SEMINAR 2 - SERII DE NUMERE REALE.

$\boxed{6}$ b) $\displaystyle\sum_{n\geq 1} \ln\left(1+\frac{1}{n}\right)$

$$s_n = \sum_{k=1}^{n} \ln\left(\frac{k}{1}+\frac{1}{k}\right) = \sum_{k=1}^{n} \ln \frac{k+1}{k} = \sum_{k=1}^{n}\left[\ln(k+1) - \ln k\right] =$$

$$= \cancel{\ln 2} - \ln 1 + \cancel{\ln 3} - \cancel{\ln 2} + \cancel{\ln 4} - \cancel{\ln 3} + \ldots + \ln(n+1) - \cancel{\ln n} =$$

$$= \ln(n+1)$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \ln(n+1) = +\infty$$

$\Rightarrow \displaystyle\sum_{n\geq 1} \ln\left(1+\frac{1}{n}\right)$ este divergentă și $\displaystyle\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) = +\infty$

c) $\sum_{n\geq 0}\left[\operatorname{arctg}(n+1)-\operatorname{arctg} n\right]$

$$s_n=\sum_{k=0}^{n}\left[\operatorname{arctg}(k+1)-\operatorname{arctg} k\right]=$$

$$=\cancel{\operatorname{arctg} 1}-\operatorname{arctg} 0+\cancel{\operatorname{arctg} 2}-\cancel{\operatorname{arctg} 1}+\dots+\operatorname{arctg}(n+1)-\cancel{\operatorname{arctg} n}$$

$$=\operatorname{arctg}(n+1)-\operatorname{arctg} 0$$

$$=\operatorname{arctg}(n+1)$$

$\Longrightarrow \lim\limits_{n\to\infty} s_n=\lim\limits_{n\to\infty}\operatorname{arctg}(n+1)=\operatorname{arctg}(+\infty)=\frac{\pi}{2} \Longrightarrow (s_n)$ - convergent

$\Longrightarrow \sum\limits_{n\geq 0}\left[\operatorname{arctg}(n+1)-\operatorname{arctg} n\right]$ este convergentă și $s=\lim\limits_{n\to\infty} s_n=\frac{\pi}{2}$

$\boxed{7}$ b) $\sum_{n\geq 1} \frac{1}{\sqrt[n]{n}}$

Notăm $u_n = \frac{1}{\sqrt[n]{n}}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = \lim_{n\to\infty} \sqrt[n]{\underbrace{\frac{1}{n}}_{a_n}} \overset{\text{C. Răd.}}{=\!=} \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

$\Rightarrow \sum_{n\geq 1} \frac{1}{\sqrt[n]{n}}$ este divergentă

$\boxed{7}$ e) $\displaystyle\sum_{n\geq 1} (-1)^n \frac{1}{n^2+(-1)^n}$

Notăm $a_n = \dfrac{1}{n^2+(-1)^n} > \forall n\geq 1$ ↘

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{n^2+(-1)^n} = 0$

$\xrightarrow{\text{C. Dirichlet}} \displaystyle\sum_{n\geq 1} (-1)^n \frac{1}{n^2+(-1)^n}$ - convergentă

$\boxed{8}$ $\sum_{n\geq 1} \frac{\sin nx}{n(n+1)}$, $x \in \mathbb{R}$

Fie $p\in\mathbb{N}$ și $\varepsilon>0$. Trebuie să arătăm că există $n_1(\varepsilon)\in\mathbb{N}$ astfel încât $\left|\sum_{k=n+1}^{n+p} \frac{\sin kx}{k(k+1)}\right| < \varepsilon$, $\forall\, n \geq n_1(\varepsilon)$

$$\left|\sum_{k=n+1}^{n+p} \frac{\sin kx}{k(k+1)}\right| \leq \sum_{k=n+1}^{n+p} \frac{|\sin kx|}{k(k+1)} \leq \sum_{k=n+1}^{n+p} \frac{1}{k(k+1)} = \sum_{k=n+1}^{n+p}\left(\frac{1}{k}-\frac{1}{k+1}\right) =$$

$$= \frac{1}{n+1} - \cancel{\frac{1}{n+2}} + \cancel{\frac{1}{n+2}} - \cancel{\frac{1}{n+3}} + \ldots + \cancel{\frac{1}{n+p}} - \frac{1}{n+p+1}$$

$$= \frac{1}{n+1} - \frac{1}{n+1+p} < \frac{1}{n+1}$$

Dar, $\frac{1}{n+1} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon} - 1$, $\forall\, \varepsilon > 0$

Luăm, $n_1 = n_1(\varepsilon) = \left[\frac{1}{\varepsilon} - 1\right] + 1 = \left[\frac{1}{\varepsilon}\right] \in \mathbb{N}$

Deci $\forall \varepsilon>0 \Rightarrow \left| \sum_{k=n+1}^{n+p} \frac{\sin kx}{k(k+1)} \right| < \varepsilon, \ \forall n \in \mathbb{N}, \forall n \geq n_1, \forall \varepsilon>0$

$\forall p \in \mathbb{N}, \forall x \in \mathbb{R}$

$\xRightarrow[\text{Cauchy serii}]{\text{c. gen.}} \sum_{n \geq 1} \frac{\sin nx}{n(n+1)}, \ x \in \mathbb{R}$ este convergentă