SEMINAR 3 - SERII CU TERMENI POZITIVI.

$\boxed{1}$ a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{7^n+3^n}$

$$0 < \underbrace{\frac{1}{7^n+3^n}}_{u_n} \le \underbrace{\frac{1}{3^n}}_{v_n}, \quad \forall n \ge 1$$

Dar $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ este convergentă deoarece este o serie geometrică cu $r = \frac{1}{3}$ $\xRightarrow{\text{c.c.1}}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{7^n+3^n} \text{ este convergentă}$$

! **Seria geometrică**

$\displaystyle\sum_{n=0}^{\infty} r^n$, $r \in \mathbb{R}$ este convergentă

$\Leftrightarrow |r| < 1 \Leftrightarrow r \in (-1, 1)$

$$S = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

! **Seria armonică**

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$, $\alpha > 0$ $\begin{cases} \text{convergentă dacă } \alpha > 1 \\ \text{divergentă dacă } \alpha \le 1 \end{cases}$

d) $\sum\limits_{n=1}^{\infty} 2^n \sin\frac{1}{3^n}$

$$0 \le \underbrace{2^n \sin\frac{1}{3^n}}_{u_n} \le 2^n \cdot \underbrace{\frac{\sin\frac{1}{3^n}}{\frac{1}{3^n}}}_{\le 1} \cdot \frac{1}{3^n} \le \frac{2^n}{3^n} = \underbrace{\left(\frac{2}{3}\right)^n}_{v_n}, \forall n \ge 1$$

Dar $\sum\limits_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ este convergentă deoarece este o serie geometrică cu $x = \frac{2}{3}$ $\xrightarrow{C.C.1}$ $\sum\limits_{n=1}^{\infty} 2^n \sin\frac{1}{3^n}$ este convergentă

$\boxed{2}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \quad (*)$$

Notăm $u_n = \dfrac{n!}{n^n}$.

$$l = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{\cancel{n!}\cdot\cancel{(n+1)}}{(n+1)^n\cdot\cancel{(n+1)}} \cdot \frac{n^n}{\cancel{n!}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n \overset{(*)}{=} \frac{1}{e}$$

$$l = \frac{1}{e} < 1 \xRightarrow{\text{C.Rap.}} \sum_{n=1}^{\infty} \frac{n!}{n^n} \text{ este convergentă}$$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(a+1)(a+2)\dots(a+n)}$, $a > -1$

$$u_n = \frac{n!}{(a+1)(a+2)\dots(a+n)}, \quad a>-1$$

$$\frac{u_{n+1}}{u_n} = \frac{\cancel{(n+1)!}^{\,n+1}}{\cancel{(a+1)}\cancel{(a+2)}\dots(a+n+1)} \cdot \frac{\cancel{(a+1)}\cancel{(a+2)}\dots\cancel{(a+n)}}{\cancel{n!}} = \frac{n+1}{a+n+1} \xrightarrow[n\to\infty]{} 1$$

Deci, c. Raportului nu oferă un rezultat. Aplicăm Raabe - Duhamel.

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\cdot\left(\frac{a+n+1}{n+1} - \frac{n+1}{n+1}\right) = \lim_{n\to\infty} n\cdot\frac{a+\cancel{n}+\cancel{1}-\cancel{n}-\cancel{1}}{n+1} =$$

$$= \lim_{n\to\infty} \frac{an}{n+1} = a$$

Discuție

- Dc. $a \in (-1,1) \overset{R.D.}{\Longrightarrow} \sum_{n=1}^{\infty} u_n$ — divergentă

- Dc. $a = 1 \Longrightarrow u_n = \frac{1}{n+1} \Longrightarrow \sum_{n\geq 1} u_n$ este divergentă deoarece o comparăm cu $\sum_{n=1}^{\infty} \frac{1}{n}$ care este o serie armonică cu $\alpha = 1$, deci divergentă

- Dc. $a > 1 \overset{R.D.}{\Longrightarrow} \sum_{n\geq 1} u_n$ — convergentă

e) $\sum_{n=1}^{\infty} \frac{\sin^n a}{n}, \quad a \in \left(0, \frac{\pi}{2}\right)$

Aplicăm C. Rădăcinii ; $u_n = \frac{\sin^n a}{n}$

$$l = \lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \sqrt[n]{\frac{\sin^n a}{n}} = \lim_{n\to\infty} \frac{\sin a}{\sqrt[n]{n}} = \sin a$$

$$\left.\begin{array}{l} l = \sin a \\ a \in \left(0, \frac{\pi}{2}\right) \end{array}\right\} \Rightarrow l < 1 \Longrightarrow$$

$\xRightarrow{C.Răd} \sum_{n=1}^{\infty} \frac{\sin^n a}{n}$ este convergentă , $\forall\, a \in \left(0, \frac{\pi}{2}\right)$