SEMINAR 5 - SERII FOURIER.

$\overset{0}{\sin}\, n\pi = 0$

$\cos n\pi = (-1)^n$

$\boxed{1}$ a) $f(x) = x^2, \quad x \in [-\pi, \pi]$

$$f(-x) = (-x)^2 = x^2 = f(x) \Rightarrow f - \text{pară} \Rightarrow b_n = 0, \ \forall n \geq 0$$

$$\underline{\underline{a_0}} = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \cos\frac{0\cdot\pi\cdot x}{\pi}\, dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\cdot \frac{x^3}{3}\Big|_0^{\pi} = \frac{2\pi^{\cancel{3}}}{\cancel{3\pi}} = \underline{\underline{\frac{2\pi^2}{3}}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos\frac{n\cancel{\pi} x}{\cancel{\pi}}\, dx = \frac{2}{\pi}\int_0^{\pi} \underbrace{x^2}_{f}\cdot \underbrace{\cos nx}_{g'}\, dx = \frac{2}{\pi}\int_0^{\pi} x^2\cdot\left(\frac{\sin nx}{n}\right)' dx$$

$$= \frac{2}{\pi}\left(x^2\cdot\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi} 2x\cdot\frac{\sin nx}{n}\, dx\right)$$

$$= \frac{2}{\pi}\left(\underbrace{\pi^2\cdot\frac{\overset{0}{\sin n\pi}}{n}}_{0} - \frac{2}{n}\int_0^{\pi} x\sin nx\, dx\right)$$

$$a_n = -\frac{4}{n\pi}\int_0^{\pi} \underbrace{x}_{f}\,\underbrace{\sin nx}_{g'}\,dx = \frac{4}{n\pi}\left(x\cdot\frac{\cos nx}{n}\Bigg|_0^{\pi} - \int_0^{\pi}\frac{\cos nx}{n}\,dx\right)$$

$$= \frac{4}{n\pi}\left(\frac{\pi}{n}\cdot\underbrace{\cos n\pi}_{(-1)^n} - \frac{1}{n}\cdot\underbrace{\frac{\sin nx}{n}}_{=0}\Bigg|_0^{\pi}\right)$$

$$= \frac{4}{n\cancel{\pi}}\cdot\frac{\cancel{\pi}}{n}\cdot(-1)^n$$

$$\rightarrow \quad a_n = \frac{4\cdot(-1)^n}{n^2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\cancel{\pi} x}{\cancel{\pi}}$$

$$f(x) = \frac{\frac{2\pi^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{4\cdot(-1)^n}{n^2} \cdot \cos nx$$

$$S(-\pi) = S(\pi) = \frac{f(-\pi+0) + f(\pi+0)}{2} = \pi^2$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4\cdot(-1)^n}{n^2} \cdot \cos nx$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4\cdot(-1)^n}{n^2} \cos nx \qquad (*)$$

Arătați că $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

$$x = \pi \overset{(*)}{\Longrightarrow} \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4\cdot(-1)^n}{n^2} \underbrace{\cos n\pi}_{=(-1)^n} \Longleftrightarrow \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \Longleftrightarrow 4\sum_{n=1}^{\infty} \frac{1}{n^2} = {}^{3/}\pi^2 - \frac{\pi^2}{3}$$

$$\Longleftrightarrow 4\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{3} \,\Big|\cdot \frac{1}{4} \Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

c) $f(x) = e^{ax}, \quad x \in [-1, 1]$

$f(-x) = e^{a(-x)} = e^{-ax} \neq f(x) \Rightarrow f$ - nu este pară

$f(-x) = e^{-ax} \neq -f(x) \Rightarrow f$ - nu este impară

$$\underline{\underline{a_0}} = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} e^{ax}\,dx = \frac{e^{ax}}{a}\Big|_{-1}^{1} = \underline{\underline{\frac{e^{a} - e^{-a}}{a}}}$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{1} e^{ax}\cos(n\pi x)\,dx = \int_{-1}^{1}\left(\frac{e^{ax}}{a}\right)' \cos(n\pi x)\,dx = \frac{e^{ax}}{a}\cos(n\pi x)\Big|_{-1}^{1} + \int_{-1}^{1}\frac{e^{ax}}{a}\cdot \sin(n\pi x)(n\pi)\,dx$$

$$= \frac{e^{a}}{a}\underbrace{\cos(n\pi)}_{(-1)^n} - \frac{e^{-a}}{a}\underbrace{\cos(-n\pi)}_{(-1)^n} + \frac{n\pi}{a}\int_{-1}^{1} e^{ax}\sin(n\pi x)\,dx = \frac{(-1)^n\left(e^{a} - e^{-a}\right)}{a} + \frac{n\pi}{a}\underbrace{\int_{-1}^{1} e^{ax}\sin(n\pi x)\,dx}_{b_n}$$

$$\Rightarrow a_n = \frac{(-1)^n\left(e^{a} - e^{-a}\right)}{a} + \frac{n\pi}{a} b_n \qquad (1)$$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{-1}^{1} e^{ax}\cdot \sin(n\pi x)\,dx = \int_{-1}^{1} \left(\frac{e^{ax}}{a}\right)' \sin(n\pi x)\,dx =$$

$$= \frac{e^{ax}}{a}\underbrace{\sin(n\pi x)}_{0=}\Big|_{-1}^{1} - \int_{-1}^{1} \frac{e^{ax}}{a}\cdot \cos(n\pi x)\cdot(n\pi)\,dx$$

$$= -\frac{n\pi}{a}\int_{-1}^{1} \underbrace{e^{ax}\cdot\cos(n\pi x)\,dx}_{a_n} \quad \Rightarrow b_n = -\frac{n\pi}{a}\cdot a_n \qquad (2)$$

$$a_n \overset{(1)}{=} \frac{(-1)^n(e^a - e^{-a})}{a} + \frac{n\pi}{a}\cdot\left(-\frac{n\pi}{a}\right)\cdot a_n$$

$$a_n = \frac{(-1)^n(e^a - e^{-a})}{a} - \frac{n^2\pi^2}{a^2}\cdot a_n \iff a_n + \frac{n^2\pi^2}{a^2}a_n = \frac{(-1)^n(e^a - e^{-a})}{a}$$

$$\iff a_n\left(\frac{a^2}{1} + \frac{n^2\pi^2}{a^2}\right) = \frac{(-1)^n(e^a - e^{-a})}{a} \Rightarrow a_n = \frac{(-1)^n(e^a - e^{-a})}{\cancel{a}}\cdot\frac{a^{\cancel{2}}}{a^2+n^2\pi^2} \Rightarrow a_n = \frac{(-1)^n(e^a-e^{-a})\cdot a}{a^2+n^2\pi^2}$$

$$b_m \overset{(1)}{=\!=} -\frac{m\pi}{\cancel{a}} \cdot \frac{(-1)^m\left(e^a - e^{-a}\right)\cdot \cancel{a}}{a^2+m^2\pi^2}$$

$$\boxed{b_m = \frac{(-1)^{m+1}\left(e^a - e^{-a}\right)\cdot m\pi}{a^2+m^2\pi^2}}$$

$$f(x) = \frac{e^a - e^{-a}}{2a} + \sum_{m=1}^{\infty}\left(\frac{(-1)^m\left(e^a - e^{-a}\right)\cdot a}{a^2+m^2\pi^2}\cdot \cos(m\pi x) + \frac{(-1)^{m+1}\left(e^a - e^{-a}\right)\cdot m\pi}{a^2+m^2\pi^2}\sin(m\pi x)\right)$$

$$e^{ax} = \frac{e^a - e^{-a}}{2a} + \left(e^a - e^{-a}\right)\sum_{m=1}^{\infty}\left(\frac{(-1)^m a}{a^2+m^2\pi^2}\cos(m\pi x) + \frac{(-1)^{m+1}\cdot m\pi}{a^2+m^2\pi^2}\sin(m\pi x)\right), \forall\, x\in[-1,1]$$

d) $f(x) = |x|, \quad x \in [-1, 1]$

$f(-x) = |-x| = |x| = f(x) \Rightarrow f$ - pară $\Rightarrow$ $b_n = 0$

$$a_0 = 2\int_0^1 f(x)\cos(0 \cdot \pi x)\,dx = 2\int_0^1 x\,dx = \cancel{2} \cdot \frac{x^2}{\cancel{2}}\Big|_0^1 = 1$$

$$a_n = 2\int_0^1 f(x)\cos(n\pi x)\,dx = 2\int_0^1 x \cdot \cos(n\pi x)\,dx = 2\left(x \cdot \frac{\overset{\circ}{\sin}(n\pi x)}{n\pi}\Big|_0^1 - \int_0^1 \frac{\overset{\circ}{\sin}(n\pi x)}{n\pi}\,dx \right)$$

$$= 2\left(0 - \frac{1}{n\pi} \cdot \int_0^1 \overset{\circ}{\sin}(n\pi x)\,dx \right) = \frac{2}{n\pi} \cdot \frac{\cos(n\pi x)}{n\pi}\Big|_0^1 = \frac{2 \cdot (-1)^n}{n^2\pi^2} - \frac{2}{n^2\pi^2} =$$

$$= \frac{2}{n^2\pi^2}\left((-1)^n - 1\right) \Rightarrow a_n = \begin{cases} 0, & \text{dacă } n \text{ - par} \\ -\dfrac{4}{n^2\pi^2}, & \text{dacă } n \text{ - impar} \end{cases}$$

$$f(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2} \cos\left((2k+1)\pi x\right) \quad , \quad \forall\, x \in [-1,1]$$