integral converges, one can use directly $D = 4$. Otherwise, setting $D = 4-2\epsilon$, we can study the behavior near $D = 4$ by expanding in ϵ. In our case we set $m = 4$ and since $\Gamma(m-D/2) = \Gamma(2) = 1! = 1$, we find it is converging and we can write:

$$I_4(\Delta) = \int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^4} = \frac{i}{16\pi^2}\frac{1}{6}\frac{1}{\Delta^2} \tag{12.41}$$

where we have used $\Gamma(4) = 3! = 6$. As noted before, the integral of an odd number of powers of ℓ^μ at the numerator will vanish:

$$I_4^{odd}(\Delta) = \int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{\ell^\mu\ell^\nu\cdots\ell^\rho}{(\ell^2-\Delta)^4} = 0 \tag{12.42}$$

where $\ell^\mu\ell^\nu\cdots\ell^\rho$ represents the product of an odd number of ℓ^μ four-vectors. We should consider products with an even number of ℓ^μ four-vectors. The first possibility is the tensor $I^{\mu\nu}$ given in Eq. (F.37) of the book:

$$I_4^{\mu\nu}(\Delta) = \int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{\ell^\mu\ell^\nu}{(\ell^2-\Delta)^4} = i\frac{(-1)^{4-1}}{2^D\pi^{D/2}}\frac{g^{\mu\nu}}{2}\frac{\Gamma(4-D/2-1)}{\Gamma(4)}\frac{1}{\Delta^{4-D/2-1}} \tag{12.43}$$

which still converges since $\Gamma(4-D/2-1) = \Gamma(4-2-1) = \Gamma(1) = 1$. We can then set $D = 4$ and write:

$$I_4^{\mu\nu}(\Delta) = \int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{\ell^\mu\ell^\nu}{(\ell^2-\Delta)^4} = -\frac{i}{16\pi^2}\frac{g^{\mu\nu}}{2}\frac{1}{6}\frac{1}{\Delta} \tag{12.44}$$

which implies, multiplying by $g_{\mu\nu}$, that:

$$I_4^2(\Delta) = \int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{\ell^2}{(\ell^2-\Delta)^4} = \frac{-i}{16\pi^2}\frac{1}{3}\frac{1}{\Delta} \tag{12.45}$$

We then observe that the integral proportional to the fourth power of ℓ will be logarithmically divergent:

$$\int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{\ell^4}{(\ell^2-\Delta)^4} \propto \int \mathrm{d}^4\ell\frac{\ell^4}{\ell^8} \propto \int \ell^3\mathrm{d}\ell\frac{\ell^4}{\ell^8} \propto \int \mathrm{d}\ell\frac{1}{\ell} \propto \ln(\ell) \to \infty \tag{12.46}$$

Indeed we can use the following result[1]:

$$\begin{aligned} I_4^{\mu\nu\rho\sigma}(\Delta) = \int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{\ell^\mu\ell^\nu\ell^\rho\ell^\sigma}{(\ell^2-\Delta)^4} &= \frac{i}{2^D\pi^{D/2}}\frac{\Gamma(4-D/2-2)}{\Gamma(4)}\frac{1}{\Delta^{4-D/2-2}} \\ &\times \frac{1}{4}\left(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) \end{aligned} \tag{12.47}$$

1 See Eq. (A.48) in M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory.* Reading, MA: Addison-Wesley, 1995.

which diverges as $D \to 4$. This result also implies, multiplying by $g_{\mu\nu}g_{\rho\sigma}$, that:

$$I_4^4(\Delta) = \int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{\ell^4}{(\ell^2-\Delta)^4} = \frac{i}{2^D\pi^{D/2}}\frac{\Gamma(4-D/2-2)}{\Gamma(4)}\frac{D(D+2)}{4}\frac{1}{\Delta^{4-D/2-2}} \tag{12.48}$$

and that

$$\begin{aligned} I_4^{2\mu\nu}(\Delta) = \int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{\ell^2\ell^\mu\ell^\nu}{(\ell^2-\Delta)^4} = g_{\rho\sigma}I_4^{\mu\nu\rho\sigma}(\Delta) &= \frac{i}{2^D\pi^{D/2}}\frac{\Gamma(4-D/2-2)}{\Gamma(4)}\frac{1}{\Delta^{4-D/2-2}} \\ &\times \frac{g_{\rho\sigma}}{4}\left(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}\right) \\ = \frac{i}{2^D\pi^{D/2}}\frac{\Gamma(4-D/2-2)}{\Gamma(4)}\frac{1}{\Delta^{4-D/2-2}}\frac{D+2}{4}g^{\mu\nu} \end{aligned} \tag{12.49}$$

For clarity we can summarize the divergent results as:

$$I_4^{\mu\nu\rho\sigma}(\Delta) = I_4^0(D)\times\frac{1}{4}\left(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}\right) \tag{12.50}$$

$$I_4^{2\mu\nu}(\Delta) = I_4^0(D)\times\left(\frac{D+2}{4}\right)g^{\mu\nu} \simeq I_4^0(D)\frac{3}{2}g^{\mu\nu} \tag{12.51}$$

$$I_4^4(\Delta) = I_4^0(D)\times\frac{D(D+2)}{4} \simeq I_4^0(D)\times 6 \tag{12.52}$$

where the logarithmically divergent part is included in:

$$I_4^0(D) \equiv \frac{i}{2^D\pi^{D/2}}\frac{\Gamma(4-D/2-2)}{\Gamma(4)}\frac{1}{\Delta^{4-D/2-2}} \tag{12.53}$$

• **Organizing the traces for integration over** $\mathrm{d}^D\ell$**.** This is the expression we need to integrate:

$$\begin{aligned}\mathcal{M}_A^{\mu\sigma\lambda\nu} &= 3!\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\int_0^1\mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\int_0^{x_2}dx_3 \\ &\times\frac{\mathrm{Tr}\left[\gamma^\mu(\not\ell+\not a+m_e)\gamma^\sigma(\not\ell+\not b+m_e)\gamma^\lambda(\not\ell+\not c+m_e)\gamma^\nu(\not\ell+\not d+m_e)\right]}{\left[\ell^2-\Delta\right]^4}\end{aligned} \tag{12.54}$$

where $\Delta \equiv A^2 - B$.

Since we need to integrate over $\mathrm{d}^D\ell$, it is important to organize the traces according to their power in ℓ. This leads to 16 terms:

- $T_1^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\left[\gamma^\mu\not\ell\gamma^\sigma\not\ell\gamma^\lambda\not\ell\gamma^\nu\not\ell\right]$
- $T_2^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\left[\gamma^\mu(\not a+m_e)\gamma^\sigma\not\ell\gamma^\lambda\not\ell\gamma^\nu\not\ell\right] \to 0$ when $\int\mathrm{d}^4\ell$
- $T_3^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\left[\gamma^\mu\not\ell\gamma^\sigma(\not b+m_e)\gamma^\lambda\not\ell\gamma^\nu\not\ell\right] \to 0$ when $\int\mathrm{d}^4\ell$
- $T_4^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\left[\gamma^\mu(\not a+m_e)\gamma^\sigma(\not b+m_e)\gamma^\lambda\not\ell\gamma^\nu\not\ell\right]$
- $T_5^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\left[\gamma^\mu\not\ell\gamma^\sigma\not\ell\gamma^\lambda(\not c+m_e)\gamma^\nu\not\ell\right] \to 0$ when $\int\mathrm{d}^4\ell$
- $T_6^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\left[\gamma^\mu(\not a+m_e)\gamma^\sigma\not\ell\gamma^\lambda(\not c+m_e)\gamma^\nu\not\ell\right]$
- $T_7^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\left[\gamma^\mu\not\ell\gamma^\sigma(\not b+m_e)\gamma^\lambda(\not c+m_e)\gamma^\nu\not\ell\right]$

- $T_8^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\big[\gamma^\mu(\not{a}+m_e)\gamma^\sigma(\not{b}+m_e)\gamma^\lambda(\not{c}+m_e)\gamma^\nu\not{\ell}\big] \to 0$ when $\int \mathrm{d}^4\ell$
- $T_9^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\big[\gamma^\mu\not{\ell}\gamma^\sigma\not{\ell}\gamma^\lambda\not{\ell}\gamma^\nu(\not{d}+m_e)\big] \to 0$ when $\int \mathrm{d}^4\ell$
- $T_{10}^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\big[\gamma^\mu(\not{a}+m_e)\gamma^\sigma\not{\ell}\gamma^\lambda\not{\ell}\gamma^\nu(\not{d}+m_e)\big]$
- $T_{11}^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\big[\gamma^\mu\not{\ell}\gamma^\sigma(\not{b}+m_e)\gamma^\lambda\not{\ell}\gamma^\nu(\not{d}+m_e)\big]$
- $T_{12}^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\big[\gamma^\mu(\not{a}+m_e)\gamma^\sigma(\not{b}+m_e)\gamma^\lambda\not{\ell}\gamma^\nu(\not{d}+m_e)\big] \to 0$ when $\int \mathrm{d}^4\ell$
- $T_{13}^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\big[\gamma^\mu\not{\ell}\gamma^\sigma\not{\ell}\gamma^\lambda(\not{c}+m_e)\gamma^\nu(\not{d}+m_e)\big]$
- $T_{14}^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\big[\gamma^\mu(\not{a}+m_e)\gamma^\sigma\not{\ell}\gamma^\lambda(\not{c}+m_e)\gamma^\nu(\not{d}+m_e)\big] \to 0$ when $\int \mathrm{d}^4\ell$
- $T_{15}^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\big[\gamma^\mu\not{\ell}\gamma^\sigma(\not{b}+m_e)\gamma^\lambda(\not{c}+m_e)\gamma^\nu(\not{d}+m_e)\big] \to 0$ when $\int \mathrm{d}^4\ell$
- $T_{16}^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\big[\gamma^\mu(\not{a}+m_e)\gamma^\sigma(\not{b}+m_e)\gamma^\lambda(\not{c}+m_e)\gamma^\nu(\not{d}+m_e)\big]$

where we used Eq. (12.42) to set to zero the terms that contain an odd number of ℓ^μ's:

$$\int \frac{\mathrm{d}^D\ell}{(2\pi)^D} \underbrace{\ell^\mu\ell^\sigma\cdots\ell^\lambda}_{\text{odd}} \frac{1}{(\ell^2-\Delta)^4} = 0 \tag{12.55}$$

• **Computing and integrating $T_1^{\mu\sigma\lambda\nu}$.** T_1 is the only term to contain a product with four ℓ^μ of the type:

$$\mathrm{Tr}\big[\gamma^\mu\not{\ell}\gamma^\sigma\not{\ell}\gamma^\lambda\not{\ell}\gamma^\nu\not{\ell}\big] \tag{12.56}$$

We should compute the trace of the product of 8 γ^μ matrices! For clarity, let us use the shorthand notation:

$$\mathrm{Tr}\big(\gamma^a\gamma^b\gamma^c\gamma^d\gamma^e\gamma^f\gamma^g\gamma^h\big) \equiv \mathrm{Tr}(abcdefgh) \tag{12.57}$$

From the anticommutation relations of the gamma matrices, $\{\gamma^a,\gamma^b\} = 2g^{ab}$, we find:

$$\begin{aligned}\mathrm{Tr}(abcdefgh) =& 2g^{ab}Tr(cdefgh) - \mathrm{Tr}(bacdefgh)\\ =& 2g^{ab}Tr(cdefgh) - 2g^{ac}\,\mathrm{Tr}(bdefgh) + \mathrm{Tr}(bcadefgh)\\ =& 2g^{ab}Tr(cdefgh) - 2g^{ac}\,\mathrm{Tr}(bdefgh) + 2g^{ad}\,\mathrm{Tr}(bcefgh) - \mathrm{Tr}(bcdaefgh)\\ =& 2g^{ab}Tr(cdefgh) - 2g^{ac}\,\mathrm{Tr}(bdefgh) + 2g^{ad}\,\mathrm{Tr}(bcefgh) - 2g^{ae}\,\mathrm{Tr}(bcdfgh) + \mathrm{Tr}(bcdeafgh)\\ =& 2g^{ab}Tr(cdefgh) - 2g^{ac}\,\mathrm{Tr}(bdefgh) + 2g^{ad}\,\mathrm{Tr}(bcefgh) - 2g^{ae}\,\mathrm{Tr}(bcdfgh)\\ &+ 2g^{af}\,\mathrm{Tr}(bcdegh) - 2g^{ag}\,\mathrm{Tr}(bcdefh) + 2g^{ah}\,\mathrm{Tr}(bcdefg) - \mathrm{Tr}(bcdefgha)\end{aligned} \tag{12.58}$$

Since traces are cyclic:

$$\begin{aligned}\mathrm{Tr}(abcdefgh) =& g^{ab}Tr(cdefgh) - g^{ac}\,\mathrm{Tr}(bdefgh) + g^{ad}\,\mathrm{Tr}(bcefgh) - g^{ae}\,\mathrm{Tr}(bcdfgh)\\ &+ g^{af}\,\mathrm{Tr}(bcdegh) - g^{ag}\,\mathrm{Tr}(bcdefh) + g^{ah}\,\mathrm{Tr}(bcdefg)\end{aligned} \tag{12.59}$$

We should now compute the trace of the product of 6 γ^μ matrices. We find:

$$\begin{aligned}\mathrm{Tr}(abcdef) =& 2g^{ab}\,\mathrm{Tr}(cdef) - \mathrm{Tr}(bacdef)\\ =& 2g^{ab}\,\mathrm{Tr}(cdef) - 2g^{ac}\,\mathrm{Tr}(bdef) + 2g^{ad}\,\mathrm{Tr}(bcef)\\ &- 2g^{ae}\,\mathrm{Tr}(bcdf) + 2g^{af}\,\mathrm{Tr}(bcde) - \mathrm{Tr}(bcdefa)\end{aligned} \tag{12.60}$$

Since traces are cyclic:

$$\mathrm{Tr}(abcdef) = g^{ab}\,\mathrm{Tr}(cdef) - g^{ac}\,\mathrm{Tr}(bdef) + g^{ad}\,\mathrm{Tr}(bcef) - g^{ae}\,\mathrm{Tr}(bcdf) + g^{af}\,\mathrm{Tr}(bcde) \tag{12.61}$$

And finally, we have seen in the trace theorem **[T5]**ăin Appendix E.2 of the book, that:

$$\mathrm{Tr}(abcd) = 4(g^{ab}g^{cd} - g^{ac}g^{bd} + g^{ad}g^{bc}) \tag{12.62}$$

So, let us do the counting! A trace of four γ^μ matrices is written as the sum of three products of $g^{ab}g^{cd}$ type of terms. A trace of six γ^μ matrices has $3 \times 5 = 15$ such products. And a trace of eight γ^μ matrices has $3 \times 5 \times 7 = 105$ such products!! For T_1 we are interested in the trace of the form:

$$\mathrm{Tr}[\gamma^a \not{\ell} \gamma^c \not{\ell} \gamma^e \not{\ell} \gamma^g \not{\ell}] = \ell_b \ell_d \ell_f \ell_h\,\mathrm{Tr}[abcdefgh] \tag{12.63}$$

So the straightforward method here is to use Eq. (12.59), Eq. (12.61) and Eq. (12.62) to reduce the traces to a products of $g^{\mu\nu}$ terms. This is very tedious (and error prone!)! Instead, we are using the power of the FeynCalc[2] package discussed in Section 11.13 of the book. We use the functions `GA[...]`, `GS[...]`, `DiracTrace[...]` and `DiracSimplify[...]`. `GA[a]` represents a γ matrix with index a. For instance, we can type:

```
In:= DiracSimplify[DiracTrace[GA[a, b, c, d]]]
```

The Feyncalc output is

$$4g^{ab}g^{cd} - 4g^{ac}g^{bd} + 4g^{ad}g^{bc} \tag{12.64}$$

which the same as Eq. (12.62). In order to directly compute $T_1^{\mu\sigma\lambda\nu}$, we use `GS[...]`. `GS[a]` represents $\not{a}$. Hence, we type:

```
In:= DiracSimplify[DiracTrace[GA[a]. GS[l].GA[c].GS[l].GA[e].GS[l].GA[g].GS[l]]]
```

and FeynCalc immediately gives us:

$$32\ell^a\ell^c\ell^e\ell^g - 4\ell^4 g^{cg}g^{ae} + 4\ell^4 g^{ag}g^{ce} + 4\ell^4 g^{eg}g^{ac} - 8\ell^2\ell^e\ell^g g^{ac} - 8\ell^2 g^{ag}\ell^c\ell^e - 8\ell^2\ell^a\ell^g g^{ce} - 8\ell^2\ell^a\ell^c g^{eg} \tag{12.65}$$

We can rewrite this as:

$$\mathrm{Tr}[\gamma^a \not{\ell} \gamma^c \not{\ell} \gamma^e \not{\ell} \gamma^g \not{\ell}] = 32 l^a l^c l^e l^g - 4\ell^4\left(g^{cg}g^{ae} - g^{ag}g^{ce} + g^{eg}g^{ac}\right) - 8\ell^2\left(\ell^e\ell^g g^{ac} + \ell^c\ell^e g^{ag} + \ell^a\ell^g g^{ce} + \ell^a\ell^c g^{eg}\right) \tag{12.66}$$

Consequently, we enter the following command in Feyncalc to use the Greek indices:

```
In:= DiracSimplify[DiracTrace[GA[\[Mu]].GS[l].GA[\[Sigma]].GS[l].
                              GA[\[Lambda]].GS[l].GA[\[Nu]].GS[l]]]
```

to obtain the final expression for $T_1^{\mu\sigma\lambda\nu}$:

$$\begin{aligned} T_1^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\left[\gamma^\mu \not{\ell} \gamma^\sigma \not{\ell} \gamma^\lambda \not{\ell} \gamma^\nu \not{\ell}\right] =& 32\ell^\mu\ell^\sigma\ell^\lambda\ell^\nu - 8\ell^2\left(\ell^\lambda\ell^\nu g^{\mu\sigma} + \ell^\sigma\ell^\lambda g^{\mu\nu} + \ell^\mu\ell^\nu g^{\sigma\lambda} + \ell^\mu\ell^\sigma g^{\lambda\nu}\right) \\ &- 4\ell^4\left(g^{\sigma\nu}g^{\mu\lambda} - g^{\mu\nu}g^{\sigma\lambda} + g^{\lambda\nu}g^{\mu\sigma}\right) \end{aligned} \tag{12.67}$$

This expression can be written as a function of the number of Lorentz indices of ℓ as:

$$T_1^{\mu\sigma\lambda\nu} = 32\mathcal{S}_0^{\mu\sigma\lambda\nu} - 8\ell^2\mathcal{S}_2^{\mu\sigma\lambda\nu} + 4\ell^4\mathcal{S}_4^{\mu\sigma\lambda\nu} \tag{12.68}$$

where

$$\mathcal{S}_0^{\mu\sigma\lambda\nu} \equiv \ell^\mu\ell^\sigma\ell^\lambda\ell^\nu \tag{12.69}$$

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and

$$\mathcal{S}_2^{\mu\sigma\lambda\nu} \equiv \ell^\lambda \ell^\nu g^{\mu\sigma} + \ell^\sigma \ell^\lambda g^{\mu\nu} + \ell^\mu \ell^\nu g^{\sigma\lambda} + \ell^\mu \ell^\sigma g^{\lambda\nu} \tag{12.70}$$

and

$$\mathcal{S}_4^{\mu\sigma\lambda\nu} \equiv g^{\lambda\sigma} g^{\mu\nu} + g^{\lambda\nu} g^{\mu\sigma} - g^{\lambda\mu} g^{\nu\sigma} \tag{12.71}$$

We now perform the integration over $\mathrm{d}^D\ell$:

$$\begin{aligned}\int \frac{\mathrm{d}^D\ell}{(2\pi)^D} \frac{T_1^{\mu\sigma\lambda\nu}}{[\ell^2 - \Delta]^4} = \int \frac{\mathrm{d}^D\ell}{(2\pi)^D} & \frac{32\mathcal{S}_0^{\mu\sigma\lambda\nu} - 8\ell^2 \mathcal{S}_2^{\mu\sigma\lambda\nu} + 4\ell^4 \mathcal{S}_4^{\mu\sigma\lambda\nu}}{[\ell^2 - \Delta]^4} \\ =& 32 I_4^{\mu\sigma\lambda\nu} - 8\left(I_4^{2\lambda\nu} g^{\mu\sigma} + I_4^{2\sigma\lambda} g^{\mu\nu} + I_4^{2\mu\nu} g^{\sigma\lambda} + I_4^{2\mu\sigma} g^{\lambda\nu}\right) \\ & + 4 I_4^4 \left(g^{\lambda\sigma} g^{\mu\nu} + g^{\lambda\nu} g^{\mu\sigma} - g^{\lambda\mu} g^{\nu\sigma}\right) \\ =& 4 I_4^0 \Big[2\left(g^{\mu\sigma} g^{\lambda\nu} + g^{\mu\lambda} g^{\sigma\nu} + g^{\mu\nu} g^{\sigma\lambda}\right) - 6\left(g^{\sigma\lambda} g^{\mu\nu} + g^{\mu\sigma} g^{\lambda\nu}\right) \\ & + 6\left(g^{\lambda\sigma} g^{\mu\nu} + g^{\lambda\nu} g^{\mu\sigma} - g^{\lambda\mu} g^{\nu\sigma}\right)\Big] \\ =& 8 I_4^0 \left[g^{\mu\sigma} g^{\lambda\nu} - 2 g^{\mu\lambda} g^{\sigma\nu} + g^{\mu\nu} g^{\sigma\lambda}\right]\end{aligned} \tag{12.72}$$

After all this effort, the result is actually quite compact. We recall that it is logarithmically divergent. The divergence is contained in the term $I_4^0(D)$ as we take the limit $D \to 4$.

• **Computing $T_{16}^{\mu\sigma\lambda\nu}$.** This term does not depend on ℓ. It reads:

$$T_{16}^{\mu\sigma\lambda\nu} \equiv \mathrm{Tr}\left[\gamma^\mu (\not{a} + m_e) \gamma^\sigma (\not{b} + m_e) \gamma^\lambda (\not{c} + m_e) \gamma^\nu (\not{d} + m_e)\right] \tag{12.73}$$

This is going to lead to a sum of three terms proportional to 1, m_e^2 and m_e^4. We can write it as:

$$T_{16}^{\mu\sigma\lambda\nu} = \mathcal{R}_0^{\mu\sigma\lambda\nu} + m_e^2 \mathcal{R}_2^{\mu\sigma\lambda\nu} + m_e^4 \mathcal{R}_4^{\mu\sigma\lambda\nu} \tag{12.74}$$

where

$$\mathcal{R}_0^{\mu\sigma\lambda\nu}(a,b,c,d) = \mathrm{Tr}\left[\gamma^\mu \not{a} \gamma^\sigma \not{b} \gamma^\lambda \not{c} \gamma^\nu \not{d}\right] \tag{12.75}$$

and

$$\begin{aligned}\mathcal{R}_2^{\mu\sigma\lambda\nu}(a,b,c,d) =& \mathrm{Tr}\left[\gamma^\mu \not{a} \gamma^\sigma \not{b} \gamma^\lambda \gamma^\nu\right] + \mathrm{Tr}\left[\gamma^\mu \not{a} \gamma^\sigma \gamma^\lambda \not{c} \gamma^\nu\right] + \mathrm{Tr}\left[\gamma^\mu \not{a} \gamma^\sigma \gamma^\lambda \gamma^\nu \not{d}\right] \\ & + \mathrm{Tr}\left[\gamma^\mu \gamma^\sigma \not{b} \gamma^\lambda \not{c} \gamma^\nu\right] + \mathrm{Tr}\left[\gamma^\mu \gamma^\sigma \not{b} \gamma^\lambda \gamma^\nu \not{d}\right] + \mathrm{Tr}\left[\gamma^\mu \gamma^\sigma \gamma^\lambda \not{c} \gamma^\nu \not{d}\right]\end{aligned} \tag{12.76}$$

and

$$\mathcal{R}_4^{\mu\sigma\lambda\nu} = \mathrm{Tr}\left[\gamma^\mu \gamma^\sigma \gamma^\lambda \gamma^\nu\right] = 4\left(g^{\lambda\nu} g^{\mu\sigma} - g^{\lambda\mu} g^{\nu\sigma} + g^{\lambda\sigma} g^{\mu\nu}\right) \tag{12.77}$$

We now perform the integration over $\mathrm{d}^4\ell$ where we have set $D = 4$ since it is convergent:

$$\int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \frac{T_{16}^{\mu\sigma\lambda\nu}}{[\ell^2 - \Delta]^4} = T_{16}^{\mu\sigma\lambda\nu} \times I_4(\Delta) = T_{16}^{\mu\sigma\lambda\nu} \times \left(\frac{i}{16\pi^2} \frac{1}{6} \frac{1}{\Delta^2}\right) \tag{12.78}$$

• **Computing $T_4^{\mu\sigma\lambda\nu}$.** There are 6 terms which contain two $\not{\ell}$ terms $T_4^{\mu\sigma\lambda\nu}$, $T_6^{\mu\sigma\lambda\nu}$, $T_7^{\mu\sigma\lambda\nu}$, $T_{10}^{\mu\sigma\lambda\nu}$, $T_{11}^{\mu\sigma\lambda\nu}$ and T_{13}. Let us analyse the specific case of $T_4^{\mu\sigma\lambda\nu}$:

$$\begin{aligned}T_4^{\mu\sigma\lambda\nu} &\equiv \mathrm{Tr}\left[\gamma^\mu (\not{a} + m_e) \gamma^\sigma (\not{b} + m_e) \gamma^\lambda \not{\ell} \gamma^\nu \not{\ell}\right] \\ &= \mathrm{Tr}\left[\gamma^\mu \not{a} \gamma^\sigma \not{b} \gamma^\lambda \not{\ell} \gamma^\nu \not{\ell}\right] + m_e^2\, \mathrm{Tr}\left[\gamma^\mu \gamma^\sigma \gamma^\lambda \not{\ell} \gamma^\nu \not{\ell}\right]\end{aligned} \tag{12.79}$$

where we used the property **[T3]** of Appendix A.2 of the book which states the trace of the product of an odd number of γ^μ matrices vanishes. We again consequently need to compute a trace of 8 products of γ matrices for the first term and 6 products for the second term. We can write $\mathrm{Tr}\left[\gamma^\mu \not{a} \gamma^\sigma \not{b} \gamma^\lambda \not{\ell} \gamma^\nu \not{\ell}\right]$ as:

$$\mathrm{Tr}\left[\gamma^\mu \not{a} \gamma^\sigma \not{b} \gamma^\lambda \not{\ell} \gamma^\nu \not{\ell}\right] = a_a b_b \ell_c \ell_d\, \mathrm{Tr}\left[\gamma^\mu \gamma^a \gamma^\sigma \gamma^b \gamma^\lambda \gamma^c \gamma^\nu \gamma^d\right] \equiv a_a b_b \ell_c \ell_d \mathcal{T}^{\mu a\sigma b\lambda c\nu d} \tag{12.80}$$

We integrate over $\mathrm{d}^4\ell$ where we have set $D=4$ since it is convergent:

$$a_a b_b \int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \frac{\ell_c \ell_d}{\left[\ell^2-\Delta\right]^4} \mathcal{T}^{\mu a \sigma b \lambda c \nu d} = -\frac{i}{16\pi^2} \frac{1}{6} \frac{a_a b_b g_{cd}}{2} \frac{1}{\Delta} \mathcal{T}^{\mu a \sigma b \lambda c \nu d} \tag{12.81}$$

The second trace proportional to m_e^2 is more tractable, since it only involves the trace of 6 γ matrices. Using Feyncalc, we find:

$$m_e^2 \operatorname{Tr}\left[\gamma^\mu \gamma^\sigma \gamma^\lambda \not\ell \gamma^\nu \not\ell\right] = 8m_e^2 \left[\ell^\lambda \ell^\nu g^{\mu\sigma} - \ell^\nu \ell^\sigma g^{\lambda\mu} + \ell^\mu \ell^\nu g^{\lambda\sigma}\right] - 4m_e^2 \ell^2 \left[g^{\lambda\nu} g^{\mu\sigma} - g^{\lambda\mu} g^{\nu\sigma} + g^{\lambda\sigma} g^{\mu\nu}\right] \tag{12.82}$$

We integrate over $\mathrm{d}^4\ell$ where we have set $D=4$ since it is convergent:

$$\begin{aligned}
m_e^2 \int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \frac{\operatorname{Tr}\left[\gamma^\mu \gamma^\sigma \gamma^\lambda \not\ell \gamma^\nu \not\ell\right]}{\left[\ell^2-\Delta\right]^4} &= 8m_e^2 \int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \frac{\ell^\lambda \ell^\nu g^{\mu\sigma} - \ell^\nu \ell^\sigma g^{\lambda\mu} + \ell^\mu \ell^\nu g^{\lambda\sigma}}{\left[\ell^2-\Delta\right]^4} \\
&\quad - 4m_e^2 \int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \frac{\ell^2}{\left[\ell^2-\Delta\right]^4} \left[g^{\lambda\nu} g^{\mu\sigma} - g^{\lambda\mu} g^{\nu\sigma} + g^{\lambda\sigma} g^{\mu\nu}\right] \\
&= 8m_e^2 \frac{-i}{16\pi^2} \frac{1}{\Delta} \frac{1}{2} \left[g^{\lambda\nu} g^{\mu\sigma} - g^{\nu\sigma} g^{\lambda\mu} + g^{\mu\nu} g^{\lambda\sigma}\right] \\
&\quad - 4m_e^2 \frac{-i}{16\pi^2} \frac{1}{\Delta} \frac{1}{3} \left[g^{\lambda\nu} g^{\mu\sigma} - g^{\lambda\mu} g^{\nu\sigma} + g^{\lambda\sigma} g^{\mu\nu}\right] \\
&= 4m_e^2 \frac{-i}{16\pi^2} \frac{1}{\Delta} \frac{2}{3} \left[g^{\lambda\nu} g^{\mu\sigma} - g^{\nu\sigma} g^{\lambda\mu} + g^{\mu\nu} g^{\lambda\sigma}\right]
\end{aligned} \tag{12.83}$$

So, ultimately, we find:

$$\int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \frac{T_4^{\mu\sigma\lambda\nu}}{\left[\ell^2-\Delta\right]^4} = \frac{-i}{16\pi^2} \frac{1}{\Delta} \frac{1}{3} \left[\frac{a_a b_b g_{cd}}{4} \mathcal{T}^{\mu a \sigma b \lambda c \nu d} + 8m_e^2 \left(g^{\lambda\nu} g^{\mu\sigma} - g^{\nu\sigma} g^{\lambda\mu} + g^{\mu\nu} g^{\lambda\sigma}\right)\right] \tag{12.84}$$

• **Computing and adding the other traces with $\not\ell$ terms.** We should repeat the above calculation of $T_4^{\mu\sigma\lambda\nu}$ for the other terms $T_i^{\mu\sigma\lambda\nu}$ where $i=6,7,10,11,13$ and add them all together. A lot of work has already been done. Let us proceed by defining the sum of the 6 traces:

$$\begin{aligned}
T_\Sigma^{\mu\sigma\lambda\nu} \equiv \sum_{i=4,6,7,10,11,13} T_i^{\mu\sigma\lambda\nu} &= \operatorname{Tr}\left[\gamma^\mu(\not a + m_e)\gamma^\sigma(\not b + m_e)\gamma^\lambda \not\ell \gamma^\nu \not\ell\right] + \operatorname{Tr}\left[\gamma^\mu(\not a + m_e)\gamma^\sigma \not\ell \gamma^\lambda(\not c + m_e)\gamma^\nu \not\ell\right] \\
&+ \operatorname{Tr}\left[\gamma^\mu \not\ell \gamma^\sigma(\not b + m_e)\gamma^\lambda(\not c + m_e)\gamma^\nu \not\ell\right] + \operatorname{Tr}\left[\gamma^\mu(\not a + m_e)\gamma^\sigma \not\ell \gamma^\lambda \not\ell \gamma^\nu(\not d + m_e)\right] + \\
&+ \operatorname{Tr}\left[\gamma^\mu \not\ell \gamma^\sigma(\not b + m_e)\gamma^\lambda \not\ell \gamma^\nu(\not d + m_e)\right] + \operatorname{Tr}\left[\gamma^\mu \not\ell \gamma^\sigma \not\ell \gamma^\lambda(\not c + m_e)\gamma^\nu(\not d + m_e)\right]
\end{aligned} \tag{12.85}$$

As before, $\mathcal{T}_6^{\mu\sigma\lambda\nu}$ can be decomposed in two terms, one trace of the product of 8 γ matrices proportional to 1, and the second trace of the product of 6 γ matrices proportional to m_e^2. We have:

$$\begin{aligned}
\operatorname{Tr}\left[\gamma^\mu \not a \gamma^\sigma \not b \gamma^\lambda \not\ell \gamma^\nu \not\ell\right] &\Longrightarrow a_a b_b \ell_c \ell_d \mathcal{T}^{\mu a \sigma b \lambda c \nu d} \\
\operatorname{Tr}\left[\gamma^\mu \not a \gamma^\sigma \not\ell \gamma^\lambda \not c \gamma^\nu \not\ell\right] &\Longrightarrow a_a \ell_b c_c \ell_d \mathcal{T}^{\mu a \sigma b \lambda c \nu d} \\
\operatorname{Tr}\left[\gamma^\mu \not\ell \gamma^\sigma \not b \gamma^\lambda \not c \gamma^\nu \not\ell\right] &\Longrightarrow \ell_a b_b c_c \ell_d \mathcal{T}^{\mu a \sigma b \lambda c \nu d} \\
&\text{etc...}
\end{aligned} \tag{12.86}$$

We use the following Feyncalc command to calculate the sum of the terms proportional to m_e^2:

```
In[]:= DiracSimplify[DiracTrace[GA[\[Mu]].GA[\[Sigma]].GA[\[Lambda]].GS[l].GA[\[Nu]].GS[l]]] +
 DiracSimplify[DiracTrace[GA[\[Mu]].GA[\[Sigma]].GS[l].GA[\[Lambda]].GA[\[Nu]].GS[l]]] +
 DiracSimplify[DiracTrace[GA[\[Mu]].GS[l].GA[\[Sigma]].GA[\[Lambda]].GA[\[Nu]].GS[l]]] +
 DiracSimplify[DiracTrace[GA[\[Mu]].GA[\[Sigma]].GS[l].GA[\[Lambda]].GS[l].GA[\[Nu]]]] +
 DiracSimplify[DiracTrace[GA[\[Mu]].GS[l].GA[\[Sigma]].GA[\[Lambda]].GS[l].GA[\[Nu]]]] +
 DiracSimplify[DiracTrace[GA[\[Mu]].GS[l].GA[\[Sigma]].GS[l].GA[\[Lambda]].GA[\[Nu]]]]
```

and get:

$$\begin{aligned}&8m_e^2\left[\ell^\lambda\ell^\nu g^{\mu\sigma}+\ell^\lambda\ell^\sigma g^{\mu\nu}+\ell^\mu\ell^\nu g^{\lambda\sigma}+\ell^\mu\ell^\sigma g^{\lambda\nu}-\ell^2 g^{\lambda\sigma}g^{\mu\nu}-\ell^2 g^{\lambda\nu}g^{\mu\sigma}+\ell^2 g^{\lambda\mu}g^{\nu\sigma}\right]\\&=8m_e^2\left[\ell^\lambda\ell^\nu g^{\mu\sigma}+\ell^\lambda\ell^\sigma g^{\mu\nu}+\ell^\mu\ell^\nu g^{\lambda\sigma}+\ell^\mu\ell^\sigma g^{\lambda\nu}\right]-8m_e^2\ell^2\left[g^{\lambda\sigma}g^{\mu\nu}+g^{\lambda\nu}g^{\mu\sigma}-g^{\lambda\mu}g^{\nu\sigma}\right]\end{aligned}\tag{12.87}$$

Putting all pieces together, we find:

$$\begin{aligned}T_\Sigma^{\mu\sigma\lambda\nu}=&\Big[a_ab_b\ell_c\ell_d+a_a\ell_bc_c\ell_d+\ell_ab_bc_c\ell_d+a_a\ell_b\ell_cd_d+\ell_ab_b\ell_cd_d+\ell_a\ell_bc_cd_d\Big]\mathcal{T}^{\mu a\sigma b\lambda c\nu d}\\&8m_e^2\left[\ell^\lambda\ell^\nu g^{\mu\sigma}+\ell^\lambda\ell^\sigma g^{\mu\nu}+\ell^\mu\ell^\nu g^{\lambda\sigma}+\ell^\mu\ell^\sigma g^{\lambda\nu}\right]-8m_e^2\ell^2\left[g^{\lambda\sigma}g^{\mu\nu}+g^{\lambda\nu}g^{\mu\sigma}-g^{\lambda\mu}g^{\nu\sigma}\right]\end{aligned}\tag{12.88}$$

We integrate over $\mathrm{d}^4\ell$ where we set $D=4$ because the result is finite since all terms are proportional to the second power of ℓ. Integration of the first term using $I_4^{\mu\nu}(\Delta)$ (see Eq. (12.44)) leads to:

$$-\frac{i}{16\pi^2}\frac{1}{6}\frac{1}{\Delta}\frac{1}{2}\mathcal{T}^{\mu a\sigma b\lambda c\nu d}\left[a_ab_bg_{cd}+a_ac_cg_{bd}+b_bc_cg_{ad}+a_ad_dg_{bc}+b_bd_dg_{ac}+c_cd_dg_{ab}\right]\tag{12.89}$$

Similarly, integration of the second term is performed using $I_4^{\mu\nu}(\Delta)$ to find:

$$\begin{aligned}8m_e^2\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{\ell^\lambda\ell^\nu g^{\mu\sigma}+\ell^\lambda\ell^\sigma g^{\mu\nu}+\ell^\mu\ell^\nu g^{\lambda\sigma}+\ell^\mu\ell^\sigma g^{\lambda\nu}}{\left[\ell^2-\Delta\right]^4}&=-\frac{i}{16\pi^2}\frac{8m_e^2}{6}\frac{1}{\Delta}\frac{1}{2}\left[g^{\lambda\nu}g^{\mu\sigma}+g^{\lambda\sigma}g^{\mu\nu}+g^{\mu\nu}g^{\lambda\sigma}+g^{\mu\sigma}g^{\lambda\nu}\right]\\&=-\frac{i}{16\pi^2}\frac{8m_e^2}{6}\frac{1}{\Delta}\left[g^{\lambda\nu}g^{\mu\sigma}+g^{\mu\nu}g^{\lambda\sigma}\right]\end{aligned}\tag{12.90}$$

and similarly using $I_4^2(\Delta)$ (see Eq. (12.45)), we find for the third term:

$$-8m_e^2\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{\ell^2}{\left[\ell^2-\Delta\right]^4}\left[g^{\lambda\sigma}g^{\mu\nu}+g^{\lambda\nu}g^{\mu\sigma}-g^{\lambda\mu}g^{\nu\sigma}\right]=+\frac{i}{16\pi^2}\frac{8m_e^2}{3}\frac{1}{\Delta}\left[g^{\lambda\sigma}g^{\mu\nu}+g^{\lambda\nu}g^{\mu\sigma}-g^{\lambda\mu}g^{\nu\sigma}\right]\tag{12.91}$$

Incidentally, the sum of the last two terms is just:

$$\begin{aligned}&-\frac{i}{16\pi^2}\frac{8m_e^2}{3}\frac{1}{\Delta}\frac{1}{2}\left[g^{\lambda\nu}g^{\mu\sigma}+g^{\mu\nu}g^{\lambda\sigma}\right]+\frac{i}{16\pi^2}\frac{8m_e^2}{3}\frac{1}{\Delta}\left[g^{\lambda\sigma}g^{\mu\nu}+g^{\lambda\nu}g^{\mu\sigma}-g^{\lambda\mu}g^{\nu\sigma}\right]\\&=\frac{i}{16\pi^2}\frac{8m_e^2}{3}\frac{1}{\Delta}\frac{1}{2}\left[g^{\lambda\nu}g^{\mu\sigma}+g^{\mu\nu}g^{\lambda\sigma}-2g^{\lambda\mu}g^{\nu\sigma}\right]\end{aligned}\tag{12.92}$$

The complete result is therefore:

$$\begin{aligned}\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{T_\Sigma^{\mu\sigma\lambda\nu}}{\left[\ell^2-\Delta\right]^4}=&-\frac{i}{16\pi^2}\frac{1}{6}\frac{1}{\Delta}\frac{1}{2}\mathcal{T}^{\mu a\sigma b\lambda c\nu d}\left[a_ab_bg_{cd}+a_ac_cg_{bd}+b_bc_cg_{ad}+a_ad_dg_{bc}+b_bd_dg_{ac}+c_cd_dg_{ab}\right]\\&+\frac{i}{16\pi^2}\frac{8m_e^2}{6}\frac{1}{\Delta}\left[g^{\lambda\nu}g^{\mu\sigma}+g^{\mu\nu}g^{\lambda\sigma}-2g^{\lambda\mu}g^{\nu\sigma}\right]\end{aligned}\tag{12.93}$$

• **Reduction of $\mathcal{T}^{\mu a\sigma b\lambda c\nu d}$.** We need to compute

$$\mathcal{T}^{\mu\sigma\lambda\nu}\equiv\mathrm{Tr}\left[\gamma^\mu\gamma^a\gamma^\sigma\gamma^b\gamma^\lambda\gamma^c\gamma^\nu\gamma^d\right]\left[a_ab_bg_{cd}+a_ac_cg_{bd}+b_bc_cg_{ad}+a_ad_dg_{bc}+b_bd_dg_{ac}+c_cd_dg_{ab}\right]\tag{12.94}$$

As mentioned before, the trace of the product of 8 γ matrices can be reduced to the sum of 105 terms of the type $g^{\mu a}g^{\sigma b}g^{\lambda c}g^{\nu d}$! These must be then contracted with $\left[a_ab_bg_{cd}+a_ac_cg_{bd}+b_bc_cg_{ad}+a_ad_dg_{bc}+b_bd_dg_{ac}+c_cd_dg_{ab}\right]$. We calculate these with the following Feyncalc commands making use of `Contract`:

```
In[]:= tt:= DiracSimplify[DiracTrace[GA[\[Mu]].GA[a].GA[\[Sigma]].GA[b].
                                          GA[\[Lambda]].GA[c].GA[\[Nu]].GA[d]]]
In[]:= ss := FV[a, a]*FV[b, b]*MT[c, d] +
  FV[a, a]*FV[c, c]*MT[b, d] +
  FV[b, b]*FV[c, c]*MT[a, d] +
  FV[a, a]*FV[d, d]*MT[b, c] +
  FV[b, b]*FV[d, d]*MT[a, c] +
  FV[c, c]*FV[d, d]*MT[a, b]
In[]:= Contract[ss*tt]
```

The result is shown here for curiosity:

$$\begin{aligned}
\mathcal{T}^{\mu\sigma\lambda\nu}(a,b,c,d) = -8\Big[& - g^{\mu\sigma} c^\nu a^\lambda - g^{\mu\sigma} d^\nu a^\lambda + b^\sigma g^{\mu\nu} a^\lambda + c^\sigma g^{\mu\nu} a^\lambda + d^\sigma g^{\mu\nu} a^\lambda + b^\nu g^{\mu\sigma} a^\lambda \\
& - b^\mu g^{\nu\sigma} a^\lambda - c^\mu g^{\nu\sigma} a^\lambda - d^\mu g^{\nu\sigma} a^\lambda - g^{\lambda\sigma} b^\nu a^\mu + b^\sigma g^{\lambda\nu} a^\mu + d^\sigma g^{\lambda\nu} a^\mu \\
& - g^{\lambda\nu} c^\sigma a^\mu + c^\nu g^{\lambda\sigma} a^\mu + d^\nu g^{\lambda\sigma} a^\mu - d^\lambda g^{\nu\sigma} a^\mu - g^{\lambda\mu} b^\sigma a^\nu - g^{\lambda\mu} c^\sigma a^\nu \\
& - g^{\lambda\mu} d^\sigma a^\nu + b^\mu g^{\lambda\sigma} a^\nu + c^\mu g^{\lambda\sigma} a^\nu + d^\mu g^{\lambda\sigma} a^\nu + d^\lambda g^{\mu\sigma} a^\nu - g^{\lambda\nu} c^\mu a^\sigma \\
& + c^\nu g^{\lambda\mu} a^\sigma + d^\nu g^{\lambda\mu} a^\sigma - g^{\lambda\mu} b^\nu a^\sigma + b^\mu g^{\lambda\nu} a^\sigma + d^\mu g^{\lambda\nu} a^\sigma - d^\lambda g^{\mu\nu} a^\sigma \\
& + b^\sigma c^\nu g^{\lambda\mu} + b^\sigma d^\nu g^{\lambda\mu} - g^{\lambda\sigma} d^\mu b^\nu - g^{\lambda\sigma} b^\mu c^\nu - g^{\lambda\sigma} b^\mu d^\nu + b^\sigma d^\mu g^{\lambda\nu} \\
& + b^\mu c^\sigma g^{\lambda\nu} + b^\mu d^\sigma g^{\lambda\nu} + c^\mu d^\sigma g^{\lambda\nu} + c^\sigma d^\lambda g^{\mu\nu} - g^{\lambda\nu} c^\mu b^\sigma - d^\lambda g^{\mu\nu} b^\sigma \\
& - g^{\lambda\nu} d^\mu c^\sigma - g^{\lambda\mu} b^\nu c^\sigma - g^{\lambda\mu} d^\nu c^\sigma - g^{\lambda\mu} b^\nu d^\sigma - g^{\lambda\mu} c^\nu d^\sigma + b^\nu c^\mu g^{\lambda\sigma} \\
& + c^\nu d^\mu g^{\lambda\sigma} + c^\mu d^\nu g^{\lambda\sigma} + (b\cdot d)\, g^{\mu\nu} g^{\lambda\sigma} + b^\nu d^\lambda g^{\mu\sigma} + c^\nu d^\lambda g^{\mu\sigma} + (a\cdot c)\, g^{\lambda\nu} g^{\mu\sigma} \\
& - d^\lambda b^\mu g^{\nu\sigma} - d^\lambda c^\mu g^{\nu\sigma} + (a\cdot b)\, g^{\lambda\mu} g^{\nu\sigma} + (a\cdot c)\, g^{\lambda\mu} g^{\nu\sigma} + (b\cdot c)\, g^{\lambda\mu} g^{\nu\sigma} \\
& + (a\cdot d)\, g^{\lambda\mu} g^{\nu\sigma} + (b\cdot d)\, g^{\lambda\mu} g^{\nu\sigma} + (c\cdot d)\, g^{\lambda\mu} g^{\nu\sigma} - g^{\lambda\sigma} g^{\mu\nu} (a\cdot b) \\
& - g^{\lambda\nu} g^{\mu\sigma} (a\cdot b) - g^{\lambda\sigma} g^{\mu\nu} (a\cdot c) - g^{\lambda\sigma} g^{\mu\nu} (b\cdot c) - g^{\lambda\nu} g^{\mu\sigma} (b\cdot c) \\
& - g^{\lambda\sigma} g^{\mu\nu} (a\cdot d) - g^{\lambda\nu} g^{\mu\sigma} (a\cdot d) - g^{\lambda\nu} g^{\mu\sigma} (b\cdot d) - g^{\lambda\sigma} g^{\mu\nu} (c\cdot d) - g^{\lambda\nu} g^{\mu\sigma} (c\cdot d) \\
& + c^\lambda \left(g^{\nu\sigma} a^\mu - g^{\mu\sigma} a^\nu + g^{\mu\nu} a^\sigma + b^\sigma g^{\mu\nu} - g^{\mu\nu} d^\sigma + b^\nu g^{\mu\sigma} + d^\nu g^{\mu\sigma} - b^\mu g^{\nu\sigma} + d^\mu g^{\nu\sigma} \right) \\
& + b^\lambda \left(g^{\nu\sigma} a^\mu - g^{\mu\sigma} a^\nu + g^{\mu\nu} a^\sigma + c^\sigma g^{\mu\nu} - g^{\mu\nu} d^\sigma + c^\nu g^{\mu\sigma} + d^\nu g^{\mu\sigma} - c^\mu g^{\nu\sigma} + d^\mu g^{\nu\sigma} \right) \Big]
\end{aligned} \tag{12.95}$$

Finally, we find:

$$\begin{aligned}
\int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \frac{T_\Sigma^{\mu\sigma\lambda\nu}}{[\ell^2-\Delta]^4} &= \frac{i}{16\pi^2}\frac{1}{6}\frac{1}{\Delta}\frac{1}{2}(8)\mathcal{T}^{\mu\sigma\lambda\nu} + \frac{i}{16\pi^2}\frac{8m_e^2}{6}\frac{1}{\Delta}\left[g^{\lambda\nu}g^{\mu\sigma} + g^{\mu\nu}g^{\lambda\sigma} - 2g^{\lambda\mu}g^{\nu\sigma}\right] \\
&= \left(\frac{8i}{16\pi^2}\frac{1}{\Delta}\right)\left[\frac{1}{12}\mathcal{T}^{\mu\sigma\lambda\nu} + \frac{m_e^2}{6}\left[g^{\lambda\nu}g^{\mu\sigma} + g^{\mu\nu}g^{\lambda\sigma} - 2g^{\lambda\mu}g^{\nu\sigma}\right]\right]
\end{aligned} \tag{12.96}$$

- **Amplitude after integration over $\mathrm{d}^D\ell$ – summarizing.**

$$\mathcal{M}_A^{\mu\sigma\lambda\nu} = 3!\int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \int_0^{x_2} dx_3 \left[\mathcal{T}_1^{\mu\sigma\lambda\nu} + \mathcal{T}_\Sigma^{\mu\sigma\lambda\nu} + \mathcal{T}_{16}^{\mu\sigma\lambda\nu}\right] \tag{12.97}$$

where

$$\begin{aligned}
\mathcal{T}_1^{\mu\sigma\lambda\nu} &\equiv \frac{8i}{16\pi^2}\frac{\Gamma(4-D/2-2)}{\Gamma(4)}\frac{1}{\Delta^{4-D/2-2}}\left[g^{\mu\sigma}g^{\lambda\nu}-2g^{\mu\lambda}g^{\sigma\nu}+g^{\mu\nu}g^{\sigma\lambda}\right] && (12.98)\\
\mathcal{T}_\Sigma^{\mu\sigma\lambda\nu}(a,b,c,d) &\equiv \left(\frac{8i}{16\pi^2}\frac{1}{\Delta}\right)\left[\frac{1}{12}\mathcal{T}^{\mu\sigma\lambda\nu}+\frac{m_e^2}{6}\left[g^{\lambda\nu}g^{\mu\sigma}+g^{\mu\nu}g^{\lambda\sigma}-2g^{\lambda\mu}g^{\nu\sigma}\right]\right] && (12.99)\\
\mathcal{T}_{16}^{\mu\sigma\lambda\nu}(a,b,c,d) &\equiv \left(\frac{i}{16\pi^2}\frac{1}{6}\frac{1}{\Delta^2}\right)\left(\mathcal{R}_0^{\mu\sigma\lambda\nu}+m_e^2\mathcal{R}_2^{\mu\sigma\lambda\nu}+m_e^4\mathcal{R}_4^{\mu\sigma\lambda\nu}\right) && (12.100)
\end{aligned}$$

We recall that $\mathcal{T}_1^{\mu\sigma\lambda\nu}$ is logarithmically divergent, while the other terms are finite. For reasons that will become clear when we add all possible diagrams (see later), we are going to ignore the $\mathcal{T}_1^{\mu\sigma\lambda\nu}$ contribution and focus on the finite terms. These latter depend on the kinematics.

• **Kinematics of the reaction.** We consider the reaction in the center-of-mass system. We can write:

$$k_1^\mu=(\omega,0,0,\omega),\quad k_2^\mu=(\omega,0,0,-\omega),\quad k_3^\mu=(\omega,0,\omega\sin\theta,\omega\cos\theta),\quad k_4^\mu=(\omega,0,-\omega\sin\theta,-\omega\cos\theta) \quad (12.101)$$

where ω is the energy of the photons and θ is the scattering angle. We immediately find:

$$k_1\cdot k_2=2\omega^2,\quad k_1\cdot k_3=\omega^2(1-\cos\theta),\quad k_2\cdot k_3=\omega^2(1+\cos\theta) \quad (12.102)$$

• **Denominator Δ.** We use Eq. (12.33) and Eq. (12.32) to find:

$$\Delta\equiv B-A^2=2x_3(k_1\cdot k_2)-m_e^2-A^2 \quad (12.103)$$

where

$$\begin{aligned}
A^2 &= \left(x_2(k_3^\mu-k_1^\mu)+x_3k_2^\mu+(1-x_1)k_1^\mu\right)^2\\
&= 2x_2x_3\left(k_2\cdot(k_3-k_1)\right)+2(1-x_1)x_3\left(k_1\cdot k_2\right)+2(1-x_1)x_2\left(k_1\cdot k_3-k_1^2\right)\\
&\qquad +x_2^2\left(-2\left(k_1\cdot k_3\right)+k_1^2+k_3^2\right)+(1-x_1)^2k_1^2+x_3^2k_2^2\\
&= 2x_2x_3\left(k_2\cdot(k_3-k_1)\right)+2(1-x_1)x_3\left(k_1\cdot k_2\right)+2(1-x_1)x_2\left(k_1\cdot k_3\right)-2x_2^2\left(k_1\cdot k_3\right) \quad (12.104)
\end{aligned}$$

Hence:

$$\begin{aligned}
\Delta &= 2x_3(k_1\cdot k_2)-m_e^2-2x_2x_3\left(k_2\cdot(k_3-k_1)\right)-2(1-x_1)x_3\left(k_1\cdot k_2\right)-2(1-x_1)x_2\left(k_1\cdot k_3\right)+2x_2^2\left(k_1\cdot k_3\right)\\
&= -2x_2x_3\left(k_2\cdot(k_3-k_1)\right)+2x_1x_3\left(k_1\cdot k_2\right)+2x_1x_2\left(k_1\cdot k_3\right)+2x_2^2\left(k_1\cdot k_3\right)-2x_2\left(k_1\cdot k_3\right)-m_e^2
\end{aligned} \quad (12.105)$$

In the center-of-mass kinematical frame, this translates to:

$$\begin{aligned}
\Delta = \omega^2\Big[&-2x_2x_3\left(\cos\theta-1\right))+4x_1x_3+2x_1x_2(1-\cos\theta)\\
&+2x_2^2(1-\cos\theta)-2x_2(1-\cos\theta)\Big]-m_e^2
\end{aligned} \quad (12.106)$$

In the low-energy regime ($\omega/m_e \ll 1$), we can approximate:

$$\Delta\approx -m_e^2 \quad (12.107)$$

• **Integration over x_1, x_2 and x_3.** Having approximated the denominator, we can straightforwardly integrate the finite terms of the amplitude.

$$\mathcal{M}_{A,\text{finite}}^{\mu\sigma\lambda\nu}=3!\int_0^1 \mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\int_0^{x_2}dx_3\left[\mathcal{T}_\Sigma^{\mu\sigma\lambda\nu}+\mathcal{T}_{16}^{\mu\sigma\lambda\nu}\right] \quad (12.108)$$

where

$$\mathcal{T}_{\Sigma}^{\mu\sigma\lambda\nu}(a,b,c,d) \approx \left(\frac{8i}{16\pi^2}\right)\left[\frac{\mathcal{T}^{\mu\sigma\lambda\nu}}{12m_e^2} + \frac{1}{6}\left[g^{\lambda\nu}g^{\mu\sigma} + g^{\mu\nu}g^{\lambda\sigma} - 2g^{\lambda\mu}g^{\nu\sigma}\right]\right] \tag{12.109}$$

$$\mathcal{T}_{16}^{\mu\sigma\lambda\nu}(a,b,c,d) \approx \left(\frac{i}{16\pi^2}\frac{1}{6}\right)\left(\frac{\mathcal{R}_0^{\mu\sigma\lambda\nu}}{m_e^4} + \frac{\mathcal{R}_2^{\mu\sigma\lambda\nu}}{m_e^2} + \mathcal{R}_4^{\mu\sigma\lambda\nu}\right) \tag{12.110}$$

In addition, we have (see Eq. (12.38)):

$$a = x_2(k_3 - k_1) + x_3 k_2 + (1 - x_1)k_1, \quad b \equiv a - k_3, \quad c \equiv a - k_1 - k_2, \quad d \equiv a - k_1 \tag{12.111}$$

Hence, we expect (see Eq. (12.95)):

$$\mathcal{T}^{\mu\sigma\lambda\nu}(a,b,c,d) \propto \omega^2, \quad \mathcal{R}_0^{\mu\sigma\lambda\nu}(a,b,c,d) \propto \omega^4 \quad \text{and} \quad \mathcal{R}_2^{\mu\sigma\lambda\nu}(a,b,c,d) \propto \omega^2 \tag{12.112}$$

In the low energy regime, we are going to neglect all of them! Consequently:

$$\begin{aligned}\mathcal{M}_{A,\text{finite,LER}}^{\mu\sigma\lambda\nu} \approx 3! \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \int_0^{x_2} dx_3 \\ \times \left(\frac{i}{16\pi^2}\right)\left(\frac{1}{6}\right)\left[8\left(g^{\lambda\nu}g^{\mu\sigma} + g^{\mu\nu}g^{\lambda\sigma} - 2g^{\lambda\mu}g^{\nu\sigma}\right) + 4\left(g^{\lambda\nu}g^{\mu\sigma} - g^{\lambda\mu}g^{\nu\sigma} + g^{\lambda\sigma}g^{\mu\nu}\right)\right]\end{aligned} \tag{12.113}$$

Then we simply have:

$$3! \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \int_0^{x_2} dx_3 = \frac{3!}{6} = 1 \tag{12.114}$$

Hence:

$$\mathcal{M}_{A,\text{finite,LER}}^{\mu\sigma\lambda\nu} \approx \left(\frac{2i}{16\pi^2}\right)\left[g^{\lambda\nu}g^{\mu\sigma} + g^{\lambda\sigma}g^{\mu\nu} - 3g^{\lambda\mu}g^{\nu\sigma}\right] \tag{12.115}$$

• **Additional diagrams at lowest order.** In addition to the diagram (A) we just consider, there are two other diagrams (B) and (C) which contribute at lowest order. These can be obtained by realising that the photons can attach to any of the internal vertices. We therefore have the three distinguishable configurations shown in Figure 12.3, where the first one of the left is just diagram (A). From a comparison of the diagrams, it is

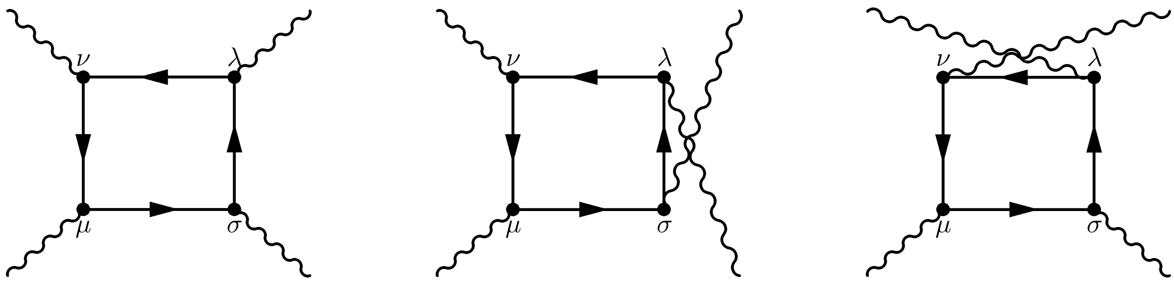

Figure 12.3 Lowest order Feynman diagrams for light by light scattering. (left) Diagram (A), (middle) Diagram B, (right) Diagram C.

evident that the amplitudes are equivalent up to a swap of indices. Indeed, starting from the amplitude for diagram (A) we can find the other amplitudes by the following swap of Lorentz indices:

$$\mathcal{M}_B\colon \lambda \leftrightarrow \sigma \quad \text{and} \quad \mathcal{M}_C\colon \lambda \leftrightarrow \nu \tag{12.116}$$

Hence:

$$\begin{aligned}\mathcal{M}^{\mu\sigma\lambda\nu}_{A,\text{finite,LER}} &\approx \left(\frac{2i}{16\pi^2}\right)\left[g^{\lambda\nu}g^{\mu\sigma}+g^{\lambda\sigma}g^{\mu\nu}-3g^{\lambda\mu}g^{\nu\sigma}\right]\\ \mathcal{M}^{\mu\sigma\lambda\nu}_{B,\text{finite,LER}} &\approx \left(\frac{2i}{16\pi^2}\right)\left[g^{\sigma\nu}g^{\mu\lambda}+g^{\lambda\sigma}g^{\mu\nu}-3g^{\sigma\mu}g^{\nu\lambda}\right]\\ \mathcal{M}^{\mu\sigma\lambda\nu}_{C,\text{finite,LER}} &\approx \left(\frac{2i}{16\pi^2}\right)\left[g^{\lambda\nu}g^{\mu\sigma}+g^{\nu\sigma}g^{\mu\lambda}-3g^{\nu\mu}g^{\lambda\sigma}\right]\end{aligned} \tag{12.117}$$

Adding the three contributions we find:

$$\mathcal{M}^{\mu\sigma\lambda\nu}_{ABC,\text{finite,LER}} \approx \left(-\frac{2i}{16\pi^2}\right)2\left[g^{\lambda\nu}g^{\mu\sigma}+g^{\lambda\sigma}g^{\mu\nu}+g^{\lambda\mu}g^{\nu\sigma}\right] \tag{12.118}$$

• **Loop in opposite direction.** We have arbitrarily chosen the direction that the internal electron traverses the loop. There is a second diagram (A') which differs from this one only in that the internal electron loop is traversed in the opposite direction. We note that the Feynman amplitude is not affected by the direction of the loop momentum. Hence, the amplitudes of the two diagrams are identical:

$$\mathcal{M}_A = \mathcal{M}_{A'} \tag{12.119}$$

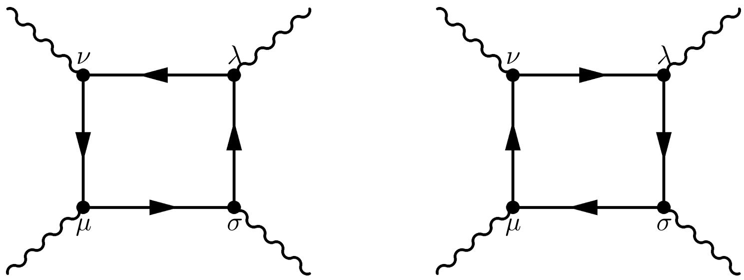

Figure 12.4 Lowest order Feynman diagrams for light by light scattering and same diagram which differs from first one only in that the internal electron loop is traversed in the opposite direction .

Hence, we can simply write:

$$\mathcal{M} = \mathcal{M}_A+\mathcal{M}_{A'}+\mathcal{M}_B+\mathcal{M}_{B'}+\mathcal{M}_C+\mathcal{M}_{C'} = 2\left(\mathcal{M}_A+\mathcal{M}_B+\mathcal{M}_C\right) \tag{12.120}$$

or

$$\mathcal{M}^{\mu\sigma\lambda\nu}_{ABCA'B'C',\text{finite,LER}} \approx \left(-\frac{2i}{16\pi^2}\right)4\left[g^{\lambda\nu}g^{\mu\sigma}+g^{\lambda\sigma}g^{\mu\nu}+g^{\lambda\mu}g^{\nu\sigma}\right] \tag{12.121}$$

• **Introducing photon polarizations.** Start from our original amplitude Eq. (12.26), we write:

$$\begin{aligned}\mathcal{M} &= (ie)^4\mathcal{M}^{\mu\sigma\lambda\nu}_{ABCA'B'C',\text{finite,LER}}\epsilon_\mu(k_1)\epsilon^*_\sigma(k_3)\epsilon^*_\lambda(k_4)\epsilon_\nu(k_2)\\ &= -ie^4\left(\frac{2}{16\pi^2}\right)4\left[g^{\lambda\nu}g^{\mu\sigma}+g^{\lambda\sigma}g^{\mu\nu}+g^{\lambda\mu}g^{\nu\sigma}\right]\epsilon_\mu(k_1)\epsilon^*_\sigma(k_3)\epsilon^*_\lambda(k_4)\epsilon_\nu(k_2)\\ &= -ie^4\left(\frac{2}{16\pi^2}\right)4\Big[\left(\epsilon^*(k_4)\cdot\epsilon(k_2)\right)\left(\epsilon^*(k_3)\cdot\epsilon(k_1)\right)+\left(\epsilon^*(k_4)\cdot\epsilon^*(k_3)\right)\left(\epsilon(k_2)\cdot\epsilon(k_1)\right)\\ &\qquad +\left(\epsilon^*(k_4)\cdot\epsilon(k_1)\right)\left(\epsilon^*(k_3)\cdot\epsilon(k_2)\right)\Big]\end{aligned} \tag{12.122}$$

In order to introduce the photon polarization into the amplitude, we follow the procedure used for Compton scattering in Section 11.16 of the book. Accordingly, we select the Coulomb gauge where the photons are transverse. Then, for each photon, we must choose two linear transverse vectors $\epsilon_i(k_1, \lambda = 1,2) = (0, \vec{\epsilon}_i)$ $(i = 1, \ldots, 4)$ with $\epsilon_i^2 = -1$ that satisfy $\epsilon_i \cdot k_i = 0$, where the k_i^μ are defined in Eq. (12.101). In general, we can always choose one polarization state of the photons to be perpendicular to the plane of the interaction. We can define the identically for the four photons in the following way:

$$\vec{\epsilon}_1(\lambda = 1) = \vec{\epsilon}_2(\lambda = 1) = \vec{\epsilon}_3(\lambda = 1) = \vec{\epsilon}_4(\lambda = 1) = \frac{\vec{k}_1 \times \vec{k}_3}{\left|\vec{k}_1 \times \vec{k}_3\right|} \tag{12.123}$$

In our choice of reference coordinates, this would correspond to polarizations parallel to the x-axis, since the interaction has been chosen to be in the y–z plane (see Eq. (12.101)). Then, the second polarization vector should be perpendicular to the first and to the momentum vector $\vec{k}_i$, i.e.:

$$\vec{\epsilon}_i(\lambda = 2) = \frac{\vec{k}_i \times \vec{\epsilon}_i(\lambda = 1)}{\left|\vec{k}_i\right|} = \frac{1}{\omega}\vec{k}_i \times \vec{\epsilon}_i(\lambda = 1) \tag{12.124}$$

For instance, using our coordinate system, we take for the two possible transverse polarizations of the photon 1 to be:

$$\epsilon_1^\mu(\lambda_1 = 1) = (0,1,0,0) \quad \text{and} \quad \epsilon_1^\mu(\lambda_1 = 2) = (0,0,1,0) \tag{12.125}$$

where we indeed have:

$$\epsilon_1(\lambda_1 = 1,2) \cdot k_1 = 0 \quad \text{and} \quad \epsilon_1^2(\lambda_1 = 1,2) = -1 \tag{12.126}$$

For the second photon, we find:

$$\epsilon_2^\mu(\lambda_2 = 1) = (0,1,0,0) \quad \text{and} \quad \epsilon_2^\mu(\lambda_2 = 2) = (0,0,-1,0) \tag{12.127}$$

We can find the final-state polarizations by rotating the initial-state polarization vectors by an angle θ around the z-axis in order to keep the transversality with the final-state momenta. Therefore:

$$\epsilon_3^\mu(\lambda_3 = 1) = (0,1,0,0) \quad \text{and} \quad \epsilon_3^\mu(\lambda_3 = 2) = (0,0,\cos\theta,-\sin\theta) \tag{12.128}$$

and

$$\epsilon_4^\mu(\lambda_4 = 1) = (0,1,0,0) \quad \text{and} \quad \epsilon_4^\mu(\lambda_4 = 2) = (0,0,-\cos\theta,\sin\theta) \tag{12.129}$$

With this notation, the amplitude becomes:

$$\begin{aligned}\mathcal{M}(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \left(\frac{-ie^4}{2\pi^2}\right) 4\Big[&\left(\epsilon_4(\lambda_4)\cdot\epsilon_2(\lambda_2)\right)\left(\epsilon_3(\lambda_3)\cdot\epsilon_1(\lambda_1)\right) + \left(\epsilon_4(\lambda_4)\cdot\epsilon_3(\lambda_3)\right)\left(\epsilon_2(\lambda_2)\cdot\epsilon_1(\lambda_1)\right) \\ &+ \left(\epsilon_4(\lambda_4)\cdot\epsilon_1(\lambda_1)\right)\left(\epsilon_3(\lambda_3)\cdot\epsilon_2(\lambda_2)\right)\Big]\end{aligned} \tag{12.130}$$

We note that we have:

$$\begin{aligned} \epsilon_i(1)\cdot\epsilon_j(1) = -1, \qquad \epsilon_i(1)\cdot\epsilon_j(2) = \epsilon_i(2)\cdot\epsilon_j(1) = 0 \quad \text{and} \\ \epsilon_1(2)\cdot\epsilon_1(2) = -1, \quad \epsilon_1(2)\cdot\epsilon_2(2) = 1, \quad \epsilon_1(2)\cdot\epsilon_3(2) = -c, \quad \epsilon_1(2)\cdot\epsilon_4(2) = c, \\ \epsilon_2(2)\cdot\epsilon_3(2) = c, \quad \epsilon_2(2)\cdot\epsilon_4(2) = -c, \quad \epsilon_3(2)\cdot\epsilon_4(2) = 1 \end{aligned} \tag{12.131}$$

We can compute explicit configurations of polarizations:

$$\begin{aligned} &\mathcal{M}(1,1,1,1) \propto (-1)^2 + (-1)^2 + (-1)^2 = 3 \\ &\mathcal{M}(2,2,2,2) \propto (-c)(-c) + (1)^2 + (c)^2 = 1 + 2c^2 \\ &\mathcal{M}(1,1,2,2) = \mathcal{M}(2,2,1,1) \propto 0 + (1)^2 + 0 = 1 \\ &\mathcal{M}(1,2,1,2) = \mathcal{M}(2,1,2,1) \propto (-c)(-1) + 0 + 0 = c \\ &\mathcal{M}(1,2,2,1) = \mathcal{M}(2,1,1,2) \propto 0 + 0 + (1)(c) = c \end{aligned} \tag{12.132}$$

where $c = \cos\theta$, and where we used T-invariance for $\mathcal{M}(1,1,2,2) = \mathcal{M}(2,2,1,1)$ and the fact that the amplitude is invariant under the interchange of the two initial or the two final state photons. The other combinations vanish (as can be readily verified).

In order to compute the unpolarized matrix element, the amplitude squared should be averaged over the initial-state polarizations and summed over the final-state ones:

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\frac{1}{2}\sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} |\mathcal{M}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)|^2 \tag{12.133}$$

We then get:

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\left[|\mathcal{M}(1,1,1,1)|^2 + |\mathcal{M}(2,2,2,2)|^2 + 2\,|\mathcal{M}(1,1,2,2)|^2 + 2\,|\mathcal{M}(1,2,1,2)|^2 + 2\,|\mathcal{M}(1,2,2,1)|^2\right] \\ &= \frac{1}{4}\left(\frac{-ie^4}{2\pi^2}\right)^2 4^2\left[3^2 + (1+2c^2)^2 + 2 + 2c^2 + 2c^2\right] \\ &= \left(\frac{-ie^4}{2\pi^2}\right)^2 16\left[3 + 2c^2 + c^4\right]\end{aligned} \tag{12.134}$$

We include the phase space factor using Eq. (5.145) of the book and arrive at the differential cross-section:

$$\begin{aligned}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CMS} &= \frac{1}{64\pi^2(4\omega^2)}\langle|\mathcal{M}|^2\rangle = \frac{1}{64\pi^2(4\omega^2)}\left(\frac{-ie^4}{2\pi^2}\right)^2 16\left[3+2c^2+c^4\right] \\ &= \frac{\alpha^4}{4\pi^2\omega^2}\left[3 + 2\cos^2\theta + \cos^4\theta\right]\end{aligned} \tag{12.135}$$

where we introduced the fine structure constant $\alpha = e^2/4\pi$. It should be stressed that this result is valid only in the case where $\omega \ll m_e$.

The proper treatment within the Standard Model can be found in M. Bohm, M. and R. Schuster, "Scattering of light by light in the electroweak standard model", Z. Phys. C63 (1994), 219–225 (`https://doi.org/10.1007/BF01411013`) and references therein. This process has been recently experimentally observed by the ATLAS Collaboration at the CERN LHC (see G. Aad *et al.* (ATLAS Collaboration), Phys. Rev. Lett. 123, 052001 (`https://doi.org/10.1103/PhysRevLett.123.052001`)).

13 Tests of QED at High Energy

13.1 Lifetime of the ℓ^*

We consider the decay $\ell^* \to \ell + \gamma$. Compute the partial decay width. In the process $e^+e^- \to \ell\ell^* \to \ell\ell\gamma$ at the Z^0 pole, estimate the lifetime and the mean free path of the ℓ^*.

a) *Starting with the Lagrangian Eq. (32.20) of the book with $|a_\gamma| \approx |b_\gamma| \approx 1$, show that the matrix element can be expressed as:*

$$\mathcal{M} = \frac{e\lambda}{2m_{\ell^*}}\left[\bar{u}(p_2)\sigma^{\mu\nu}(1-\gamma^5)q_\nu\epsilon^*_\mu u(p_1)\right] \tag{13.1}$$

b) *Show that the corresponding squared matrix element averaged over incoming spins and summed over outgoing spins is given by:*

$$\frac{1}{2}\sum_{spins}\mathcal{M}\mathcal{M}^* = \frac{1}{2}\left(\frac{e\lambda}{4m_{\ell^*}}\right)^2 q_\mu q_\alpha \epsilon^*_\nu \epsilon_\beta \,\mathrm{Tr}\left[\not{p}_2(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(1-\gamma^5)\not{p}_1(1+\gamma^5)\left(\gamma^\beta\gamma^\alpha - \gamma^\alpha\gamma^\beta\right)\right] \tag{13.2}$$

c) *Use the photon completeness relation to find that:*

$$\langle|\mathcal{M}|^2\rangle = \left(\frac{e\lambda}{4m_{\ell^*}}\right)^2\left[32(p_1\cdot q)(p_2\cdot q)\right] \tag{13.3}$$

d) *Choose the rest frame of the ℓ^*, thus assign $p_1 = (m_{\ell^*}, \vec{0})$ and $p_2 = (E_2, \vec{p}_2)$. Obtain the partial width of the decay:*

$$\mathrm{d}\Gamma = \frac{1}{8m_{\ell^*}}\frac{1}{(2\pi)^2}\left(\frac{e\lambda}{4m_{\ell^*}}\right)^2\left[32(m^3_{\ell^*}E_2 - m^2_{\ell^*}E_2^2)\right]\frac{p_2^2\mathrm{d}p_2\mathrm{d}\Omega\mathrm{d}^3\vec{k}}{\omega E_2}\delta(m_{\ell^*} - E_2 - \omega)\delta^3(-\vec{p}_2 - \vec{k}) \tag{13.4}$$

e) *Integrate, neglecting the mass of the ordinary lepton, to find:*

$$\Gamma(\ell^* \to \ell\gamma) = \frac{\alpha}{4}\left(\frac{\lambda}{m_{\ell^*}}\right)^2 m^3_{\ell^*} \tag{13.5}$$

f) *Estimate numerically the lifetime and the mean free path of a 45 GeV ℓ^* for a coupling $\lambda/m_{\ell^*} \approx 10^{-4}$ GeV^{-1}, produced in the reaction $e^+e^- \to Z^0 \to \ell\ell^*$.*

Solution:

a) The general form of the magnetic interaction $\ell^*\ell\gamma$ between excited leptons, ordinary leptons, and photons is written as (see Eq. (32.20) of the book):

$$\mathcal{L}_{\ell^*\ell\gamma} = \frac{e\lambda}{2m_{\ell^*}}\overline{\Psi_{\ell^*}}\sigma^{\mu\nu}(a_\gamma - b_\gamma\gamma^5)\Psi_\ell F_{\mu\nu} + \mathrm{h.c.} \tag{13.6}$$

where λ is the unitless coupling constant, m_{ℓ^*} the mass of the excited lepton, and a_γ, b_γ are the vector and axial-vector couplings, and (see Eqs. (E.5) and (E.6) in Appendix E of the book):

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \quad \text{and} \quad \sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] = \frac{i}{2}\left[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right] \tag{13.7}$$

In the following we set $a_\gamma = b_\gamma = 1$.

We define the kinematics and the spins involved in the decay process:

$$\ell^*(p_1, s_1) \to \ell(p_2, s_2) + \gamma(q, \epsilon) \tag{13.8}$$

where presumably $m_\ell \ll m_{\ell^*}$. The Feynman diagram for the decay easily derived from the Lagrangian Eq. (13.6) is shown in Figure 13.1. Accordingly, the amplitude is just:

$$\mathcal{M} = \frac{e\lambda}{2m_{\ell^*}}\left[\bar{u}(p_2, s_2)\sigma^{\mu\nu}(\mathbb{1} - \gamma^5)q_\nu\epsilon^*_\mu u(p_1, s_1)\right] \tag{13.9}$$

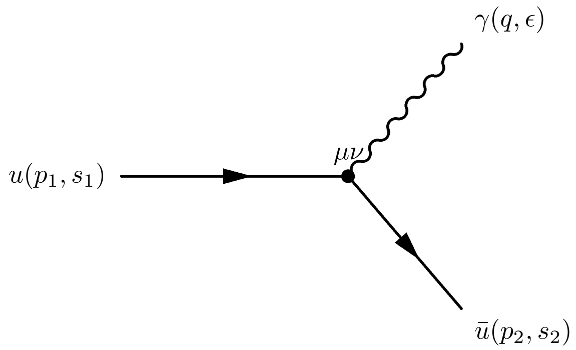

Figure 13.1 Feynman diagram for the decay $\ell^* \to \ell + \gamma$.

b) Using the definition of $\sigma^{\mu\nu}$, the amplitude can be written as:

$$\mathcal{M} = \frac{ie\lambda}{4m_{\ell^*}}\left[\bar{u}(p_2, s_2)\left[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right](\mathbb{1} - \gamma^5)q_\mu\epsilon^*_\nu u(p_1, s_1)\right] \tag{13.10}$$

We need to compute $\mathcal{M}\mathcal{M}^*$:

$$\begin{aligned}\mathcal{M}\mathcal{M}^* = \left(\frac{e\lambda}{4m_{\ell^*}}\right)^2 &\left[\bar{u}(p_2, s_2)\left[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right](\mathbb{1} - \gamma^5)q_\mu\epsilon^*_\nu u(p_1, s_1)\right] \\ &\left[\bar{u}(p_2, s_2)\left[\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right](\mathbb{1} - \gamma^5)q_\alpha\epsilon^*_\beta u(p_1, s_1)\right]^*\end{aligned} \tag{13.11}$$

Now let's define

$$\Gamma \equiv \left[\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right](\mathbb{1} - \gamma^5) \tag{13.12}$$

then, using $\gamma^0 = \gamma^{0\dagger}$, we find:

$$\begin{aligned}\left[\bar{u}(p_2, s_2)\Gamma q_\alpha\epsilon^*_\beta u(p_1, s_1)\right]^* &= \left[u^\dagger(p_1, s_1)\epsilon_\beta q_\alpha\Gamma^\dagger\gamma^{0\dagger}u(p_2, s_2)\right] \\ &= \left[u^\dagger(p_1, s_1)\gamma^0\gamma^0\Gamma^\dagger\gamma^{0\dagger}\epsilon_\beta q_\alpha u(p_2, s_2)\right] \\ &= \left[\bar{u}(p_1, s_1)\overline{\Gamma}\epsilon_\beta q_\alpha u(p_2, s_2)\right]\end{aligned} \tag{13.13}$$

where, using also $\gamma^5 = \gamma^{5\dagger}$, we find:

$$\begin{aligned}\overline{\Gamma} \equiv \gamma^0\Gamma^\dagger\gamma^0 &= \gamma^0(\mathbb{1}-\gamma^5)\left[\gamma^{\beta,\dagger}\gamma^{\alpha,\dagger} - \gamma^{\alpha,\dagger}\gamma^{\beta,\dagger}\right]\gamma^0 \\ &= \gamma^0(\mathbb{1}-\gamma^5)\left[\gamma^0\gamma^\beta\gamma^0\gamma^0\gamma^\alpha\gamma^0 - \gamma^0\gamma^\alpha\gamma^0\gamma^0\gamma^\beta\gamma^0\right]\gamma^0 \\ &= (\mathbb{1}+\gamma^5)\left[\gamma^\beta\gamma^\alpha - \gamma^\alpha\gamma^\beta\right]\end{aligned} \tag{13.14}$$

where we used Eq. (E.4) from Appendix E.1 of the book which states $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

Summing over outgoing spins and averaging over the incoming spin of the ℓ^* we find using Casimir's trick:

$$\begin{aligned}\frac{1}{2}\sum_{s_1,s_2}\mathcal{M}\mathcal{M}^* &= \frac{1}{2}\left(\frac{e\lambda}{4m_{\ell^*}}\right)^2 q_\mu q_\alpha \epsilon^*_\nu\epsilon_\beta \times \\ &\quad \mathrm{Tr}\left[(\not{p}_2 + m_\ell\mathbb{1})\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)(\mathbb{1}-\gamma^5)(\not{p}_1 + m_{\ell^*}\mathbb{1})(\mathbb{1}+\gamma^5)\left(\gamma^\beta\gamma^\alpha - \gamma^\alpha\gamma^\beta\right)\right] \\ &= \frac{1}{2}\left(\frac{e\lambda}{4m_{\ell^*}}\right)^2 q_\mu q_\alpha \epsilon^*_\nu\epsilon_\beta\,\mathrm{Tr}\left[\not{p}_2\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)(\mathbb{1}-\gamma^5)(\not{p}_1 + m_{\ell^*}\mathbb{1})(\mathbb{1}+\gamma^5)\left(\gamma^\beta\gamma^\alpha - \gamma^\alpha\gamma^\beta\right)\right]\end{aligned} \tag{13.15}$$

where in the last step we assumed that $m_\ell \ll m_{\ell^*}$. We recall that the trace of an odd number of γ matrices vanishes and given its definition, the γ^5 counts as one. The term proportional to m_{ℓ^*} is the following:

$$m_{\ell^*}\,\mathrm{Tr}\left[\not{p}_2\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)\underbrace{(\mathbb{1}-\gamma^5)(\mathbb{1}+\gamma^5)}_{=0}\left(\gamma^\beta\gamma^\alpha - \gamma^\alpha\gamma^\beta\right)\right] = 0 \tag{13.16}$$

Consequently:

$$\frac{1}{2}\sum_{s_1,s_2}\mathcal{M}\mathcal{M}^* = \frac{1}{2}\left(\frac{e\lambda}{4m_{\ell^*}}\right)^2 q_\mu q_\alpha \epsilon^*_\nu\epsilon_\beta\,\mathrm{Tr}\left[\not{p}_2\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)(\mathbb{1}-\gamma^5)\not{p}_1(\mathbb{1}+\gamma^5)\left(\gamma^\beta\gamma^\alpha - \gamma^\alpha\gamma^\beta\right)\right] \tag{13.17}$$

c) The trace can be rewritten as:

$$\frac{1}{2}\sum_{s_1,s_2}\mathcal{M}\mathcal{M}^* = \frac{1}{2}\left(\frac{e\lambda}{4m_{\ell^*}}\right)^2 \epsilon^*_\nu\epsilon_\beta\,\mathrm{Tr}\left[\not{p}_2\left(\not{q}\gamma^\nu - \gamma^\nu\not{q}\right)(\mathbb{1}-\gamma^5)\not{p}_1(\mathbb{1}+\gamma^5)\left(\gamma^\beta\not{q} - \not{q}\gamma^\beta\right)\right] \tag{13.18}$$

Using the fact that the trace of an odd number of γ matrices vanishes, the last expression can be split into the two following parts:

$$\begin{aligned}&\mathrm{Tr}\left[\not{p}_2\left(\not{q}\gamma^\nu - \gamma^\nu\not{q}\right)\not{p}_1\left(\gamma^\beta\not{q} - \not{q}\gamma^\beta\right)\right] - \mathrm{Tr}\left[\not{p}_2\left(\not{q}\gamma^\nu - \gamma^\nu\not{q}\right)\gamma^5\not{p}_1\gamma^5\left(\gamma^\beta\not{q} - \not{q}\gamma^\beta\right)\right] \\ &\quad = \mathrm{Tr}\left[\not{p}_2\left(\not{q}\gamma^\nu - \gamma^\nu\not{q}\right)\not{p}_1\left(\gamma^\beta\not{q} - \not{q}\gamma^\beta\right)\right] + \mathrm{Tr}\left[\not{p}_2\left(\not{q}\gamma^\nu - \gamma^\nu\not{q}\right)\not{p}_1\left(\gamma^\beta\not{q} - \not{q}\gamma^\beta\right)\right] \\ &\quad = 2\,\mathrm{Tr}\left[\not{p}_2\left(\not{q}\gamma^\nu - \gamma^\nu\not{q}\right)\not{p}_1\left(\gamma^\beta\not{q} - \not{q}\gamma^\beta\right)\right]\end{aligned} \tag{13.19}$$

where, in the second line, we used $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$ and $(\gamma^5)^2 = \mathbb{1}$. We therefore have the following four traces to compute (using the short notation $\mathrm{Tr}(ab\dots f) = \mathrm{Tr}(\gamma_a\gamma_b\dots\gamma_f)$):

- $T_1^{\nu\beta} \equiv \mathrm{Tr}\left[\not{p}_2\not{q}\gamma^\nu\not{p}_1\gamma^\beta\not{q}\right] = p_2^a q^b \delta_\nu^c p_1^d \delta_\beta^e q^f\,\mathrm{Tr}(abdecf)$
- $T_2^{\nu\beta} \equiv -\mathrm{Tr}\left[\not{p}_2\gamma^\nu\not{q}\not{p}_1\gamma^\beta\not{q}\right] = -p_2^a \delta_\nu^b q^c p_1^d \delta_\beta^e q^f\,\mathrm{Tr}(abdecf)$
- $T_3^{\nu\beta} \equiv -\mathrm{Tr}\left[\not{p}_2\not{q}\gamma^\nu\not{p}_1\not{q}\gamma^\beta\right] = -p_2^a q^b \delta_\nu^c p_1^d q^e \delta_\beta^f\,\mathrm{Tr}(abdecf)$
- $T_4^{\nu\beta} \equiv \mathrm{Tr}\left[\not{p}_2\gamma^\nu\not{q}\not{p}_1\not{q}\gamma^\beta\right] = p_2^a \delta_\nu^b q^c p_1^d q^e \delta_\beta^f\,\mathrm{Tr}(abdecf)$

We have to compute the trace of the product of six γ matrices. We are going make use of Eq. (12.61) derived in **Ex. 12.2**:

$$\mathrm{Tr}(abcdef) = g_{ab}\,\mathrm{Tr}(cdef) - g_{ac}\,\mathrm{Tr}(bdef) + g_{ad}\,\mathrm{Tr}(bcef) - g_{ae}\,\mathrm{Tr}(bcdf) + g_{af}\,\mathrm{Tr}(bcde) \tag{13.20}$$

and also the trace theorem **[T5]**ăin Appendix E.2 of the book:

$$\mathrm{Tr}(abcd) = 4(g_{ab}g_{cd} - g_{ac}g_{bd} + g_{ad}g_{bc}) \tag{13.21}$$

Consequently:

$$\begin{aligned}\mathrm{Tr}(abcdef) =& 4g_{ab}(g_{cd}g_{ef} - g_{ce}g_{df} + g_{cf}g_{de}) \\ & - 4g_{ac}(g_{bd}g_{ef} - g_{be}g_{df} + g_{bf}g_{de}) \\ & + 4g_{ad}(g_{bc}g_{ef} - g_{be}g_{cf} + g_{bf}g_{ce}) \\ & - 4g_{ae}(g_{bc}g_{df} - g_{bd}g_{cf} + g_{bf}g_{cd}) \\ & + 4g_{af}(g_{bc}g_{de} - g_{bd}g_{ce} + g_{be}g_{cd})\end{aligned} \tag{13.22}$$

We can then show that:

$$\begin{aligned}T_1^{\nu\beta} =& 4(p_2\cdot q)(p_1^\nu q^\beta - g^{\nu\beta}(p_1\cdot q) + q^\nu p_1^\beta) \\ & - 4p_2^\nu((p_1\cdot q)q^\beta - q^\beta(p_1\cdot q) + q^2 p_1^\beta) \\ & + 4(p_2\cdot p_1)(q^\nu q^\beta - q^\beta q^\nu + q^2 g^{\nu\beta}) \\ & - 4p_2^\beta(q^\nu(p_1\cdot q) - (q\cdot p_1)q^\nu + q^2 p_1^\nu) \\ & + 4(p_2\cdot q)(q^\nu p_1^\beta - (q\cdot p_1)g^{\nu\beta} + q^\beta p_1^\nu) \\ =& 8(p_2\cdot q)(p_1^\nu q^\beta + q^\nu p_1^\beta) - 8(p_1\cdot q)(p_2\cdot q)g^{\nu\beta}\end{aligned} \tag{13.23}$$

where we used that $q^2 = 0$, and

$$\begin{aligned}T_2^{\nu\beta} =& -4p_2^\nu((q\cdot p_1)q^\beta - q^\beta(q\cdot p_1) + q^2 p_1^\beta) \\ & + 4(p_2\cdot q)(p_1^\nu q^\beta - g^{\nu\beta}(q\cdot p_1) + q^\nu p_1^\beta) \\ & - 4(p_1\cdot p_2)(q^\nu q^\beta - g^{\nu\beta}q^2 + q^\nu q^\beta) \\ & + 4p_2^\beta(q^\nu(p_1\cdot q) - p_1^\nu q^2 + q^\nu(q\cdot p_1)) \\ & - 4(q\cdot p_2)(q^\nu p_1^\beta - p_1^\nu q^\beta + g^{\nu\beta}(q\cdot p_1)) \\ =& 8(p_2\cdot q)p_1^\nu q^\beta - 8(p_1\cdot p_2)q^\nu q^\beta + 8(p_1\cdot q)q^\nu p_2^\beta - 8(p_2\cdot q)(q\cdot p_1)g^{\nu\beta}\end{aligned} \tag{13.24}$$

and

$$\begin{aligned}T_3^{\nu\beta} =& -4(q\cdot p_2)(p_1^\nu q^\beta - q^\nu p_1^\beta + g^{\nu\beta}(p_1\cdot q)) \\ & + 4p_2^\nu((q\cdot p_1)q^\beta - q^2 p_1^\beta + q^\beta(p_1\cdot q)) \\ & - 4(p_2\cdot p_1)(q^\nu q^\beta - q^2 g^{\nu\beta} + q^\beta q^\nu) \\ & + 4(p_2\cdot q)(q^\nu p_1^\beta - (q\cdot p_1)g^{\nu\beta} + q^\beta p_1^\nu) \\ & - 4p_2^\beta(q^\nu(q\cdot p_1) - (q\cdot p_1)q^\nu + q^2 p_1^\nu) \\ =& 8(q\cdot p_1)p_2^\nu q^\beta - 8(p_2\cdot p_1)q^\nu q^\beta + 8(p_2\cdot q)q^\nu p_1^\beta - 8(q\cdot p_1)(p_2\cdot q)g^{\nu\beta}\end{aligned} \tag{13.25}$$

and similarly

$$T_4^{\nu\beta} = 8(q\cdot p_1)(p_2^\nu q^\beta + p_2^\beta q^\nu) - 8(p_1\cdot q)(p_2\cdot q)g^{\nu\beta} \tag{13.26}$$

We can check these results with Feyncalc (see Section 11.13 of the book) by typing:

```
T1 := DiracSimplify[DiracTrace[GS[p2].GS[q].GA[\[Nu]].GS[p1].GA[\[Beta]].GS[q]]]
T2 := -DiracSimplify[DiracTrace[GS[p2].GA[\[Nu]].GS[q].GS[p1].GA[\[Beta]].GS[q]]]
T3 := -DiracSimplify[DiracTrace[GS[p2].GS[q].GA[\[Nu]].GS[p1].GS[q].GA[\[Beta]]]]
T4 := DiracSimplify[DiracTrace[GS[p2].GA[\[Nu]].GS[q].GS[p1].GS[q].GA[\[Beta]]]]
```

and directly verify the correctness of our expressions. Then we can sum all four traces and regroup smartly to get:

$$\frac{1}{2}\sum_{s_1,s_2}\mathcal{M}\mathcal{M}^* = \frac{1}{2}\left(\frac{e\lambda}{4m_{\ell^*}}\right)^2 \epsilon^*_\nu\epsilon_\beta\sum_i T_i^{\nu\beta} \tag{13.27}$$

where

$$\sum_i T_i^{\nu\beta} = 16(q\cdot p_1)(p_2^\beta q^\nu + p_2^\nu q^\beta - (q\cdot p_2)g^{\nu\beta}) + 16(q\cdot p_2)(p_1^\beta q^\nu + p_1^\nu q^\beta - (q\cdot p_1)g^{\nu\beta}) - 16(p_1\cdot p_2)q^\nu q^\beta \tag{13.28}$$

which is quite neatly and beautifully symmetric. We now want to sum over the final-state photon polarization. To do this, we use the following completeness relation (see Eq. (10.76) of the book):

$$\sum_\lambda \epsilon^*_\nu\epsilon_\beta \leftrightarrow -g^{\nu\beta} \tag{13.29}$$

Hence:

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\sum_{s_1,s_2,\lambda}\mathcal{M}\mathcal{M}^* = -\frac{1}{2}\left(\frac{e\lambda}{4m_{\ell^*}}\right)^2 g_{\nu\beta}\sum_i T_i^{\nu\beta} \tag{13.30}$$

where

$$\begin{aligned} g_{\nu\beta}\sum_i T_i^{\nu\beta} =& 16(q\cdot p_1)(p_2\cdot q + p_2\cdot q - 4(q\cdot p_2)) + 16(q\cdot p_2)(p_1\cdot q + p_1\cdot q - 4(q\cdot p_1)) - 16(p_1\cdot p_2)q^2 \\ =& -64(q\cdot p_1)(q\cdot p_2) \end{aligned} \tag{13.31}$$

Hence:

$$\langle|\mathcal{M}|^2\rangle = \left(\frac{e\lambda}{4m_{\ell^*}}\right)^2 [32(q\cdot p_1)(q\cdot p_2)] \tag{13.32}$$

d) We choose the reference frame of the ℓ^* particle. Hence, we can assign:

$$p_1^\mu = (m_{\ell^*},\vec{0}) \quad \text{and} \quad p_2^\mu = (E_2,\vec{p}_2) \quad \text{and} \quad q^\mu = (\omega,\vec{k}) \tag{13.33}$$

Energy-momentum conservation implies:

$$p_1 = p_2 + q \quad \Longrightarrow \quad q = p_1 - p_2 \tag{13.34}$$

Hence:

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle &= \left(\frac{e\lambda}{4m_{\ell^*}}\right)^2 [32((p_1-p_2)\cdot p_1)((p_1-p_2)\cdot p_2)] = \left(\frac{e\lambda}{4m_{\ell^*}}\right)^2 \left[32(p_1^2 - p_2\cdot p_1)(p_1\cdot p_2 - p_2^2))\right] \\ &= \left(\frac{e\lambda}{4m_{\ell^*}}\right)^2 \left[32(m_{\ell^*}^2 - m_{\ell^*}E_2)(m_{\ell^*}E_2))\right] = \left(\frac{e\lambda}{4m_{\ell^*}}\right)^2 \left[32(m_{\ell^*}^3E_2 - m_{\ell^*}^2E_2^2)\right] \end{aligned} \tag{13.35}$$

where we used $p_1\cdot p_2 = m_{\ell^*}E_2$ and assumed $p_2^2 = m_\ell^2 = 0$. The partial decay width is obtained using Eq. (5.152) of the book:

$$\begin{aligned} d\Gamma &= \frac{\langle|\mathcal{M}|^2\rangle}{2m_{\ell^*}}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\frac{d^3\vec{k}}{(2\pi)^3 2\omega}(2\pi)^4\delta^4(p_1-p_2-q) \\ &= \frac{1}{8m_{\ell^*}}\frac{1}{(2\pi)^2}\left(\frac{e\lambda}{4m_{\ell^*}}\right)^2\left[32(m_{\ell^*}^3E_2 - m_{\ell^*}^2E_2^2)\right]\frac{p_2^2 dp_2 d\Omega d^3\vec{k}}{\omega E_2}\delta(m_{\ell^*}-E_2-\omega)\delta^3(-\vec{p}_2-\vec{k}) \end{aligned} \tag{13.36}$$

e) We have $\omega = |\vec{k}|$ and neglecting the mass m_ℓ, since $m_\ell \ll m_{\ell^*}$, we have $E_2 = |\vec{p}_2|$. The partial decay width is obtained using Eq. (5.152) of the book:

$$\begin{aligned} d\Gamma &= \frac{1}{8m_{\ell^*}}\frac{1}{(2\pi)^2}\left(\frac{e\lambda}{4m_{\ell^*}}\right)^2\left[32(m_{\ell^*}^3 E_2 - m_{\ell^*}^2 E_2^2)\right]\frac{p_2^2 dp_2 d\Omega d^3\vec{k}}{\omega E_2}\delta(m_{\ell^*} - E_2 - \omega)\delta^3(-\vec{p}_2 - \vec{k}) \\ &= \frac{1}{8m_{\ell^*}}\frac{1}{(2\pi)^2}\left(\frac{e\lambda}{4m_{\ell^*}}\right)^2\left[32(m_{\ell^*}^3 E_2 - m_{\ell^*}^2 E_2^2)\right]\frac{p_2^2 dp_2 d\Omega}{\omega E_2}\delta(m_{\ell^*} - E_2 - \omega) \end{aligned} \tag{13.37}$$

where after integration over $d^3\vec{k}$, we have $\vec{k} = -\vec{p}_2$, and hence $E_2 = \omega = m_{\ell^*}/2$. The two final-state particles are back-to-back and possess the same energy when we neglect the mass m_ℓ, equal to half of the mass of the excited lepton. We now integrate over $dp_2 d\Omega$ to find:

$$\Gamma = \frac{1}{8m_{\ell^*}}\frac{4\pi}{(2\pi)^2}\left(\frac{e\lambda}{4m_{\ell^*}}\right)^2\left[32\left(\frac{m_{\ell^*}^3 m_{\ell^*}}{2} - \frac{m_{\ell^*}^2 m_{\ell^*}^2}{4}\right)\right] = \frac{1}{4}\frac{1}{4\pi}\left(\frac{e\lambda}{m_{\ell^*}}\right)^2 m_{\ell^*}^3 \tag{13.38}$$

Finally,

$$\Gamma(\ell^* \to \ell\gamma) = \frac{1}{4}\frac{e^2}{4\pi}\left(\frac{\lambda}{m_{\ell^*}}\right)^2 m_{\ell^*}^3 = \frac{\alpha}{4}\left(\frac{\lambda}{m_{\ell^*}}\right)^2 m_{\ell^*}^3 \tag{13.39}$$

where $\alpha = e^2/4\pi$ is the fine structure constant.

f) We assume $m_{\ell^*} = 45$ GeV/c^2. The decay width in GeV can be expressed as:

$$\Gamma = \frac{\alpha}{4}\lambda^2 m_{\ell^*} \tag{13.40}$$

The lifetime τ is given by:

$$\tau = \frac{1}{\Gamma}\breve{a} = \frac{4}{\alpha\lambda^2}m_{\ell^*}^{-1} \approx \frac{12}{\lambda^2}\text{ GeV}^{-1} \approx \frac{8\times 10^{-24}}{\lambda^2}\text{ s} \tag{13.41}$$

where we used that 1 GeV$^{-1} = 6.6\times 10^{-25}$ s (see Table 1.6 in Section 1.8 of the book). For the reaction $e^+e^- \to Z^0 \to \ell\ell^*$, we assume the configuration is an e^+e^- collider, hence we have:

$$E_{\ell^*} + E_\ell = M_Z \qquad \text{and} \qquad \vec{p}_{\ell^*} + \vec{p}_\ell = 0 \Longrightarrow p \equiv |\vec{p}_{\ell^*}| = |\vec{p}_\ell| \tag{13.42}$$

This gives us (neglecting the ℓ rest mass):

$$\sqrt{p^2 + m_{\ell^*}^2} + p = M_Z \quad \Longrightarrow \quad p = \frac{1}{2}\left(\frac{M_Z^2 - m_{\ell^*}^2}{M_Z}\right) \tag{13.43}$$

Numerically,

$$M_Z \simeq 91.2\text{ GeV} \Longrightarrow p \simeq 34.5\text{ GeV}, \quad E_{\ell^*} = \sqrt{p^2 + m_{\ell^*}^2} \simeq 56.7\text{ GeV} \tag{13.44}$$

The mean free path L is given by:

$$L = \gamma\beta c\tau \simeq \frac{1.84\times 10^{-13}\text{ cm}}{\lambda^2} \tag{13.45}$$

where $\gamma = E_{\ell^*}/m_{\ell^*} \simeq 1.26$ is the Lorentz boost and $\beta c = pc/E_{\ell^*} \simeq 0.61c$ is the velocity of the ℓ^*. Numerically, assuming $\lambda/m_{\ell^*} \approx 10^{-4}$ GeV^{-1}, we find:

$$\lambda \approx 10^{-4}\text{ GeV}^{-1} m_{\ell^*} \quad \Longrightarrow \quad \lambda^2 \simeq 2\times 10^{-5} \tag{13.46}$$

Hence:

$$L \simeq 10^{-8}\text{ cm} \tag{13.47}$$

13.2 Dark photon production in e^+e^- collisions

In addition to gravity, dark matter could interact feebly to the ordinary particles through some mediators. We shall study the production of a hypothetical massive spin-1 A' boson (dark photon) in electron–positron collisions and its corresponding decay properties for the following effective Lagrangian:

$$\mathcal{L} \supset -\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}m_{A'}^2 X_\mu X^\mu - \epsilon e Q_f X_\mu \overline{\psi}\gamma^\mu\psi + (D_\mu\phi)^* D^\mu\phi - \mu^2\phi^*\phi - \frac{1}{2}\rho^2(\phi\phi + \phi^*\phi^*) \tag{13.48}$$

where $\tilde{F}^{\mu\nu} = \partial^\mu X^\nu - \partial^\nu X^\mu$ is the A' boson field strength tensor, $D_\mu = \partial_\mu + ig_D X_\mu$ with g_D the dark coupling parameter, and $\phi = 1/\sqrt{2}(\phi_1 + i\phi_2)$ the complex dark-matter scalar field; ϵ is the mixing parameter between the $U(1)_Y$ and $U(1)_{A'}$ gauge fields.

1. *Draw the lowest-order Feynman diagrams corresponding to the process $e^+e^- \to A'\gamma$ and compute the differential cross-section $\mathrm{d}\sigma/\mathrm{d}\cos\theta$ where θ is the angle between the incoming electron and the outgoing A' boson. How does it depend on ϵ and $m_{A'}$?*
2. *Compute the total cross-section for the above process. What is the center-of-mass energy dependence? To which collider energy would the experiment be the most sensitive?*
3. *Given the above Lagrangian, determine what are the possible decay channels of the A' boson.*
4. *Compute the spin-average decay width for each of the above final states. How do the branching ratios depend on the parameters ϵ, g_D and the dark photon masses?*
5. *How would you detect the final states in a fixed-target experiment and thus infer the presence of the A' boson?*

Solution:

a) This exercise is inspired from the article X. Chen *et al.*, "Search for dark photon and dark matter signatures around electron-positron colliders", Phys. Lett. **B814** 136076 (2021). The existence of Dark Matter (DM) is well motivated by astrophysical observation, but its composition in terms of elementary particles remain an unresolved puzzle. If the DM is composed of elementary particles, these latter will not only interact gravitationally, but they could possibly also interact weakly with "ordinary" matter (i.e. the particles of the Standard Model) through some "mediators". One such mediator is called the "Dark Photon", generally labeled A'. The presence of Dark Matter interacting via a Dark Photon is commonly introduced with an effective Lagrangian of the form to be added to the SM Lagrangian:

$$\mathcal{L} \supset \underbrace{-\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}}_{\text{kinetic}} + \underbrace{\frac{1}{2}m_{A'}^2 X_\mu X^\mu}_{A'\text{mass term}} - \underbrace{\epsilon e Q_f X_\mu \overline{\psi}\gamma^\mu\psi}_{A'\text{coupling to SM}} + \underbrace{(D_\mu\phi)^* D^\mu\phi}_{\text{DM kinetic and interaction}} - \underbrace{\mu^2\phi^*\phi}_{\text{DM mass}} - \underbrace{\frac{1}{2}\rho^2(\phi\phi + \phi^*\phi^*)}_{U(1)_{A'}\text{gauge violation}} \tag{13.49}$$

where X^μ is the Dark Photon A' boson field, $\tilde{F}^{\mu\nu} = \partial^\mu X^\nu - \partial^\nu X^\mu$ is the Dark Photon field strength tensor, $m_{A'}$ is the rest mass of the Dark Photon, f is a SM fermion with charge Q_f, $D_\mu = \partial_\mu + ig_D X_\mu$ is the covariant derivative with g_D the dark coupling parameter, and $\phi = 1/\sqrt{2}(\phi_1 + i\phi_2)$ the complex dark-matter scalar field; the term proportional to ρ is a Majorana-like term violating $U(1)_{A'}$ symmetry. This violation causes a mass splitting between the two scalar fields ϕ_1 and ϕ_2, which acquire the masses:

$$m_{1,2} = \sqrt{\mu^2 \pm \rho^2} \tag{13.50}$$

As a hypothetical particle, the mass of the Dark Photon is not well established. The mass range of interest depends on the specific theoretical model and experimental constraints. Dark Photons can have masses ranging from sub-eV to several GeV, depending on the model parameters and experimental constraints. Here we are considering the production of the Dark Photon at an e^+e^- collider, so we are implicitly focusing on the GeV-range. In other words, $m_{A'} \gg m_e$ and the center-of-mass energy of the collider must be sufficient to produce the Dark Photon. This will allow us to approximate $m_e \approx 0$, thereby greatly simplifying the calculations.

Let us look at the interaction of the Dark Photon with SM fermions. When compared to the ordinary QED coupling we see that ϵ behaves as a mixing parameter between the $U(1)_Y$ and $U(1)_{A'}$ gauge fields. Indeed, the interactions are similar in form except that the interaction with the Dark Photon is suppressed by the small factor ϵ:

$$\text{QED:} \quad eQ_f A_\mu \overline{\psi}\gamma^\mu\psi \qquad\qquad \text{Dark Photon } A': \quad \epsilon e Q_f X_\mu \overline{\psi}\gamma^\mu\psi \tag{13.51}$$

Hence, the Feynman rules for the Dark Photon are equivalent to that for the ordinary photon except that:

- The vertex factor must be changed $-ie\gamma^\mu \to \epsilon e\gamma^\mu Q_f$
- The A' is assumed to be a massive spin-1 boson, hence has three independent polarization states, while the photon is massless and accordingly has only two possible polarizations.

Accordingly, the $e^+ + e^- \to \gamma + A'$ process (A' on-shell) can be studied in a similar way as the $e^+ + e^- \to \gamma + \gamma$, after the appropriate changes. Pair annihilation into two photons in QED was discussed in detail in Section 11.15 of the book. We make use of the results derived there to study the $e^+ + e^- \to \gamma + A'$ process. The tree-level Feynman diagrams for the $e^+ + e^- \to \gamma + \gamma$ process are shown in Figure 11.14 of the book. The corresponding Feynman diagrams at tree level for $e^+ + e^- \to \gamma + A'$ are shown in Figure 13.2.

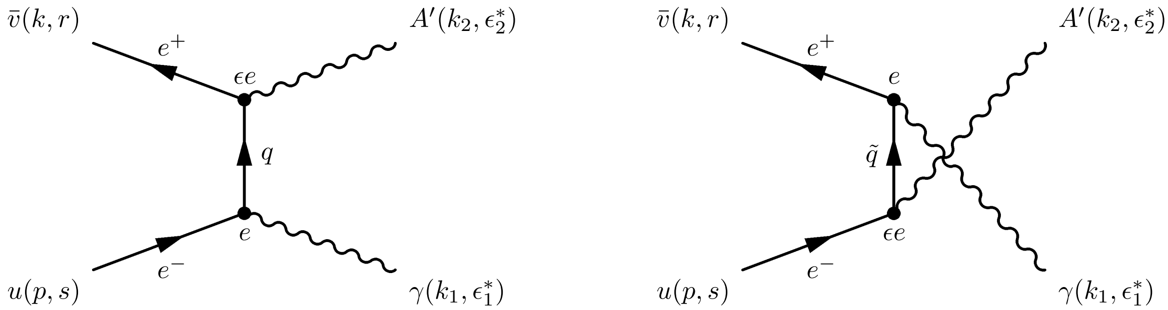

Figure 13.2 Feynman diagrams for $e^+e^- \to \gamma A'$ at tree level.

The kinematics is given by:

$$e^-(p,s) + e^+(k,r) \to \gamma(k_1,\epsilon_1) + A'(k_2,\epsilon_2) \tag{13.52}$$

We note that $p^2 = k^2 = m_e^2$, $k_1^2 = 0$, $k_2^2 = m_{A'}^2$, $q = p - k_1 = k_2 - k$ and $\tilde{q} = p - k_2 = k_1 - k$. The fermion propagators therefore have the form:

$$\frac{i(\not{q} + m_e)}{q^2 - m_e^2} = \frac{i(\not{p} - \not{k}_1 + m_e)}{(p-k_1)^2 - m_e^2} = \frac{i(\not{p} - \not{k}_1 + m_e)}{-2p\cdot k_1} \tag{13.53}$$

where we used $((p-k_1)^2 - m_e^2 = p^2 - 2p\cdot k_1 + k_1^2 - m_e^2 = -2p\cdot k_1$ since $p^2 = m_e^2$ and $k_1^2 = 0$. And similarly for $\tilde{q} = p - k_2$,

$$\frac{i(\not{\tilde{q}} + m_e)}{\tilde{q}^2 - m_e^2} = \frac{i(\not{p} - \not{k}_2 + m_e)}{m_{A'}^2 - 2p\cdot k_2} \tag{13.54}$$

Following the procedure used in Section 11.15 of the book, the net amplitude of the two diagrams can be expressed by factorizing the outgoing photon polarization vectors:

$$i\mathcal{M} = \epsilon^*_{2\mu}(k_2)\epsilon^*_{1\nu}(k_1)\left(i\mathcal{M}_1^{\mu\nu} + i\mathcal{M}_2^{\mu\nu}\right) \tag{13.55}$$

where (compare Eq. (11.193) of the book):

$$i\mathcal{M}_1^{\mu\nu} = \frac{(-i\epsilon e^2)}{-2p\cdot k_1}\bar{v}(k,r)\gamma^\mu(\not{p} - \not{k}_1 + m_e)\gamma^\nu u(p,s) \tag{13.56}$$

and (compare Eq. (11.194) of the book):

$$i\mathcal{M}_2^{\mu\nu} = \frac{(-i\epsilon e^2)}{m^2_{A'} - 2p\cdot k_2}\bar{v}(k,r)\gamma^\nu(\not{p} - \not{k}_2 + m_e)\gamma^\mu u(p,s) \tag{13.57}$$

Note the opposite orders of the γ^μ and γ^ν vertices in the $\mathcal{M}_1^{\mu\nu}$ and $\mathcal{M}_2^{\mu\nu}$ amplitudes, as well as the extra mixing factor ϵ compared to the QED case. We can express these two amplitudes in compact form by defining $i\mathcal{M}^{\mu\nu} \equiv i\mathcal{M}_1^{\mu\nu} + i\mathcal{M}_2^{\mu\nu}$, i.e.:

$$i\mathcal{M}^{\mu\nu} \equiv (-i\epsilon e^2)\bar{v}(k,r)\left(\frac{\gamma^\mu(\not{p} - \not{k}_1 + m_e)\gamma^\nu}{-2p\cdot k_1} + \frac{\gamma^\nu(\not{p} - \not{k}_2 + m_e)\gamma^\mu}{m^2_{A'} - 2p\cdot k_2}\right)u(p,s) \tag{13.58}$$

Then

$$i\mathcal{M} = \epsilon^*_{2\mu}(k_2)\epsilon^*_{1\nu}(k_1)i\mathcal{M}^{\mu\nu} \tag{13.59}$$

We are interested in the unpolarized cross-section. We therefore need to average over the initial-state spins of the fermions and sum over the final-state polarization states of the photon and Dark Photon. The photon possesses *two*ăindependent polarization states, while the Dark Photon being massive possesses *three* independent polarization states (although both photon and Dark Photon have spin-1). Hence, the matrix element squared will be given by:

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\frac{1}{2}\sum_{s,s'}\sum_{\lambda_1=1,2}\sum_{\lambda_2=1,2,3}|\mathcal{M}|^2 \tag{13.60}$$

For the photon, the completeness relation yields:

$$\sum_{\lambda_1=1}^{2}\epsilon_\lambda^{\mu,*}\epsilon_\lambda^\nu \leftrightarrow -g^{\mu\nu} \tag{13.61}$$

while for the Dark Photon we have (see Eq. (10.153) of the book):

$$\sum_{\lambda_2=1}^{3}\epsilon_\lambda^{\mu,*}\epsilon_\lambda^\nu \leftrightarrow -g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2} = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2_{A'}} \tag{13.62}$$

Then:

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{s,s'}\sum_{\lambda_1,\lambda_2}\epsilon^*_{2\mu}(k_2)\epsilon^*_{1\nu}(k_1)\mathcal{M}^{\mu\nu}\epsilon_{2\rho}(k_2)\epsilon_{1\sigma}(k_1)\left(\mathcal{M}^{\rho\sigma}\right)^* \\
&= \frac{1}{4}\sum_{s,s'}g_{\nu\sigma}\left(g_{\mu\rho} - \frac{k_{2\mu}k_{2\rho}}{m^2_{A'}}\right)\mathcal{M}^{\mu\nu}\left(\mathcal{M}^{\rho\sigma}\right)^* \\
&= \underbrace{\frac{1}{4}\sum_{s,s'}\mathcal{M}^{\mu\nu}\left(\mathcal{M}_{\mu\nu}\right)^*}_{=\mathcal{A}} - \underbrace{\frac{1}{4m^2_{A'}}\sum_{s,s'}k_{2\mu}k_{2\rho}\mathcal{M}^{\mu\nu}\left(\mathcal{M}^\rho{}_\nu\right)^*}_{=\mathcal{B}}
\end{aligned} \tag{13.63}$$

where the first term corresponds to the two transverse polarizations of the Dark Photon, which reduces to the case of a massless photon.

The term $\mathcal{A}$ is going to give us the same amplitudes as for the $e^+ + e^- \to \gamma\gamma$ process with an additional ϵ factor.

$$\mathcal{A} = \frac{1}{4}\sum_{s,s'}(-i\epsilon e^2)\bar{v}(k,r)\Gamma^{\mu\nu}u(p,s)(+i\epsilon e^2)\bar{u}(p,s)\overline{\Gamma_{\mu\nu}}v(k,r) \tag{13.64}$$

where

$$\Gamma^{\mu\nu} = \left(\frac{\gamma^\mu(\not{p} - \not{k}_1 + m_e)\gamma^\nu}{-2p\cdot k_1} + \frac{\gamma^\nu(\not{p} - \not{k}_2 + m_e)\gamma^\mu}{m_{A'}^2 - 2p\cdot k_2}\right) \tag{13.65}$$

and

$$\overline{\Gamma^{\mu\nu}} = \left(\frac{\gamma^\nu(\not{p} - \not{k}_1 + m_e)\gamma^\mu}{-2p\cdot k_1} + \frac{\gamma^\mu(\not{p} - \not{k}_2 + m_e)\gamma^\nu}{m_{A'}^2 - 2p\cdot k_2}\right) \tag{13.66}$$

where we used that $\overline{\gamma^a\gamma^b\ldots\gamma^f} = \overline{\gamma^f}\ldots\overline{\gamma^b}\overline{\gamma^a}$ and that $\overline{\gamma^\mu} = \gamma^\mu$. We can now use Casimir's trick to replace the sum over the spins with a trace:

$$\begin{aligned}\mathcal{A} =& \frac{\epsilon^2 e^4}{4}\operatorname{Tr}\left((\not{k} - m_e)\Gamma^{\mu\nu}(\not{p} + m_e)\overline{\Gamma_{\mu\nu}}\right)\\ =& \frac{\epsilon^2 e^4}{4}\left[\frac{T_1}{(-2p\cdot k_1)^2} + \frac{T_2}{(-2p\cdot k_1)(m_{A'}^2 - 2p\cdot k_2)} + \frac{T_3}{(-2p\cdot k_1)(m_{A'}^2 - 2p\cdot k_2)} + \frac{T_4}{(m_{A'}^2 - 2p\cdot k_2)^2}\right]\end{aligned} \tag{13.67}$$

where

- $T_1 = \operatorname{Tr}\left((\not{k} - m_e)\gamma^\mu(\not{p} - \not{k}_1 + m_e)\gamma^\nu(\not{p} + m_e)\gamma_\nu(\not{p} - \not{k}_1 + m_e)\gamma_\mu\right)$
- $T_2 = \operatorname{Tr}\left((\not{k} - m_e)\gamma^\mu(\not{p} - \not{k}_1 + m_e)\gamma^\nu(\not{p} + m_e)\gamma_\mu(\not{p} - \not{k}_2 + m_e)\gamma_\nu\right)$
- $T_3 = \operatorname{Tr}\left((\not{k} - m_e)\gamma^\mu(\not{p} - \not{k}_2 + m_e)\gamma^\nu(\not{p} + m_e)\gamma_\nu(\not{p} - \not{k}_1 + m_e)\gamma_\mu\right)$
- $T_4 = \operatorname{Tr}\left((\not{k} - m_e)\gamma^\nu(\not{p} - \not{k}_2 + m_e)\gamma^\mu(\not{p} + m_e)\gamma_\mu(\not{p} - \not{k}_2 + m_e)\gamma_\nu\right)$

We make use of FeynCalc in order to compute these traces:

```
FCClearScalarProducts[];
ScalarProduct[k1, k1] = 0;
ScalarProduct[k2, k2] = M;
ScalarProduct[k, k] = m;
ScalarProduct[p, p] = m;
T1 := DiracSimplify[
  DiracTrace[(GS[k] - m).GA[\[Mu]].(GS[p] - GS[k1] + m).
  GA[\[Nu]].(GS[p] + m).GA[\[Nu]].(GS[p] - GS[k1] + m).GA[\[Mu]]]]
T2 := DiracSimplify[
  DiracTrace[(GS[k] - m).GA[\[Mu]].(GS[p] - GS[k1] + m).
  GA[\[Nu]].(GS[p] + m).GA[\[Mu]].(GS[p] - GS[k2] + m).GA[\[Nu]]]]
T3 := DiracSimplify[
  DiracTrace[(GS[k] - m).GA[\[Nu]].(GS[p] - GS[k2] + m).
  GA[\[Mu]].(GS[p] + m).GA[\[Nu]].(GS[p] - GS[k1] + m).GA[\[Mu]]]]
T4 := DiracSimplify[
  DiracTrace[(GS[k] - m).GA[\[Nu]].(GS[p] - GS[k2] + m).
  GA[\[Mu]].(GS[p] + m).GA[\[Mu]].(GS[p] - GS[k2] + m).GA[\[Nu]]]]
```

We find:

$$T_1 = 32\,(k\cdot k_1)\,(k_1\cdot p) + 32m_e^2\,((k\cdot k_1) - (k\cdot p) + 2\,(k_1\cdot p)) - 64m_e^4 \tag{13.68}$$

We note that the expression is simplified if we assume $p^2 = m_e^2 \simeq 0$. Hence, from now on we approximate:

$$T_1 = 32\,(k\cdot k_1)\,(k_1\cdot p) \tag{13.69}$$

and

$$T_2 = T_3 = 32\,(k\cdot p)\,((k_1\cdot p) + (k_2\cdot p) - (k_1\cdot k_2)) \tag{13.70}$$

and finally

$$T_4 = 32\,(k\cdot k_2)\,(k_2\cdot p) - 16m_{A'}^2\,(k\cdot p) \tag{13.71}$$

We may further simplify these formulae by expressing all the momenta products in terms of the Mandelstam's variables s, t, and u. Recalling $s+t+u = 2m_e^2 + m_\gamma^2 + m_{A'}^2 \simeq m_{A'}^2$, we have:

$$s = (p+k)^2 = p^2 + k^2 + 2(p\cdot k) \simeq 2(p\cdot k) \quad\rightarrow\quad p\cdot k \simeq \frac{s}{2} \tag{13.72}$$

also

$$s = (k_1+k_2)^2 = m_{A'}^2 + 2(k_1\cdot k_2) \quad\rightarrow\quad k_1\cdot k_2 = \frac{s - m_{A'}^2}{2} \tag{13.73}$$

and

$$t = (p-k_1)^2 = p^2 + k_1^2 - 2(p\cdot k_1) \simeq -2(p\cdot k_1) \quad\rightarrow\quad p\cdot k_1 = -\frac{t}{2} \tag{13.74}$$

Further

$$u = (p-k_2)^2 = p^2 + k_2^2 - 2(p\cdot k_2) \simeq m_{A'}^2 - 2(p\cdot k_2) \quad\rightarrow\quad p\cdot k_2 = -\frac{1}{2}(u - m_{A'}^2) \tag{13.75}$$

and also

$$k\cdot k_1 = (k_2 + k_1 - p)\cdot k_1 = k_2\cdot k_1 + k_1^2 - p\cdot k_1 = \frac{1}{2}(s + t - m_{A'}^2) = -\frac{u}{2} \tag{13.76}$$

and

$$k\cdot k_2 = (k_2 + k_1 - p)\cdot k_2 = k_2^2 + k_1\cdot k_2 - p\cdot k_2 = \frac{1}{2}(s+u) = -\frac{1}{2}(t - m_{A'}^2) \tag{13.77}$$

Then, we can replace the relevant products into our matrix elements to find:

$$\mathcal{A} = \frac{\epsilon^2 e^4}{4}\left[\frac{T_1}{t^2} + 2\frac{T_2}{ut} + \frac{T_4}{u^2}\right] \tag{13.78}$$

where

$$T_1 = 32\left(-\frac{t}{2}\right)\left(-\frac{u}{2}\right) = 8ut \tag{13.79}$$

and

$$T_2 = 32\left(\frac{s}{2}\right)\left[\left(-\frac{t}{2}\right) - \left(\frac{u - m_{A'}^2}{2}\right) - \left(\frac{s - m_{A'}^2}{2}\right)\right] = 8s\left[-t - u - s + 2m_{A'}^2\right] = 8m_{A'}^2 s \tag{13.80}$$

and

$$T_4 = 8(t - m_{A'}^2)(u - m_{A'}^2) - 8m_{A'}^2 s = 8(ut - m_{A'}^2(s+t+u) + m_{A'}^4) = 8ut \tag{13.81}$$

Hence:

$$\mathcal{A} = 2\epsilon^2 e^4\left[\frac{tu}{t^2} + 2m_{A'}^2\frac{s}{ut} + \frac{ut}{u^2}\right] = 2\epsilon^2 e^4\left[\frac{u}{t} + \frac{t}{u} + 2m_{A'}^2\frac{s}{ut}\right] \tag{13.82}$$

which indeed reduces to Eq. (11.233) of the book for $\epsilon = 1$ and $m_{A'} \to 0$:

$$\langle |\mathcal{M}|^2 \rangle \, (e^+e^- \to \gamma\gamma) = 2e^4 \left[\frac{u}{t} + \frac{t}{u} \right] \qquad (m_A, m_e \longrightarrow 0) \tag{13.83}$$

In order to compute $\mathcal{B}$, we need to evaluate:

$$\begin{aligned} \sum_{s,s'} k_{2\mu} k_{2\rho} \mathcal{M}^{\mu\nu} \left(\mathcal{M}^{\rho}{}_{\nu}\right)^* =& \epsilon^2 e^4 \sum_{s,s'} k_{2\mu} k_{2\rho} \left[\bar{v}(k,r)\Gamma^{\mu\nu} u(p,s)\right] \left[\bar{u}(p,s)\overline{\Gamma^{\rho}{}_{\nu}} v(k,r)\right] \\ =& \epsilon^2 e^4 k_{2\mu} k_{2\rho} \, \mathrm{Tr} \left[(\not{k} - m_e) \Gamma^{\mu\nu} (\not{p} + m_e) \overline{\Gamma^{\rho}{}_{\nu}} \right] \end{aligned} \tag{13.84}$$

Consequently, we find:

$$\begin{aligned} \mathcal{B} =& -\frac{1}{4m_{A'}^2} \epsilon^2 e^4 k_{2\mu} k_{2\rho} \, \mathrm{Tr} \Bigg[(\not{k} - m_e) \left(\frac{\gamma^\mu (\not{p} - \not{k}_1 + m_e) \gamma^\nu}{-2p \cdot k_1} + \frac{\gamma^\nu (\not{p} - \not{k}_2 + m_e) \gamma^\mu}{m_{A'}^2 - 2p \cdot k_2} \right) \\ & \times (\not{p} + m_e) \left(\frac{\gamma_\nu (\not{p} - \not{k}_1 + m_e) \gamma^\rho}{-2p \cdot k_1} + \frac{\gamma^\rho (\not{p} - \not{k}_2 + m_e) \gamma_\nu}{m_{A'}^2 - 2p \cdot k_2} \right) \Bigg] \\ =& -\frac{1}{4m_{A'}^2} \epsilon^2 e^4 k_{2\mu} k_{2\rho} \left[\frac{T_1^{\mu\rho}}{(2p \cdot k_1)^2} + \frac{T_2^{\mu\rho}}{(2p \cdot k_1)(m_{A'}^2 - 2p \cdot k_2)} + \frac{T_3^{\mu\rho}}{(2p \cdot k_1)(m_{A'}^2 - 2p \cdot k_2)} + \frac{T_4^{\mu\rho}}{(m_{A'}^2 - 2p \cdot k_2)^2} \right] \end{aligned} \tag{13.85}$$

where we have the following traces of products of up to 8 γ matrices:

- $T_1^{\mu\rho} \equiv \mathrm{Tr} \left[(\not{k} - m_e) \gamma^\mu (\not{p} - \not{k}_1 + m_e) \gamma^\nu (\not{p} + m_e) \gamma_\nu (\not{p} - \not{k}_1 + m_e) \gamma^\rho \right]$
- $T_2^{\mu\rho} \equiv \mathrm{Tr} \left[(\not{k} - m_e) \gamma^\mu (\not{p} - \not{k}_1 + m_e) \gamma^\nu (\not{p} + m_e) \gamma^\rho (\not{p} - \not{k}_2 + m_e) \gamma_\nu \right]$
- $T_3^{\mu\rho} \equiv \mathrm{Tr} \left[(\not{k} - m_e) \gamma^\nu (\not{p} - \not{k}_2 + m_e) \gamma^\mu (\not{p} + m_e) \gamma_\nu (\not{p} - \not{k}_1 + m_e) \gamma^\rho \right]$
- $T_4^{\mu\rho} \equiv \mathrm{Tr} \left[(\not{k} - m_e) \gamma^\nu (\not{p} - \not{k}_2 + m_e) \gamma^\mu (\not{p} + m_e) \gamma^\rho (\not{p} - \not{k}_2 + m_e) \gamma_\nu \right]$

We make use of FeynCalc in order to compute these traces:

```
T1 := DiracSimplify[
  DiracTrace[(GS[k] - m).GA[\[Mu]].(GS[p] - GS[k1] + m).GA[\[Nu]].(GS[p] + m).
  GA[\[Nu]].(GS[p] - GS[k1] + m).GA[\[Rho]]]]
T2 := DiracSimplify[
  DiracTrace[(GS[k] - m).GA[\[Mu]].(GS[p] - GS[k1] + m).
      GA[\[Nu]].(GS[p] + m).GA[\[Rho]].(GS[p] - GS[k2] + m).GA[\[Nu]]]]
T3 := DiracSimplify[
  DiracTrace[(GS[k] - m).GA[\[Nu]].(GS[p] - GS[k2] + m).
  GA[\[Mu]].(GS[p] + m).GA[\[Nu]].(GS[p] - GS[k1] + m).GA[\[Rho]]]]
T4 := DiracSimplify[
  DiracTrace[(GS[k] - m).GA[\[Nu]].(GS[p] - GS[k2] + m).GA[\[Mu]].
  (GS[p] + m).GA[\[Rho]].(GS[p] - GS[k2] + m).GA[\[Nu]]]]
```

to find for instance for $T_1^{\mu\rho}$ (setting $k_1^2 = 0$):

$$\begin{aligned} T_1^{\mu\rho} =& 16 g^{\mu\rho} (k \cdot k_1)(k_1 \cdot p) - 16 (k_1 \cdot p) \left[k^\mu k_1^\rho + k^\rho k_1^\mu \right] + 16 p^2 \left[k^\rho k_1^\mu + k^\mu k_1^\rho - g^{\mu\rho} (k \cdot k_1) \right] \\ & - 8p^2 \left[k^\mu p^\rho + k^\rho p^\mu - g^{\mu\rho} (k \cdot p) \right] \\ & - 32 m_e^2 \left[k^\mu k_1^\rho + k^\rho k_1^\mu - g^{\mu\rho} (k \cdot k_1) \right] \\ & + 24 m_e^2 \left[k^\rho p^\mu + k^\mu p^\rho - g^{\mu\rho} (k \cdot p) \right] \\ & + 16 m_e^2 g^{\mu\rho} (k_1 \cdot p) \\ & - 16 m_e^4 g^{\mu\rho} \end{aligned} \tag{13.86}$$

We again note that the expression is indeed greatly simplified if we assume $p^2 = m_e^2 \simeq 0$. Hence, we take:

$$T_1^{\mu\rho} \simeq 16\,(k_1 \cdot p)\,[g^{\mu\rho}\,(k \cdot k_1) - k^\mu k_1^\rho - k^\rho k_1^\mu] \tag{13.87}$$

We now contract $T_1^{\mu\rho}$ with $k_{2\mu}k_{2\rho}$, recalling that $k_2^2 = m_{A'}^2$ to find:

$$k_{2\mu}k_{2\rho}T_1^{\mu\rho} = 16\,(k_1 \cdot p)\left[m_{A'}^2\,(k \cdot k_1) - 2(k_2 \cdot k)(k_2 \cdot k_1)\right] \tag{13.88}$$

Similarly, one finds:

$$\begin{aligned} k_{2\mu}k_{2\rho}T_2^{\mu\rho} =& 16\,(k_2 \cdot p)\,[-\,(k \cdot k_1)\,(k_2 \cdot p) + (k \cdot p)\,(k_1 \cdot k_2) + (k \cdot k_2)\,(k_1 \cdot p)] \\ & - 8m_{A'}^2\,[(k \cdot p)\,(k_1 \cdot k_2) + (k \cdot k_2)\,(k_1 \cdot p) - (k \cdot k_1)\,(k_2 \cdot p)] \\ =& 8\left[(2(k_2 \cdot p) - m_{A'}^2)\,[-\,(k \cdot k_1)\,(k_2 \cdot p) + (k \cdot p)\,(k_1 \cdot k_2) + (k \cdot k_2)\,(k_1 \cdot p)]\right] \end{aligned} \tag{13.89}$$

and

$$k_{2\mu}k_{2\rho}T_3^{\mu\rho} = k_{2\mu}k_{2\rho}T_2^{\mu\rho} \tag{13.90}$$

and

$$k_{2\mu}k_{2\rho}T_4^{\mu\rho} = -8\,(k \cdot p)\left[4\,(k_2 \cdot p)^2 - 4m_{A'}^2\,(k_2 \cdot p) + m_{A'}^4\right] \tag{13.91}$$

We further simplify these formulae replacing the scalar products with the corresponding expressions with the Mandelstam variables:

$$\begin{aligned} k_{2\mu}k_{2\rho}T_1^{\mu\rho} =& 8\left(-\frac{t}{2}\right)\left[m_{A'}^2\,(-u) + (t - m_{A'}^2)(s - m_{A'}^2)\right] = -4t\left[-m_{A'}^2(s+t+u) + st + m_{A'}^4\right] \\ =& -4st^2 \end{aligned} \tag{13.92}$$

and

$$k_{2\mu}k_{2\rho}T_2^{\mu\rho} = k_{2\mu}k_{2\rho}T_3^{\mu\rho} = 2u\left(m_{A'}^2(s+t-u) - s^2 - t^2 + u^2\right) = 4stu \tag{13.93}$$

and finally

$$k_{2\mu}k_{2\rho}T_4^{\mu\rho} = -4su^2 \tag{13.94}$$

Now we are ready to combine all pieces together:

$$\begin{aligned} \mathcal{B} =& -\frac{1}{4m_{A'}^2}\epsilon^2 e^4 k_{2\mu}k_{2\rho}\left[\frac{T_1^{\mu\rho}}{(2p \cdot k_1)^2} + \frac{T_2^{\mu\rho}}{(2p \cdot k_1)(m_{A'}^2 - 2p \cdot k_2)} + \frac{T_3^{\mu\rho}}{(2p \cdot k_1)(m_{A'}^2 - 2p \cdot k_2)} + \frac{T_4^{\mu\rho}}{(m_{A'}^2 - 2p \cdot k_2)^2}\right] \\ =& -\frac{1}{4m_{A'}^2}\epsilon^2 e^4\left[\frac{-4st^2}{t^2} + 2\frac{4stu}{ut} + \frac{-4su^2}{u^2}\right] \\ =& -\frac{1}{m_{A'}^2}\epsilon^2 e^4\,[-s + 2s - s] = 0 \qquad (!) \end{aligned} \tag{13.95}$$

Consequently, the matrix element is finally just:

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle\,(e^+e^- \to \gamma A') = \mathcal{A} + \mathcal{B} =& 2\epsilon^2 e^4\left[\frac{u}{t} + \frac{t}{u} + 2m_{A'}^2\frac{s}{ut}\right] \\ =& \frac{2\epsilon^2 e^4}{ut}\left[u^2 + t^2 + 2m_{A'}^2 s\right] \end{aligned} \tag{13.96}$$

We define the kinematics in the center-of-mass system of the reaction. We have (for $m_{A'} \neq 0$, $m_e \neq 0$):

$$p^\mu = (E, 0, 0, p) \quad \text{and} \quad k^\mu = (E, 0, 0, -p) \tag{13.97}$$

and

$$k_1^\mu = (\omega, 0, -\omega\sin\theta, -\omega\cos\theta) \quad \text{and} \quad k_2^\mu = (E_{A'}, 0, \omega\sin\theta, \omega\cos\theta) \tag{13.98}$$

where

$$E^2 = p^2 + m_e^2 \quad \text{and} \quad E_{A'}^2 = \omega^2 + m_{A'}^2 \tag{13.99}$$

We obtain:

$$s = (k+p)^2 = (2E)^2 = 4E^2 \quad \Longrightarrow \quad E = \frac{\sqrt{s}}{2} \tag{13.100}$$

Therefore:

$$\omega + E_{A'} = \omega + \sqrt{\omega^2 + m_{A'}^2} = 2E = \sqrt{s} \quad \Longrightarrow \quad \omega = \frac{s - m_{A'}^2}{2\sqrt{s}} \tag{13.101}$$

and

$$E_{A'}^2 = \omega^2 + m_{A'}^2 \quad \Longrightarrow \quad E_{A'} = \frac{s + m_{A'}^2}{2\sqrt{s}} \tag{13.102}$$

and finally

$$E^2 = p^2 + m_e^2 \quad \Longrightarrow \quad p = \frac{\sqrt{s - 4m_e^2}}{2} \tag{13.103}$$

With these results, we obtain:

$$t = (p - k_1)^2 = m_e^2 - 2(E\omega + p\omega\cos\theta) = m_e^2 - \left(\frac{s - m_{A'}^2}{2}\right)\left(1 + \sqrt{1 - \frac{4m_e^2}{s}}\cos\theta\right) \tag{13.104}$$

and

$$u = (p - k_2)^2 = m_e^2 + m_{A'}^2 - 2(EE_{A'} - p\omega\cos\theta) = m_e^2 - \left(\frac{s - m_{A'}^2}{2}\right)\left(1 - \sqrt{1 - \frac{4m_e^2}{s}}\cos\theta\right) \tag{13.105}$$

Then, from now on neglecting the electron rest mass $m_e \to 0$, we find:

$$\begin{aligned} ut &\approx \left(\frac{s - m_{A'}^2}{2}\right)^2 (1 - \cos\theta)(1 + \cos\theta) = \frac{1}{4}(s - m_{A'}^2)^2 \sin^2\theta \\ u^2 + t^2 &\approx \left(\frac{s - m_{A'}^2}{2}\right)^2 \left[(1 - \cos\theta)^2 + (1 + \cos\theta)^2\right] = \frac{1}{2}(s - m_{A'}^2)^2(1 + \cos^2\theta) \end{aligned} \tag{13.106}$$

Putting all the pieces together, we find:

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle (e^+e^- \to \gamma A') &\approx \frac{8\epsilon^2 e^4}{(s - m_{A'}^2)^2 \sin^2\theta}\left[\frac{1}{2}(s - m_{A'}^2)^2(1 + \cos^2\theta) + 2m_{A'}^2 s\right] \\ &= \frac{4\epsilon^2 e^4}{\sin^2\theta}\left[(1 + \cos^2\theta) + \frac{4m_{A'}^2 s}{(s - m_{A'}^2)^2}\right] \end{aligned} \tag{13.107}$$

We include the phase space factor using Eq. (5.145) of the book and arrive at the differential cross-section:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 s} = \frac{\epsilon^2 e^4}{16\pi^2 s\sin^2\theta}\left[(1 + \cos^2\theta) + \frac{4m_{A'}^2 s}{(s - m_{A'}^2)^2}\right] \tag{13.108}$$

Hence:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CMS} (e^+e^- \to \gamma A') = \frac{\epsilon^2\alpha^2}{s\sin^2\theta}\left[(1 + \cos^2\theta) + \frac{4m_{A'}^2 s}{(s - m_{A'}^2)^2}\right] \tag{13.109}$$

As expected, this expression reduces to the cross-section for $e^+e^- \to \gamma\gamma$ for $m_A \to 0$ and $\epsilon = 1$ up to a symmetry factor (1/2) to account for the identical particles in the final state (see Eq. (11.239) of the book):

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CMS}(e^+e^- \to \gamma\gamma) = \frac{\alpha^2}{2s}\left[\frac{1+\cos^2\theta}{\sin^2\theta}\right] \tag{13.110}$$

The cross-section is strongly peaked in the forward direction, as shown in Figure 13.3.

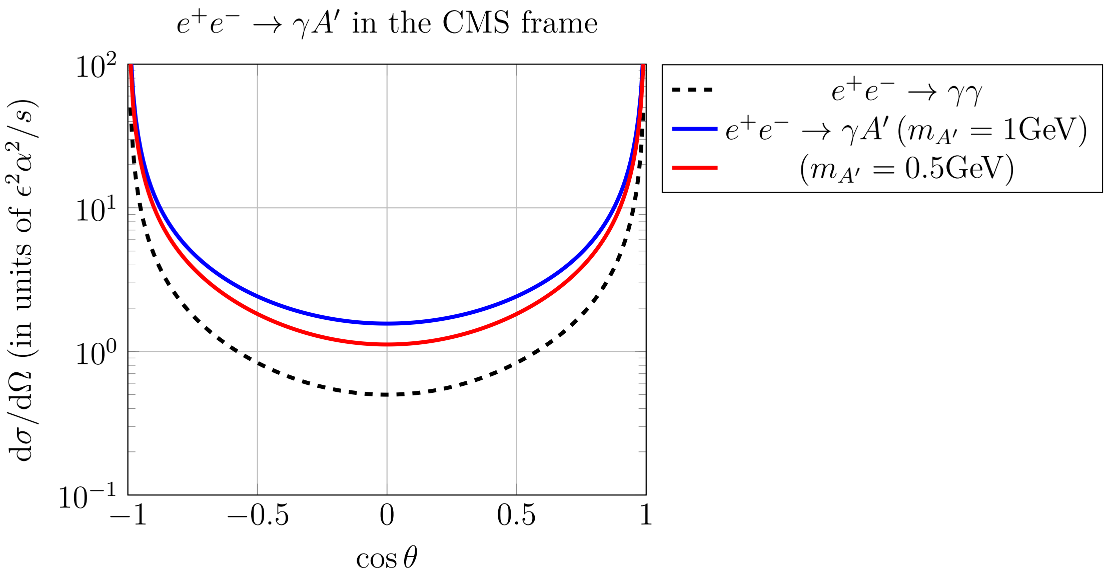

Figure 13.3 Differential cross-section for $e^+e^- \to \gamma A'$ for $\sqrt{s} = 3$ GeV for various values of $m_{A'}$.

Finally,

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\right)_{CMS}(e^+e^- \to \gamma A') = \frac{2\pi\epsilon^2\alpha^2}{s\sin^2\theta}\left[(1+\cos^2\theta) + \frac{4m_{A'}^2 s}{(s-m_{A'}^2)^2}\right] \tag{13.111}$$

The cross-section is proportional to the dark sector mixing factor ϵ^2, and it grows with increasing $m_{A'}$.

b) We need to integrate the expression for the differential cross-section above over $\mathrm{d}\cos\theta$. In the approximation that $m_e \to 0$, the total cross-section diverges due to the $\sin^2\theta$ denominator! The total cross-section for the process must be calculated in the case where $m_e \neq 0$. This is a tedious process, as can be appreciated from part a) of our problem, hence we fully rely on Feycalc to resolve the problem. We start generating the tree-level diagrams with the following command:

```
    diags = InsertFields[CreateTopologies[0, 2 -> 2], {F[2, {1}], -F[2, {1}]} ->
{V[1], V[1]}, InsertionLevel -> {Classes}, Restrictions->QEDOnly];
```

We can easily check with the following command:

```
Paint[diags, ColumnsXRows -> {2, 1}, Numbering -> Simple,
SheetHeader->None,ImageSize->{512,256}];
```

that *diags* contains exactly the two diagrams depicted in Figure 13.2.

We then convert the diagrams into an amplitude with the following command:

```
amp = FCFAConvert[CreateFeynAmp[diags], IncomingMomenta->{p,k},
OutgoingMomenta->{k1,k2},UndoChiralSplittings->True,ChangeDimension->4,
TransversePolarizationVectors->{k1,k2}, List->False, SMP->True,
Contract->True]
```

We define our scalar products:

```
FCClearScalarProducts[];
SetMandelstam[s, t, u, p, k, -k1, -k2, SMP["m_e"], SMP["m_e"], 0, M];
```

Note that the mass of the A' boson is labelled M in the above command. We now perform the main calculation:

```
(amp[0] (ComplexConjugate[amp[0]])) //   /* compute M multiplied by complex conjugate of M */

      /* replace denominators with the Mandelstam variables */
      FeynAmpDenominatorExplicit //

      /* sum over the photon polarizations (massless case) */
      DoPolarizationSums[#, k1, 0] & //
      /* sum over the A' boson polarizations (massive case) */
      DoPolarizationSums[#, k2] & //

      /* average over the initial-state electron-positron spins */
      FermionSpinSum[#, ExtraFactor -> 1/2^2] & //

      /* solve traces */
      DiracSimplify //

      /* rewrite s as a function of t and u using s+t+u = 2m^2 + M^2 */
      TrickMandelstam[#, {s, t, u, 2 SMP["m_e"]^2 + M^2}] & //
      /* simplify expression */
      Simplify
```

to get after inserting the ϵ^2 factor:

$$
\begin{aligned}
\langle|\mathcal{M}|^2\rangle\left(e^+e^- \to \gamma A'\right) = &-\frac{2\epsilon^2 e^4}{\left(t-m_e^2\right)^2\left(u-m_e^2\right)^2}\times \\
&\Big[m_e^2\left(2m_{A'}^4(t+u)-8m_{A'}^2tu+t^3+7t^2u+7tu^2+u^3\right) \\
&-m_e^4\left(2m_{A'}^4-2m_{A'}^2(t+u)+3t^2+14tu+3u^2\right)+6m_e^8 \\
&-tu\left(2m_{A'}^4-2m_{A'}^2(t+u)+t^2+u^2\right)\Big]
\end{aligned}
\tag{13.112}
$$

One immediately sees that by setting $m_e = 0$ in the above expression, we recover:

$$
\langle|\mathcal{M}|^2\rangle\left(e^+e^- \to \gamma A'\right) = \frac{2\epsilon^2 e^4}{tu}\left(u^2+t^2+2m_{A'}^2 s\right) \qquad (m_e \longrightarrow 0) \tag{13.113}
$$

which is exactly the result found in Eq. (13.96). We can now replace t and u in the general expression to find:

$$
4\epsilon^2 e^4\left[\frac{8m_e^2 s\left(-4m_e^2 s\sin^2\theta+m_{A'}^4\cos^2\theta+2s(s-m_{A'}^2)\right)+2s^2\sin^2\theta\left(m_{A'}^4+s^2\right)}{\left(m_{A'}^2-s\right)^2\left(4m_e^2\cos^2\theta+s\sin^2\theta\right)^2}-1\right] \tag{13.114}
$$

We can rewrite this expression as:

$$\langle |\mathcal{M}|^2 \rangle (e^+e^- \to \gamma A') = 4\epsilon^2 e^4 [\mathcal{S}_1 + \mathcal{S}_2] \tag{13.115}$$

where

$$\mathcal{S}_1 \equiv \frac{8m_e^2 s \left(-4m_e^2 s \sin^2\theta + m_{A'}^4 \cos^2\theta + 2s(s - m_{A'}^2)\right)}{\left(m_{A'}^2 - s\right)^2 \left(4m_e^2 \cos^2\theta + s\sin^2\theta\right)^2} \tag{13.116}$$

and

$$\mathcal{S}_2 \equiv \frac{2s^2 \sin^2\theta \left(m_{A'}^4 + s^2\right)}{\left(m_{A'}^2 - s\right)^2 \left(4m_e^2 \cos^2\theta + s\sin^2\theta\right)^2} - 1 \tag{13.117}$$

We note that for $m_e \to 0$, we have $\mathcal{S}_1 \approx 0$ and:

$$\begin{aligned}
\mathcal{S}_2 &\approx \frac{2\left(m_{A'}^4 + s^2\right)}{\left(m_{A'}^2 - s\right)^2 \sin^2\theta} - 1 = \frac{2\left(m_{A'}^4 + s^2\right) - \left(m_{A'}^4 - 2m_{A'}^2 s + s^2\right)\sin^2\theta}{\left(m_{A'}^2 - s\right)^2 \sin^2\theta} \\
&= \frac{(s^2 + m_{A'}^4)(1 + \cos^2\theta) + 2m_{A'}^2 s \sin^2\theta}{\left(m_{A'}^2 - s\right)^2 \sin^2\theta} = \frac{((s - m_{A'}^2)^2 + 2m_{A'} s)(1 + \cos^2\theta) + 2m_{A'}^2 s \sin^2\theta}{\left(m_{A'}^2 - s\right)^2 \sin^2\theta} \\
&= \frac{(s - m_{A'}^2)^2(1 + \cos^2\theta) + 4m_{A'}^2 s}{\left(m_{A'}^2 - s\right)^2 \sin^2\theta} = \frac{1}{\sin^2\theta}\left[(1 + \cos^2\theta) + \frac{4m_{A'}^2 s}{\left(m_{A'}^2 - s\right)^2}\right]
\end{aligned} \tag{13.118}$$

which exactly yields Eq. (13.109)!

In summary, the differential equation without any approximation on m_e is given by:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{\epsilon^2 e^4}{16\pi^2 s} [\mathcal{S}_1 + \mathcal{S}_2] = \frac{\epsilon^2 \alpha^2}{s} [\mathcal{S}_1 + \mathcal{S}_2] \tag{13.119}$$

The above equation can be integrated numerically for $s > m_{A'}^2$. The Python code is available on GitHub[1]. The results are shown in Figure 13.4.

c) We consider the following terms which are part of our effective Lagrangian:

$$\begin{aligned}
\mathcal{L} \supset &- \underbrace{\epsilon e Q_f X_\mu \overline{\psi}\gamma^\mu \psi}_{A'\text{coupling to SM}} + \underbrace{(D_\mu \phi)^* D^\mu \phi}_{\text{DM kinetic and interaction}} \qquad \text{where } D_\mu = \partial_\mu + i g_D X_\mu \\
= &- \underbrace{\epsilon e Q_f X_\mu \overline{\psi}\gamma^\mu \psi}_{A'\text{coupling to SM}} + \underbrace{(\partial_\mu \phi)^* \partial^\mu \phi}_{\text{DM kinetic}} + \underbrace{i g_D X_\mu \left[(\partial^\mu \phi^*)\phi - \phi^* \partial^\mu \phi\right]}_{\text{direct coupling}} + \underbrace{g_D^2 X_\mu X^\mu \phi \phi^*}_{\text{four point interaction}}
\end{aligned} \tag{13.120}$$

We immediately see that the A' boson can decay via two main channels:

- Decay to SM model fermions, $A' \to f\bar{f}$, with a vector coupling with vertex factor $-\epsilon e Q_f \gamma^\mu$
- Decay to two distinguishable dark matter scalar particles, $A' \to \phi_1 \phi_2$ (of mass m_1 and m_2 given by Eq. (13.50)), with a scalar coupling with vertex factor $-ig_D(p_1 - p_2)^\mu$ for p_1 and p_2 are the momenta of the scalar particles. Note the minus sign between p_1 and p_2 to account for the flow of the particles in the decay (one can arbitrary choose the sign of the momenta but they must be subtracted from each other in the vertex term).

The corresponding Feynman diagrams are shown in Figure 13.5.

1 https://github.com/CambridgeUniversityPress/Phenomenology-Particle-Physics

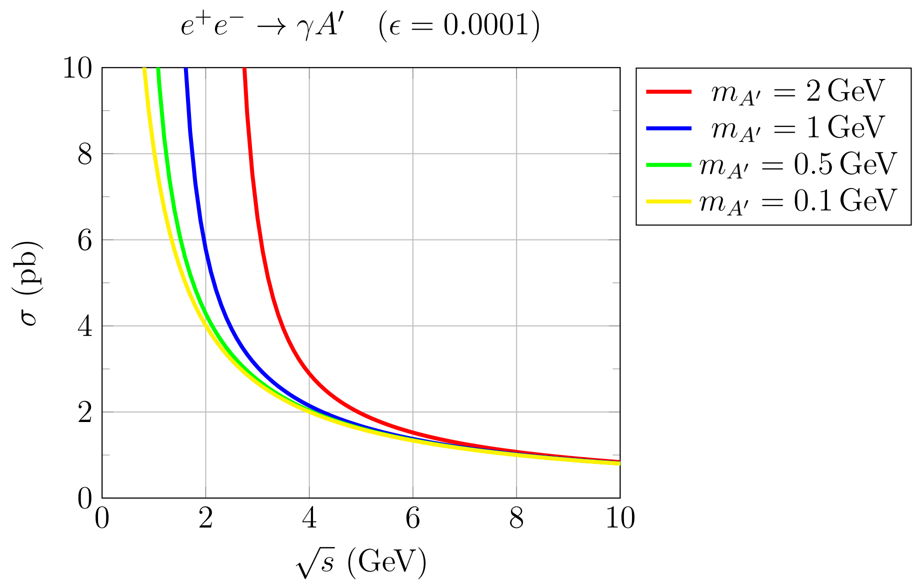

Figure 13.4 Total cross-section for $e^+e^- \to \gamma A'$ as a function of $\sqrt{s}$ for various values of $m_{A'}$.

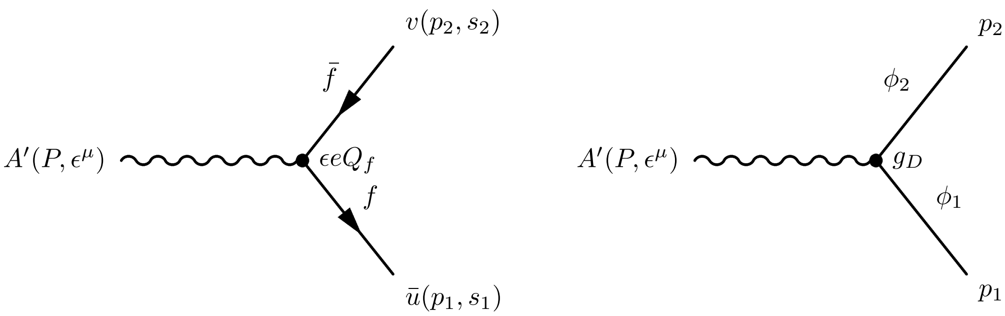

Figure 13.5 Feynman diagrams for $A' \to f\bar{f}$ and $A' \to \phi_1\phi_2$ at tree level.

d) We estimate the two decay widths for the process outlined in part c).

- **Decay to SM fermions.** The kinematics is given by:

$$A'(P) \to f(p_1, s_1) + \bar{f}(p_2, s_2) \tag{13.121}$$

The corresponding amplitude to the diagram in Figure 13.5(left) is:

$$i\mathcal{M} = -i\epsilon e Q_f \epsilon_\mu^{(\lambda)}(P)\bar{u}^{(s_1)}(p_1)\gamma^\mu v^{(s_2)}(p_2) \tag{13.122}$$

Multiplying by its complex conjugate, we find:

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^* = (\epsilon e Q_f)^2 \epsilon_\mu^{(\lambda)} \epsilon_\nu^{(\lambda)*} \bar{u}^{(s_1)}(p_1)\gamma^\mu v^{(s_2)}(p_2)\bar{v}^{(s_2)}(p_2)\gamma^\nu u^{(s_1)}(p_1) \tag{13.123}$$

In order to compute the unpolarized decay width, we need to average over the initial-state dark photon polarizations and sum over the final-state fermion spins. A' is a massive spin-1 particle, so it possesses three independent states. Therefore:

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{3} \sum_{\lambda=1,2,3} \sum_{s_1, s_2} |\mathcal{M}|^2 \tag{13.124}$$

Using Casimir's trick we find:

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{3}\sum_{\lambda=1,2,3}\epsilon_\mu^{(\lambda)}\epsilon_\nu^{(\lambda)*}\,\mathrm{Tr}\big((\not{p}_1 + m_f)\gamma^\mu(\not{p}_2 - m_f)\gamma^\nu\big) \tag{13.125}$$

The sum over the Dark photon polarization states can be done using the completeness relation:

$$\sum_{\lambda=1,2,3}\epsilon_\mu^{(\lambda)}\epsilon_\nu^{(\lambda)*} \leftrightarrow \left(-g_{\mu\nu} + \frac{P_\mu P_\nu}{m_{A'}^2}\right) \tag{13.126}$$

Hence:

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{4}{3}\left(-g_{\mu\nu} + \frac{P_\mu P_\nu}{m_{A'}^2}\right)\left((p_1^\nu p_2{}^\mu + p_2^\nu p_1^\mu) - (m_f^2 + p_1\cdot p_2)g^{\mu\nu}\right)\\
&= \frac{4}{3}\left(4(m_f^2 + p_1\cdot p_2) - 2(p_1\cdot p_2) + \frac{2}{m_{A'}^2}(P\cdot p_1)(P\cdot p_2) - m_f^2 - (p_1\cdot p_2)\right)\\
&= \frac{4}{3}\left(3m_f^2 + (p_1\cdot p_2) + \frac{2}{m_{A'}^2}(P\cdot p_1)(P\cdot p_2)\right)
\end{aligned} \tag{13.127}$$

In the center-of-mass system of the A', we can readily write noting that $\sqrt{s}$ă $= m_{A'} = E_f + E_{\bar{f}}$:

$$P^\mu = (m_{A'}, \vec{0}), \quad p_1^\mu = (E_f, \vec{p}) = \left(\frac{m_{A'}}{2}, \vec{p}\right) \quad \text{and} \quad p_2^\mu = \left(E_{\bar{f}}, -\vec{p}\right) = \left(\frac{m_{A'}}{2}, -\vec{p}\right) \tag{13.128}$$

Therefore:

$$P\cdot p_1 = P\cdot p_2 = \frac{m_{A'}^2}{2} \quad \text{and} \quad p_1\cdot p_2 = \frac{m_{A'}^2}{4} + |\vec{p}|^2 = \frac{m_{A'}^2}{4} + \frac{m_{A'}^2}{4} - m_f^2 = \frac{m_{A'}^2}{2} - m_f^2 \tag{13.129}$$

where we used $|\vec{p}|^2 = E_f^2 - m_f^2 = E_{\bar{f}}^2 - m_f^2$. Consequently, the averaged matrix element becomes:

$$\langle|\mathcal{M}|^2\rangle = \frac{4}{3}\epsilon^2 e^2 Q_f^2(m_{A'}^2 + 2m_f^2) = \frac{4}{3}\epsilon^2 e^2 Q_f^2 m_{A'}^2\left(1 + \frac{2m_f^2}{m_{A'}^2}\right) \tag{13.130}$$

We use Eq. (5.163) of the book to compute the decay width:

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle}{32\pi^2}\frac{|\vec{p}|}{m_{A'}^2}\mathrm{d}\Omega \tag{13.131}$$

where

$$E_f^2 = |\vec{p}|^2 + m_f^2 \quad \Longrightarrow \quad |\vec{p}| = \sqrt{\frac{m_{A'}^2}{4} - m_f^2} = \frac{m_{A'}}{2}\sqrt{1 - \frac{4m_f^2}{m_{A'}^2}} \tag{13.132}$$

Hence:

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 m_{A'}}\sqrt{1 - \frac{4m_f^2}{m_{A'}^2}}\,d\Omega$$

After integration over the solid angle, the decay width to SM fermions becomes:

$$\Gamma(A' \to \bar{f}f) = (4\pi)\frac{4}{3}\frac{\epsilon^2 e^2 Q_f^2 m_{A'}^2}{64\pi^2 m_{A'}}\left(1 + \frac{2m_f^2}{m_{A'}^2}\right)\sqrt{1 - \frac{4m_f^2}{m_{A'}^2}} = \frac{\epsilon^2 Q_f^2\alpha m_{A'}}{3}\left(1 + \frac{2m_f^2}{m_{A'}^2}\right)\sqrt{1 - \frac{4m_f^2}{m_{A'}^2}} \tag{13.133}$$

where $\alpha = e^2/4\pi$. The decay width is proportional to ϵ^2 as expected, and grows with increasing $m_{A'}$.

• **Decay to dark scalar particles.** The kinematics is given by:

$$A'(P) \to \phi_1(p_1) + \phi_2(p_2) \tag{13.134}$$

where the rest masses m_1 and m_2 of ϕ_1 and ϕ_2 are given by Eq. (13.50). The corresponding amplitude to the diagram in Figure 13.5(right) is just:

$$i\mathcal{M} = \epsilon_\mu^{(\lambda)}(P)(-ig_D(p_1 - p_2)^\mu) \tag{13.135}$$

The matrix element is then:

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^* = \epsilon_\mu^{(\lambda)*}\epsilon_\nu^{(\lambda)} g_D^2 (p_1 - p_2)^\mu (p_1 - p_2)^\nu \tag{13.136}$$

We compute the averaged matrix element squared over the spin of the incoming Dark photon (the scalar particles are obviously spinless):

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{3}\sum_{\lambda=1,2,3} |\mathcal{M}|^2 \\
&= \frac{g_D^2}{3}\left(-g_{\mu\nu} + \frac{P_\mu P_\nu}{m_{A'}^2}\right)(p_1 - p_2)^\mu (p_1 - p_2)^\nu \\
&= \frac{g_D^2}{3}\left(-(p_1 - p_2)^2 + \frac{1}{m_{A'}^2}(P\cdot(p_1 - p_2))^2\right) \\
&= \frac{g_D^2}{3}\left(-m_1^2 - m_2^2 + 2p_1\cdot p_2 + \frac{(P\cdot p_1)^2}{m_{A'}^2} + \frac{(P\cdot p_2)^2}{m_{A'}^2} - \frac{2(P\cdot p_1)(P\cdot p_2)}{m_{A'}^2}\right)
\end{aligned} \tag{13.137}$$

where in the second line we used the Dark photon completeness relation.
We define the kinematics for the case where the decay products have different masses m_1 and m_2:

$$P^\mu = (m_{A'}, \vec{0}), \quad p_1^\mu = (E_1, \vec{p}) \quad \text{and} \quad p_2^\mu = (E_2, -\vec{p}) \tag{13.138}$$

where we use Eq. (5.133) to compute $|\vec{p}|$:

$$|\vec{p}| = \frac{\lambda^{1/2}\left(m_{A'}^2, m_1^2, m_2^2\right)}{2m_{A'}} \tag{13.139}$$

and $\lambda(x, y, z)$ is the Källén function:

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx = (x - y - z)^2 - 4yz \tag{13.140}$$

We find:

$$\begin{aligned}
\lambda(m_{A'}^2, m_1^2, m_2^2) &= m_{A'}^4 + m_1^4 + m_2^4 - 2m_{A'}^2 m_1^2 - 2m_{A'}^2 m_2^2 - 2m_1^2 m_2^2 \\
&= m_{A'}^4 - m_{A'}^2\left(m_1^2 + m_2^2\right) - m_{A'}^2\left(m_1^2 + m_2^2\right) + \left(m_1^4 - 2m_1^2 m_2^2 + m_2^4\right) \\
&= m_{A'}^4 - m_{A'}^2\left(m_1^2 - 2m_1 m_2 + m_2^2\right) - m_{A'}^2\left(m_1^2 + 2m_1 m_2 + m_2^2\right) + \left(m_1^2 - m_2^2\right)^2 \\
&= m_{A'}^4 - m_{A'}^2 (m_1 - m_2)^2 - m_{A'}^2 (m_1 + m_2)^2 + ((m_1 - m_2)(m_1 + m_2))^2 \\
&= \left(m_{A'}^2 - (m_1 - m_2)^2\right)\left(m_{A'}^2 - (m_1 + m_2)^2\right)
\end{aligned} \tag{13.141}$$

Finally:

$$|\vec{p}| = \frac{m_{A'}}{2}\sqrt{1-\frac{(m_1-m_2)^2}{m_{A'}^2}}\sqrt{1-\frac{(m_1+m_2)^2}{m_{A'}^2}} \tag{13.142}$$

The energies are given by:

$$\begin{aligned}
E_1^2 = p^2 + m_1^2 &= \frac{1}{4m_{A'}^2}\left[\lambda(m_{A'}^2, m_1^2, m_2^2) + 4m_{A'}^2 m_1^2\right] = \frac{1}{4m_{A'}^2}\left[(m_{A'}^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2 + 4m_{A'}^2 m_1^2\right] \\
&= \frac{1}{4m_{A'}^2}\left[m_{A'}^4 + m_1^4 + m_2^4 - 2m_{A'}^2 m_1^2 - 2m_{A'}^2 m_2^2 + 2m_1^2 m_2^2 - 4m_1^2 m_2^2 + 4m_{A'}^2 m_1^2\right] \\
&= \frac{1}{4m_{A'}^2}\left[m_{A'}^4 + m_1^4 + m_2^4 + 2m_{A'}^2 m_1^2 - 2m_{A'}^2 m_2^2 - 2m_1^2 m_2^2\right] \\
&= \frac{1}{4m_{A'}^2}\left[m_{A'}^2 + m_1^2 - m_2^2\right]^2 \qquad \Longrightarrow \qquad E_1 = \frac{m_{A'}^2 + m_1^2 - m_2^2}{2m_{A'}}
\end{aligned} \tag{13.143}$$

and similarly for E_2:

$$E_2 = \sqrt{p^2 + m_2^2} = \frac{m_{A'}^2 + m_2^2 - m_1^2}{2m_{A'}} \tag{13.144}$$

Then:

$$\frac{(P\cdot p_1)^2}{m_{A'}^2} = E_1^2, \quad \frac{(P\cdot p_2)^2}{m_{A'}^2} = E_2^2, \quad \frac{2(P\cdot p_1)(P\cdot p_2)}{m_{A'}^2} = 2E_1E_2 \tag{13.145}$$

and

$$2p_1\cdot p_2 = 2(E_1E_2 + |\vec{p}|^2) \tag{13.146}$$

With these definitions, the averaged matrix element squared simplifies to:

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{g_D^2}{3}\left[-m_1^2 - m_2^2 + 2(E_1E_2 + |\vec{p}|^2) + E_1^2 + E_2^2 - 2E_1E_2\right] \\
&= \frac{g_D^2}{3}\left[4|\vec{p}|^2\right] = \frac{g_D^2}{3}4\left[\frac{m_{A'}^2}{4}\left(1-\frac{(m_1-m_2)^2}{m_{A'}^2}\right)\left(1-\frac{(m_1+m_2)^2}{m_{A'}^2}\right)\right] \\
&= \frac{g_D^2}{3m_{A'}^2}\left[\left(m_{A'}^2 - (m_1-m_2)^2\right)\left(m_{A'}^2 - (m_1+m_2)^2\right)\right] \\
&= \frac{g_D^2}{3m_{A'}^2}\left[\left(m_{A'}^4 \underbrace{-m_{A'}^2(m_1-m_2)^2 - m_{A'}^2(m_1+m_2)^2}_{=-2m_{A'}^2(m_1^2+m_2^2)} + \underbrace{(m_1-m_2)^2(m_1+m_2)^2}_{=(m_1^2-m_2^2)^2}\right)\right] \\
&= \frac{g_D^2}{3}m_{A'}^2\left(1 - \frac{2(m_1^2+m_2^2)}{m_{A'}^2} + \frac{(m_1^2-m_2^2)^2}{m_{A'}^4}\right)
\end{aligned} \tag{13.147}$$

To compute the total decay width we use Eq. (5.163) of the book:

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle}{32\pi^2}\frac{p}{m_{A'}^2}\mathrm{d}\Omega \quad \Longrightarrow \quad \Gamma = \frac{\langle|\mathcal{M}|^2\rangle}{8\pi}\frac{p}{m_{A'}^2}$$

Regrouping the terms we find:

$$\Gamma(A'\to\phi_1\phi_2) = \frac{g_D^2}{48\pi}m_{A'}\left(1 - \frac{2(m_1^2+m_2^2)}{m_{A'}^2} + \frac{(m_1^2-m_2^2)^2}{m_{A'}^4}\right)\sqrt{1-\frac{(m_1-m_2)^2}{m_{A'}^2}}\sqrt{1-\frac{(m_1+m_2)^2}{m_{A'}^2}} \tag{13.148}$$

The decay width is proportional to g_D^2 as expected, and grows with increasing $m_{A'}$.

e) In a fixed target experiment, high energy electrons are shot at a target and the Dark photon can be produced via the so-called Bremsstrahlung process in the field of a target nucleus:

$$e^- Z \to e^- Z A' \qquad A' \to e^+ e^- \text{ or } A' \to \text{invisible} \tag{13.149}$$

where Z is the atomic number of the target nucleus, and it is assumed that the $m_{A'}$ is light, such that it can only decay into e^+e^- or into invisibly into dark matter scalars. The decay into e^+e^- can be studied in an apparatus as sketched in Figure 13.6. For small enough ϵ, the decay of the A' can be seen as a displaced vertex from the production point. Tracking stations and an electromagnetic calorimeter can be used to reconstruction the kinematics of the e^+e^- pair, and thereby the decay point and invariant mass of the A'. See e.g. J D. Bjorken, R. Essig, P. Schuster, N. Toro, "New Fixed-Target Experiments to Search for Dark Gauge Forces", Phys. Rev. D80:075018, 2009 (`https://doi.org/10.1103/PhysRevD.80.075018`) for a detailed discussion.

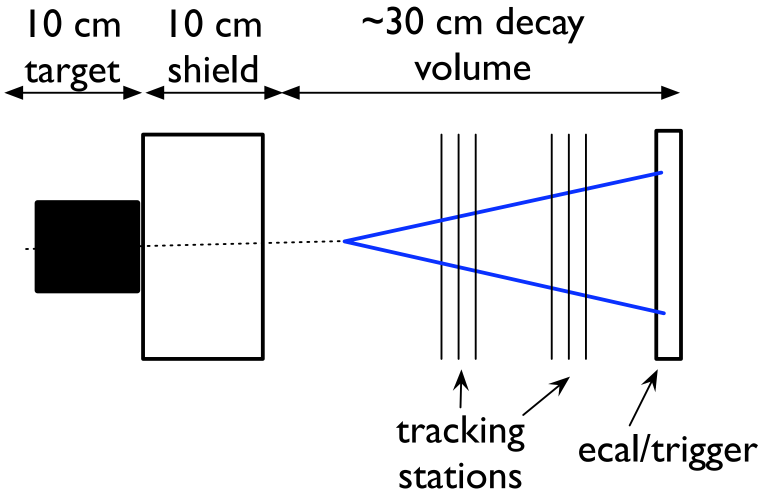

Figure 13.6 Figure from J D. Bjorken, R. Essig, P. Schuster, N. Toro, "New Fixed-Target Experiments to Search for Dark Gauge Forces", Phys. Rev. D80:075018, 2009.

14 Tests of QED at Low Energy

14.1 Motion in a Penning trap

We consider a Penning trap within a uniform magnetic field given by $\vec{B} = B_0\hat{z}$ and an electric potential given by Eq. (14.29) of the book. The equation of motion of a charged particle with charge e is given by:

$$m\ddot{\vec{x}} = e\left(\vec{E} + \dot{\vec{x}} \times \vec{B}\right) \tag{14.1}$$

(a) Show that the electric field is given by:

$$\vec{E} = -\frac{V_0}{r_0^2}(x, y, -2z) \tag{14.2}$$

(b) Show that the equation of motion setting $\vec{x} = (x, y, z)$ then becomes:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = -\frac{eV_0}{mr_0^2}\begin{pmatrix} x \\ y \\ -2z \end{pmatrix} + \omega_0\begin{pmatrix} \dot{y} \\ -\dot{x} \\ 0 \end{pmatrix} \tag{14.3}$$

where $\omega_0 = eB_0/m$ is the non-relativistic cyclotron frequency (see Eq. (14.26) of the book).

(c) Note that in the above expression the z coordinate is decoupled from the radial ones. The movement along the z coordinate leads to the ***axial oscillation****. Use the following ansatz:*

$$z(t) = z_0 e^{i\omega_E t} \tag{14.4}$$

to prove Eq. (14.30) of the book.

(d) We now need to solve the radial motion. Show that it is determined by the equation:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \frac{\omega_E^2}{2}\begin{pmatrix} x \\ y \end{pmatrix} + \omega_0\begin{pmatrix} \dot{y} \\ -\dot{x} \end{pmatrix} \tag{14.5}$$

Introduce the complex function $u(t) = x(t) + iy(t)$ and show that it follows the following equation:

$$\ddot{u} = \frac{\omega_E^2}{2}u - i\omega_0\dot{u} \tag{14.6}$$

(e) Make the following ansatz $u(t) = u_0 e^{-i\omega t}$ and find the two solutions $\omega_\pm$:

$$\omega_\pm = \frac{1}{2}\left(\omega_0 \pm \sqrt{\omega_0^2 - 2\omega_E^2}\right) \tag{14.7}$$

(f) *Show that the slow frequency defined as the* ***magnetron frequency*** ω_{EB} *in Eq. (14.31) of the book can be identified with* ω_-. *While the fast* ***cyclotron frequency*** *is given by:*

$$\omega_B \equiv \omega_+ = \omega_0 - \omega_{EB} \tag{14.8}$$

(g) *Compute the three frequencies for an electron in a Penning trap with the following parameters:* $B = 0.4$ *T,* $V_0 = 10$ *V, and* $r_0 = 0.8$ *cm.*

Solution:

(a) The electric potential given in Eq. (14.29) of the book is defined as:

$$V(r, z) = \left(\frac{V_0}{r_0^2}\right)\left(\frac{r^2}{2} - z^2\right) \tag{14.9}$$

where r_0 is a geometrical parameter. To create this potential inside the Penning trap, there are three cylindrically symmetric electrodes that follow the equipotential lines of V: two endcaps, called the upper and lower endcaps, and one ring electrode. Following the figure, the endcaps are set at a potential equal to $-V_0/2$, while the ring electrode has a potential equal to $+V_0/2$.

In order to create this potential, the endcaps must hence follow the contour defined by:

$$V(r, z) = -\frac{V_0}{2} \implies \left(\frac{V_0}{r_0^2}\right)\left(\frac{r^2}{2} - z^2\right) = -\frac{V_0}{2} \implies \frac{r^2}{2} - z^2 = -\frac{r_0^2}{2} \tag{14.10}$$

or

$$z(r) = \pm\sqrt{\frac{r^2}{2} + \frac{r_0^2}{2}} \equiv \pm\sqrt{\frac{r^2}{2} + z_0^2} \tag{14.11}$$

where $z_0 \equiv r_0/\sqrt{2}$. $\pm z_0$ define the closest points of the upper and lower endcaps to the trap center.

Similarly, the ring electrode follows the contour given by:

$$V(r, z) = +\frac{V_0}{2} \implies \frac{V_0}{r_0^2}\left(\frac{r^2}{2} - z^2\right) = \frac{V_0}{2} \implies \frac{r^2}{2} - z^2 = \frac{r_0^2}{2} \tag{14.12}$$

or

$$r(z) = \pm\sqrt{r_0^2 + 2z^2} \tag{14.13}$$

where we recognize that r_0 is the closest distance between the trap center and the ring electrode. Hence, the diameter of the ring is $2r_0$ as shown in Figure 14.2 of the book.

This potential defines the electric field following the relation $\vec{E} = -\nabla V$, hence, introducing the x and y coordinates, and with $r^2 = x^2 + y^2$, we find:

$$V(x, y, z) = \left(\frac{V_0}{r_0^2}\right)\left(\frac{x^2 + y^2}{2} - z^2\right) \implies \vec{E}(x, y, z) = -\frac{V_0}{r_0^2}(x, y, -2z) \tag{14.14}$$

The force felt by an electric charge e is then:

$$\vec{F}(x, y, z) = -\frac{eV_0}{r_0^2}(x, y, -2z) = \frac{eV_0}{r_0^2}(-x, -y, 2z) \tag{14.15}$$

The important part is happening in the component F_z, parallel to the magnetic field: the electric force should push the particle towards $z = 0$, hence if we wish to trap an electron ($e < 0$) then $V_0 > 0$. Alternatively, for a positive charge, we would have $V_0 < 0$. Therefore, the condition for the potential V_0 is that

$$eV_0 < 0 \tag{14.16}$$

Since the electric force is proportional to the displacement z, it is a restoring force similar to a spring. It leads to a harmonic oscillation.

(b) The Lorentz force on a charge e coupled to the Newton equation yields:

$$m\vec{a} = e(\vec{E} + \vec{v} \times \vec{B}) \quad \Longrightarrow \quad \ddot{\vec{x}} = \frac{e}{m}(\vec{E} + \dot{\vec{x}} \times \vec{B}) \tag{14.17}$$

where $\vec{x} = (x, y, z)$. Setting $\vec{B} = B_0\hat{z}$, we find in the Euclidian coordinates:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = -\frac{eV_0}{mr_0^2}\begin{pmatrix} x \\ y \\ -2z \end{pmatrix} + \frac{eB_0}{m}\begin{pmatrix} \dot{y} \\ -\dot{x} \\ 0 \end{pmatrix} = -\frac{eV_0}{mr_0^2}\begin{pmatrix} x \\ y \\ -2z \end{pmatrix} + \omega_0\begin{pmatrix} \dot{y} \\ -\dot{x} \\ 0 \end{pmatrix} \tag{14.18}$$

where the cyclotron frequency is $\omega_0 \equiv eB_0/m$.

(c) The equation of motion for the z coordinates is given by:

$$\ddot{z} = -\frac{eV_0}{mr_0^2}(-2z) = \frac{2eV_0}{mr_0^2}z \tag{14.19}$$

With the suggested ansatz we find:

$$z(t) = z_0 e^{i\omega_E t} \quad \Longrightarrow \quad \ddot{z}(t) = (i\omega_E)^2 z_0 e^{i\omega_E t} = -\omega_E^2 z(t) \tag{14.20}$$

Hence the axial frequency ω_E is given by:

$$-\omega_E^2 z(t) = \frac{2eV_0}{mr_0^2}z(t) \quad \Longrightarrow \quad \omega_E^2 = \frac{-2eV_0}{mr_0^2} \tag{14.21}$$

We recall that we have the condition $eV_0 < 0$. Generally, we prefer to write the frequency using only positive quantities and therefore we set

$$\omega_E = \sqrt{\frac{2eV_0}{mr_0^2}} \tag{14.22}$$

which is Eq. (14.30) of the book.

(d) The radial equation of motion then becomes

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = -\frac{eV_0}{mr_0^2}\begin{pmatrix} x \\ y \end{pmatrix} + \omega_0\begin{pmatrix} \dot{y} \\ -\dot{x} \end{pmatrix} = \frac{\omega_E^2}{2}\begin{pmatrix} x \\ y \end{pmatrix} + \omega_0\begin{pmatrix} \dot{y} \\ -\dot{x} \end{pmatrix} \tag{14.23}$$

where we used Eq. (14.21). We introduce the complex function $u(t) = x(t) + iy(t)$. The first and second time derivates are then:

$$i\dot{u}(t) = i\dot{x}(t) - \dot{y}(t) = -\dot{y}(t) + i\dot{x}(t) \qquad \text{and} \qquad \ddot{u}(t) = \ddot{x}(t) + i\ddot{y}(t) \tag{14.24}$$

The two equations of motion can then be rewritten as a single equation for u:

$$\ddot{u}(t) = \frac{\omega_E^2}{2}u(t) - i\omega_0\dot{u}(t) \tag{14.25}$$

(e) We apply the suggested ansatz:

$$u(t) = u_0 e^{-i\omega t} \quad \Longrightarrow \quad \dot{u}(t) = (-i\omega)u(t) \quad \Longrightarrow \quad \ddot{u}(t) = (-i\omega)^2 u(t) = -\omega^2 u(t) \tag{14.26}$$

This yields:

$$-\omega^2 u(t) = \frac{\omega_E^2}{2}u(t) - i\omega_0(-i\omega)u(t) \quad \Longrightarrow \quad -\omega^2 = \frac{\omega_E^2}{2} - \omega_0\omega \tag{14.27}$$

We need to solve the second degree equation:

$$\omega^2 - \omega_0\omega + \frac{\omega_E^2}{2} = 0 \quad \Longrightarrow \quad \omega_\pm = \frac{\omega_0 \pm \sqrt{\omega_0^2 - 4\frac{\omega_E^2}{2}}}{2} = \frac{1}{2}\left(\omega_0 \pm \sqrt{\omega_0^2 - 2\omega_E^2}\right) \tag{14.28}$$

We find a condition for the frequencies to be real-valued, otherwise the trajectories will not be harmonic, in order words, unstable. We need to have:

$$\omega_0^2 > 2\omega_E^2 \quad \Longrightarrow \quad B_0 > \sqrt{\frac{4mV_0}{er_0^2}} \tag{14.29}$$

where we again expressed the result using only positive quantities (as in Eq. (14.22)). If the voltage between the trap electrodes is too big compared to the magnetic field, then the electric forces become too large and the particle is no longer contained radially by the magnetic field. This is called the stability limit.

(f) The slow frequency ω_- is the magnetron frequency, labelled ω_{EB}:

$$\omega_{EB}\text{ă} = \frac{1}{2}\left(\omega_0 - \sqrt{\omega_0^2 - 2\omega_E^2}\right) = \frac{\omega_0}{2} - \sqrt{\frac{\omega_0^2}{4} - \frac{\omega_E^2}{2}} = \frac{\omega_0}{2} - \frac{\omega_0}{2}\sqrt{1 - \frac{2\omega_E^2}{\omega_0^2}} \tag{14.30}$$

The cyclotron frequency ω_B is given by ω_+:

$$\omega_B = \omega_+ = \frac{1}{2}\left(\omega_0 + \sqrt{\omega_0^2 - 2\omega_E^2}\right) = \omega_0 - \frac{1}{2}\left(\omega_0 - \sqrt{\omega_0^2 - 2\omega_E^2}\right) = \omega_0 - \omega_{EB} \tag{14.31}$$

(g) We compute the three frequencies for $B = 0.4$ T, $V_0 = 10$ V, and $r_0 = 0.8$ cm for an electron. We recall that 1 T $= 1\ \mathrm{kg\,s^{-2}A^{-1}}$ and 1 V $= 1\ \mathrm{kg\,m^2s^{-3}\,A^{-1}}$.

- The cyclotron frequency is given by:

$$\omega_0 = \frac{eB}{m} = \frac{(1.602\times10^{-19}\ \mathrm{A\,s})(0.4\ \mathrm{kg\,s^{-2}A^{-1}})}{(9.109\times10^{-31}\ \mathrm{kg})} \approx 7\times10^{10}\,\mathrm{rad\,s^{-1}} \tag{14.32}$$

- The axial frequency is given by Eq. (14.21):

$$\omega_E = \sqrt{\frac{2eV_0}{mr_0^2}} = \sqrt{\frac{2(1.602\times10^{-19}\ \mathrm{A\,s})(10\ \mathrm{V})}{(9.109\times10^{-31}\ \mathrm{kg})(0.008\ \mathrm{m})^2}} \approx 2.4\times10^8\,\mathrm{rad\,s^{-1}} \quad \Longrightarrow \quad \nu_E \approx 37\ \mathrm{MHz} \tag{14.33}$$

- The magnetron frequency is given by:

$$\omega_{EB} = \frac{\omega_0}{2}\left(1 - \sqrt{1 - \frac{2(2.4\times10^8\,\mathrm{rad\,s^{-1}})^2}{(7\times10^{10}\,\mathrm{rad\,s^{-1}})^2}}\right) \approx 4.2\times10^5\,\mathrm{rad\,s^{-1}} \quad \Longrightarrow \quad \nu_{EB} \approx 67\ \mathrm{kHz} \tag{14.34}$$

- The (reduced) cyclotron frequency

$$\omega_B = \omega_0 - \omega_{EB} \approx 7\times10^{10}\,\mathrm{rad\,s^{-1}} \quad \Longrightarrow \quad \nu_B \approx 11\ \mathrm{GHz} \tag{14.35}$$

14.2 True muonium decay

True muonium is the bound state comprising a muon (μ^-) and its antiparticle (μ^+)[1]. Since both muons are fermions, there are two spin configurations the bound state can have. One is called para-true-muonium (henceforth pMu) and has total spin 0, while the total spin-1 configuration is called ortho-true-muonium (henceforth oMu).

1. *C-parity for the bound state is given by $(-1)^{L+S}$, where L is the orbital angular momentum and S the spin angular momentum of the bound state. Assuming the atom is in the s state (spectroscopic notation), how many photons need to be created in the decay for C-parity to be conserved?*
2. *Draw the lowest-order Feynman diagrams for pMu and oMu to decay which respect C-parity conservation.* Hint: *It can also decay to lighter fermions!*
3. *Calculate the total cross-section at tree level for the decay of pMu and oMu.* Hint: *Calculate the unpolarized total cross-section in the low-energy limit of the respective graphs of part (b).*
4. *Using the Pirenne–Wheeler factorization formula, Eq. (14.112) of the book:*

$$\Gamma \approx \frac{1}{2J+1}|\psi(0)|^2 \times (4v_{rel}\sigma_{LO})$$

   where Γ is the decay rate, $\psi(0)$ is the hydrogen wave function (given by the solution of the Schrödinger equation for $n=1$ and $l=m=0$ and using the appropriate reduced mass of the true muonium atom) at the origin, and v_{rel} is the relative velocity of the constituents, calculate the lifetime of oMu and pMu, respectively.
5. *Calculate the ratio of the pMu to the oMu lifetimes. Argue why this value is so different from the case of positronium, which is on the order of 10^{-3}.*

Solution:

a) The states of true muonium (TM) are classified in a similar fashion than hydrogen-like atoms atomic states, namely according to the notation $n^{2s+1}l_j$ with n being the principal quantum number, s the sum of spins quantum number, $j=l+s$ the total angular momentum quantum number (l is the orbital momentum quantum number such that $(l=0)\to S$, $(l=1)\to P$, $(l=2)\to D$, ...).

   Because the spins of the muon and antimuon can combine in both parallel or antiparallel configurations, the ground state of TM is either the singlet, para-true-muonium $(p-TM)$, 1^1S_0 with $J^{PC}=0^{-+}$, or the triplet, ortho-true-muonium $(o-TM)$, 1^3S_1 with $J^{PC}=1^{--}$. TM singlet and triplet states are C-eigenstates, respectively C-even ($C=1$) and C-odd ($C=-1$) states, where $C=(-1)^{l+s}$.

   According to Eq. (4.87) of the book, the photon as C-parity $C_\gamma=-1$, hence a state of n photons has $C_{n\gamma}=(-1)^n$, thus the following selection rule then holds:

$$(-1)^{l+s}=(-1)^n \tag{14.36}$$

   From the above we conclude that the singlet state $(p-TM)$ decays dominantly into an even number of photons, while the triplet state $(o-TM)$ can only an odd number of photons.

b) We have seen under a) that the singlet state $(p-TM)$ can only decay into an even number of photons. Hence, the simplest decay is $p-TM\to\gamma\gamma$. The corresponding lowest-order QED Feynman diagrams are shown in Figure 14.1(top). For the triplet state $(o-TM)$, the number of photons must be odd, so the simplest decay to photons is $p-TM\to\gamma\gamma\gamma$ with an amplitude of the order $\mathcal{O}(e^3)$. On the other

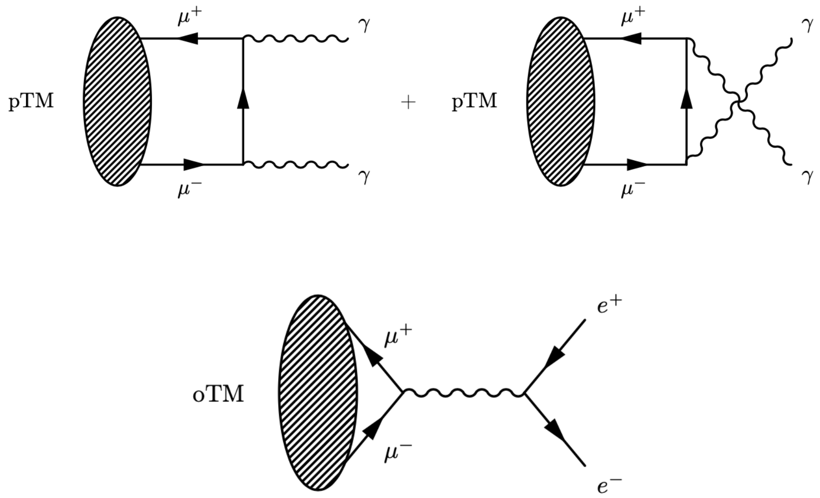

Figure 14.1 Lowest order Feynman diagram for the decay of (top) singlet state $p-TM \to \gamma\gamma$ and (bottom) triplet state $o-TM \to e^+e^-$.

hand, the channel $p-TM \to e^+e^-$ is also kinematically open and the amplitude is order $\mathcal{O}(e^2)$. Its Feynman diagram is shown in Figure 14.1(bottom).

c) We want to calculate the total cross-section at tree level for the processes $\mu^+ + \mu^- \to \gamma + \gamma$ and $\mu^+ + \mu^- \to e^+ + e^-$.

• **The** $\mu^+ + \mu^- \to \gamma + \gamma$ **case.** The $\mu^+ + \mu^- \to \gamma + \gamma$ process can be computed in a similar way as the $e^+ + e^- \to \gamma + \gamma$, after the appropriate changes. Pair annihilation into two photons in QED was discussed in detail in Section 11.15 of the book. We make use of the results derived there to compute the $\mu^+ + \mu^- \to \gamma + \gamma$ cross-section. The tree-level Feynman diagrams for the $e^+ + e^- \to \gamma + \gamma$ process are shown in Figure 11.14 of the book.

We define the kinematics of the reaction:

$$\mu^-(p,s) + \mu^+(p',s') \to \gamma(k_1,\epsilon_1) + \gamma(k_2,\epsilon_2) \tag{14.37}$$

Following the procedure used in Section 11.15 of the book, the net amplitude of the two diagrams can be expressed by factorizing the outgoing photon polarization vectors:

$$\begin{aligned} i\mathcal{M} &= \epsilon^*_{1\mu}(k_1)\epsilon^*_{2\nu}(k_2)(-ie^2)\overline{v}^{(s')}(p')\left(\frac{\gamma^\mu(\not{p}-\not{k}_1+m_\mu)\gamma^\nu}{(p-k)^2-m_\mu^2} + \frac{\gamma^\nu(\not{p}-\not{k}_2+m_\mu)\gamma^\mu}{(p-k')^2-m_\mu^2}\right)u^{(s)}(p) \\ &= (-ie^2)\epsilon^*_{1\mu}(k_1)\epsilon^*_{2\nu}(k_2)\mathcal{M}^{\mu\nu} \end{aligned} \tag{14.38}$$

Because $p^2 = p'^2 = m_\mu^2$ and $k_1^2 = k_2^2 = 0$, the four-tensor $\mathcal{M}^{\mu\nu}$ simplifies to:

$$\mathcal{M}^{\mu\nu} = \overline{v}^{(s')}(p')\left(\frac{\gamma^\mu(\not{p}-\not{k}_1+m_\mu)\gamma^\nu}{-2p\cdot k_1} + \frac{\gamma^\nu(\not{p}-\not{k}_2+m_\mu)\gamma^\mu}{-2p\cdot k_2}\right)u^{(s)}(p) \tag{14.39}$$

We are interested in the unpolarized cross-section. We therefore need to average over the initial-state spins of the fermions and sum over the final-state polarization states of the photons. The photon possesses

1 For historical reasons, the bound state of an electron and a positive muon is called Muonium, even if "-onium" atoms actually describe atoms of particles and their corresponding antiparticles, hence the qualifier "true."

*two*ăindependent polarization states, hence the matrix element squared will be given by:

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\frac{1}{2}\sum_{s,s'}\sum_{\lambda_1=1,2}\sum_{\lambda_2=1,2}|\mathcal{M}|^2 = \frac{1}{4}\sum_{s,s'}\sum_{\lambda_1,\lambda_2}(-ie^2)\epsilon^*_{1\mu}(k_1)\epsilon^*_{2\nu}(k_2)\mathcal{M}^{\mu\nu}(ie^2)\epsilon_{1\rho}(k_1)\epsilon_{2\sigma}(k_2)\left(\mathcal{M}^{\rho\sigma}\right)^* \tag{14.40}$$

For the photons, the completeness relation yields:

$$\sum_{\lambda_i=1}^{2}\epsilon_\lambda^{\mu,*}\epsilon_\lambda^\nu \leftrightarrow -g^{\mu\nu} \qquad (i=1,2) \tag{14.41}$$

Therefore, the matrix element can be written as:

$$\langle|\mathcal{M}|^2\rangle = \frac{e^2}{4}g_{\mu\rho}\mathcal{M}^{\mu\nu}g_{\nu\sigma}\left(\mathcal{M}^{\rho\sigma}\right)^* = \frac{e^4}{4}\sum_{s,s'}\mathcal{M}^{\mu\nu}\left(\mathcal{M}_{\mu\nu}\right)^* \tag{14.42}$$

The sum over the spins of the muons reduces to:

$$\langle|\mathcal{M}|^2\rangle = \sum_{s,s'}|\mathcal{M}|^2 = e^4\left(\frac{T_1}{(2p\cdot k_1)^2} + \frac{T_2}{(2p\cdot k_1)(2p\cdot k_2)} + \frac{T_3}{(2p\cdot k_1)(2p\cdot k_2)} + \frac{T_4}{(2p\cdot k_2)^2}\right) \tag{14.43}$$

with the traces over the γ-matrices:

- $T_1 = \mathrm{Tr}\big((\not{p}' - m_\mu)(\gamma^\mu(\not{p} - \not{k}_1 + m_\mu)\gamma^\nu)(\not{p} + m_\mu)(\gamma_\nu(\not{p} - \not{k}_1 + m_\mu)\gamma_\mu)\big)$
- $T_2 = \mathrm{Tr}\big((\not{p}' - m_\mu)(\gamma^\mu(\not{p} - \not{k}_1 + m_\mu)\gamma^\nu)(\not{p} + m_\mu)(\gamma_\mu(\not{p} - \not{k}_2 + m_\mu)\gamma_\nu)\big)$
- $T_3 = \mathrm{Tr}\big((\not{p}' - m_\mu)(\gamma^\nu(\not{p} - \not{k}_2 + m_\mu)\gamma^\mu)(\not{p} + m_\mu)(\gamma_\nu(\not{p} - \not{k}_1 + m_\mu)\gamma_\mu)\big)$
- $T_4 = \mathrm{Tr}\big((\not{p}' - m_\mu)(\gamma^\nu(\not{p} - \not{k}_2 + m_\mu)\gamma^\mu)(\not{p} + m_\mu)(\gamma_\mu(\not{p} - \not{k}_2 + m_\mu)\gamma_\nu)\big)$

We make use of FeynCalc in order to compute these traces:

```
 /* set the Mandelstam variables for the incoming muons with mass m and outgoing photons */
 SetMandelstam[s, t, u, p, k, -k1, -k2, m, m, 0, 0]
 T1 := DiracSimplify[
  DiracTrace[(GS[k] - m).GA[\[Mu]].(GS[p] - GS[k1] + m).GA[\[Nu]].(GS[p] + m).
  GA[\[Nu]].(GS[p] - GS[k1] + m).GA[\[Mu]]]]
T2 := DiracSimplify[
  DiracTrace[(GS[k] - m).GA[\[Mu]].(GS[p] - GS[k1] + m).GA[\[Nu]].(GS[p] + m).
  GA[\[Mu]].(GS[p] - GS[k2] + m).GA[\[Nu]]]]
T3 := DiracSimplify[
  DiracTrace[(GS[k] - m).GA[\[Nu]].(GS[p] - GS[k2] + m).GA[\[Mu]].(GS[p] + m).
  GA[\[Nu]].(GS[p] - GS[k1] + m).GA[\[Mu]]]]
T4 := DiracSimplify[
  DiracTrace[(GS[k] - m).GA[\[Nu]].(GS[p] - GS[k2] + m).GA[\[Mu]].(GS[p] + m).
  GA[\[Mu]].(GS[p] - GS[k2] + m).GA[\[Nu]]]]
  /* reduces the expressions by imposing s+t+u = 2m^2 */
TrickMandelstam[T1, {s, t, u, 2 m^2}] // Simplify
TrickMandelstam[T2, {s, t, u, 2 m^2}] // Simplify
TrickMandelstam[T3, {s, t, u, 2 m^2}] // Simplify
TrickMandelstam[T4, {s, t, u, 2 m^2}] // Simplify
```

We find:

$$T_1 = -8\left(m^4 + m_\mu^2(3t+u) - tu\right) = 8((t-m_\mu^2)(u-3m_\mu^2) - 4m_\mu^4) \tag{14.44}$$

and

$$T_2 = T_3 = 8m_\mu^2\left(s - 4m^2\right) \tag{14.45}$$

and finally

$$T_4 = -8\left(m_\mu^4 + m_\mu^2(t+3u) - tu\right) = 8((u-m_\mu^2)(t-3m_\mu^2) - 4m_\mu^4) \tag{14.46}$$

Hence, putting all the pieces together we find:

$$\langle|\mathcal{M}|^2\rangle = \frac{8((t-m_\mu^2)(u-3m_\mu^2)-4m_\mu^4)}{(t-m_\mu^2)^2} + 2\frac{8m_\mu^2\left(s-4m_\mu^2\right)}{(t-m_\mu^2)(u-m_\mu^2)} + \frac{8((u-m_\mu^2)(t-3m_\mu^2)-4m_\mu^4)}{(u-m_\mu^2)^2} \tag{14.47}$$

As shown in Section 11.15 of the book, this expression can be cast into the common form (compare with Eq. (11.232) of the book):

$$\langle|\mathcal{M}|^2\rangle\left(\mu^+\mu^- \to \gamma\gamma\right) = 2e^4\left(\frac{u-m_\mu^2}{t-m_\mu^2} + \frac{t-m_\mu^2}{u-m_\mu^2} + 1 - \left(1 + \frac{2m_\mu^2}{t-m_\mu^2} + \frac{2m_\mu^2}{u-m_\mu^2}\right)^2\right) \tag{14.48}$$

We now define the kinematics in the center-of-mass frame where

$$p = (E,0,0,p), \quad k = (E,0,0,-p) \quad \text{and} \quad k_{1,2} = (\omega, \pm\vec{k}) \tag{14.49}$$

Note that $\omega = \left|\vec{k}\right| = E$. Consequently, the Mandelstam variables are equal to:

$$\begin{aligned} s &= (p+k)^2 = (2E)^2 = 4E^2 \\ t - m_e^2 &= -2(p\cdot k_1) = -2(E^2 - \vec{p}\cdot\vec{k}_1) = -2(E^2 - pE\cos\theta) = -2E(E - p\cos\theta) \\ u - m_e^2 &= -2(p\cdot k_2) = -2E(E + p\cos\theta) \end{aligned} \tag{14.50}$$

where θ is the scattering angle (compare with Figure 11.15 of the book). The matrix element squared then simplifies to:

$$\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{3m_\mu^2 + p^2(3+\cos^2\theta)}{m_\mu^2 + p^2\sin^2\theta} - \left(\frac{p^2\sin^2\theta - m_\mu^2}{p^2\sin^2\theta + m_\mu^2}\right)^2\right) \tag{14.51}$$

Finally, we put this result in the phase-space element using Eq. (5.142) of the book to find:

$$\left(\frac{d\sigma(\mu^+\mu^- \to \gamma\gamma)}{d\Omega}\right)_{\text{CMS}} = \frac{1}{2}\frac{\langle|\mathcal{M}|^2\rangle}{F}\frac{1}{16\pi^2}\frac{\omega}{2E} = \frac{1}{4F}\frac{2e^4}{16\pi^2}\Big(\cdots\Big) = \frac{\alpha^2}{2F}\Big(\cdots\Big) \tag{14.52}$$

where we have introduced a symmetry factor $S = 1/2$ to account for the identical particles in the final state, and the fine structure constant $e^2 = 4\pi\alpha$. The flux factor F is equal to (see Eq. (5.121) of the book):

$$F = 4\sqrt{(p\cdot k)^2 - m_\mu^4} = 4\sqrt{(E^2+p^2)^2 - m_\mu^4} = 4\sqrt{(2p^2+m_\mu^2)^2 - m_\mu^4} = 8pE \tag{14.53}$$

In the low energy limit, $p \ll m$, we find:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CMS,non rel.}}\left(\mu^+\mu^- \to \gamma\gamma\right) \simeq \frac{\alpha^2}{2(8pE)}(3-1) = \frac{\alpha^2}{8m_\mu p} \tag{14.54}$$

After integration over the solid angle $d\Omega$, the total cross-section of the process is:

$$\sigma_{\rm LO}(\mu^+\mu^- \to \gamma\gamma) = \frac{\pi\alpha^2}{2m_\mu p} \tag{14.55}$$

• **The** $\mu^+ + \mu^- \to e^+ + e^-$ **case.** Let us now consider the case of $o - TM$. We define the kinematics as:

$$\mu^-(p,s) + \mu^+(p',s') \to e^-(k,r) + e^+(k',r') \tag{14.56}$$

At leading order the Feynman diagram contributing to the Born cross-section $\sigma_{\rm LO}(\mu^+\mu^- \to e^+e^-)$ is the s-channel shown in Figure 14.2.

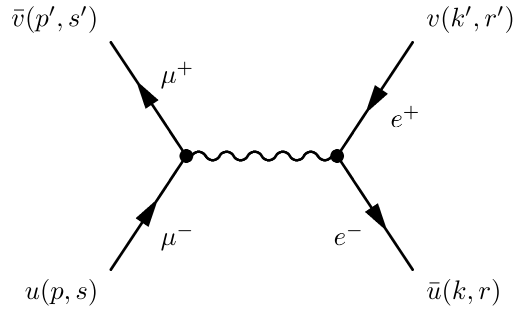

Figure 14.2 Lowest order Feynman digram contributing to $\mu^+ + \mu^- \to e^+ + e^-$.

The corresponding matrix element is given by:

$$\begin{aligned} i\mathcal{M} &= \overline{v}^{(s')}(p')(-ie\gamma^\mu)u^{(s)}(p)\frac{-ig_{\mu\nu}}{(p+p')^2+i\epsilon}\overline{u}^{(r)}(k)(-ie\gamma^\nu)v^{(r')}(k') \\ &= (ie^2)\frac{\overline{v}^{(s')}(p')\gamma^\mu u^{(s)}(p)\overline{u}^{(r)}(k)\gamma_\mu v^{(r')}(k')}{(p+p')^2} \end{aligned} \tag{14.57}$$

The complex conjugate amplitude is then (compare with Eq. (11.109) of the book):

$$i\mathcal{M}^* = (-ie^2)\frac{\overline{v}^{(r')}(k')\gamma_\nu u^{(r)}(k)\overline{u}^{(s)}(p)\gamma^\nu v^{(s')}(p')}{(p+p')^2} \tag{14.58}$$

The matrix element squared can then be conveniently written as (see Eqs. (11.111) and (11.112) of the book):

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^* = \frac{e^4}{(p+p')^4}L_\mu^{\mu\nu}(p,p')L^e_{\mu\nu}(k,k') \tag{14.59}$$

where

$$L_\mu^{\mu\nu}(p,p') = \overline{v}^{(s')}(p')\gamma^\mu u^{(s)}(p)\overline{u}^{(s)}(p)\gamma^\nu v^{(s')}(p') \tag{14.60}$$

and

$$L^e_{\mu\nu}(k,k') = \overline{u}^{(r)}(k)\gamma_\mu v^{(r')}(k')\overline{v}^{(r')}(k')\gamma_\nu u^{(r)}(k) \tag{14.61}$$

For an unpolarized cross-section, the average matrix element squared is computed by summing over final states spin configurations and averaging over the initial states spins. Then:

$$\langle|\mathcal{M}|^2\rangle = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\sum_{s,s'}\sum_{r,r'}|\mathcal{M}|^2 = \frac{1}{4}\frac{e^4}{(p+p')^4}\sum_{s,s'}\sum_{r,r'}L_\mu^{\mu\nu}(p,p')L^e_{\mu\nu}(k,k') \tag{14.62}$$

We have (compare with Eq. (11.115) of the book):

$$\sum_{s,s'} L_\mu^{\mu\nu}(p,p') = \mathrm{Tr}\left(\gamma^\mu(\not{p}+m_\mu)\gamma^\nu(\not{p}'-m_\mu)\right) \quad \text{and} \quad \sum_{r,r'} L^e_{\mu\nu}(k,k') = \mathrm{Tr}\left(\gamma_\mu(\not{k}'-m_e)\gamma_\nu(\not{k}+m_e)\right) \tag{14.63}$$

The reduction of these traces has been performed in Section 11.10 of the book. We find (see Eq. (11.119) of the book):

$$\sum_{s,s'} L_\mu^{\mu\nu}(p,p') = 4\left(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}\left(p'\cdot p + m_\mu^2\right)\right) \tag{14.64}$$

and

$$\sum_{r,r'} L^e_{\mu\nu}(k,k') = 4\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}\left(k'\cdot k + m_e^2\right)\right) \tag{14.65}$$

Therefore,

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle =& \frac{16}{4}\frac{e^4}{(p+p')^4}\left(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}\left(p'\cdot p + m_\mu^2\right)\right)\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}\left(k'\cdot k + m_e^2\right)\right)\\ =&\frac{8e^4}{(p+p')^4}\Big[(p\cdot k')(p'\cdot k) + (p\cdot k)(p'\cdot k')\\ &\quad - (p'\cdot p)(k\cdot k' + m_e^2) - (k'\cdot k)(p\cdot p' + m_\mu^2) + 2\left(p'\cdot p + m_\mu^2\right)\left(k\cdot k' + m_e^2\right)\Big]\\ =&\frac{8e^4}{(p+p')^4}\left[(p\cdot k')(p'\cdot k) + (p\cdot k)(p'\cdot k') + m_\mu^2(k\cdot k') + m_e^2(p\cdot p') + 2m_e^2 m_\mu^2\right]\end{aligned} \tag{14.66}$$

We now note that

$$(p+p')^2 = p^2 + p'^2 + 2(p\cdot p') \quad \Longrightarrow \quad (p\cdot p') = \frac{1}{2}(p+p')^2 - m_\mu^2 \tag{14.67}$$

and

$$(p+p')^2 = (k+k')^2 = k^2 + k'^2 + 2(k\cdot k') \quad \Longrightarrow \quad (k\cdot k') = \frac{1}{2}(p+p')^2 - m_e^2 \tag{14.68}$$

Finally:

$$\langle|\mathcal{M}|^2\rangle = \frac{8e^4}{(p+p')^4}\left((p\cdot k)(p'\cdot k') + (p'\cdot k)(p\cdot k') + \frac{m_\mu^2 + m_e^2}{2}(p+p')^2\right) \tag{14.69}$$

We now introduce the Mandelstam variables:

$$s = (p+p')^2 = (k+k')^2, \quad t = (p-k)^2 = (p'-k')^2, \quad u = (p'-k)^2 = (p-k')^2 \tag{14.70}$$

Hence:

$$t = (p-k)^2 = p^2 + k^2 - 2p\cdot k \quad \Longrightarrow \quad (p\cdot k) = (p'\cdot k') = -\frac{1}{2}(t - (m_\mu^2 + m_e^2)) \tag{14.71}$$

and

$$u = (p'-k)^2 = p'^2 + k^2 - 2p'\cdot k \quad \Longrightarrow \quad (p'\cdot k) = (p\cdot k') = -\frac{1}{2}(u - (m_\mu^2 + m_e^2)) \tag{14.72}$$

The matrix element squared can then be expressed as:

$$\langle|\mathcal{M}|^2\rangle = \frac{2e^4}{s^2}\left((t-(m_\mu^2+m_e^2))^2 + (u-(m_\mu^2+m_e^2))^2 + 2(m_\mu^2+m_e^2)s\right) \tag{14.73}$$

From now on, we are going to neglect the rest mass of the electron relative to that of the muon, i.e. $m_e \ll m_\mu$. Hence:

$$\langle|\mathcal{M}|^2\rangle \simeq \frac{2e^4}{s^2}\left((t-m_\mu^2)^2+(u-m_\mu^2)^2+2m_\mu^2 s\right) \tag{14.74}$$

We define the kinematics in the center-of-mass frame:

$$p=(E,0,0,p) \quad \text{and} \quad p'=(E,0,0,-p) \quad \text{where} \quad E^2=p^2+m_\mu^2 \tag{14.75}$$

The final state electrons can be defined as:

$$k=(E,0,E\sin\theta,E\cos\theta) \quad \text{and} \quad k'=(E,0,-E\sin\theta,-E\cos\theta) \tag{14.76}$$

where θ is the scattering angle. Consequently, the Mandelstam variables are equal to:

$$\begin{aligned} s &= (p+p')^2=(2E)^2=4E^2 \\ t-m_\mu^2 &= -2(p\cdot k)=-2(E^2-pE\cos\theta)=-2E^2(E-\frac{p}{E}\cos\theta)=-\frac{s}{2}(1-\beta\cos\theta) \\ u-m_\mu^2 &= -2(p\cdot k')=-\frac{s}{2}(1+\beta\cos\theta) \end{aligned} \tag{14.77}$$

where $\beta=p/E$ is the velocity of the muons. Consequently:

$$\langle|\mathcal{M}|^2\rangle=\frac{e^4}{2}\left((1-\beta\cos\theta)^2+(1+\beta\cos\theta)^2+\frac{8m_\mu^2}{s}\right)=e^4\left(1+\beta^2\cos^2\theta+\frac{m_\mu^2}{E^2}\right) \tag{14.78}$$

Finally, we multiply this result by the phase-space element using Eq. (5.141) of the book to find:

$$\left(\frac{d\sigma(\mu^+\mu^-\to e^+e^-)}{d\Omega}\right)_{\text{CMS}}=\frac{\langle|\mathcal{M}|^2\rangle}{F16\pi^2}\frac{E}{2E}=\frac{1}{16pE}\frac{e^4}{16\pi^2}\Big(\cdots\Big)=\frac{\alpha^2}{16pE}\Big(\cdots\Big) \tag{14.79}$$

where we have introduced the flux $F=8pE$ and the fine structure constant $e^2=4\pi\alpha$. In the low-energy limit, $p\ll m_\mu$, $\beta\to 0$ and $E\to m_\mu$, the differential cross-section is then given by:

$$\left(\frac{d\sigma(\mu^+\mu^-\to e^+e^-)}{d\Omega}\right)_{\text{CMS}}\simeq\frac{\alpha^2}{16pm_\mu}\Big(1+1\Big)=\frac{\alpha^2}{8pm_\mu} \tag{14.80}$$

We can now integrate over the solid angle $d\Omega$ to find:

$$\sigma_{\text{LO}}(\mu^+\mu^-\to e^+e^-)\simeq\frac{\pi\alpha^2}{2pm_\mu} \tag{14.81}$$

d) The decay rates of both $p-TM$ to two photons, $(\mu^+\mu^-)\to 2\gamma$, and $o-TM$ to a pair of electron-positron, $(\mu^+\mu^-)\to e^+e^-$, are computed at leading order (LO) using the Wheeler-Pirenne factorisation formula (see Eq. (14.112) of the book):

$$\Gamma_{\text{LO}}=\frac{1}{2J+1}|\psi(0)|^2\times(4v_{\text{rel}}\sigma_{\text{LO}}) \tag{14.82}$$

where $\psi(0)$ is the hydrogen wave function at the origin[2], appropriately modified to take into account the reduced mass of the TM system compared to hydrogen , v_{rel} the relative velocity of the muon with

2 We recall that the hydrogen wave function as a solution to the Schrödinger equation can be written as a factorisation between a radial part and an angular part. In particular for the ground state $(n,m,l)=(1,0,0)$):

$$\psi_{1,0,0}(r)=\frac{1}{\sqrt{\pi(a_0^{TM})^3}}e^{-r/a_0^{TM}} \tag{14.83}$$

See Appendix C.10 of the book.

respect to the antimuon and $\sigma_{\rm LO}$ the LO cross-section in the non-relativistic limit. The relative velocity can be extracted from the bound state total momentum in the center-of-mass frame:

$$\vec{p} = -\vec{p'} \quad \Longrightarrow \quad |\vec{p}_{\rm rel}| = 2|\vec{p}| = 2p \tag{14.84}$$

The Bohr radius of the TM is given by (see Eq. (C.9) in Appendix C of the book):

$$a_0^{TM} = \frac{1}{\alpha\mu_R} = \frac{2}{\alpha m_\mu} \tag{14.85}$$

where we have used the reduced mass $\mu_R = m_\mu/2$. Hence:

$$|\psi(0)|^2 = \left(\frac{1}{\sqrt{\pi(a_0^{TM})^3}}\right)^2 = \frac{1}{\pi(a_0^{TM})^3} = \frac{\alpha^3 m_\mu^3}{8\pi} \tag{14.86}$$

The decay rate of the ground state para-true-muonium, $1^1S_0 \to 2\gamma$, is then given by:

$$\Gamma_{\rm LO}(1^1S_0 \to 2\gamma) = \frac{\alpha^3 m_\mu^3}{8\pi} \times \left(4\left(\frac{2p}{m_\mu}\right)\sigma_{\rm LO}(\mu^+\mu^- \to 2\gamma)\right) = \frac{\alpha^3 m_\mu^3}{8\pi} \times \left(4\left(\frac{2p}{m_\mu}\right)\frac{\pi\alpha^2}{2m_\mu p}\right) \tag{14.87}$$

Finally the decay rate at leading order and its associated lifetime are:

$$\begin{aligned}
\Gamma_{\rm LO}(1^1S_0 \to 2\gamma) &= \frac{\alpha^5 m_\mu}{2} \approx 1.1 \times 10^{-12}\ \text{GeV} \\
\tau_{\rm LO}(1^1S_0 \to 2\gamma) &= \frac{1}{\Gamma_{\rm LO}^{1^1S_0\to 2\gamma}} \approx 9.11 \times 10^{11}\ \text{GeV}^{-1} = 0.600\ \text{ps}
\end{aligned} \tag{14.88}$$

where we used $m_\mu \approx 0.106$ GeV,$\alpha \approx 1/137$ and 1 GeV^{-1} = 6.58×10^{-25} s (see Table 1.6 of the book).
Similarly, we compute the lifetime for ortho-true-muonium decay to a pair of electron-positron, $1^3S_1 \to e^+e^-$, with $J = 1$ to find:

$$\Gamma_{\rm LO}(1^3S_1 \to e^+e^-) = \frac{1}{3}\left(\frac{\alpha^3 m_\mu^3}{8\pi}\right) \times \left(4\left(\frac{2p}{m_\mu}\right)\frac{\pi\alpha^2}{2pm_\mu}\right) = \frac{\alpha^5 m_\mu}{6} \tag{14.89}$$

Consequently:

$$\begin{aligned}
\Gamma_{\rm LO}(1^3S_1 \to e^+e^-) &= \frac{\alpha^5 m_\mu}{6} = \frac{1}{3}\Gamma_{\rm LO}(1^1S_0 \to 2\gamma) \approx 3.66 \times 10^{-13}\ \text{GeV} \\
\tau_{\rm LO}(1^3S_1 \to e^+e^-) &= 3\tau_{\rm LO}(1^1S_0 \to 2\gamma) \approx 1.80\ \text{ps}
\end{aligned} \tag{14.90}$$

e) The ratio of the pMu and oMu lifetimes is about 1/3. This is very different from the case of positronium, for which the ratio is of the order of 10^{-3}. The reason is that in the case of positronium, the decay into e^+e^- is obviously kinematically prohibited, hence the lowest order allowed decay is $1^3S_1 \to 3\gamma$, which contains an additional QED vertex, and is therefore suppressed by an additional order α compared to the $1^1S_0 \to 2\gamma$ decay.

15 Hadrons

15.1 π^0 lifetime

Estimate the $\pi^0 \to \gamma\gamma$ decay width and corresponding lifetime.

Solution: The π^0 is a bound state member of the isospin $I = 1$ triplet, and can be written as (see Sections 15.6 and 17.2.1 of the book):

$$|\pi^0\rangle = |I = 1, I_3 = 0\rangle = \frac{1}{2}(u\bar{u} - d\bar{d}) \tag{15.1}$$

The electromagnetic decay of the π^0 can be seen as the annihilation of the quark–antiquark pair into a photon, which then can convert into two photons via the so-called triangle diagram, as shown in Figure 15.1(left). Naively, the amplitude would then depend on the VVV triangle diagram shown in Figure 15.1(right), where V denotes a vector coupling, as is the case for the photon. The problem is that in general the loop integral diverges as

$$\int \frac{\mathrm{d}^4 p}{p^3} \propto \int \frac{p^3 \mathrm{d}p}{p^3} = \int \mathrm{d}p \to \infty \tag{15.2}$$

which is violently ill-behaved. Fortunately, there is no problem in QED: the VVV diagram vanishes because the couplings are odd under charge conjugation. Hence, a fundamental symmetry is needed to cure the VVV diagram. But on the other hand, this creates a problem for the π^0 decay!

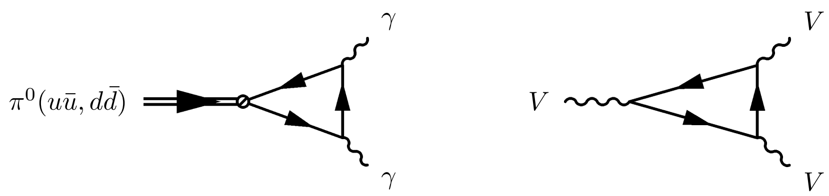

Figure 15.1 (left) The tree-level diagram for the $\pi^0 \to \gamma\gamma$ decay. (right) Triangle diagram VVV (note that there are identical diagrams where the final state photons are swapped).

This cancellation does not occur if we consider axial couplings, such as AVV or AAA. In theories where these coupling exist, these divergences are called "**chiral anomalies**". It turns out that in the Standard Model, where the coupling for left-handed and right-handed particles are different, each fundamental fermion makes a separate contribution to such a chiral anomaly. The theory is *anomaly free*ăonly if the sum of the contributions from each fermion cancels exactly. This is indeed the case in the Standard Model because the electric charges in each generation (e.g. (u,d,e^-, ν_e), ...) sum to zero – which incidentally requires the existence of three quark colors!

Let us return to the $\pi^0 \to \gamma\gamma$ decay. We define the kinematics as follows:

$$\pi^0(P) \to \gamma(k_1, \epsilon_1) + \gamma(k_2, \epsilon_2) \tag{15.3}$$

The QCD Lagrangian has a chiral symmetry ($U(3)_L \times U(3)_R$) where the three lightest quark masses are set to zero. This means that for vanishing quark masses, the decay rate for $\pi^0 \to \gamma\gamma$ should vanish. The symmetry is believed to be spontaneously broken via the non-vanishing expectation value of the quark–antiquark state. Then the $\pi^0 \to \gamma\gamma$ decay amplitude is completely related to the axial anomaly. The amplitude can only depend on Lorentz invariant quantities formed with k_1^μ, ϵ_1^μ, k_2^μ and ϵ_2^μ. Its general form requiring Lorentz invariance, parity conservation and gauge invariance can be written as:

$$\mathcal{M}(k_1, \epsilon_1, k_2, \epsilon_2) = -iA_{\gamma\gamma}(P^2)\epsilon_{\mu\nu\rho\sigma}\epsilon_1^{*\mu}k_1^\nu\epsilon_2^{*\rho}k_2^\sigma \tag{15.4}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor, and $A_{\gamma\gamma}(P^2)$ is the axial anomaly. This latter can only depend on P^2, since the pion is spinless. The matrix element squared is then given by:

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^* = \left|A_{\gamma\gamma}(P^2)\right|^2 \epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}\epsilon_1^{*\mu}k_1^\nu\epsilon_2^{*\rho}k_2^\sigma\epsilon_{1,\alpha}k_{1,\beta}\epsilon_{2,\gamma}k_{2,\delta} \tag{15.5}$$

We are interested in the unpolarized cross-section so we should sum over the photon polarizations:

$$\langle|\mathcal{M}|^2\rangle = \sum_{\lambda_1,\lambda_2} \left|A_{\gamma\gamma}(P^2)\right|^2 \epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}\epsilon_1^{*\mu}k_1^\nu\epsilon_2^{*\rho}k_2^\sigma\epsilon_{1,\alpha}k_{1,\beta}\epsilon_{2,\gamma}k_{2,\delta} \tag{15.6}$$

We use the photon completeness relation Eq. (10.76) of the book:

$$\sum_\lambda \epsilon^{\mu,*}_{(\lambda)}\epsilon^\nu_{(\lambda)} \leftrightarrow -g^{\mu\nu} \tag{15.7}$$

We find:

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \left|A_{\gamma\gamma}(P^2)\right|^2 \epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}g^\mu_\alpha g^\rho_\gamma k_1^\nu k_2^\sigma k_{1,\beta}k_{2,\delta} = \left|A_{\gamma\gamma}(P^2)\right|^2 \epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\beta\rho\delta}k_1^\nu k_2^\sigma k_{1,\beta}k_{2,\delta} \\ &= -2\left|A_{\gamma\gamma}(P^2)\right|^2 \left(\delta^\beta_\nu\delta^\delta_\sigma - \delta^\sigma_\nu\delta^\beta_\sigma\right)k_1^\nu k_2^\sigma k_{1,\beta}k_{2,\delta} \\ &= -2\left|A_{\gamma\gamma}(P^2)\right|^2 \left(k_1^2k_2^2 - (k_1\cdot k_2)^2\right)\end{aligned} \tag{15.8}$$

where in the second line we used Eq. (D.46) in Appendix D of the book. We can define the kinematics in the center-of-mass system of the decaying pion:

$$P^\mu = (m_{\pi^0}, \vec{0}), \quad k_1 = (\omega, 0, 0, \omega), \quad k_2 = (\omega, 0, 0, -\omega) \tag{15.9}$$

where $k_1^2 = k_2^2 = 0$, and $\omega = m_{\pi^0}/2$ by energy conservation. Hence:

$$\langle|\mathcal{M}|^2\rangle = 2\left|A_{\gamma\gamma}(P^2)\right|^2(k_1\cdot k_2)^2 = \left|A_{\gamma\gamma}(P^2)\right|^2\left(2\omega^2\right)^2 = \left|A_{\gamma\gamma}(P^2)\right|^2\frac{m^4_{\pi^0}}{2} \tag{15.10}$$

We include the phase-space factor with Eq. (5.163) of the book, to find:

$$\mathrm{d}\Gamma = \frac{1}{2}\frac{\langle|\mathcal{M}|^2\rangle}{32\pi^2}\frac{\omega}{P^2}\mathrm{d}\Omega = \frac{1}{2}\frac{1}{32\pi^2}\frac{m_{\pi^0}}{2m^2_{\pi^0}}\left|A_{\gamma\gamma}(P^2)\right|^2\frac{m^4_{\pi^0}}{2}\mathrm{d}\Omega = \frac{1}{4}\frac{m^3_{\pi^0}}{64\pi^2}\left|A_{\gamma\gamma}(P^2)\right|^2 \tag{15.11}$$

where we introduced the statistical factor 1/2 to account for the two indistinguishable particles in the final state. Integration over the solid angle $\mathrm{d}\Omega$ yields:

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{m^3_{\pi^0}}{64\pi}\left|A_{\gamma\gamma}\right|^2 \tag{15.12}$$

From a dimensional analysis, we notice that $A_{\gamma\gamma}$ has the units GeV^{-1}. Since there are two photons in the final state, it should be proportional to e^2. Finally, it should be proportional to the number of quark colors N_C. Hence, we can write:

$$A_{\gamma\gamma} = \frac{Ce^2N_c}{f(P^2)} = \frac{Ce^2N_c}{m_{\pi^0}} = \frac{C\alpha N_c}{m_{\pi^0}} \tag{15.13}$$

where C is a unitless constant, which should be $\mathcal{O}(1)$ if the model is correct, and $f(q^2)$ is a function with the units of an energy, called the decay constant of the pion. We tentatively set $f(m_{\pi^0}) \approx m_{\pi^0}$, since there are no other scales involved in the problem. Numerically, we obtain:

$$A_{\gamma\gamma} \simeq 0.162 \times C \text{ GeV}^{-1} \tag{15.14}$$

Consequently:

$$\Gamma(\pi^0 \to \gamma\gamma) \simeq \frac{(0.135)^3}{64\pi}(0.162)^2 C^2 \approx 3.21 \times 10^{-7} \times C^2 \,\text{GeV} \tag{15.15}$$

If we assume that the decay branching ratio is about 100%, then we find that the lifetime of the π^0 is just:

$$\tau = \frac{1}{\Gamma(\pi^0 \to \gamma\gamma)} \approx \frac{3.11 \times 10^6 \,\text{GeV}^{-1}}{C^2} \approx \frac{2.05 \times 10^{-18} \text{ s}}{C^2} \tag{15.16}$$

where we used 1 $\text{GeV}^{-1} \approx 6.58 \times 10^{-25}$ s (see Table 1.6 of the book). Consequently, we estimate that the π^0 is short lived. Experimentally, one finds (see Eq. (15.21) of the book):

$$\tau_{exp} = (8.52 \pm 0.18) \times 10^{-17} \,\text{s} \tag{15.17}$$

which is very close to our estimate. Indeed, we find that:

$$C \approx 0.16 \tag{15.18}$$

which is indeed $\mathcal{O}(1)$.

A theoretical discussion of the $\pi^0 \to \gamma\gamma$ lifetime can be found in e.g. J.L. Goity, A.M. Bernstein, B.R. Holstein, "The decay pi0 to gamma gamma to next to leading order in Chiral Perturbation Theory", Phys.Rev.D66:076014,2002 `https://journals.aps.org/prd/abstract/10.1103/PhysRevD.66.076014` and references therein.

The theoretical decay rate is given in Eq. (17.68) of the book:

$$\Gamma(\pi^0 \to \gamma\gamma) = \left(\frac{\alpha}{3\pi}\right)^2 N_C^2 (Q_u^2 - Q_d^2) \frac{m_\pi^3}{8\pi f_\pi^2} \tag{15.19}$$

where $f_\pi \approx m_\pi$ is the pion decay constant. By direct comparison with our previous result, we find that:

$$C \approx \frac{2\sqrt{2}}{3\pi}\sqrt{Q_u^2 - Q_d^2} = \frac{2\sqrt{2}}{9\pi} \approx 0.10 \tag{15.20}$$

in agreement with the value found above. This calculation and its comparison to experimental data represent an important achievement, as they provide a direct confirmation that the number of quark colors is three ($N_C = 3$).

15.2 Deuteron production

The deuteron d is a bound state composed of a neutron and a proton (see Section 15.5 of the book). Its isospin is equal to $I = 0$. Compute the ratio of the cross-sections of the following processes:

$$p + p \to d + \pi^+, \qquad p + n \to d + \pi^0, \qquad n + n \to d + \pi^- \tag{15.21}$$

Solution:
We consider the isospin eigenstates $|I, I_3\rangle$ of the different particles:

- Deuteron: $d = |0, 0\rangle$
- Nucleons: $p = |\frac{1}{2}, \frac{1}{2}\rangle$, $n = |\frac{1}{2}, -\frac{1}{2}\rangle$
- Pions: $\pi^+ = |1, 1\rangle$, $\pi^0 = |1, 0\rangle$, $\pi^- = |1, -1\rangle$

We should now write the two-particles charge eigenstates as a function of eigenstates of the isospins. Let us for instance consider the $p + p$ initial state. We must combine the two $|1/2, 1/2\rangle$ states. We make use of the Clebsch–Gordan coefficients (see Section C.11 of the book and `https://pdg.lbl.gov/2018/reviews/rpp2018-rev-clebsch-gordan-coefs.pdf`). We look at the $1/2 \times 1/2$ table for $m_1 = m_2 = 1/2$, we find a coefficient 1 for the $|J, M\rangle = |1, 1\rangle$ state. Accordingly:

$$|pp\rangle = |1, 1\rangle \tag{15.22}$$

Similarly, for the other initial-state and final-state configurations, we find:

$$\begin{aligned} |pn\rangle = \frac{1}{\sqrt{2}}\,|1,0\rangle + \frac{1}{\sqrt{2}}\,|0,0\rangle\,, \qquad |nn\rangle = |1,-1\rangle \\ |d\pi^+\rangle = |1,1\rangle\,, \qquad |d\pi^0\rangle = |1,0\rangle\,, \qquad |d\pi^-\rangle = |1,-1\rangle \text{ă} \end{aligned} \tag{15.23}$$

We assume that the strong interaction is isospin invariant (i.e. the commutator $[H_{strong}, \vec{I}] = 0$). For orthogonal and normalized isospin eigenstates, we can then write:

$$\langle I', I'_3 | H_{strong} | I, I_3\rangle = \delta_{I,I'}\delta_{I_3,I'_3}\mathcal{M}_I \tag{15.24}$$

Consequently, we only need to consider two independent amplitudes $\mathcal{M}_1$ and $\mathcal{M}_0$ for the three processes. We actually find:

$$\begin{aligned} \langle d\pi^+|H_{strong}|pp\rangle &= \langle 1,1|H_{strong}|1,1\rangle = \mathcal{M}_1 \\ \langle d\pi^0|H_{strong}|pn\rangle &= \langle 1,0|H_{strong}|\left(\frac{1}{\sqrt{2}}\,|1,0\rangle + \frac{1}{\sqrt{2}}\,|0,0\rangle\right)\rangle = \\ &= \frac{1}{\sqrt{2}}\,\langle 1,0|H_{strong}|1,0\rangle + \frac{1}{\sqrt{2}}\,\underbrace{\langle 1,0|H_{strong}|0,0\rangle}_{=0} \\ &= \frac{1}{\sqrt{2}}\mathcal{M}_1 \\ \langle d\pi^-|H_{strong}|nn\rangle &= \langle 1,-1|H_{strong}|1,-1\rangle = \mathcal{M}_1 \end{aligned} \tag{15.25}$$

The ratio of the cross-sections of the three processes can then be estimated as follows:

$$\sigma(p+p \to d+\pi^+) : \sigma(p+n \to d+\pi^0) : \sigma(n+n \to d+\pi^-) = |\mathcal{M}_1|^2 : \frac{1}{2}\,|\mathcal{M}_1|^2 : |\mathcal{M}_1|^2 = 2:1:2 \tag{15.26}$$

15.3 The α particle

The bound state of two protons and two neutrons, in other words the nucleus of ^4He, is called an α particle.

(a) We know that the isotopes of hydrogen or lithium with an atomic weight of four (^4H, ^4Li) do not exist. What can we conclude on the isospin of the α particle?

(b) The reaction $d + d \to \alpha + \pi^0$ was never observed. Explain why.

(c) Would you expect that the isotope ^4Be exists?

(d) What about the bound state of four neutrons?

Solution:
An α particle is a bound state of 4 nucleons, composed of 2 protons and 2 neutrons. Since nucleons have isospin $I = 1/2$, by the rule of addition of isospin which follows that of angular momentum, the bound state of 4 nucleons can possess an isospin eigenstate $I = 0, 1, 2$.

(a) We would have the following isospin eigenstates:

- ${}^4\mathrm{H} = |pnnn\rangle$: in this case $I_3 = -1$, hence the total isospin $I_{{}^4\mathrm{H}} \geq 1$.
- ${}^4\mathrm{Li} = |pppn\rangle$: in this case $I_3 = +1$, hence the total isospin $I_{{}^4\mathrm{Li}} \geq 1$.

If these two states are not observed in Nature, they presumably do not exist and we can conclude that four nucleons bound states with $I \geq 1$ are not stable. We therefore conclude that the total isospin of the α particle must vanish:

$$I_\alpha = 0 \tag{15.27}$$

Note that the proton and the neutron are fermions, hence follow the Pauli exclusion principle. No two protons or neutrons can rest in the same eigenstate in the bound state. Hence, in the $|pnnn\rangle$ and $|pppn\rangle$ eigenstates, some nucleon would necessarily find itself in an excited state. In contrast, the α particle is the $|ppnn\rangle$ which can exist in the ground state of the four nucleons, since the $2p$ and $2n$ systems are allowed by placing each nucleon in spin-up and spin-down configuration.

(b) Let us analyze the total isospin of the reaction $d + d \to \alpha + \pi^0$. We found that $I_d = I_\alpha = 0$ and $I_\pi = 1$. Hence, we have:

$$\underbrace{d + d}_{I=0} \to \underbrace{\alpha + \pi^0}_{I=1} \tag{15.28}$$

The reaction is therefore prohibited by the conservation of isospin in strong interactions. This observation is consistent with the fact that the α has a vanishing isospin.

(c) We consider the state ${}^4\mathrm{Be} = |pppp\rangle$. We have $I_3 = 2$ hence the total isospin is maxed at $I = 2$. We note that as was already the case in part a), the four protons cannot exist in the same state, and we noted that only the α is observed. Hence, we expect the ${}^4\mathrm{Be} = |pppp\rangle$ to be even less probable than the ${}^4\mathrm{H} = |pnnn\rangle$ and ${}^4\mathrm{Li} = |pppn\rangle$ states.

(c) We consider the bound state $|nnnn\rangle$. For the same reasons as in part c), we do not expect this state to exist.

15.4 ^{3}H and ^{3}He production in the pd scattering

The isospin I of a nucleus must follow the rule $|I_3| \leq I \leq \frac{1}{2}\,A$. In the ground state of the nucleus, the isospin must always take the smallest possible value. Compute the following ratio of the cross-sections:

$$\sigma(p + d \to \pi^+ + {}^3\mathrm{H})\ /\ \sigma(p + d \to \pi^0 + {}^3\mathrm{He}) \tag{15.29}$$

Solution: We first write the isospin eigenstates of the elementary components involved in the reactions:

$$p = |\frac{1}{2}, \frac{1}{2}\rangle\,, \quad d = |0, 0\rangle\,, \quad \pi^+ = |1, 1\rangle\,, \quad \pi^0 = |1, 0\rangle \tag{15.30}$$

Then for the ground-states of the nuclei we expect:

$$|^3\mathrm{H}\rangle = |pnn\rangle\,\text{ă} = |\frac{1}{2}, -\frac{1}{2}\rangle\,, \quad |^3\mathrm{He}\rangle = |ppn\rangle\,\text{ă} = |\frac{1}{2}, \frac{1}{2}\rangle \tag{15.31}$$

We make use of the Clebsch–Gordan coefficients (see Section C.11 of the book and `https://pdg.lbl.gov/2018/reviews/rpp2018-rev-clebsch-gordan-coefs.pdf`). We find:

$$\begin{aligned} |pd\rangle &= |\frac{1}{2}, \frac{1}{2}\rangle \\ |\pi^+\ {}^3\mathrm{H}\rangle &= \sqrt{\frac{1}{3}}\,|\frac{3}{2}, \frac{1}{2}\rangle + \sqrt{\frac{2}{3}}\,|\frac{1}{2}, \frac{1}{2}\rangle \\ |\pi^0\ {}^3\mathrm{He}\rangle &= \sqrt{\frac{2}{3}}\,|\frac{3}{2}, \frac{1}{2}\rangle - \sqrt{\frac{1}{3}}\,|\frac{1}{2}, \frac{1}{2}\rangle \end{aligned} \tag{15.32}$$

For orthogonal and normalized isospin eigenstates, we can then write:

$$\langle I', I'_3|H_{strong}|I, I_3\rangle = \delta_{I,I'}\delta_{I_3,I'_3}\mathcal{M}_I \tag{15.33}$$

Consequently:

$$\langle\pi^+\ {}^3\mathrm{H}|H_{strong}|pd\rangle = \sqrt{\frac{2}{3}}\mathcal{M}_{1/2} \quad \text{and} \quad \langle\pi^0\ {}^3\mathrm{He}|H_{strong}|pd\rangle = -\sqrt{\frac{1}{3}}\mathcal{M}_{1/2} \tag{15.34}$$

The ratio of the cross-sections are then expected to behave as:

$$\sigma(p + d \to \pi^+ + ^3\mathrm{H}) : \sigma(p + d \to \pi^0 + ^3\mathrm{He}) = \frac{2}{3}\left|\mathcal{M}_{1/2}\right|^2 : \frac{1}{3}\left|\mathcal{M}_{1/2}\right|^2 = 2 : 1 \tag{15.35}$$

15.5 Pion nucleus scattering

We consider the six following elastic πN scattering processes:

(a) $\pi^+ + p \to \pi^+ + p$ (b) $\pi^0 + p \to \pi^0 + p$

(c) $\pi^- + p \to \pi^- + p$ (d) $\pi^+ + n \to \pi^+ + n$

(e) $\pi^0 + n \to \pi^0 + n$ (f) $\pi^- + n \to \pi^- + n$

and the four following charge-exchange processes:

(g) $\pi^+ + n \to \pi^0 + p$ (h) $\pi^0 + p \to \pi^+ + n$

(i) $\pi^0 + n \to \pi^- + p$ (j) $\pi^- + p \to \pi^0 + n$

Since a πN state can have either a total isospin $I = 3/2$ or $I = 1/2$, the 10 amplitudes for the processes above can be described as a function of two independent amplitudes $\mathcal{M}_3$ ($I = 3/2$) and $\mathcal{M}_1$ ($I = 1/2$).

(a) Express the 10 amplitudes $\mathcal{M}_a$, ..., $\mathcal{M}_j$ as a function of $\mathcal{M}_1$ and $\mathcal{M}_3$.

(b) The ratios of the cross-sections σ_a, σ_c, and σ_j are given by:

$$\sigma_a : \sigma_c : c_j = 9|\mathcal{M}_3|^2 : |\mathcal{M}_3 + 2\mathcal{M}_1|^2 : 2|\mathcal{M}_3 - \mathcal{M}_1|^2$$

Compute the ratios of all 10 cross-sections $\sigma_a : \sigma_b : \ldots : \sigma_j$.

(c) *The measured cross-sections summarized by the Particle Data Group's Review can be found at* **`http://pdg.lbl.gov/current/xsect/`**. *Search the tables corresponding to πN scattering. At a center-of-mass energy of 1232 MeV the πN-scattering cross-sections exhibit a pronounced peak, which corresponds to the excitation of the Δ resonance with isospin $I = 3/2$. At the resonance, one expects $\mathcal{M}_3 \gg \mathcal{M}_1$, hence $\sigma_a : \sigma_c : \sigma_j = 9 : 1 : 2$. Compute the ratios of all 10 cross-sections at the resonance and compare with experimental data.*

Solution:

With the help of the Clebsch–Gordan coefficients (see Section C.11 of the book and `https://pdg.lbl.gov/2018/reviews/rpp2018-rev-clebsch-gordan-coefs.pdf`), the charge eigenstates of the initial and final states can be written as isospin eigenstates. We find:

$$
\begin{aligned}
|\pi^+ p\rangle ă &= |\frac{3}{2}, \frac{3}{2}\rangle \\
|\pi^+ n\rangle &= \sqrt{\frac{1}{3}}\,|\frac{3}{2}, \frac{1}{2}\rangle + \sqrt{\frac{2}{3}}\,|\frac{1}{2}, \frac{1}{2}\rangle \\
|\pi^0 p\rangle &= \sqrt{\frac{2}{3}}\,|\frac{3}{2}, \frac{1}{2}\rangle - \sqrt{\frac{1}{3}}\,|\frac{1}{2}, \frac{1}{2}\rangle \\
|\pi^0 n\rangle &= \sqrt{\frac{2}{3}}\,|\frac{3}{2}, -\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}\,|\frac{1}{2}, -\frac{1}{2}\rangle \\
|\pi^- p\rangle &= \sqrt{\frac{1}{3}}\,|\frac{3}{2}, -\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|\frac{1}{2}, -\frac{1}{2}\rangle \\
|\pi^- n\rangle ă &= |\frac{3}{2}, -\frac{3}{2}\rangle
\end{aligned}
\tag{15.36}
$$

For orthogonal and normalized isospin eigenstates, we can write:

$$
\langle I', I_3'|H_{strong}|I, I_3\rangle = \delta_{I,I'}\delta_{I_3,I_3'}\mathcal{M}_{2I} \tag{15.37}
$$

(a) For the elastic scattering processes, we find:

$$
\begin{aligned}
\mathcal{M}_a &= \langle \pi^+ p|H_{strong}|\pi^+ p\rangle = \mathcal{M}_3, \qquad \mathcal{M}_b = \frac{2}{3}\mathcal{M}_3 + \frac{1}{3}\mathcal{M}_1 \\
\mathcal{M}_c &= \frac{1}{3}\mathcal{M}_3 + \frac{2}{3}\mathcal{M}_1, \qquad \mathcal{M}_d = \frac{1}{3}\mathcal{M}_3 + \frac{2}{3}\mathcal{M}_1 \\
\mathcal{M}_e &= \frac{2}{3}\mathcal{M}_3 + \frac{1}{3}\mathcal{M}_1, \qquad \mathcal{M}_f = \mathcal{M}_3
\end{aligned}
\tag{15.38}
$$

Similarly, for the charge exchange processes:

$$
\begin{aligned}
\mathcal{M}_g &\overset{(T\text{ inv.})}{=} \mathcal{M}_h = \frac{\sqrt{2}}{3}\mathcal{M}_3 - \frac{\sqrt{2}}{3}\mathcal{M}_1, \\
\mathcal{M}_i &\overset{(T\text{ inv.})}{=} \mathcal{M}_j = \frac{\sqrt{2}}{3}\mathcal{M}_3 - \frac{\sqrt{2}}{3}\mathcal{M}_1
\end{aligned}
\tag{15.39}
$$

where we used T-invariance.

(b) The multiple ratios of the cross-sections are then given by (multiplying each cross-section by $3^2 = 9$):

$$
\begin{aligned}
\sigma_a : \sigma_b : \cdots \sigma_j =& 9|\mathcal{M}_3|^2 : |2\mathcal{M}_3 + \mathcal{M}_1|^2 : |\mathcal{M}_3 + 2\mathcal{M}_1|^2 : |\mathcal{M}_3 + 2\mathcal{M}_1|^2 : |2\mathcal{M}_3 + \mathcal{M}_1|^2 : 9|\mathcal{M}_3|^2 : \\
& |2\mathcal{M}_3 - \mathcal{M}_1|^2 : |\mathcal{M}_3 - \mathcal{M}_1|^2 : |\mathcal{M}_3 - \mathcal{M}_1|^2 : |\mathcal{M}_3 - \mathcal{M}_1|^2
\end{aligned}
\tag{15.40}
$$

(c) The Δ resonance is observed in four electric charge states via the reactions:

- Δ^{++}: $\pi^+ p$
- Δ^{+}: $\pi^+ n$ and $\pi^0 p$
- Δ^{0}: $\pi^- p$ and $\pi^0 n$
- Δ^{-}: $\pi^- n$

Since there are $2I_\Delta + 1 = 4$ isospin states, then the total isospin of the Δ is $I_\Delta = 3/2$. At the resonance, we therefore indeed expect $|\mathcal{M}_3| \gg |\mathcal{M}_1|$. Hence, the expected ratio of the cross-section at the resonance (i.e. when the center-of-mass energy of the reaction is in the region of m_Δ) should be:

$$\sigma_a : \sigma_b : \cdots \sigma_j \approx 9 : 4 : 1 : 1 : 4 : 9 : 2 : 2 : 2 : 2 \tag{15.41}$$

In particular,

$$\sigma(\pi^+ p \to \pi^+ p) : \sigma(\pi^- p \to \pi^- p) : \sigma(\pi^- p \to \pi^0 n) \approx 9 : 1 : 2 \tag{15.42}$$

We use the measured cross-sections summarized by the Particle Data Group's Review (`http://pdg.lbl.gov/current/xsect/`). At the Δ resonance peak, one reads:

$$\sigma_{tot}(\pi^+ p) \approx 200 \text{ mb} \quad \text{and} \quad \sigma_{tot}(\pi^- p) \approx 70 \text{ mb} \tag{15.43}$$

which is consistent with our expectation:

$$\sigma(\pi^+ p \to \pi^+ p) : (\sigma(\pi^- p \to \pi^- p) + \sigma(\pi^- p \to \pi^0 n)) \approx 9 : 3 \tag{15.44}$$

16 Electron–Proton Scattering

16.1 The right electron energy

The proton has a size of about 1 fm. Discuss what is the "optimal" energy of an electron beam needed to study a system of such dimensions.

Solution:

We want to study the proton via the elastic reaction:

$$e^- + p \to e^- + p \tag{16.1}$$

We have seen in Section 16.5 of the book that the elastic differential cross-section is given by the Rosenbluth formula (see Eq. (16.64) of the book):

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{Rosenbluth} = \frac{\alpha^2}{4E_1^2 \sin^4(\theta/2)} \frac{E_3}{E_1} \left[\frac{G_E^2 + \tau G_M^2}{1+\tau} \cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2}\right] \tag{16.2}$$

where $G_E(Q^2)$ and $G_M(Q^2)$ are the Sachs form factors. The form factors are empirical functions that are extracted from experimental data.

In order to estimate the "optimal energy", let us simplify the expression, ignore the magnetic form factor, and just assume that the proton is represented by a static, extended spherical charge distribution of dimension 1 fm. Then, we can assume that the differential cross-section is just given by Eq. (16.25) of the book:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right) \simeq \frac{Z^2\alpha^2}{4E^2 \sin^4(\theta/2)} \cos^2\frac{\theta}{2} |F(\vec{q}^{\,2})|^2 \tag{16.3}$$

where the form factor $F(\vec{q}^{\,2})$ is Fourier transform of the charge distribution (see Eq. (16.23) of the book):

$$F(\vec{q}^{\,2}) \equiv \int \mathrm{d}^3\vec{x} e^{-i\vec{q}\cdot\vec{x}} \rho(\vec{x}) \tag{16.4}$$

The form factor for a spherical target with constant density with radius a is given by Eq. (16.28) of the book:

$$|F_{sphere}(qa)| = \left|\frac{3\cos qa}{(qa)^3}\left[qa - \tan qa\right]\right| \tag{16.5}$$

The form factor is zero when the expression in square brackets is zero, i.e., when $\tan(qa) = qa$, which is true for various values of q. These correspond to various scattering angles and we would expect a diffraction-like pattern from electron scattering off this target. This is illustrated in Figure 16.1(left). From the figure, we can then derive the following condition on qa which maximizes the effect:

$$qa \lesssim 4 \quad \longrightarrow \quad q^2 \lesssim \left(\frac{4}{a}\right)^2 = \left(\frac{4\hbar c}{1 \text{ fm}}\right)^2 \approx 0.62 \text{ GeV}^2 \tag{16.6}$$

We then have that:

$$|\vec{q}|^2 = (\vec{p}_i - \vec{p}_f)^2 = p_i^2 + p_f^2 - 2p_i p_f \cos\theta \approx 2p^2(1-\cos\theta) \tag{16.7}$$

where θ is the electron scattering angle. Hence for $a = 1$ fm, we find:

$$p^2 \lesssim \frac{0.62 \text{ GeV}^2}{2(1-\cos\theta)} \quad \Longrightarrow \quad p \lesssim p_{max}(\theta) = \sqrt{\frac{0.31 \text{ GeV}^2}{1-\cos\theta}} \tag{16.8}$$

where $p_{max}(\theta)$ is plotted in Figure 16.1(right)

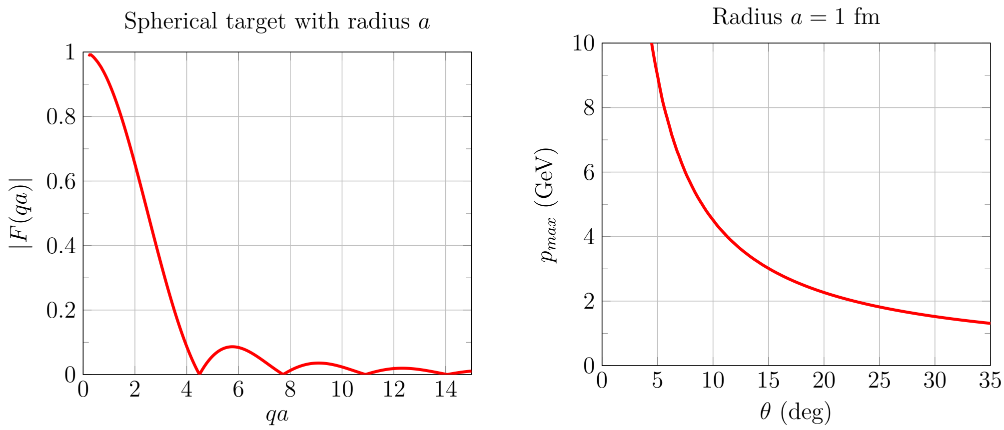

Figure 16.1 (left) Form factor for a spherical target with radius a; (right) "Optimal" momentum as a function of the scattering angle θ.

We notice that p_{max} decreases for increasing scattering angle θ. However, it should be noted that the scattering cross-section has an overall factor $1/\sin^4(\theta/2)$ reminiscent of the Rutherford scattering formula, which makes larger angles scatters rapidly more and more unlikely as the angle increases. Experimentally, one must therefore make a trade-off between electron energy and event rate. The SLAC-MIT experiment, discussed in Section 16.6 of the book, selected a range of electron energies between 4.5 and 18 GeV and scattering angles between 6° and 34°. See Table 16.1 of the book for more details.

At higher energies and hence $|\vec{q}|^2$, the deep inelastic scattering cross-section dominates and the proton therefore appears via its constituents (quarks) and can no longer be studied as an individual proton with dimension about 1 fm.

16.2 The most general form factor

In the lowest-order QED the electron–proton scattering $e^-(p) + p(k) \to e^-(p') + p(k')$, the spin-averaged amplitude is given by:

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{q^4} L^{\mu\nu}_{electron} W_{\mu\nu,proton} \tag{16.9}$$

where $W_{\mu\nu,proton}$ is an unknown function describing the photon–proton vertex, and

$$L^{\mu\nu}_{electron} = 2\left(p^\mu \cdot p'^\nu + p^\nu \cdot p'^\mu + g^{\mu\nu}[m_e^2 - (p \cdot p^{'})]\right) \tag{16.10}$$

The most general form of the $W_{\mu\nu}$ tensor is given by (neglecting the W_3 term which enters into neutrino–proton scattering):

$$W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{M_p^2} k^\mu k^\nu + \frac{W_4}{M_p^2} q^\mu q^\nu + \frac{W_5}{M_p^2}(k^\mu q^\nu + k^\nu q^\mu) \tag{16.11}$$

(a) *Prove that $q_\mu W^{\mu\nu} = 0$ and $q_\mu L^{\mu\nu} = 0$.*

(b) *Prove that it follows from (a) that the following are true:*

$$W_4 = \frac{M_p^2}{q^2} W_1 + \frac{1}{4} W_2 \qquad \text{and} \qquad W_5 = \frac{1}{2} W_2 \tag{16.12}$$

that is, $W^{\mu\nu}$ can be expressed in terms of just two functions, $W_1(q^2)$ and $W_2(q^2)$.

(c) *Derive the spin-averaged amplitude by contracting Eq. (16.9), substituting the obtained expression for $W^{\mu\nu}$.*

Solution:

The lowest-order QED, spin-averaged, electron–proton $e^-(p) + p(k) \to e^-(p') + p(k')$ scattering amplitude is given in Eq. (16.37) of the book:

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{q^4} \langle L^{\mu\nu} \rangle \langle W_{\mu\nu} \rangle \tag{16.13}$$

where $\langle W_{\mu\nu} \rangle$ is an unknown function describing the photon–proton vertex, and where the lepton tensor is given by (see Eq. (16.40) of the book):

$$\langle L^{\mu\nu} \rangle = \frac{1}{2} \operatorname{Tr}((\not{p}' + m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu) = 2\left(p^\mu p'^\nu + p^\nu p'^\mu + g^{\mu\nu}[m_e^2 - (p \cdot p')]\right) \tag{16.14}$$

(a) We want to show that $q_\mu \langle W^{\mu\nu} \rangle = 0$ and $q_\mu \langle L^{\mu\nu} \rangle = 0$ where $q^\mu = p^\mu - p'^\mu$.

- $q_\mu \langle L^{\mu\nu} \rangle = 0$**.** We find that:

$$q_\mu \langle L^{\mu\nu} \rangle = 2\left((p \cdot q)p'^\nu + p^\nu (q \cdot p') + q^\nu (m_e^2 - p \cdot p')\right) \tag{16.15}$$

In addition, we have:

$$q^2 = (p - p')^2 = p^2 + p'^2 - 2p \cdot p' = 2m_e^2 - 2p \cdot p' \quad \Longrightarrow \quad p \cdot p' = -q^2/2 + m_e^2 \tag{16.16}$$

and

$$q \cdot p = (p - p') \cdot p = p^2 - p' \cdot p = q^2/2 \tag{16.17}$$

and also

$$q \cdot p' = (p - p') \cdot p' = p \cdot p' - m_e^2 = -q^2/2 \tag{16.18}$$

Then the contraction of the leptonic current with q^μ reduces to:

$$q_\mu \langle L^{\mu\nu} \rangle = 2\left(\frac{q^2}{2} p'^\nu - p^\nu \frac{q^2}{2} + q^\nu \frac{q^2}{2}\right) = q^2 \left(p'^\nu - p^\nu + p^\nu - p'^\nu\right) = 0 \tag{16.19}$$

Note that since $L^{\mu\nu}$ is symmetric, we obviously have :

$$q_\mu \langle L^{\mu\nu} \rangle = q_\nu \langle L^{\mu\nu} \rangle = 0 \tag{16.20}$$

- $q_\mu \langle W^{\mu\nu} \rangle = 0$**.** We should consider the conservation properties of the $W^{\mu\nu}$ tensor. Let us go back to the proton current. The electromagnetic current for the electron is very well-defined since this latter is a Dirac particles, while for the proton we can only say that its current should be a 4-vector, constructed from the kinematical quantities involved k, k', q and Dirac γ matrices sandwiched between spinors. Consequently, we have:

$$\begin{aligned} j^\mu_{electron} &= e\overline{\Psi}_e\gamma^\mu\Psi_e = e\bar{u}(p')\gamma^\mu u(p)e^{i(p'-p)\cdot x} \\ j^\mu_{proton} &= e\overline{\Psi}_p\Big[\cdots\Big]^\mu\Psi_p = e\bar{u}(k')\Big[\cdots\Big]^\mu u(k)e^{i(k'-k)\cdot x} \end{aligned} \tag{16.21}$$

The conservation of the hadronic current implies that:

$$\partial_\mu j^\mu_{proton} = 0 \quad \Longrightarrow \quad (k'-k)_\mu\Big[\cdots\Big]^\mu = q_\mu\Big[\cdots\Big]^\mu = 0 \tag{16.22}$$

Consequently, since:

$$W^{\mu\nu} \propto j^\mu_{proton}j^{\dagger\nu}_{proton} \quad \Longrightarrow \quad q_\mu\left\langle W^{\mu\nu}\right\rangle = q_\nu\left\langle W^{\mu\nu}\right\rangle = 0 \tag{16.23}$$

(b) We immediately find that:

$$q_\mu W^{\mu\nu} = -W_1q^\nu + \frac{W_2}{M_p^2}(q\cdot k)k^\nu + \frac{W_4}{M_p^2}q^2q^\nu + \frac{W_5}{M_p^2}((q\cdot k)q^\nu + k^\nu q^2) = 0 \tag{16.24}$$

We further multiply this expression by k_ν:

$$k_\nu q_\mu W^{\mu\nu} = -W_1(q\cdot k) + W_2(q\cdot k) + \frac{W_4}{M_p^2}q^2(q\cdot k) + \frac{W_5}{M_p^2}((q\cdot k)^2 + M_p^2q^2) = 0 \tag{16.25}$$

where we used $k^2 = M_p^2$. Kinematics gives us:

$$(q+k)^2 = q^2 + M_p^2 + 2q\cdot k \quad \text{and} \quad k'^2 = M_p^2 = (q+k)^2 \tag{16.26}$$

where we used $k'^\mu = q^\mu + k^\mu$. Then:

$$q^2 + M_p^2 + 2q\cdot k = M_p^2 \quad \Longrightarrow \quad q\cdot k = -q^2/2 \tag{16.27}$$

Therefore:

$$W_1\frac{q^2}{2} - W_2\frac{q^2}{2} - \frac{W_4}{M_p^2}\frac{q^4}{2} + \frac{W_5}{M_p^2}\left(\frac{q^4}{4} + M_p^2q^2\right) = 0 \tag{16.28}$$

or equivalently:

$$M_p^2(W_1-W_2) - W_4q^2 + W_5\left(\frac{q^2}{2} + 2M_p^2\right) = 0 \quad \Longrightarrow \quad q^2W_4 = M_p^2\left(W_1 - W_2 + W_5\left(2 + \frac{q^2}{2M_p^2}\right)\right) \tag{16.29}$$

We can then write:

$$\begin{aligned} q_\mu W^{\mu\nu} =& -W_1q^\nu - \frac{W_2}{M_p^2}\frac{q^2}{2}k^\nu + q^\nu\left(W_1 - W_2 + W_5\left(2 + \frac{q^2}{2M_p^2}\right)\right) + \frac{W_5}{M_p^2}\left(-\frac{q^2}{2}q^\nu + k^\nu q^2\right) \\ =& -\frac{W_2}{M_p^2}\frac{q^2}{2}k^\nu + q^\nu\left(-W_2 + 2W_5\right) + \frac{W_5}{M_p^2}\left(k^\nu q^2\right) = 0 \end{aligned} \tag{16.30}$$

Consequently:

$$\frac{q^2}{2M_p^2}k^\nu(-W_2 + 2W_5) + q^\nu(-W_2 + 2W_5) = 0 \quad \Longrightarrow \quad W_2 = 2W_5 \tag{16.31}$$

We then immediately find that:

$$q^2 W_4 = M_p^2 \left(W_1 + W_2 \frac{q^2}{4M_p^2} \right) \quad \Longrightarrow \quad W_4 = \frac{M_p^2}{q^2} W_1 + \frac{1}{4} W_2 \tag{16.32}$$

We these two constraints on the form factors, the hadronic current can be simplified to the following expression, which only depends on the two form factors W_1 and W_2:

$$\begin{aligned} W^{\mu\nu} =& - W_1 g^{\mu\nu} + \frac{W_2}{M_p^2} k^\mu k^\nu + \frac{W_4}{M_p^2} q^\mu q^\nu + \frac{W_5}{M_p^2} (k^\mu q^\nu + k^\nu q^\mu) \\ =& - W_1 g^{\mu\nu} + \frac{W_2}{M_p^2} k^\mu k^\nu + \frac{\left(\frac{M_p^2}{q^2} W_1 + \frac{1}{4} W_2 \right)}{M_p^2} q^\mu q^\nu + \frac{W_2}{2M_p^2} (k^\mu q^\nu + k^\nu q^\mu) \\ =& W_1 \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{W_2}{M_p^2} \left(k^\mu k^\nu + \frac{q^\mu q^\nu}{4} + \frac{k^\mu q^\nu}{2} + \frac{k^\nu q^\mu}{2} \right) \text{ă} \end{aligned} \tag{16.33}$$

In summary, while we started with the most general form for the hadronic current, containing four form factors, current conservation limits the number of independent form factors to just two.

(c) We use the hadronic current derived above to compute the spin-averaged amplitude as a function of W_1 and W_2. We want to compute:

$$\langle L^{\mu\nu} \rangle \langle W_{\mu\nu} \rangle = 2 \left(p^\mu p'^\nu + p^\nu p'^\mu + g^{\mu\nu} [m_e^2 - (p \cdot p')] \right) \frac{1}{2} W_{\mu\nu} \tag{16.34}$$

where $\langle W^{\mu\nu} \rangle$ is written as a function of the spin-averaged form factors W_1 and W_2 (same form as that derived in (b) but with factor 1/2 to account for average on the spin of the initial-state proton). Let us define:

$$\langle W_{\mu\nu} \rangle = \langle W^1_{\mu\nu} \rangle + \langle W^2_{\mu\nu} \rangle \quad \text{where} \quad \langle W^1_{\mu\nu} \rangle = W_1 \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \tag{16.35}$$

We find:

$$\begin{aligned} \langle L^{\mu\nu} \rangle \langle W^1_{\mu\nu} \rangle &= W_1 \left(-2p \cdot p' - 4(m_e^2 - p \cdot p') \right) + \frac{W_1}{q^2} \left((q \cdot p)(q \cdot p') + (q \cdot p')(q \cdot p) + q^2 (m_e^2 - p \cdot p') \right) \\ &= W_1 \left[-2p \cdot p' - 3(m_e^2 - p \cdot p') + \frac{2}{q^2} (q \cdot p)(q \cdot p') \right] \\ &= W_1 \left(p \cdot p' + \frac{2}{q^2} (q \cdot p)(q \cdot p') - 3m_e^2 \right) \\ &= W_1 \left(2(p \cdot p') - 4m_e^2 \right) \end{aligned} \tag{16.36}$$

where we used $p \cdot p' = -q^2/2 + m_e^2$, $q \cdot p = q^2/2$ and $q \cdot p' = -q \cdot p$.

Similarly, we find:

$$\begin{aligned} \langle L^{\mu\nu} \rangle \langle W^2_{\mu\nu} \rangle = \frac{W_2}{M_p^2} \Bigg(& (p \cdot k)(p \cdot k') + (p \cdot k)(p \cdot k') + k^2 (m_e^2 - p \cdot p') + \frac{(q \cdot p)(p' \cdot k)}{2} \\ & + \frac{(p \cdot k)(q \cdot p')}{2} + \frac{k \cdot q}{2} (m_e^2 - p \cdot p') + \frac{(p \cdot k)(p' \cdot q)}{2} + \frac{(p \cdot q)(k \cdot p')}{2} \\ & + \frac{k \cdot q}{2} (m_e^2 - p \cdot p') + \frac{(q \cdot p)(p' \cdot q)}{4} + \frac{(p \cdot q)(q \cdot p')}{4} + \frac{q^2}{4} (m_e^2 - p \cdot p') \Bigg) \end{aligned} \tag{16.37}$$

The expression can be simplified to:

$$\begin{aligned}\langle L^{\mu\nu}\rangle\langle W^2_{\mu\nu}\rangle &= \frac{W_2}{M_p^2}\Bigg(2(p\cdot k)(p'\cdot k)+(p\cdot k)(p'\cdot q)+(q\cdot p)(k\cdot p')+\frac{(q\cdot p)(p'\cdot q)}{2}\\ &\qquad+\left((k\cdot q)+\left(k^2+\frac{q^2}{2}\right)(m^2-p\cdot p')\right)\Bigg)\\ &= \frac{W_2}{M_p^2}\left(2(p\cdot k)(p\cdot k')+\frac{q^4}{8}+\frac{q^2}{2}\left(M_p^2-\frac{q^2}{4}\right)\right)\\ &= \frac{W_2}{M_p^2}\left(2(p\cdot k)(p\cdot k')+\frac{q^2}{2}M_p^2\right)\end{aligned} \tag{16.38}$$

where we used $q\cdot k = -q^2/2$ and again $p\cdot p' = -q^2/2+m_e^2$, $q\cdot p = q^2/2$ and $q\cdot p' = -q\cdot p$. Finally, the spin-averaged amplitude can be expressed as:

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{q^4}\left[W_1(q^2)\left(2(p\cdot p')-4m_e^2\right)+\frac{W_2(q^2)}{M_p^2}\left(2(p\cdot k)(p\cdot k')+\frac{q^2}{2}M_p^2\right)\right] \tag{16.39}$$

where we explicitly wrote the q^2 dependence of the form factors W_1 and W_2.

16.3 Form factors for a Dirac proton

Consider a Dirac proton: starting from the assumption that $W^{\mu\nu}_{Dirac\,p}$ has the same form as $L^{\mu\nu}_{electron}$, calculate the form factors $W_{1,Dirac\,p}(q^2)$ and $W_{2,Dirac\,p}(q^2)$.

Solution:

The lowest-order QED, spin-averaged, electron–proton $e^-(p)+p(k)\to e^-(p')+p(k')$ scattering amplitude is given in Eq. (16.37) of the book:

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{q^4}\langle L^{\mu\nu}\rangle\langle W_{\mu\nu}\rangle \tag{16.40}$$

where $\langle W_{\mu\nu}\rangle$ is an unknown function describing the photon–proton vertex, and where the lepton tensor is given by (see Eq. (16.40) of the book):

$$\langle L^{\mu\nu}\rangle = \frac{1}{2}\,\mathrm{Tr}\left((\not{p}'+m_e)\gamma^\mu(\not{p}+m_e)\gamma^\nu\right) = 2\left(p^\mu p'^\nu+p^\nu p'^\mu+g^{\mu\nu}[m_e^2-(p\cdot p')]\right) \tag{16.41}$$

We derived in **Ex. 16.2** that the most general hadronic form factor can be expressed as:

$$W^{\mu\nu} = W_1\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)+\frac{W_2}{M_p^2}\left(k^\mu k^\nu+\frac{q^\mu q^\nu}{4}+\frac{k^\mu q^\nu}{2}+\frac{k^\nu q^\mu}{2}\right)\text{ă} \tag{16.42}$$

where $q^\mu = k'^\mu - k^\mu$ and $W_1(q^2)$, $W_2(q^2)$ are the form factors of the proton. In the case of the proton being considered a Dirac particle, the hadronic current will have the same form as the leptonic one. After proper replacement of the 4-vectors and the mass, we find:

$$\langle W^{\mu\nu}\rangle = 2\left(k^\mu k'^\nu+k^\nu k'^\mu+g^{\mu\nu}\left[M_p^2-(k\cdot k')\right]\right) \tag{16.43}$$

Kinematics gives us:

$$(q+k)^2 = q^2+M_p^2+2q\cdot k \quad\text{and}\quad k'^2 = M_p^2 = (q+k)^2 \quad\Longrightarrow\quad k\cdot q = -\frac{q^2}{2} \tag{16.44}$$

where we used $k'^{\mu} = q^{\mu} + k^{\mu}$. Then, one finds:

$$\begin{aligned}
\langle W^{\mu\nu} \rangle &= 2 \left(k^{\mu} \left(q^{\nu} + k^{\nu} \right) + k^{\nu} \left(q^{\mu} + k^{\mu} \right) + g^{\mu\nu} \left[M_p^2 - k \cdot (q+k) \right] \right) \\
&= 2 \left(k^{\mu} q^{\nu} + k^{\nu} q^{\mu} + 2 k^{\mu} k^{\nu} + g^{\mu\nu} \left[M_p^2 + \frac{q^2}{2} - M_p^2 \right] \right) \\
&= 2 \left(k^{\mu} q^{\nu} + k^{\nu} q^{\mu} \right) + 4 k^{\mu} k^{\nu} + g^{\mu\nu} q^2
\end{aligned} \tag{16.45}$$

The most general hadronic form factor derived in **Ex. 16.2**, taking into account a factor 1/2 for the averaging over the initial-state proton spin, can be recast into:

$$\begin{aligned}
\langle W^{\mu\nu} \rangle &= \frac{W_1}{2} \left(-g^{\mu\nu} + \frac{q^{\mu} q^{\nu}}{q^2} \right) + \frac{W_2}{2M_p^2} \left(k^{\mu} k^{\nu} + \frac{q^{\mu} q^{\nu}}{4} + \frac{k^{\mu} q^{\nu}}{2} + \frac{k^{\nu} q^{\mu}}{2} \right) ă \\
&= -g^{\mu\nu} \underbrace{\frac{W_1}{2}}_{=-q^2} + q^{\mu} q^{\nu} \underbrace{\left(\frac{W_1}{2q^2} + \frac{W_2}{8M_p^2} \right)}_{=0} + \left(k^{\mu} q^{\nu} + k^{\nu} q^{\mu} \right) \underbrace{\frac{W_2}{4M_p^2}}_{=2} + k^{\mu} k^{\nu} \underbrace{\frac{W_2}{2M_p^2}}_{=4}
\end{aligned} \tag{16.46}$$

which can be satisfied for any q^2 if:

$$W_1(q^2) = -2q^2 \quad \text{and} \quad W_2(q^2) = 8M_p^2 \tag{16.47}$$

Using the final result from **Ex. 16.2**, the spin-averaged amplitude can be expressed as:

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{q^4} \left[W_1(q^2) \left(2(p \cdot p') - 4m_e^2 \right) + \frac{W_2(q^2)}{M_p^2} \left(2(p \cdot k)(p \cdot k') + \frac{q^2}{2} M_p^2 \right) \right] \\
&= \frac{e^4}{q^4} \left[-2q^2 \left(2(p \cdot p') - 4m_e^2 \right) + 8 \left(2(p \cdot k)(p \cdot k') + \frac{q^2}{2} M_p^2 \right) \right]
\end{aligned} \tag{16.48}$$

In order to get some insight on the result, we introduce the Mandelstam variables:

$$\begin{aligned}
s &= (p+k)^2 = M_p^2 + m_e^2 + 2p \cdot k \\
t &= (p-p')^2 = 2m_e^2 - 2p \cdot p' \\
u &= (p-k')^2 = m_e^2 + M_p^2 - 2p \cdot k'
\end{aligned} \tag{16.49}$$

which imply, neglecting the electron rest mass:

$$p \cdot p' = -\frac{1}{2} t, \quad p \cdot k = \frac{1}{2}(s - M_p^2) \quad \text{and} \quad p \cdot k' = -\frac{1}{2}(u - M_p^2) \tag{16.50}$$

Therefore, with $t = q^2$:

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{t^2} \left[2t^2 + 4 \left((s - M_p^2)(-u + M_p^2) + t M_p^2 \right) \right] \\
&= \frac{e^4}{t^2} \left[2t^2 + 4 \left(s M_p^2 - su - M_p^4 + u M_p^2 + t M_p^2 \right) \right] \\
&= \frac{e^4}{t^2} \left[2t^2 + 4 \left(M_p^2 (s+t+u) - su - M_p^4 \right) \right]
\end{aligned} \tag{16.51}$$

We can now use $s + t + u = 2M_p^2$ (see Eq. (11.88) of the book) to find:

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{2e^4}{t^2} \left[t^2 + 2 \left(M_p^4 - su \right) \right] = \frac{2e^4}{t^2} \left[t^2 + 2M_p^4 - 4M_p^2 u + 2ut + 2u^2 \right] = \frac{2e^4}{t^2} \left[(t+u)^2 + 2M_p^4 - 4M_p^2 u + u^2 \right] \\
&= \frac{2e^4}{t^2} \left[(2M_p^2 - s)^2 + 2M_p^4 - 4M_p^2 u + u^2 \right] = \frac{2e^4}{t^2} \left[4M_p^4 - 4M_p^2 s + 2M_p^4 - 4M_p^2 u + s^2 + u^2 \right] \\
&= \frac{2e^4}{t^2} \left[4M_p^2 (M_p^2 - s - u) + 2M_p^4 + s^2 + u^2 \right] = \frac{2e^4}{t^2} \left[4M_p^2 \left(\frac{s+t+u}{2} - s - u \right) + 2M_p^4 + s^2 + u^2 \right]
\end{aligned} \tag{16.52}$$

Finally:

$$\langle|\mathcal{M}(e^- p \to e^- p)_{Dirac}|^2\rangle = \frac{2e^4}{t^2}\left[-2M_p^2(s-t+u)+2M_p^4+s^2+u^2\right] \tag{16.53}$$

which is exactly Eq. (11.160) of the book with the replacement $m_\ell \to M_p$ and setting $m_e = 0$!!! As expected, when the proton is considered as a Dirac particle, we exactly recover the same amplitude that was calculated in the book for the $e^-\ell^- \to e^-\ell^-$ process!

17 Partons

17.1 Mass formula of Gell-Mann–Okubo

The so-called Gell-Mann–Okubo mass formula yields a relation between the masses of the members of the baryon octet. It is expressed as:

$$2(m_N + m_\Xi) = 3m_\Lambda + m_\Sigma \tag{17.1}$$

Use the known values of m_N, m_Ξ, and m_Σ to estimate the rest mass of the Λ. How large is the deviation from the experimentally observed value?

Solution:

The Gell-MannOkubo mass formula provides a sum rule for the masses of hadrons within a specific multiplet, determined by their isospin I and strangeness S (or alternatively, hypercharge Y):

$$M = a_0 + a_1 Y + a_2 \left[I(I+1) - \frac{Y^2}{4} \right] \tag{17.2}$$

where the a_i are parameters. Using the values of relevant I and S for baryons, the Gell-MannOkubo formula can be rewritten for the baryon octet (see Eq. 4.8 of M. Gell-Mann, "The Eightfold Way: A theory of strong interaction symmetry", Pasadena, CA. (Unpublished) `https://resolver.caltech.edu/CaltechAUTHORS:20180910-145743377`):

$$\frac{1}{2}(m_N + m_\Xi) = \frac{3}{4} m_\Lambda + \frac{1}{4} m_\Sigma \tag{17.3}$$

which is equivalent to the equation given in Eq. (17.1). From the experimental values of the rest masses of the particles, we derive the following values for the average rest masses of each isospin multiplet:

$$m_N = 938.9 \text{ MeV}, \quad m_\Lambda = 1115.7 \text{ MeV}, \quad m_\Sigma = 1193.2 \text{ MeV}, \quad m_\Xi = 1318.1 \text{ MeV} \tag{17.4}$$

From the Gell-MannOkubo formula, we get for the mass of the Λ:

$$m_\Lambda^{GMO} = \frac{2(m_N + m_\Xi) - m_\Sigma}{3} \simeq 1106.9 \text{ MeV} \tag{17.5}$$

The deviation from the experimentally observed value is strikingly small:

$$\frac{m_\Lambda^{exp} - m_\Lambda^{GMO}}{m_\Lambda^{exp}} \simeq 0.8\% \tag{17.6}$$

The experimental values have been taken from P. A. Zyla *et al.* (Particle Data Group), *Prog. Theor. Exp. Phys.* **2020**, 083C01 (2020).

17.2 The ρ meson

The ρ meson has an isospin $I = 1$ and comes in three charge eigenstates: ρ^+, ρ^0 and ρ^-. It decays almost solely into two pions $\rho \to \pi\pi$. Show that the decays $\rho^\pm \to \pi^\pm\pi^0$ and $\rho^0 \to \pi^+\pi^-$ are allowed, while the decay $\rho^0 \to \pi^0\pi^0$ is forbidden.

Solution:

The isospin eigenvalues of the three charged states of the ρ are the following:

$$|\rho^+\rangle = |1,1\rangle\,, \quad |\rho^0\rangle = |1,0\rangle\,, \quad |\rho^-\rangle = |1,-1\rangle \tag{17.7}$$

For the combination of two pions we have according to addition rules:

$$\begin{aligned}
|\pi^+\pi^0\rangle &= \frac{1}{\sqrt{2}}\,|2,1\rangle + \frac{1}{\sqrt{2}}\,|1,1\rangle \\
|\pi^-\pi^0\rangle &= \frac{1}{\sqrt{2}}\,|2,-1\rangle - \frac{1}{\sqrt{2}}\,|1,-1\rangle \\
|\pi^+\pi^-\rangle &= \frac{1}{\sqrt{6}}\,|2,0\rangle + \frac{1}{\sqrt{2}}\,|1,0\rangle + \frac{1}{\sqrt{3}}\,|0,0\rangle \\
|\pi^0\pi^0\rangle &= \sqrt{\frac{2}{3}}\,|2,0\rangle - \frac{1}{\sqrt{3}}\,|0,0\rangle
\end{aligned} \tag{17.8}$$

Hence, the decays $\rho^\pm \to \pi^\pm\pi^0$ and $\rho^0 \to \pi^+\pi^-$ are allowed with the following amplitudes:

$$\begin{aligned}
\langle\pi^+\pi^0|H|\rho^+\rangle &= \frac{1}{\sqrt{2}}M_1 \\
\langle\pi^-\pi^0|H|\rho^-\rangle &= -\frac{1}{\sqrt{2}}M_1 \\
\langle\pi^+\pi^-|H|\rho^0\rangle &= \frac{1}{\sqrt{2}}M_1
\end{aligned} \tag{17.9}$$

while the decay $\rho^0 \to \pi^0\pi^0$ is forbidden from the isospin conservation rule (although it would be allowed from the electric charge conservation point of view):

$$\langle\pi^0\pi^0|H|\rho^0\rangle = 0 \tag{17.10}$$

17.3 The light vector mesons

The spin-triplet combination ($\uparrow\uparrow$) of a $q\bar{q}$ pair with $\ell = 0$ yields the vector mesons ($J = 1$). In order to explain the experimental observations, one must assume that the octet member with $I = S = 0$ and the corresponding singlet state are mixed. This can be expressed as:

$$\omega = \phi_0 \cos\theta + \phi_8 \sin\theta \quad \text{and} \quad \phi = \phi_0 \sin\theta - \phi_8 \cos\theta \tag{17.11}$$

where ϕ_0 and ϕ_8 are the singlet and octet states with $I = S = 0$. The ϕ and ω represent the physical states and θ is the mixing parameter. According to the $SU(3)$ symmetry, the ϕ_0 and ϕ_8 are classified as:

$$\phi_0 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \quad \text{and} \quad \phi_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \tag{17.12}$$

(a) *It is assumed that the expectation value of the energy operator between two eigenstates is proportional to the mass squared of the mesons. Show that the mixing parameter θ is given by:*

$$\tan^2\theta = \frac{M_\phi^2 - M_8^2}{M_8^2 - M_\omega^2} \tag{17.13}$$

(b) *Use the empirical formula $M_8^2 = \frac{1}{3}\left(4M_{K^*}^2 - M_\rho^2\right)$ to estimate the value of θ.*

(c) *Show that for $\sin\theta = 1/\sqrt{3}$, one has simply:*

$$\phi = s\bar{s} \quad \text{and} \quad \omega = \frac{1}{\sqrt{2}}\left(u\bar{u} + d\bar{d}\right) \tag{17.14}$$

(d) *Draw the corresponding quark flux diagrams (see Section 17.4 of the book) and explain qualitatively the branching ratios of the following decays:*

$$\phi(1020) \quad \left.\begin{array}{l} \to K^+K^- \\ \to K^0\bar{K}^0 \end{array}\right\} \; 84\% \qquad \to \pi^+\pi^-\pi^0 \quad 15\% \tag{17.15}$$

and

$$\omega(783) \quad \to \pi^+\pi^-\pi^0 \quad 90\% \qquad \left.\begin{array}{l} \to \pi^+\pi^- \\ \to \pi^0\gamma \end{array}\right\} \; 10\% \tag{17.16}$$

Solution:

(a) With the assumed form, the mixing between the physical and the $SU(3)$ eigenstates can be expressed as:

$$|\phi\rangle = s\,|\phi_0\rangle - c\,|\phi_8\rangle \quad \text{and} \quad |\omega\rangle = c\,|\phi_0\rangle + s\,|\phi_8\rangle \tag{17.17}$$

where $c = \cos\theta$ and $s = \sin\theta$ and θ is an angle to be determined. We use the quadratic mass relation in order to estimate it from the physical masses. The mass squared matrix in the basis of the physical states must be diagonal by construction: The mass squared matrix in the basis of the states ϕ_0 and ϕ_8 can be written as:

$$M^2 = \begin{pmatrix} \langle\phi_0|M^2|\phi_0\rangle & \langle\phi_0|M^2|\phi_8\rangle \\ \langle\phi_8|M^2|\phi_0\rangle & \langle\phi_8|M^2|\phi_8\rangle \end{pmatrix} = \begin{pmatrix} M_0^2 & M_{08}^2 \\ M_{08}^2 & M_8^2 \end{pmatrix} \tag{17.18}$$

For the physical state ϕ, we then have:

$$\begin{aligned} M_\phi^2 &= \langle\phi|M^2|\phi\rangle = \left(s\,\langle\phi_0| - c\,\langle\phi_8|\right) M^2 \left(s\,|\phi_0\rangle - c\,|\phi_8\rangle\right) \\ &= s^2M_0^2 - csM_{08}^2 - csM_{08}^2 + c^2M_8^2 \\ &= s^2M_0^2 + c^2M_8^2 - 2csM_{08}^2 \end{aligned} \tag{17.19}$$

Similarly,

$$M_\omega^2 = c^2M_0^2 + s^2M_8^2 + 2csM_{08}^2 \tag{17.20}$$

In addition, the orthogonality of the physical eigenstates implies that

$$\begin{aligned} \langle\omega|M^2|\phi\rangle &= \langle\phi|M^2|\omega\rangle = \left(c\,\langle\phi_0| + s\,\langle\phi_8|\right) M^2 \left(s\,|\phi_0\rangle - c\,|\phi_8\rangle\right) \\ &= cs\left(M_0^2 - M_8^2\right) + M_{08}^2\left(s^2 - c^2\right) = 0 \end{aligned} \tag{17.21}$$

Hence,

$$M_{08}^2 = -\frac{cs\left(M_0^2 - M_8^2\right)}{s^2 - c^2} \tag{17.22}$$

We also note that:

$$M_\phi^2 + M_\omega^2 = (s^2+c^2)M_0^2 + (s^2+c^2)M_8^2 = M_0^2 + M_8^2 \tag{17.23}$$

We consider the difference

$$\begin{aligned}
M_\phi^2 - M_8^2 &= s^2M_0^2 + c^2M_8^2 - 2csM_{08}^2 - M_8^2 = s^2M_0^2 + \underbrace{(c^2-1)}_{-s^2}M_8^2 - 2csM_{08}^2 \\
&= s^2\left(M_0^2 - M_8^2\right) - 2cs\left(-\frac{cs\left(M_0^2-M_8^2\right)}{s^2-c^2}\right) \\
&= \left(M_0^2-M_8^2\right)\left(s^2 - \frac{2c^2s^2}{s^2-c^2}\right) = \left(M_0^2-M_8^2\right)\left(\frac{s^2(s^2-c^2)-2c^2s^2}{s^2-c^2}\right) \\
&= \left(M_0^2-M_8^2\right)\left(\frac{s^4+s^2c^2}{s^2-c^2}\right) = s^2\left(M_0^2-M_8^2\right)\left(\frac{s^2+c^2}{s^2-c^2}\right)
\end{aligned} \tag{17.24}$$

Similarly, we have that:

$$M_8^2 - M_\omega^2 = c^2\left(M_0^2-M_8^2\right)\left(\frac{s^2+c^2}{s^2-c^2}\right) \tag{17.25}$$

Hence, as expected:

$$\tan^2\theta = \frac{s^2}{c^2} = \frac{M_\phi^2 - M_8^2}{M_8^2 - M_\omega^2} \tag{17.26}$$

(b) We use the empirical formula $M_8^2 = \frac{1}{3}\left(4M_{K^*}^2 - M_\rho^2\right)$. With $M_{K^*} = 892$ MeV, $M_\rho = 776$ MeV, one finds $M_8^2 = 860160$ MeV^2. Plugging in $M_\omega = 783$ MeV and $M_\phi = 1019$ MeV, one gets:

$$\tan^2\theta = 0.72 \implies \tan\theta = 0.85 \implies \theta \simeq 40.3^\circ \tag{17.27}$$

(c) If we assume that $\sin\theta = 1/\sqrt{3}$, then $\cos^2\theta = 1 - 1/3 = 2/3$, hence $\cos\theta = \sqrt{2/3}$. With the $SU(3)$ assignment, we find that

$$\phi_0 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \quad \text{and} \quad \phi_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \tag{17.28}$$

Hence,

$$\begin{aligned}
\omega &= \sqrt{\frac{2}{3}}\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) + \frac{1}{\sqrt{3}}\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \\
&= \frac{\sqrt{2}}{3}(u\bar{u} + d\bar{d} + s\bar{s}) + \frac{1}{3\sqrt{2}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \\
&= \frac{2}{3\sqrt{2}}(u\bar{u} + d\bar{d} + s\bar{s}) + \frac{1}{3\sqrt{2}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \\
&= \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) + \left(\frac{2}{3\sqrt{2}} - 2\frac{1}{3\sqrt{2}}\right)s\bar{s} = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})
\end{aligned} \tag{17.29}$$

Similarly,

$$\begin{aligned}
\phi &= \frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) - \sqrt{\frac{2}{3}}\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \\
&= \frac{1}{3}(u\bar{u} + d\bar{d} + s\bar{s}) - \frac{1}{3}(u\bar{u} + d\bar{d} - 2s\bar{s}) = s\bar{s}
\end{aligned} \tag{17.30}$$

(d) We assume that the $\phi(1020)$ is dominantly an $s\bar{s}$ state. The corresponding diagrams are shown in Figure 17.1. The dominant decay modes $\phi(1020) \to K^+K^-$ and $\phi(1020) \to K^0\bar{K}^0$ are explained by the fact that they are OZI allowed, while the decay mode $\phi(1020) \to \pi^+\pi^-\pi^0$ is OZI suppressed. The corresponding diagrams for the ω are shown in Figure 17.2.

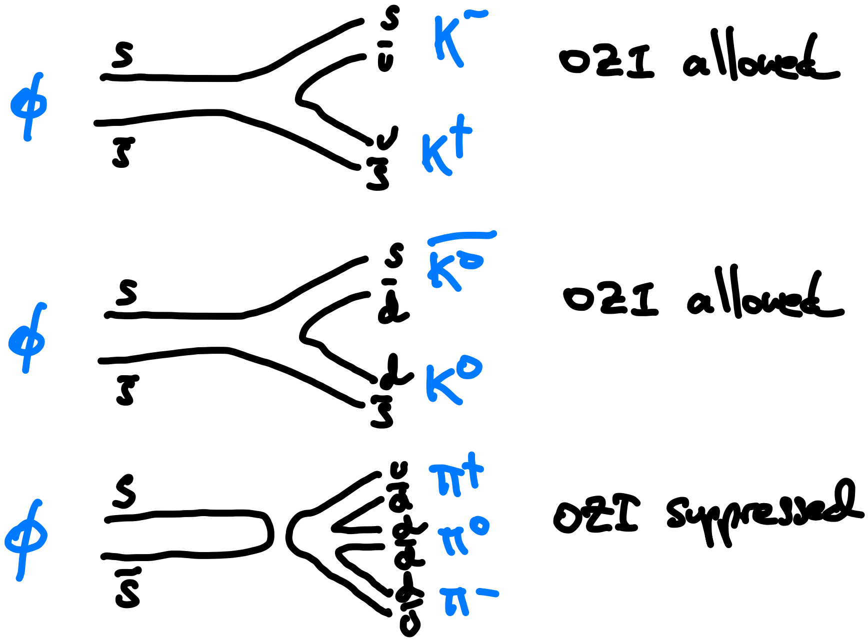

Figure 17.1 Quark flux diagrams for ϕ decays.

17.4 Electron–neutron experiments

Electron–neutron experiments are harder to do than electron–proton experiments because you cannot make a target of free neutrons. Nevertheless, the essential data can be inferred from electron–deuteron scattering, and it is found that:

$$\int_0^1 \mathrm{d}x F_2^{en} = 0.12 \tag{17.31}$$

Use this together with the proton result:

$$\int_0^1 \mathrm{d}x F_2^{ep} = 0.18 \tag{17.32}$$

to confirm $\int_0^1 \mathrm{d}x x u(x) = 2\int_0^1 \mathrm{d}x x d(x)$. *(*Hint: *How do you suppose that* $u^n(x)$ *and* $d^n(x)$ *are related to the corresponding functions of the proton?)*

Solution:

The prediction for the structure functions relevant for ep scattering, as a function of the underlying quark distribution functions in the free parton model, was given in Eq. (17.88) of the book:

$$F_2^{ep}(x, Q^2) = 2xF_1^{ep}(x, Q^2) = x\sum_q e_q^2 q^p(x) \tag{17.33}$$

Neglecting the contribution of the strange and heavier quarks, we find:

$$F_2^{ep}(x, Q^2) = x \sum_{q=u,d,\bar{u},\bar{d}} e_q^2 q^p(x) = x\left(\frac{4}{9}u^p(x) + \frac{1}{9}d^p(x) + \frac{4}{9}\bar{u}^p(x) + \frac{1}{9}\bar{d}^p(x)\right) \tag{17.34}$$

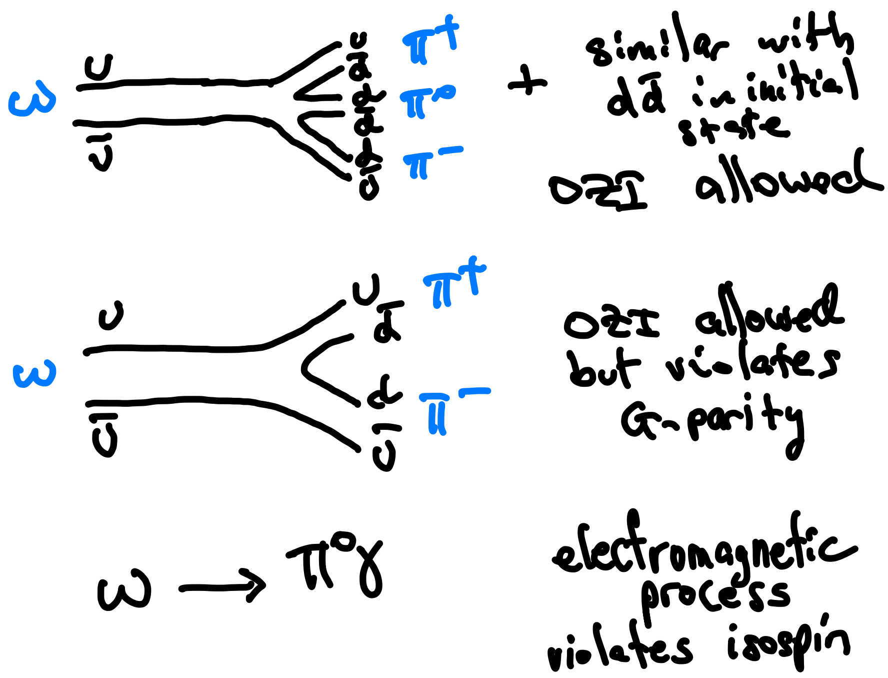

Figure 17.2 Quark flux diagrams for ω decays.

For the scattering off neutrons, we have:

$$F_2^{en}(x,Q^2) = x \sum_{q=u,d,\overline{u},\overline{d}} e_q^2 q^n(x) = x\left(\frac{4}{9}u^n(x) + \frac{1}{9}d^n(x) + \frac{4}{9}\bar{u}^n(x) + \frac{1}{9}\bar{d}^n(x)\right) \tag{17.35}$$

Because of isospin symmetry, the neutron structure function can be directly derived from that of the proton by exchanging $u \leftrightarrow d$. Therefore, the parton we can write the parton distribution functions of the neutron as a function of those of the proton:

$$u^n(x) = d^p(x) \equiv d(x) \quad \text{and} \quad d^n(x) = u^p(x) \equiv u(x) \tag{17.36}$$

and similarly for the antiquarks. Hence:

$$\begin{aligned} F_2^{ep}(x,Q^2) &= x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x) + \frac{4}{9}\bar{u}(x) + \frac{1}{9}\bar{d}(x)\right) = x\left(\frac{4}{9}(u(x)+\bar{u}(x)) + \frac{1}{9}(d(x)+\bar{d}(x))\right) \\ F_2^{en}(x,Q^2) &= x\left(\frac{1}{9}u(x) + \frac{4}{9}d(x) + \frac{1}{9}\overline{u}(x) + \frac{4}{9}\bar{d}(x)\right) = x\left(\frac{1}{9}(u(x)+\bar{u}(x)) + \frac{4}{9}(d(x)+\bar{d}(x))\right) \end{aligned} \tag{17.37}$$

Therefore:

$$\int_0^1 dx\; F_2^{ep} = \frac{4}{9}f_u + \frac{1}{9}f_d \quad \text{and} \quad \int_0^1 dx\; F_2^{en} = \frac{4}{9}f_d + \frac{1}{9}f_u \tag{17.38}$$

where

$$f_q \equiv \int_0^1 \mathrm{d}x\; x(q(x) + \overline{q}(x)) \tag{17.39}$$

The experimental values are:

$$I^{en} \equiv \int_0^1 \mathrm{d}x\ F_2^{en} \simeq 0.12 \quad \text{and} \quad I^{ep} \equiv \int_0^1 \mathrm{d}x\ F_2^{ep} \simeq 0.18 \tag{17.40}$$

We solve:

$$\begin{cases} \frac{4}{9} f_u + \frac{1}{9} f_d = I^{ep} \\ \frac{4}{9} f_d + \frac{1}{9} f_u = I^{en} \end{cases} \Longrightarrow \begin{cases} f_u = \frac{3}{5}(4I^{ep} - I^{en}) \simeq 0.36 \\ f_d = \frac{3}{5}(4I^{en} - I^{ep}) \simeq 0.18 \end{cases} \tag{17.41}$$

Indeed, experimentally one finds:

$$f_u \simeq 2 f_d \quad \Longrightarrow \quad \int_0^1 \mathrm{d}x\, x u(x) \simeq 2 \int_0^1 \mathrm{d}x\, x d(x) \tag{17.42}$$

so, as expected $f_u \approx 2 f_d$ since the two up quarks in the proton should carry twice the momentum of the down quarks. However, this result also shows that **all quarks carry just over $f_u + f_d \approx 50\%$ of the total proton momentum**. The rest is carried by *gluons*, which do not contribute to electron–nucleon scattering, since they are electrically neutral.

17.5 Sum rules for the proton.

From the known flavor content of the proton, find the value of $\int_0^1 \mathrm{d}x(u(x) - \bar{u}(x))$. State the corresponding sum rules for d and s.

Solution:

Within the quark model, the proton p is a baryon made of three valence quarks uud with a sea of quarks-anti-quarks. We can write (see Eq. (17.99) of the book):

$$u(x) = u_v(x) + q_s(x) \quad \text{and} \quad d(x) = d_v(x) + q_s(x) \quad \text{and} \quad \bar{u}(x) = \bar{d}(x) = q_s(x) \tag{17.43}$$

where the subscript v stands for the valence quarks, and s for the sea quarks. Then it directly follows that:

$$u(x) - \overline{u}(x) = u_v(x) \quad \text{and} \quad d(x) - \overline{d}(x) = d_v(x) \quad \text{and} \quad s(x) - \overline{s}(x) = 0 \tag{17.44}$$

Then:

$$\int_0^1 dx\ (u(x) - \overline{u}(x)) = \int_0^1 dx\ u_v(x) = 2 \tag{17.45}$$

and

$$\int_0^1 dx\ (d(x) - \overline{d}(x)) = \int_0^1 dx\ d_v(x) = 1 \tag{17.46}$$

and

$$\int_0^1 dx\ (s(x) - \overline{s}(x)) = 0 \qquad \square \tag{17.47}$$