SEMINAR 1 - SIRURI SI SERII DE NUMERE REALE.

$\boxed{1}$ b) $a_n = \frac{n+2}{3^n}$, $n \in \mathbb{N}$

$$a_{n+1} = \frac{n+1+2}{3^{n+1}} = \frac{n+3}{3^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{n+3}{3^{n+1}} \cdot \frac{3^n}{n+2} = \frac{n+3}{\cancel{3^n} \cdot 3^1} \cdot \frac{\cancel{3^n}}{n+2} = \frac{n+3}{3(n+2)} = \frac{n+2+1}{3(n+2)} = \frac{\cancel{n+2}}{3(\cancel{n+2})} + \frac{1}{3(n+2)} =$$

$$= \frac{1}{3} + \frac{1}{3(n+2)} \leq 1, \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \frac{a_{n+1}}{a_n} \leq 1 \Leftrightarrow a_{n+1} \leq a_n, \ \forall n \in \mathbb{N} \Rightarrow (a_n)_{n \geq 0} \searrow \text{ - descrescător}$$

$\boxed{2}$ $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

$\boxed{3}$ $a_n = \frac{2n-1}{2n+1}$, $n \geq 1$

$a_n \longrightarrow a = 1$

Pentru $\forall \varepsilon > 0$, căutăm acele nr. naturale n pentru care este satisfăcută inegalitatea $|a_n - a| < \varepsilon$.

Calculăm,

$$|a_n - a| < \varepsilon \iff \left| \frac{2n-1}{2n+1} - \overset{2n+1}{\frac{}{}1} \right| < \varepsilon \iff \left| \frac{\cancel{2n}-1-\cancel{2n}-1}{2n+1} \right| < \varepsilon \iff \left| \frac{-2}{2n+1} \right| < \varepsilon \iff$$

$$\iff \frac{2}{2n+1} < \varepsilon \iff 2 < \varepsilon(2n+1) \iff 2 < 2n\varepsilon + \varepsilon \iff 2 - \varepsilon < 2n\varepsilon \iff$$

$$\iff n > \frac{2-\varepsilon}{2\varepsilon} \iff n > \frac{1}{2}\left(\frac{2}{\varepsilon} - 1\right)$$

Considerăm $n_1 = n_1(\varepsilon) = \underbrace{\left[\frac{1}{2}\left(\frac{2}{\varepsilon} - 1\right)\right]}_{\text{parte întreagă}} + 1 \in \mathbb{N}$, $\forall n \in \mathbb{N}$, $\forall n \geq n_1$, $\forall \varepsilon > 0$

Cum $[x]+1 > x \implies n > \frac{1}{2}\left(\frac{2}{\varepsilon}-1\right)$, $\forall\ n \in \mathbb{N}$, $\forall\ n \geq n_1$, $\forall \varepsilon > 0$

$\iff |a_n - \underline{\underline{a}}| < \varepsilon$, $\forall\ n \in \mathbb{N}$, $\forall\ n \geq n_1$, $\forall\ \varepsilon > 0$

$\implies (a_n)_{n \geq 1}$ este convergent și $\lim\limits_{n\to\infty} a_n = 1$ ///

$\boxed{4}$ a) $\lim\limits_{n\to\infty} \dfrac{\cos n}{4^n}$, $n \in \mathbb{N}$

$$-1 \le \cos n \le 1, \ \forall n \in \mathbb{N} \ \Big| \cdot \frac{1}{4^n}$$

$$-\frac{1}{4^n} \le \frac{\cos n}{4^n} \le \frac{1}{4^n}$$

$$0$$

$$\xRightarrow{\text{c. cleștelui}} \lim_{n\to\infty} \frac{\cos n}{4^n} = 0, \ \forall n \in \mathbb{N}$$

d) $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n}$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e \quad (*)$$

$$\frac{\sqrt[n]{n!}}{n} = \sqrt[n]{\frac{n!}{n^n}}$$

$$a_n = \frac{n!}{n^n}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} = \lim_{n\to\infty}\frac{\cancel{n!}\,\cancel{(n+1)}}{(n+1)^n\cdot\cancel{(n+1)}}\cdot\frac{n^n}{\cancel{n!}} =$$

$$= \lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n \overset{(*)}{=} \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

c. Rad. $\Longrightarrow$ $\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{e}$

e) $\lim\limits_{n\to\infty} \frac{1}{n}\left(\frac{1}{\ln 2} + \frac{1}{\ln 3} + \dots + \frac{1}{\ln n}\right)$

Notăm $a_n = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \dots + \frac{1}{\ln n}$

$b_n = n \nearrow$ cu $b_n \to \infty \Rightarrow$ aplicăm Stolz-Cesaro.

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} \frac{\frac{1}{\ln(n+1)}}{\cancel{n}+1-\cancel{n}} = \lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$$

$$\xRightarrow{\text{Stolz-C.}} \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{\ln 2} + \frac{1}{\ln 3} + \dots + \frac{1}{\ln n}\right) = 0$$