SEMINAR 4 - SIRURI SI SERII DE FUNCTII.

BitPaper
Share Knowledge

$\boxed{1}$ a) $f_n(x) = \frac{1}{nx^2}$, $x > 0$

- $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{1}{nx^2} = 0 \Rightarrow f_n \xrightarrow{P} 0$ pe $I = (0,+\infty)$

- Luăm $x_n = \frac{1}{\sqrt{n}} \in (0,1] \Rightarrow f_n(x_n) = \frac{1}{n\cdot\left(\frac{1}{\sqrt{n}}\right)^2} = \frac{1}{\cancel{n}\cdot\frac{1}{\cancel{n}}} = 1 > \varepsilon, \ \forall \varepsilon > 0$

$$\Rightarrow f_n \overset{u}{\not\longrightarrow} 0 \text{ pe } I = (0,1]$$

Pentru $x > 1 \Rightarrow \left| f_n(x) - 0 \right| = \left| \frac{1}{nx^2} \right| = \frac{1}{nx^2} < \frac{1}{n} = a_n$, $\forall n \geq 1$

$a_n = \frac{1}{n} \xrightarrow[n\to\infty]{} 0 \overset{T.1.1}{\Longrightarrow} f_n \xrightarrow{u} 0$ pe $I = (1,\infty)$

$\boxed{2}$ $f_n : \overset{\circ}{I} \to \mathbb{R}$, $f_n(x) = \dfrac{1}{n\sqrt{x}}$, $n \geq 1$, $\overset{\circ}{I} = (0, +\infty)$

- $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{1}{n\sqrt{x}} = 0 \Rightarrow f_n \xrightarrow{P} 0$ pe $\overset{\circ}{I} = (0, +\infty)$

- Luăm $x_n = \dfrac{1}{n^2} \in (0,1] \Rightarrow f_n(x_n) = \dfrac{1}{n\cdot\sqrt{\frac{1}{n^2}}} = \dfrac{1}{\cancel{n}\cdot\frac{1}{\cancel{n}}} = 1 > \varepsilon$, $\forall \varepsilon > 0$

$$\Rightarrow f_n \not\xrightarrow{u} 0 \text{ pe } \overset{\circ}{I} = (0,1]$$

Pentru $x > 1 \Rightarrow \left| f_n(x) - 0 \right| = \left| \dfrac{1}{n\sqrt{x}} \right| = \dfrac{1}{n\sqrt{x}} < \dfrac{1}{n}$, $\forall n \geq 1$

$a_n = \dfrac{1}{n} \xrightarrow[n\to\infty]{} 0 \overset{T1.1}{\Longrightarrow} f_n \xrightarrow{u} 0$ pe $\overset{\circ}{I} = (1, \infty)$

BitPaper
Share Knowledge

$\boxed{3}$ a) $f_n(x) = \dfrac{(n+1)^n}{n^{n+x}}$, $x \in \mathbb{R}$

$$f_n(x) = \frac{(n+1)^n}{n^{n+x}} = \frac{(n+1)^n}{n^n} \cdot \frac{1}{n^x} = \left(1+\frac{1}{n}\right)^n \cdot \frac{1}{n^x}$$

• Pentru $x \le 0 \Rightarrow \sum_{n\ge 1} f_n(x)$ – divergentă

• Pentru $x \in (0,1] \Rightarrow f_n(x) = \left(1+\frac{1}{n}\right)^n \cdot \frac{1}{n^x} \ge \frac{1}{n^x}$ $\qquad \Rightarrow A = (1,+\infty)$

Dar $\sum_{n=1}^{\infty} \frac{1}{n^x}$ este divergentă $x \in (0,1]$ (serie armonică)

$\overset{C.C.1}{\Longrightarrow} \sum_{n=1}^{\infty} f_n(x)$ – divergentă

• Pentru $x > 1 \Rightarrow \lim_{n\to\infty} \dfrac{f_n(x)}{\frac{1}{n^x}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \cdot \cancel{\frac{1}{n^x}} \cdot \cancel{n^x} = e \in (0,+\infty) \Rightarrow$

$\overset{C.C.2}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{1}{n^x} \sim \sum_{n=1}^{\infty} f_n(x)$ – convergentă (serie armonică cu $x > 1$)

BitPaper
Share Knowledge

b) $\displaystyle\sum_{n=1}^{\infty} 2^n \cdot \sin\frac{x^2}{3^n}, \quad x\in\mathbb{R}$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(2^n \sin\frac{x^2}{3^n}\right) = \lim_{n\to\infty} 2^n \cdot \underbrace{\frac{\sin\frac{x^2}{3^n}}{\frac{x^2}{3^n}}}_{\to 1}\cdot\frac{x^2}{3^n} = \lim_{n\to\infty}\left(\frac{2}{3}\right)^n\cdot x^2 = 0 \Rightarrow$$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} f_n(x)$ ar putea fi convergentă

Aplicăm criteriul raportului.

$$\lim_{n\to\infty}\left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}\cdot\sin\frac{x^2}{3^{n+1}}}{\cancel{2^n}\cdot\sin\frac{x^2}{3^n}}\right| = 2\lim_{n\to\infty}\underbrace{\frac{\sin\frac{x^2}{3^{n+1}}}{\frac{x^2}{3^{n+1}}}}_{\to 1}\cdot\frac{\cancel{x^2}}{3^{n+1}}\cdot\underbrace{\frac{\frac{x^2}{3^n}}{\sin\frac{x^2}{3^n}}}_{\to 1}\cdot\frac{\cancel{3^n}}{\cancel{x^2}}$$

$$= \frac{2}{3} < 1 \xRightarrow{C.R.} \sum_{n=1}^{\infty} f_n(x) \text{ este convergentă pentru } x\in\mathbb{R} \Rightarrow A=\mathbb{R}$$

BitPaper
Share Knowledge

$\boxed{4}$

a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(nx)}{n(n+1)}$

$$f_n(x) = \frac{\cos(nx)}{n(n+1)}$$

$$|f_n(x)| = \left|\frac{\cos(nx)}{n(n+1)}\right| \le \frac{1}{n(n+1)} \le \frac{1}{n^2}, \quad n \ge 1$$

Dar $\sum_{n=1}^{\infty} a_n$ este convergentă (serie armonică cu $\alpha = 2 > 1$)

$\xRightarrow{\text{Weierstrass}}$ $\sum_{n=1}^{\infty} f_n(x)$ este uniform convergentă pe $\mathbb{R}$.

b) $a^2 + b^2 \ge 2|ab|$