A DPG method for compressible flow problems

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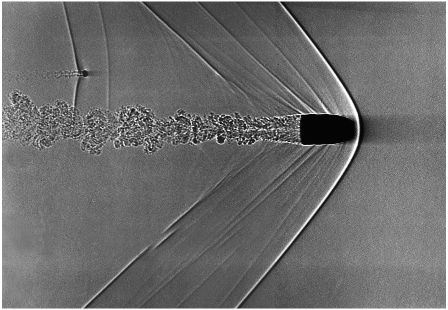
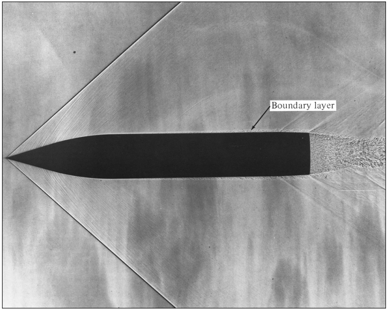
July 24, 2012

Compressible flow problems



(a) Shock wave



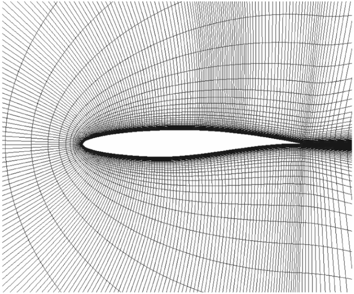
(b) Boundary layer

Compressible flow plays an important role in the aerospace and energy industries - supersonic aircraft, combustion engines, etc.

Goal: Compressible Navier-Stokes equations

Numerical difficulties:

- Resolving solutions (sharp, localized $O(\mathrm{Re}^{-1})$ phenomena)
  - Shocks, boundary layers
  - Turbulent phenomena
- Stability of numerical schemes
  - Coarse/adaptive grids
  - Higher order



Idea: begin first with the model problem of convection-diffusion.

Convection-diffusion as a model problem

$$\nabla \cdot (\beta u) - \epsilon \Delta u = f, \quad \text{on } \Omega \in \mathbb{R}^3$$

In 1D: $\beta u' - \epsilon u'' = f$. Standard continuous Galerkin variational formulation: solve

$$b(u, v) = l(v)$$

where

$$b(u, v) = \int_\Omega -\beta u v' + \epsilon u' v'$$

$$l(v) = \int_\Omega f v$$

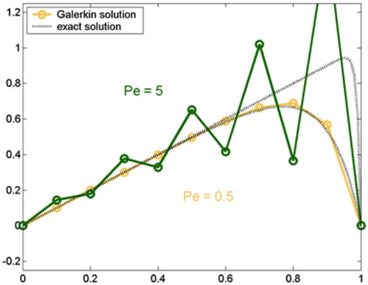


Figure: Oscillations in the standard Galerkin method for underresolved meshes and small ϵ.

Stabilization

Historical stabilization methods:

- Artificial diffusion - solve

$$\beta u' - \tilde{\epsilon} u'' = f$$

  where $\tilde{\epsilon}$ is set depending on β, ϵ, and the mesh.
- Upwind stencils - for positive β, and uniform grid points x_A,

$$u'(x_A) \approx \frac{u_h(x_A) - u_h(x_{A-1})}{h}$$

Both of these methods introduce additional numerical diffusion, and their effectiveness depends on the forcing f and parameters h, $|\beta|$, and ϵ.

Streamline-upwind Petrov-Galerkin (SUPG)

SUPG solves $b_{\mathrm{SUPG}}(u,v) = l_{\mathrm{SUPG}}(v)$, where

$$b_{\mathrm{SUPG}}(u,v) = b(u,v) + \sum_K \int_K \tau\left(L_{\mathrm{adv}} v\right) Lu$$

$$l_{\mathrm{SUPG}}(v) = l(v) + \sum_K \int_K \tau\left(L_{\mathrm{adv}} v\right) f.$$

- $L_{\mathrm{adv}} u = \nabla \cdot (\beta u)$, and τ is a parameter.
- Recovers "exact" artificial diffusion for $f = 0$.
- Effective for $f \neq 0$, unlike "exact" artificial diffusion.
- *Residual-based* stabilization.

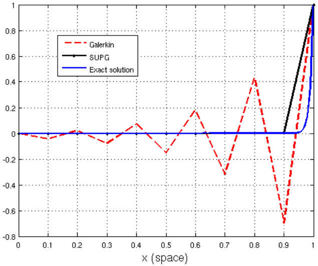


Figure: Exact, FEM, and SUPG solutions.

Can be interpreted as a *Petrov-Galerkin* method,

$$b\left(u,\tilde{v}_i\right) = l\left(\tilde{v}_i\right), \quad \forall i = 1,\ldots,N-1,$$

where the SUPG test function $\tilde{v}_i$ is defined elementwise as

$$\tilde{v}_i = \phi_i(x) + \tau L_{\mathrm{adv}}\phi_i.$$

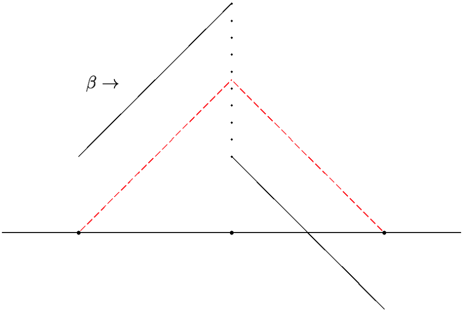


Figure: SUPG test function v_i.

Minimum residual methods and optimal testing

Identify $B : U \to V'$ such that

$$\langle Bu, v\rangle_V := b(u, v), \quad u \in U, v \in V.$$

Then, $b(u_h, v) = l(v)$ is equivalent to the equation in V'

$$Bu_h = l$$

We seek to minimize the functional on V'

$$J(u_h) = \frac{1}{2}\|Bu_h - l\|_{V'}^2 := \frac{1}{2}\sup_{v\in V\setminus\{0\}} \frac{|b(u_h, v) - l(v)|^2}{\|v\|_V^2}.$$

Let $R_V : V \to V'$ be the isometric the Riesz map st.

$$\langle R_V v, \delta v \rangle_V := (v, \delta v)_V, \quad \forall \delta v \in V.$$

Then, our functional $J(u_h)$ is equal to

$$\min_{u_h \in U_h} J(u_h) = \frac{1}{2} \left\| B u_h - l \right\|_{V'}^2 = \frac{1}{2} \left\| R_V^{-1}(B u_h - l) \right\|_V^2.$$

First order optimality: Gâteaux derivative is zero in all directions $\delta u \in U_h$

$$\begin{aligned} \left(R_V^{-1}(B u_h - l), R_V^{-1} B \delta u \right)_V &= 0, \quad \forall \delta u \in U. \\ \to \left\langle (B u_h - l), R_V^{-1} B \delta u \right\rangle &= 0, \\ \to b(u_h, R_V^{-1} B \delta u) - l(R_V^{-1} B \delta u) &= 0 \end{aligned}$$

For $\delta u \in U$, we define the *optimal test function* $v_{\delta u}$

$$v_{\delta u} := R_V^{-1} B \delta u$$

such that $J(u_h)$ is minimized by the solution of

$$b\left(u_h, v_{\delta u}\right) = l\left(v_{\delta u}\right), \quad \forall \delta u \in U$$

Details and properties of DPG

- By choosing V as a *broken* test space, test functions can be determined locally.
- In practice, $V := V_h$, where $\dim(V_h) > \dim(U_h)$ elementwise, and test functions are approximated via solving

$$\left(v_{\delta u}, \delta v\right)_V = b(\delta u, \delta v).$$

- Symmetric positive-definite stiffness matrix: for $B_{ji} = b(u_j, v_i)$, $l_i = l(v_i)$ for v_i spanning V_h, DPG solves

$$u^T B^T R_V^{-1} B = l R_V^{-1} B$$

- DPG provides the best approximation in the energy norm

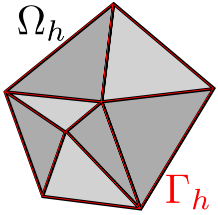
$$\|u\|_E = \|Bu\|_{V'} = \sup \frac{b(u,v)}{\|v\|_V}.$$

- The actual energy error is computable through the residual

$$\|u - u_h\|_E = \|B(u-u_h)\|_{V'} = \left\|R_V^{-1}(l - Bu_h)\right\|_V = \|e\|_V$$

where $(e, \delta v)_V = l(v) - b(u_h, v)$ for all $v \in V$.

Ultra-weak formulation



Given a first order system $Au = f$, we identify the **partition** Ω_h and **mesh skeleton** Γ_h.

The ultra-weak formulation for $Au = f$ on Ω_h is

$$b\left(\left(u, \widehat{u}\right), v\right) := \langle \widehat{u}, [\![v]\!] \rangle_{\Gamma_h} - (u, A_h^* v)_{\Omega_h} = (f, v)_{\Omega_h}.$$

where

$$u \in L^2\left(\Omega_h\right) \equiv L^2(\Omega), \quad v \in V = D(A_h^*), \quad \widehat{u} \in \gamma(D(A)),$$

where $D(A_h^*)$ is the broken graph space of A_h^*, and $\gamma(D(A))$ the trace space of the graph space of operator A.

Canonical norms for the ultra-weak formulation

Under the ultra-weak formulation, we have the following relations:

Trial norm		Test norm
$\boxed{\|u\|^2_{L^2(\Omega)} + \|\widehat{u}\|^2}$	$\Longrightarrow$	$\|A_h^* v\|^2_{L^2(\Omega)} + \left(\sup_{\widehat{u}} \frac{\langle \widehat{u}, [\![v]\!]\rangle_{\Gamma_h}}{\|\widehat{u}\|}\right)^2$
$\|u\|^2_{L^2(\Omega)} + \sup_v \left(\frac{\langle \widehat{u}, [\![v]\!]\rangle_{\Gamma_h}}{\|v\|_V}\right)^2$	$\Longleftarrow$	$\boxed{\|A_h^* v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}}$

Figure: The optimal *test* norm is naturally derived by beginning with the canonical norm on the trial space, while the quasi-optimal *trial* norm is derived from beginning with the canonical norm on the test space.

Ultra-weak formulation for convection-diffusion

In first order form, the convection-diffusion equation is

$$A(u,\sigma) := \begin{bmatrix} \nabla\cdot(\beta u - \sigma) \\ \frac{1}{\epsilon}\sigma - \nabla u \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

The variational formulation is

$$\begin{aligned} b\left(\left(u,\sigma,\widehat{u},\widehat{f}_n\right),(v,\tau)\right) &= (u, \nabla\cdot\tau - \beta\cdot\nabla v)_{\Omega_h} + \left(\sigma, \epsilon^{-1}\tau + \nabla v\right)_{\Omega_h} \\ &\quad - \langle [\![\tau\cdot n]\!], \widehat{u}\rangle_{\Gamma_h} + \left\langle \widehat{f}_n, [\![v]\!]\right\rangle_{\Gamma_h}, \end{aligned}$$

where $\widehat{f}_n := \beta_n u - \sigma_n$. Note that $\nabla\cdot$ and ∇ are taken to act elementwise.

Quasi-optimal norm under convection-diffusion

For convection-diffusion, the quasi-optimal test norm is

$$\|(v,\tau)\|_V^2 = \|\nabla\cdot\tau - \beta\cdot\nabla v\|_{L^2}^2 + \|\epsilon^{-1}\tau + \nabla v\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\tau\|_{L^2}^2.$$

Problem with this test norm: approximability of test functions.

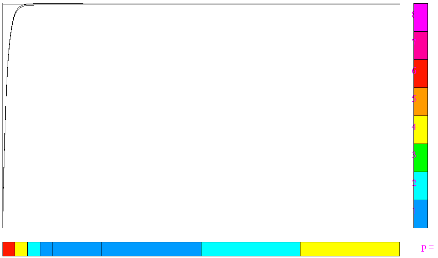


Figure: v and τ components of the 1D optimal test functions for flux $\widehat{f}_n$ on the *right-hand* side of a unit element for $\epsilon = 0.01$.

Bilinear forms and induced adjoints

Recall the convection-diffusion bilinear form

$$b\left(\left(u,\sigma,\widehat{u},\widehat{f}_n\right),(v,\tau)\right) = (u,\nabla\cdot\tau-\beta\cdot\nabla v)_{\Omega_h} + \left(\sigma,\epsilon^{-1}\tau+\nabla v\right)_{\Omega_h} - \langle [\![\tau\cdot n]\!],\widehat{u}\rangle_{\Gamma_h} + \left\langle \widehat{f}_n,[\![v]\!]\right\rangle_{\Gamma_h},$$

We recover $\|u\|_{L^2(\Omega)}$ by choosing continuous (v,τ) s.t.

$$\nabla\cdot\tau-\beta\cdot\nabla v = u$$
$$\frac{1}{\epsilon}\tau-\nabla v = 0$$

where boundary conditions are s.t. $\langle [\![\tau\cdot n]\!],\widehat{u}\rangle_{\Gamma_h}$ and $\left\langle \widehat{f}_n,[\![v]\!]\right\rangle_{\Gamma_h}$ vanish.

Determining an alternative test norm

"Necessary" conditions for robustness — let $\mathbf{U} = \left(u, \sigma, \widehat{u}, \widehat{f}_n\right)$, then, by choosing specific (v, τ),

$$\|u\|^2_{L^2(\Omega)} = b\left(\mathbf{U}, (v,\tau)\right) = \frac{b\left(\mathbf{U}, (v,\tau)\right)}{\|(v,\tau)\|_V} \|(v,\tau)\|_V \leq \|\mathbf{U}\|_E \|(v,\tau)\|_V$$

If $\|(v,\tau)\|_V \lesssim \|u\|_{L^2(\Omega)}$, then we have the robust bound

$$\|u\|^2_{L^2(\Omega)} \lesssim \|\mathbf{U}\|_E$$

“Building blocks” norm

Test norm quantities are robustly bounded from above by $\|u\|_{L^2(\Omega)}$.

$$\|(v,\tau)\|_{V,K}^2 = \|v\|^2 + \epsilon\|\nabla v\|^2 + \|\beta\cdot\nabla v\|^2 + \|\nabla\cdot\tau\|^2 + \frac{1}{\epsilon}\|\tau\|^2.$$

Problem: boundary layers in optimal test functions:

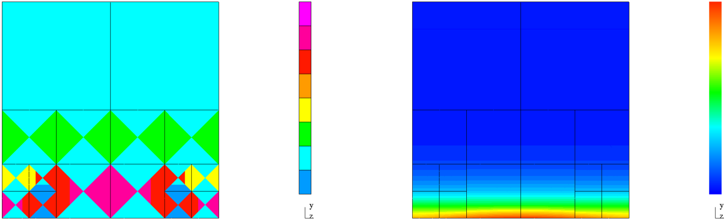


Figure: The v component of the optimal test function corresponding to flux $\widehat{u} = x(1-x)$ on the bottom side of a unit element for $\epsilon = 0.01$.

Mesh-scaled test norms

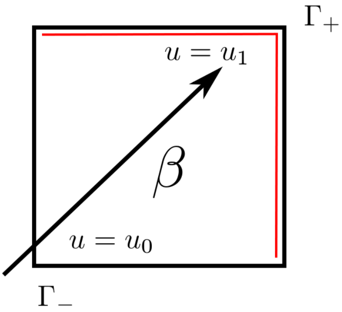
Our test norm, as defined over a single element K, is now

$$\|\,(v,\tau)\,\|_{V,K}^2 = \min\left\{\frac{\epsilon}{|K|},1\right\}\|v\|^2 + \epsilon\|\nabla v\|^2 + \|\beta\cdot\nabla v\|^2 + \|\nabla\cdot\tau\|^2 + \min\left\{\frac{1}{\epsilon},\frac{1}{|K|}\right\}\|\tau\|^2.$$

- v and τ decoupled.
- Coefficients are $O(1)$ after transforming to the unit element.

Dirichlet inflow boundary condition

Standard choice of boundary condition: $u = u_0$ on inflow boundary Γ_{in}.

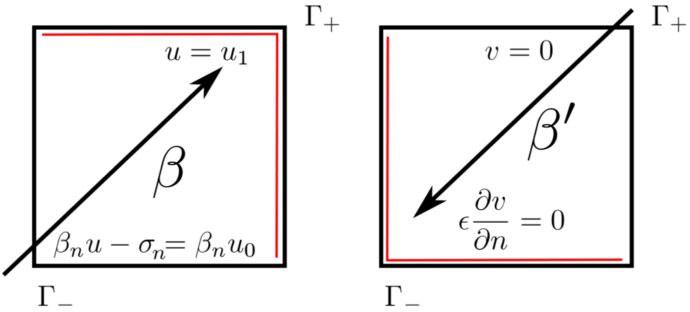


(a) Primal problem (b) Adjoint problem

Figure: For the standard Dirichlet inflow condition, the solution to the adjoint problem can develop strong boundary layers at the outflow of the adjoint problem.

New inflow boundary condition on $\widehat{f}_n$

Non-standard choice of boundary condition: $\widehat{f}_n = \beta_n u_0$ on $\Gamma_{\rm in}$.



(a) Primal problem (b) Adjoint problem

Figure: Under the new inflow condition, the wall-stop boundary condition is relaxed to a zero-stress condition at the outflow boundary of the adjoint problem.

Test norms and adjoint solutions

- Standard choice of boundary condition: $u = u_0$ on inflow boundary Γ_{in}: induces boundary layers in adjoint problems.
- Non-standard choice of boundary condition: $\widehat{f}_n = \beta_n u_0$ on Γ_{in}: induces smoother adjoint problems.

Intuition: the effectiveness of DPG under a test norm is governed by how a **specific test norm** measures the **solutions of the adjoint problem**.

Erikkson-Johnson model problem

On domain $\Omega = [0,1]^2$, with $\beta = (1,0)^T$, $f = 0$ and boundary conditions

$$\widehat{\beta_n u - \sigma_n} = \widehat{f}_n = u_0, \quad \beta_n \leq 0$$
$$\widehat{u} = 0, \quad \beta_n > 0$$

Separation of variables gives an analytic solution.

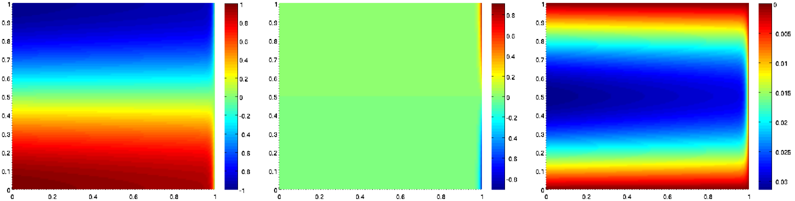


Figure: Solution for u, σ_x, and σ_y for $\epsilon = .01$, $C_1 = 1$, $C_n = 0$, $n \neq 1$

Error rates

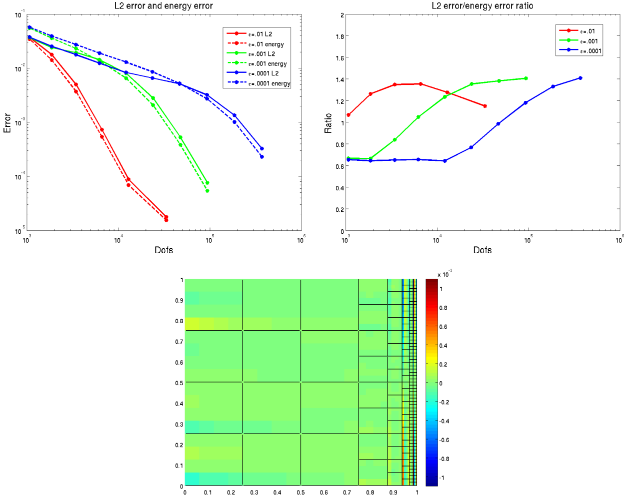


Figure: L^2/energy error, their ratio, and pointwise error in u for $\epsilon = .01$.

Regularization

For $\beta = (-y, x)^T$ on $\Omega = [-1, 1]^2$, and zero normal stress outflow conditions. Ill posed in the convection setting. Similar tests have been done with discontinuous data.

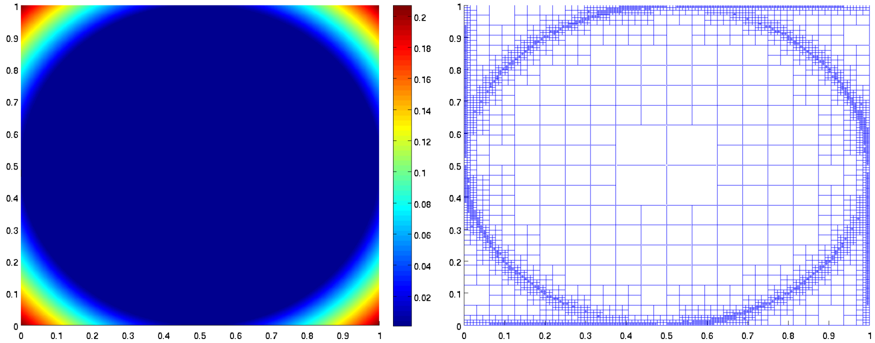


Figure: Steady-state vortex problem with $\epsilon = 1e-5$.

DPG for nonlinear problems

Given some linearization technique, we can measure

- the nonlinear residual

$$\|R(u)\|_E := \|B(u) - l\|_E = \|B(u) - l\|_{V'} = \left\|R_V^{-1} B(u) - l\right\|_V$$

- size of the linearized update Δu

$$\|\Delta u\|_E := \|B_u \Delta u\|_{V'} = \left\|R_V^{-1} B_u \Delta u\right\|_V$$

Test case: Burgers equation

$$\frac{\partial\left(u^2/2\right)}{\partial x} + \frac{\partial u}{\partial y} + \epsilon \Delta u = f$$

Burgers equation can be written with $\beta(u) = (u/2, 1)$

$$\nabla \cdot (\beta(u)u - \sigma) = f$$
$$\frac{1}{\epsilon}\sigma - \nabla u = 0.$$

i.e. nonlinear convection-diffusion.

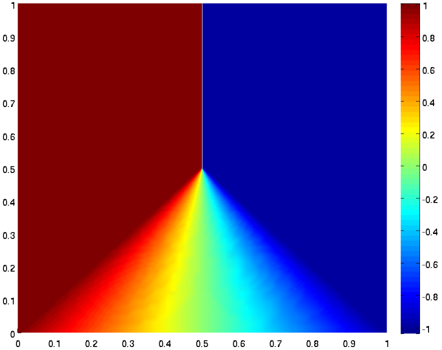


Figure: Shock solution for Burgers' equation, $\epsilon = 1e-4$.

Compressible Navier-Stokes equations

$$\nabla \cdot \begin{bmatrix} \rho u \\ \rho v \end{bmatrix} = 0$$

$$\nabla \cdot \left(\begin{bmatrix} \rho u^2 + p \\ \rho u v \end{bmatrix} - \boldsymbol{\sigma}_1 \right) = 0$$

$$\nabla \cdot \left(\begin{bmatrix} \rho u v \\ \rho v^2 + p \end{bmatrix} - \boldsymbol{\sigma}_2 \right) = 0$$

$$\nabla \cdot \left(\begin{bmatrix} ((\rho e) + p) u \\ ((\rho e) + p) v \end{bmatrix} - \boldsymbol{\sigma} \mathbf{u} + \vec{q} \right) = 0$$

$$\frac{1}{2\mu} \boldsymbol{\sigma} - \frac{\lambda}{4\mu(\mu + \lambda)} \mathrm{tr}(\boldsymbol{\sigma}) \mathbf{I} = \nabla \mathbf{u} - \mathrm{Re}\, \boldsymbol{\omega}$$

$$\frac{1}{\kappa} \vec{q} = \nabla T$$

$\boldsymbol{\sigma}$ is a Newtonian fluid $\sigma_{ij} = \mu(u_{i,j} + u_{j,i}) + \lambda u_{k,k}\delta_{ij}$.

$$\frac{1}{2}\left(\nabla U + \nabla^T U\right) = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{4\mu(\mu+\lambda)}\sigma_{kk}\delta_{ij}$$

We also have

$$\frac{1}{2}\left(\nabla U + \nabla^T U\right) = \nabla U - \boldsymbol{\omega}$$

Our final form is

$$\nabla U - \boldsymbol{\omega} = \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{4\mu(\mu+\lambda)}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I}.$$

Taking the antisymmetric part implicitly defines $\boldsymbol{\omega}$ to be the antisymmetric part of ∇u.

Extrapolation of test norms

Convection-diffusion:

$$\nabla \cdot (\beta u - \sigma) = f$$
$$\frac{1}{\epsilon}\sigma - \nabla u = 0.$$

Navier-Stokes:

$$\nabla \cdot (A_{\text{Euler}} U - A_{\text{visc}} \Sigma) = R_{\text{Euler}}(U, \Sigma)$$
$$E_{\text{visc}} \Sigma - \nabla U = R_{\text{visc}}(U, \Sigma)$$

where $R_{\text{Euler}}(U, \Sigma)$ and $R_{\text{visc}}(U, \Sigma)$ are the Eulerian/viscous residuals.

Convection-diffusion:

$$\|(v,\tau)\|_{V,K}^2 = \min\left\{\frac{\epsilon}{|K|},1\right\}\|v\|^2 + \epsilon\|\nabla v\|^2 + \|\beta\cdot\nabla v\|^2 + \|\nabla\cdot\tau\|^2 + \min\left\{\frac{1}{\epsilon},\frac{1}{|K|}\right\}\|\tau\|^2.$$

Navier-Stokes: let V and W be vectors of test functions v_i and τ_i.

$$\|(V,W)\|_{V,K}^2 = \min\left\{\frac{1}{\mathrm{Re}|\mathrm{K}|},1\right\}\|V\|^2 + \frac{1}{\mathrm{Re}}\left\|A_{\mathrm{visc}}^T\nabla V\right\|^2 + \left\|A_{\mathrm{Euler}}^T\nabla V\right\|^2 + \|\nabla\cdot W\|^2 + \min\left\{1,\frac{1}{\mathrm{Re}|\mathrm{K}|}\right\}\left\|E_{\mathrm{visc}}^T W\right\|^2.$$

Carter plate and Boundary conditions

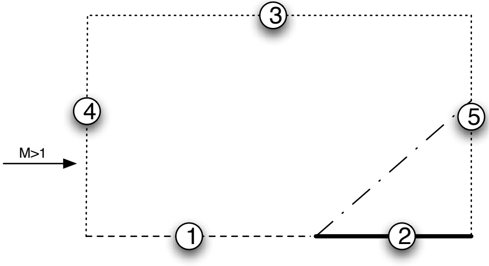
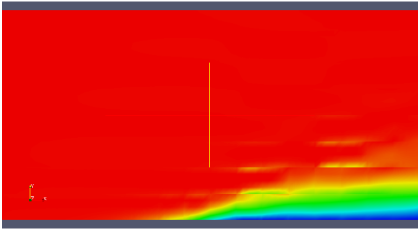


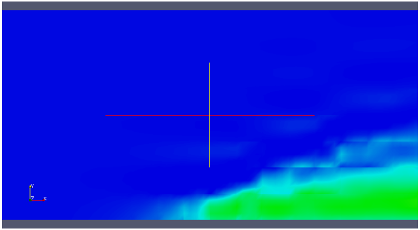
Figure: Carter flat plate problem.

- Inflow and stress boundary conditions, momentum flux boundary conditions.
- No outflow condition set.

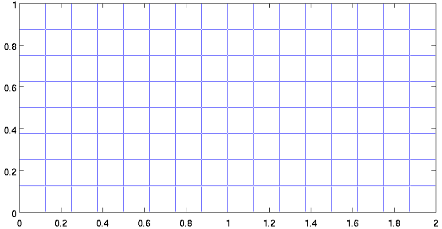
Refinement level 0



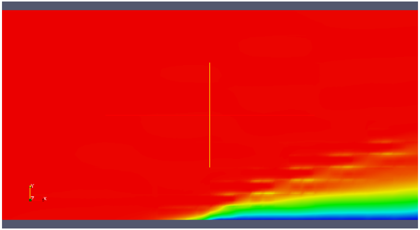
(a) u_1



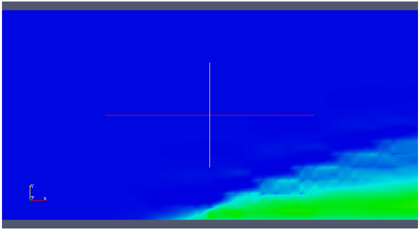
(b) T



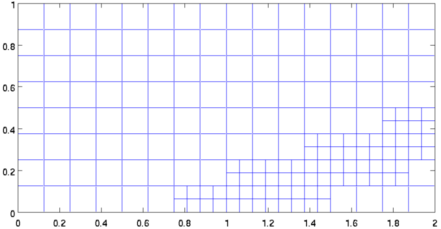
Refinement level 1



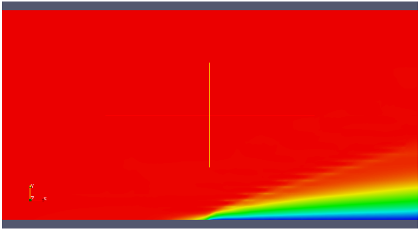
(a) u_1



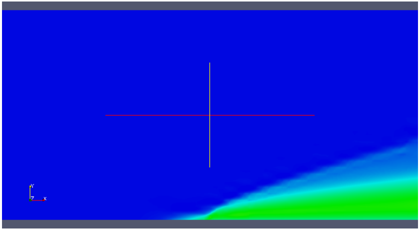
(b) T



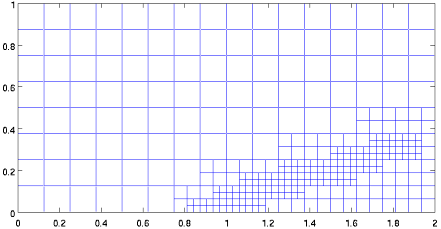
Refinement level 2



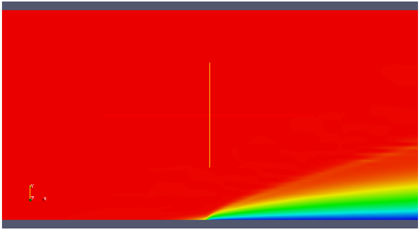
(a) u_1



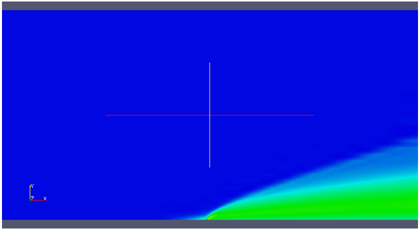
(b) T



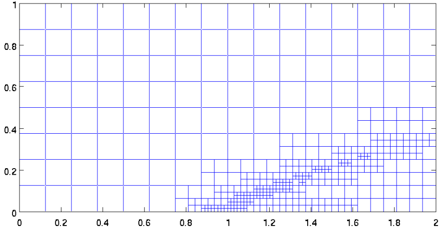
Refinement level 3



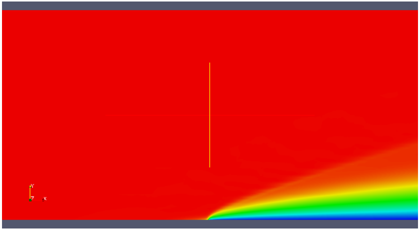
(a) u_1



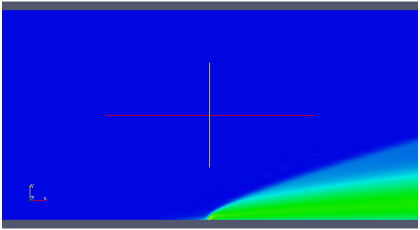
(b) T



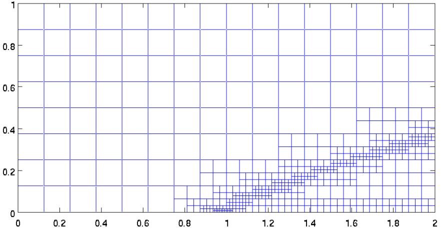
Refinement level 4



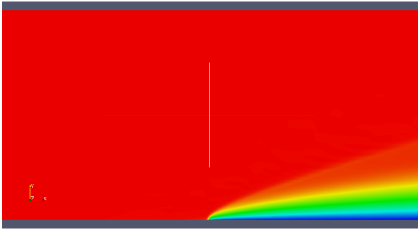
(a) u_1



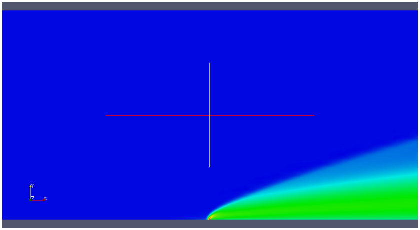
(b) T



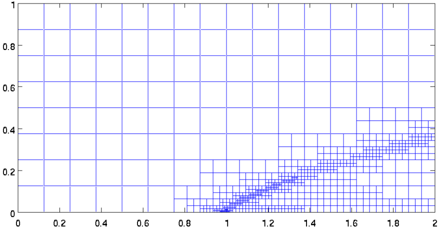
Refinement level 5



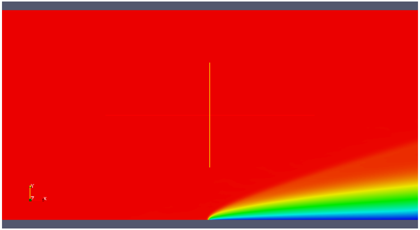
(a) u_1



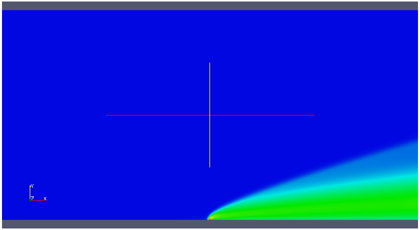
(b) T



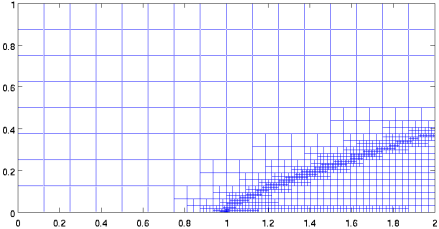
Refinement level 6



(a) u_1

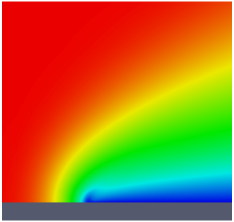


(b) T

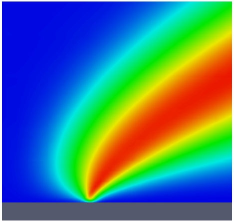


Zoomed solutions at the plate edge

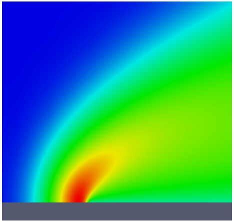
(a) ρ



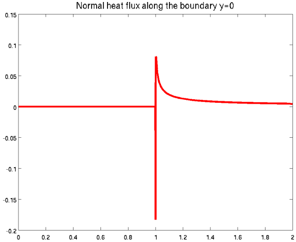
(b) u_1



(c) u_2



(d) T



(e) q_n

Proposed work: Area A

- **Completed: Prove robustness of DPG method for the scalar convection-diffusion problem.**
  We have introduced a test norm under which the DPG method robustly bounds the L^2 error in the field variables u and the scaled stress σ. Numerical results confirm the theoretical bounds given.
- **Proposed: Attempt analysis of the linearized Navier-Stokes system.**
  We hope to analyze the linearized Navier-Stokes equations to determine an optimal extrapolation of the test norm for the scalar convection-diffusion problem to systems.

Proposed work: Area B

- **Completed: Collaborative work with Nathan Roberts on the higher order parallel adaptive DPG code Camellia.**
  Numerical experiments have been done using the higher-order adaptive codebase Camellia, built upon the Trilinos library and designed by Nathan Roberts.
- **Proposed: Anisotropic refinements and *hp*-schemes.**
  The error representation function has been shown to yield an effective and natural residual with which to drive refinement, which we hope to generalize to yield anisotropic adaptive schemes. GET GRAPHICS FOR ANISOTROPY

Proposed: Distributed iterative static condensation.

$$Ku = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} u_{\text{flux}} \\ u_{\text{field}} \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} = l$$

where D has a block-diagonal structure. The system can be reduced to yield the condensed system

$$\left(A - BD^{-1}B^T\right) u_{\text{flux}} = f - BD^{-1}g$$

where D^{-1} can be inverted block-wise. For FE stiffness matrices under the Laplace equation, the Schur complement has reduced condition number of $O(h^{-1})$ as opposed to $O(h^{-2})$.

Proposed: a Nonlinear Hessian-based DPG method. Given a nonlinear variational problem $b(u, v) = l(v)$, linear in v but not in u, beginning with the *nonlinear* dual residual

$$J(u_h) = \frac{1}{2}\|B(u_h) - l\|_{V'}^2 := \frac{1}{2} \sup_{v \in V \setminus \{0\}} \frac{|b(u_h, v) - l(v)|^2}{\|v\|_V^2}.$$

produces a Hessian-based DPG method, which solves

$$b_u(\Delta u, v) + b''(\Delta u, \delta u, v_{R(u)}) = l(v) - b(u, v) = r(u, v),$$

and aims to minimize the nonlinear dual residual instead of the linearized problem residual.

Proposed work: Area C

- **Completed: convection-dominated diffusion, Burgers, and a model problem for Navier-Stokes.**
- **Proposed: Range of Reynolds/Mach numbers, ramp problem, Gaussian bump.**
- **Proposed: regularized Euler.** Graphics?