A Discontinuous Petrov-Galerkin method for compressible flow problems

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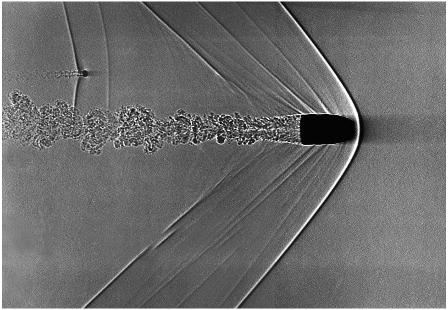
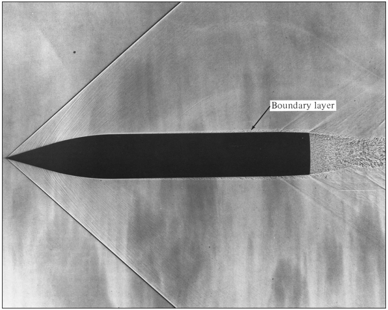
Phenomena in compressible flow



(a) Shock wave



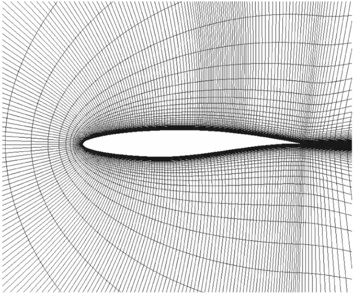
(b) Boundary layer

Compressible flow plays an important role in the aerospace and energy industries - transonic and supersonic aircraft, combustion engines, etc.

Compressible Navier-Stokes equations

Numerical difficulties:

- Resolving solution features (sharp, localized viscous-scale phenomena)
  - Shocks
  - Boundary layers - resolution needed for drag/load
  - Turbulence (non-localized)
- Nonlinear convergence and uniqueness of solutions
- Stability of numerical schemes
  - Coarse/adaptive grids
  - Higher order



Idea: begin first with the model problem of convection-diffusion.

Robustness: convection-diffusion as a model problem

$$\nabla \cdot (\beta u) - \epsilon \Delta u = f, \quad \text{on } \Omega \in \mathbb{R}^3$$

In 1D: $\beta u' - \epsilon u'' = f$. Standard continuous Galerkin variational formulation: solve

$$b(u,v) = \ell(v), \quad u, v \in H_0^1([0,1])$$

where

$$b(u,v) = \int_\Omega -\beta u v' + \epsilon u' v'$$

$$\ell(v) = \int_\Omega f v$$

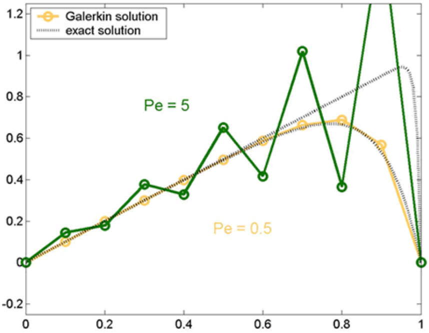


Figure: Solution for $f = 1$. Oscillations in the standard Galerkin method for large Peclet numbers $\mathrm{Pe} := \frac{h}{\epsilon}$.

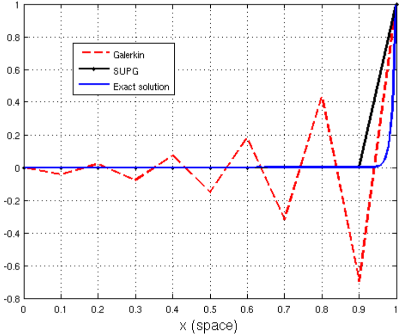
Streamline-upwind Petrov-Galerkin (SUPG)

SUPG solves $b_{\rm SUPG}(u,v) = l_{\rm SUPG}(v)$, where

$$b_{\rm SUPG}(u,v) = b(u,v) + \sum_K \int_K \tau \left(L_{\rm adv} v\right) Lu$$

$$l_{\rm SUPG}(v) = \ell(v) + \sum_K \int_K \tau \left(L_{\rm adv} v\right) f.$$

- $Lu = \nabla \cdot (\beta u) - \epsilon \triangle u$, $L_{\rm adv} u = \nabla \cdot (\beta u)$, and τ is a parameter.
- Effective for $f \neq 0$.
- *Residual-based* stabilization.



Can be interpreted as a *Petrov-Galerkin* method,

$$b(u, \tilde{v}_i) = \ell(\tilde{v}_i), \quad \forall i = 1, \ldots, N-1,$$

where the SUPG test function $\tilde{v}_i$ is defined elementwise as[1]

$$\tilde{v}_i = \phi_i(x) + \tau L_{\mathrm{adv}} \phi_i.$$

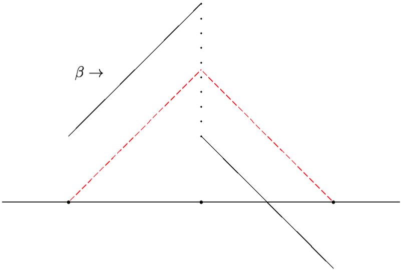


Figure: SUPG test function v_i.

[1] A. Brooks and T. Hughes. Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. *Comp. Meth. Appl. Mech. Engr*, 32:199–259, 1982

DPG: a minimum residual method via optimal testing

Given a trial space U and Hilbert test space V,

$$b(u,v) = \ell(v), \quad u \in U, \quad v \in V$$

This is equivalent to the operator equation posed in V'

$$Bu = \ell$$

if we identify $B : U \to V'$ and $\ell \in V'$ such that

$$\begin{aligned} \langle Bu, v\rangle_V &:= b(u,v), \quad u \in U, v \in V, \\ \langle \ell, v\rangle_V &:= \ell(v), \quad v \in V. \end{aligned}$$

We seek to minimize the dual functional over $U_h \subset U$

$$J(u_h) = \frac{1}{2}\|Bu_h - \ell\|_{V'}^2 := \frac{1}{2}\sup_{v\in V\setminus\{0\}} \frac{|b(u_h,v) - \ell(v)|^2}{\|v\|_V^2}.$$

Let $R_V : V \to V'$ be the isometric Riesz map st.

$$\langle R_V v, \delta v\rangle_V := (v, \delta v)_V, \quad \forall \delta v \in V.$$

Then, our functional $J(u_h)$ is equal to

$$\min_{u_h \in U_h} J(u_h) = \frac{1}{2}\left\|Bu_h - \ell\right\|_{V'}^2 = \frac{1}{2}\left\|R_V^{-1}(Bu_h - \ell)\right\|_V^2.$$

First order optimality: Gâteaux derivative is zero in all directions $\delta u \in U_h$

$$\begin{aligned}\left(R_V^{-1}(Bu_h - \ell), R_V^{-1}B\delta u\right)_V = 0, \quad \forall \delta u \in U_h.\\ \to \left\langle (Bu_h - \ell), R_V^{-1}B\delta u\right\rangle = 0,\\ \to b(u_h, R_V^{-1}B\delta u) - \ell(R_V^{-1}B\delta u) = 0\end{aligned}$$

Summary: select test functions to minimize residuals

For $\delta u \in U_h$, define the optimal test function $v_{\delta u}$.

$$v_{\delta u} := R_V^{-1} B\delta u.$$

Then, the residual $J(u_h) = \frac{1}{2}\left\| Bu_h - \ell \right\|_{V'}^2$ is minimized by the solution of

$$b\left(u_h, v_{\delta u}\right) = \ell\left(v_{\delta u}\right), \quad \forall \delta u \in U_h.$$

Practical details of DPG

Computation of $v_{\delta u} := R_V^{-1} B\delta u$ is **global** and **infinite-dimensional**.

- By choosing a broken test space V and localizable norm $\|v\|_V^2 = \sum_K \|v\|_{V(K)}^2$, test functions can be determined locally.
- In practice, we use an enriched space $V_h \subset V$, where $\dim(V_h) > \dim(U_h)$ elementwise, and optimal test functions are approximated by computing $v_{\delta u} := R_{V_h}^{-1} B\delta u \in V_h$ through[2]

$$(v_{\delta u}, \delta v)_V = b(\delta u, \delta v), \quad \delta u \in U_h, \quad \forall \delta v \in V_h$$

Typically, if $U_h = \mathcal{P}^p(\mathbb{R}^n)$, $V_h = \mathcal{P}^{p+\triangle p}(\mathbb{R}^n)$, where $\triangle p \geq n$.[3]

[2] L. Demkowicz and J. Gopalakrishnan. A class of discontinuous Petrov-Galerkin methods. II. Optimal test functions. *Num. Meth. for Partial Diff. Eq*, 27:70–105, 2011

[3] J. Gopalakrishnan and W. Qiu. An analysis of the practical DPG method. Technical report, IMA, 2011. Submitted

Properties of DPG

DPG provides a symmetric positive-definite stiffness matrix. Let $\{\phi_j\}_{j=1}^m$ be a basis for U_h, and $\{v_i\}_{i=1}^n$ a basis for V_h, such that $n > m$. Then, for

$$
\begin{aligned}
B_{ji} &= b(\phi_j, v_i), \\
l_i &= \ell(v_i),
\end{aligned}
$$

DPG solves the discrete system for degrees of freedom u

$$
\left(B^T R_V^{-1} B\right) u = \left(B^T R_V^{-1}\right) l,
$$

where, under a localizable norm and discontinuous test functions, R_V^{-1} is block diagonal.

Properties of DPG

Additional properties of DPG include[4]

- DPG provides the best approximation in the energy norm

$$\|u\|_E = \|Bu\|_{V'} = \sup_{\|v\|_V=1} |b(u,v)| .$$

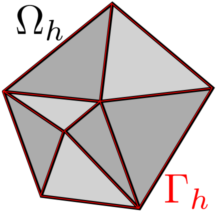
- The energy error is computable through the residual

$$\|u-u_h\|_E = \|B(u-u_h)\|_{V'} = \left\|R_V^{-1}(l - Bu_h)\right\|_V = \|e\|_V$$

where the error representation function e is defined through $(e,\delta v)_V = \ell(v) - b(u_h,\delta v)$ for all $\delta v \in V$.

[4] L. Demkowicz, J. Gopalakrishnan, and A. Niemi. A class of discontinuous Petrov-Galerkin methods. Part III: Adaptivity. *Appl. Numer. Math.*, 62(4):396–427, April 2012

Ultra-weak formulation



Given a first order system $Au = f$, we identify the **partition** Ω_h and **mesh skeleton** Γ_h.

The ultra-weak formulation for $Au = f$ on Ω_h is

$$b\left((u, \widehat{u}), v\right) := \sum_K \langle \widehat{u}, v \rangle_{\partial K} + (u, A_h^* v)_{\Omega_h} = (f, v)_{\Omega_h}.$$

Under proper assumptions, $\sum_K \langle \widehat{u}, v \rangle_{\partial K} = \langle \widehat{u}, [\![v]\!] \rangle_{\Gamma_h}$, with energy setting

$$u \in L^2(\Omega_h) \equiv L^2(\Omega), \quad v \in V = D(A_h^*), \quad \widehat{u} \in \gamma(D(A)),$$

where $D(A_h^*)$ is the broken graph space of the formal adjoint A_h^*, and $\gamma(D(A))$ the trace space of the graph space of operator A.[5]

[5] L. Demkowicz and J. Gopalakrishnan. Analysis of the DPG method for the Poisson equation. *SIAM J. Numer. Anal.*, 49(5):1788–1809, September 2011

The canonical "graph" test norm

Recall $\|u\|_E := \sup_{v\in V\setminus\{0\}} \frac{b(u,v)}{\|v\|_V}$. Under the ultra-weak formulation, the trial norm

$$\|(u,\widehat{u})\|_U := \|u\|^2_{L^2(\Omega)} + \|\widehat{u}\|^2$$

generates a test norm *equivalent* to

$$\|v\|_{V_{\rm opt}} := \|A_h^* v\|^2_{L^2(\Omega)} + \left(\sup_{\widehat{u}} \frac{\langle \widehat{u}, [\![v]\!]\rangle_{\Gamma_h}}{\|\widehat{u}\|}\right)^2.$$

This norm is not localizable, so we instead substitute the jump terms for an L^2 term, giving us the *graph norm*[6]

$$\|v\|_V := \|A_h^* v\|^2_{L^2(\Omega)} + \|v\|_{L^2}.$$

[6] T. Bui-Thanh, L. Demkowicz, and O. Ghattas. A unified discontinuous Petrov-Galerkin method and its analysis for Friedrichs' systems. *Submitted to SIAM J. Numer. Anal.*, 2011. Also ICES report 11-34, November 2011

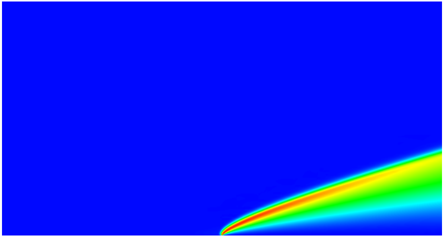
Goal: robust higher order adaptive methods

Our aim is to design an adaptive method for compressible laminar flows that is

- robust in Reynolds number
- stable for arbitrary elements

This proposal will outline

1. a robust DPG method for convection-diffusion,
2. extension to nonlinear problems and systems,
3. proposed work.



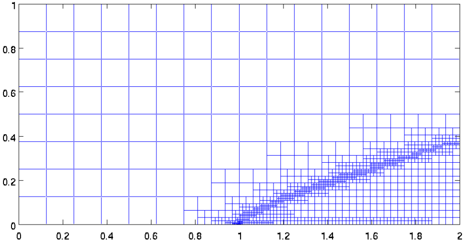


Figure: u_2 and $p = 2$ mesh for $\mathrm{Re} = 1000$.

Ultra-weak formulation for convection-diffusion

In first order form, the convection-diffusion equation is

$$A(u,\sigma) := \begin{bmatrix} \nabla\cdot(\beta u - \sigma) \\ \frac{1}{\epsilon}\sigma - \nabla u \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

The variational formulation is

$$b\left(\left(u,\sigma,\widehat{u},\widehat{f}_n\right),(v,\tau)\right) = (u, \nabla_h\cdot\tau - \beta\cdot\nabla_h v)_{\Omega_h} + \left(\sigma, \epsilon^{-1}\tau + \nabla_h v\right)_{\Omega_h} - \langle [\![\tau\cdot n]\!], \widehat{u}\rangle_{\Gamma_h} + \left\langle \widehat{f}_n, [\![v]\!]\right\rangle_{\Gamma_h},$$

where $\widehat{f}_n := \beta_n u - \sigma_n$ and $\left\langle \widehat{f}_n, [\![v]\!]\right\rangle_{\Gamma_h}$ is defined

$$\left\langle \widehat{f}_n, [\![v]\!]\right\rangle_{\Gamma_h} := \sum_K \int_{\partial K} \mathrm{sgn}(\vec{n})\, \widehat{f}_n v.$$

Graph norm under convection-diffusion

For convection-diffusion, the graph test norm is defined elementwise

$$\left\| (v, \tau) \right\|_{V(K)}^2 = \left\| \nabla \cdot \tau - \beta \cdot \nabla v \right\|_{L^2(K)}^2 + \left\| \epsilon^{-1} \tau + \nabla v \right\|_{L^2(K)}^2 + \left\| v \right\|_{L^2(K)}^2.$$

Problem with this test norm: approximability of test functions.

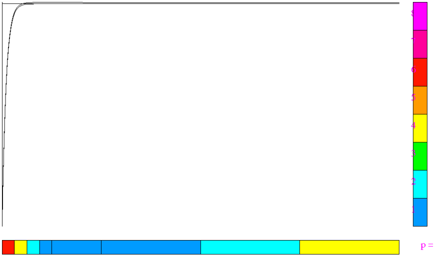


Figure: v and τ components of the 1D optimal test functions for flux $\widehat{f}_n$ on the *right-hand* side of a unit element for $\epsilon = 0.01$.

Determining an alternative test norm

Recall the convection-diffusion bilinear form

$$b\left(\left(u,\sigma,\widehat{u},\widehat{f}_n\right),(v,\tau)\right) = \left(u,\nabla\cdot\tau-\beta\cdot\nabla v\right)_{\Omega_h} + \left(\sigma,\epsilon^{-1}\tau+\nabla v\right)_{\Omega_h} - \left\langle\llbracket\tau\cdot n\rrbracket,\widehat{u}\right\rangle_{\Gamma_h} + \left\langle\widehat{f}_n,\llbracket v\rrbracket\right\rangle_{\Gamma_h},$$

We recover $\|u\|^2_{L^2(\Omega)}$ by choosing conforming (v,τ) satisfying the *adjoint equations*

$$\nabla\cdot\tau-\beta\cdot\nabla v = u$$
$$\frac{1}{\epsilon}\tau+\nabla v = 0$$

with boundary conditions s.t. $\left\langle\llbracket\tau\cdot n\rrbracket,\widehat{u}\right\rangle_{\Gamma}$ and $\left\langle\widehat{f}_n,\llbracket v\rrbracket\right\rangle_{\Gamma}$ vanish.

A robust bound

"Necessary" conditions for robustness — let $\mathbf{U} = \left(u, \sigma, \widehat{u}, \widehat{f}_n\right)$. Then, by choosing specific (v, τ) satisfying the adjoint equations,

$$\|u\|^2_{L^2(\Omega)} = b\left(\mathbf{U}, (v,\tau)\right) = \frac{b\left(\mathbf{U}, (v,\tau)\right)}{\|(v,\tau)\|_V} \|(v,\tau)\|_V \leq \|\mathbf{U}\|_E \|(v,\tau)\|_V$$

Let $\lesssim$ denote a robust bound - if $\|(v,\tau)\|_V \lesssim \|u\|_{L^2(\Omega)}$, then we have that

$$\|u\|_{L^2(\Omega)} \lesssim \|\mathbf{U}\|_E$$

Main idea: the test norm should measure adjoint solutions robustly.

Choice of inflow boundary condition

We impose the standard outflow wall boundary condition on u. For inflow boundary condition:

- The standard choice of inflow boundary condition: $u = u_0$.
- We impose the non-standard inflow condition: $\widehat{f}_n := \beta_n u - \sigma_n \approx \beta_n u_0$ on Γ_{in}.

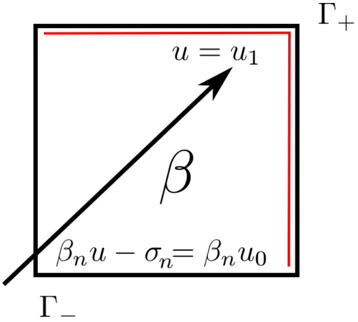
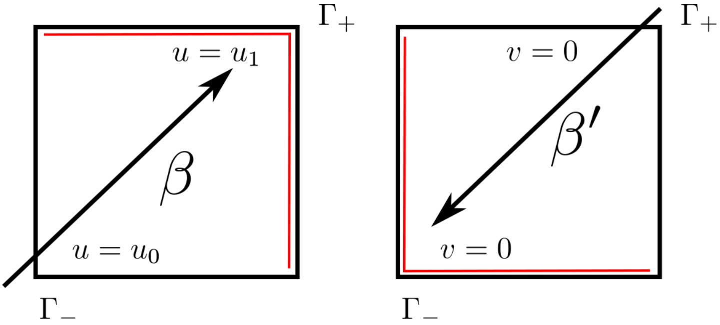


Figure: Non-standard inflow.

When $\sigma_n \approx 0$ near the inflow, condition on $\widehat{f}_n$ approximates condition on u.

Dirichlet inflow boundary condition

Standard choice of boundary condition: $u = u_0$ on inflow boundary $\Gamma_{\rm in}$, induces boundary layers in adjoint problems, $\|\beta \cdot \nabla v\|_{L^2} = O(\epsilon^{-1})$.

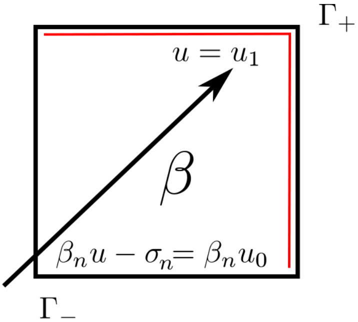


(a) Primal problem (b) Adjoint problem

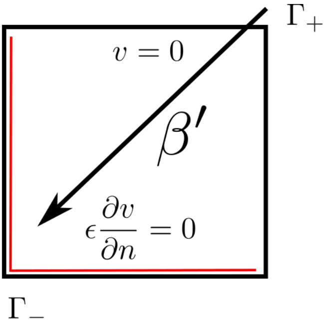
Figure: For the standard Dirichlet inflow condition, the solution to the adjoint problem can develop strong boundary layers at the outflow of the adjoint problem.

New inflow boundary condition on $\widehat{f}_n$

Non-standard choice of boundary condition: $\widehat{f}_n = \beta_n u_0$ on Γ_{in}, induces smoother adjoint problems, $\|\beta \cdot \nabla v\|_{L^2} = O(1)$.



(a) Primal problem

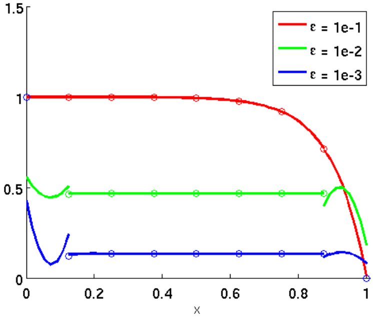


(b) Adjoint problem

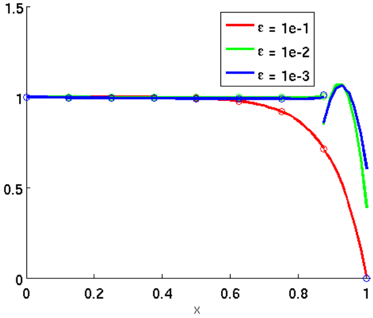
Figure: Under the new inflow condition, the wall-stop boundary condition is relaxed to a zero-stress condition at the outflow boundary of the adjoint problem.

Test norms and adjoint solutions

Intuition: the effectiveness of DPG under a test norm is governed by how a **specific test norm** measures the **solutions of the adjoint problem**.



(a) Dirichlet inflow



(b) “Convection” inflow

Figure: DPG solutions to convection-diffusion for both inflow conditions using an H^1 test norm.

Adjoint estimates, or "norm building blocks"

For solutions (v, τ) of the adjoint equations, the following quantities are robustly bounded from above by $\|u\|_{L^2(\Omega)}$.

$$\|v\|, \sqrt{\epsilon}\|\nabla v\|, \|\beta \cdot \nabla v\|$$
$$\|\nabla \cdot \tau\|, \frac{1}{\sqrt{\epsilon}}\|\tau\|.$$

We will construct a test norm through a combination of the above quantities, such that

- v and τ decoupled (no systems).
- Coefficients are of equal order after transforming to the unit element (no boundary layers).

Mesh-scaled test norms

Our test norm, as defined over a single element K, is now

$$\|\,(v,\tau)\,\|_{V,K}^2 = \min\left\{\frac{\epsilon}{|K|},1\right\}\|v\|^2 + \epsilon\|\nabla v\|^2 + \|\beta\cdot\nabla v\|^2 + \|\nabla\cdot\tau\|^2 + \min\left\{\frac{1}{\epsilon},\frac{1}{|K|}\right\}\|\tau\|^2.$$

which induces the proven *robust* bound[7]

$$\|u\|_{L^2(\Omega)} + \|\sigma\|_{L^2(\Omega)} + \epsilon\,\|\widehat{u}\| + \sqrt{\epsilon}\left\|\widehat{f}_n\right\| \lesssim \left\|\left(u,\sigma,\widehat{u},\widehat{f}_n\right)\right\|_E.$$

[7] J. Chan, N. Heuer, T. Bui Thanh, and L. Demkowicz. Robust DPG method for convection-diffusion problems II: natural inflow conditions. Technical Report 12-21, ICES, June 2012. Submitted

Erikkson-Johnson model problem

On domain $\Omega = [0,1]^2$, with $\beta = (1,0)^T$, $f = 0$ and boundary conditions

$$\widehat{\beta_n u - \sigma_n} = \widehat{f}_n = u_0, \quad \beta_n \leq 0$$
$$\widehat{u} = 0, \quad \beta_n > 0$$

All numerical experiments done using Camellia codebase.[8]

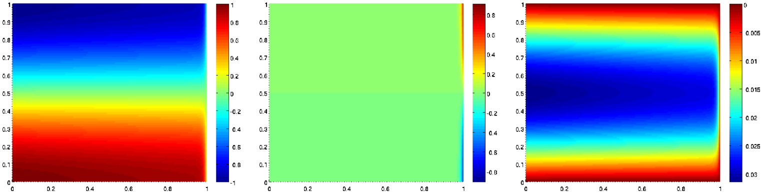


Figure: Exact solution for u, σ_x, and σ_y for $\epsilon = .01$, $C_1 = 1$, $C_n = 0$, $n \neq 1$

[8] N. Roberts, D. Ridzal, P. Bochev, and L. Demkowicz. A Toolbox for a Class of Discontinuous Petrov-Galerkin Methods Using Trilinos. Technical Report SAND2011-6678, Sandia National Laboratories, 2011

Error rates

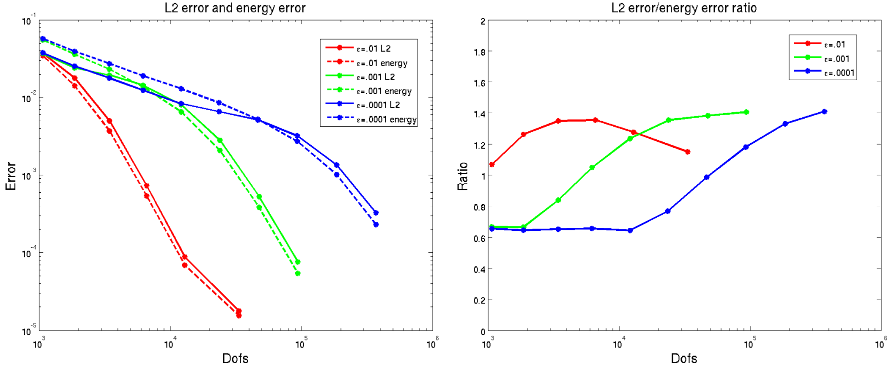
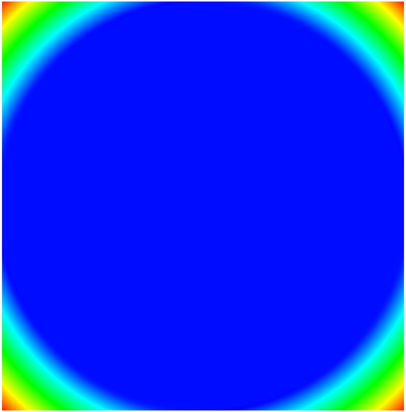


Figure: L^2/energy errors and their ratio for $\epsilon = .01, .001, .0001$.

Regularization

For $\beta = (-y, x)^T$ on $\Omega = [-1, 1]^2$. Ill posed in the convection setting. Similar tests have been done with discontinuous data.



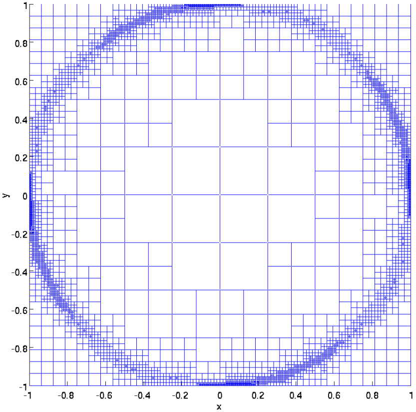


Figure: Steady vortex problem with $\epsilon = 1e-4$, using a variant of the test norm.

DPG for nonlinear problems

Given some linearization technique (typically Newton-Raphson and pseudo-timestepping linearization), we measure

- size of the linearized update Δu

$$\|\Delta u\|_E := \|B_u \Delta u\|_{V'} = \left\|R_V^{-1} B_u \Delta u\right\|_V$$

- the nonlinear residual

$$\|R(u)\|_E := \|B(u) - \ell\|_{V'} = \left\|R_V^{-1}\left(B(u) - \ell\right)\right\|_V$$

Preliminary experiments were done in 1D[9] and space-time.[10]

[9] J. Chan, L. Demkowicz, R. Moser, and N. Roberts. A New Discontinuous Petrov-Galerkin Method with Optimal Test Functions. Part V: Solution of 1D Burgers' and Navier-Stokes Equations. Technical Report 10-25, ICES, June 2010

[10] J. Chan, L. Demkowicz, and M. Shashkov. Space-time DPG for shock problems. Technical Report LA-UR 11-05511, LANL, September 2011

2D test case: Burgers equation

$$\frac{\partial\left(u^2/2\right)}{\partial x}+\frac{\partial u}{\partial y}+\epsilon\Delta u=f$$

Burgers equation can be written with $\beta(u)=(u/2,1)$

$$\nabla\cdot(\beta(u)u-\sigma)=f$$
$$\frac{1}{\epsilon}\sigma-\nabla u=0.$$

i.e. nonlinear convection-diffusion on domain $[0,1]^2$.

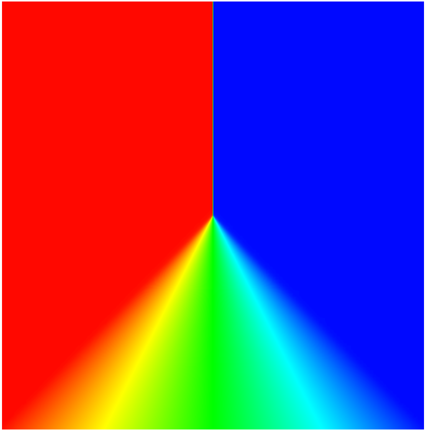


Figure: Shock solution for Burgers' equation, $\epsilon=1e-4$, using Newton-Raphson.

Adaptivity begins with a cubic 4×4 mesh.

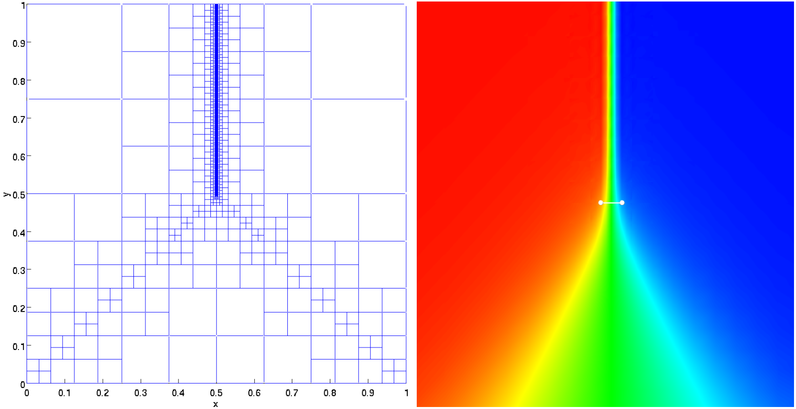


Figure: Adaptive mesh after 9 refinements, and zoom view at point (.5,.5) with shock formation and $1e-3$ width line for reference.

2D Compressible Navier-Stokes equations (ideal gas)

Given density ρ, velocities $\mathbf{u} = (u_1, u_2)$ and temperature T,

$$\nabla \cdot \begin{bmatrix} \rho u_1 \\ \rho u_2 \end{bmatrix} = 0$$

$$\nabla \cdot \left(\begin{bmatrix} \rho u_1^2 + p \\ \rho u_1 u_2 \end{bmatrix} - \boldsymbol{\sigma}_1 \right) = 0$$

$$\nabla \cdot \left(\begin{bmatrix} \rho u_1 u_2 \\ \rho u_2^2 + p \end{bmatrix} - \boldsymbol{\sigma}_2 \right) = 0$$

$$\nabla \cdot \left(\begin{bmatrix} ((\rho e) + p) u_1 \\ ((\rho e) + p) u_2 \end{bmatrix} - \boldsymbol{\sigma}\mathbf{u} + \vec{q} \right) = 0$$

$$\frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{4\mu(\mu+\lambda)}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} = \nabla \mathbf{u} - \mathrm{Re}\,\boldsymbol{\omega}$$

$$\frac{1}{\kappa}\vec{q} = \nabla T$$

Stress law

$\boldsymbol{\sigma}$ for a Newtonian fluid: $\sigma_{ij} = \mu(u_{i,j} + u_{j,i}) + \lambda u_{k,k}\delta_{ij}$.

$$\frac{1}{2}\left(\nabla U + \nabla^T U\right) = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{4\mu(\mu+\lambda)}\sigma_{kk}\delta_{ij}$$

The symmetric part of the gradient is

$$\frac{1}{2}\left(\nabla U + \nabla^T U\right) = \nabla U - \boldsymbol{\omega}$$

Our final form is

$$\nabla U - \boldsymbol{\omega} = \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{4\mu(\mu+\lambda)}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I}.$$

By enforcing strong symmetry of $\boldsymbol{\sigma}$, taking the antisymmetric part implicitly defines $\boldsymbol{\omega}$ to be the antisymmetric part of ∇u.

Extrapolation of test norms

Convection-diffusion:

$$\begin{aligned} \nabla \cdot (\beta u - \sigma) &= f \\ \frac{1}{\epsilon}\sigma - \nabla u &= 0. \end{aligned}$$

Navier-Stokes: defining vector variables $U = \{\rho, u_1, u_2, T\}$ and $\Sigma = \{\boldsymbol{\sigma}, \mathbf{q}, \omega\}$,

$$\begin{aligned} \nabla \cdot (A_{\text{invisc}} U - A_{\text{visc}} \Sigma) &= R_{\text{conserv}}(U, \Sigma) \\ E_{\text{visc}} \Sigma - \nabla U &= R_{\text{constit}}(U, \Sigma) \end{aligned}$$

where $R_{\text{conserv}}(U, \Sigma)$ and $R_{\text{constit}}(U, \Sigma)$ are the conservation/constitutive residuals.

Test norms over one element

Convection-diffusion:

$$\|\,(v,\tau)\,\|_{V,K}^2 = \min\left\{\frac{\epsilon}{|K|},1\right\}\|v\|^2 + \|\beta\cdot\nabla v\|^2 + \epsilon\|\nabla v\|^2 + \|\nabla\cdot\tau\|^2 + \min\left\{\frac{1}{\epsilon},\frac{1}{|K|}\right\}\|\tau\|^2.$$

Navier-Stokes: let V and W be vectors of test functions v_i and τ_i.

$$\|\,(V,W)\,\|_{V,K}^2 = \min\left\{\frac{1}{\mathrm{Re}|\mathrm{K}|},1\right\}\|V\|^2 + \left\|A_{\mathrm{invisc}}^T\nabla V\right\|^2 + \frac{1}{\mathrm{Re}}\left\|A_{\mathrm{visc}}^T\nabla V\right\|^2 + \|\nabla\cdot W\|^2 + \min\left\{1,\frac{1}{\mathrm{Re}|\mathrm{K}|}\right\}\left\|E_{\mathrm{visc}}^T W\right\|^2.$$

Carter's flat plate problem

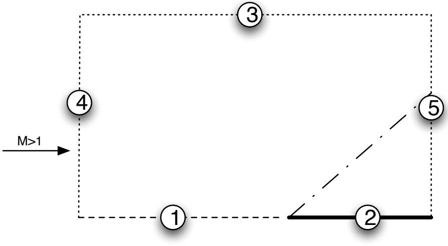
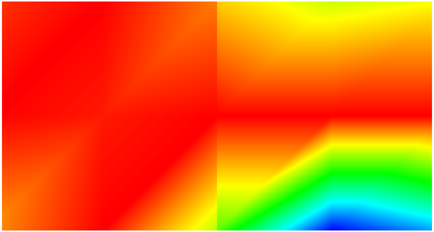


Figure: Carter flat plate problem on domain $[0,2] \times [0,1]$. Plate begins at $x = 1$, $\mathrm{Re} = 1000$.

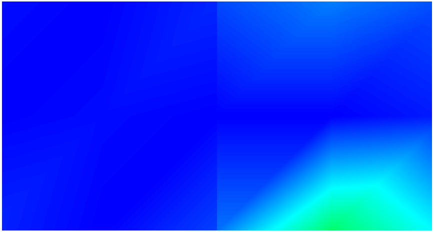
1. Symmetry boundary conditions.
2. Prescribed temperature and wall stagnation conditions.
3. Symmetry boundary conditions.
4. Inflow: conserved quantities specified using far-field values.
5. No outflow condition set.

Stress/heat flux boundary conditions are set in terms of the momentum and energy fluxes.

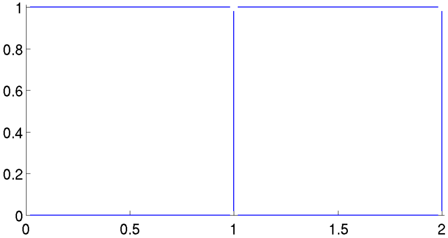
Refinement level 0



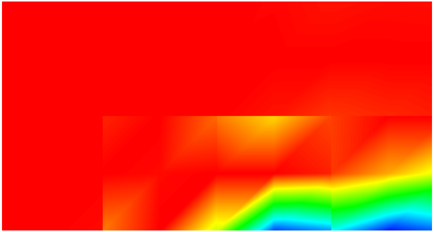
(a) u_1



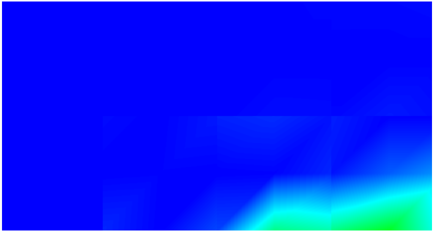
(b) T



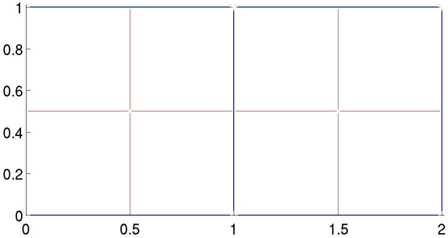
Refinement level 1



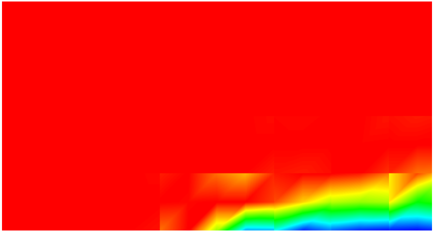
(a) u_1



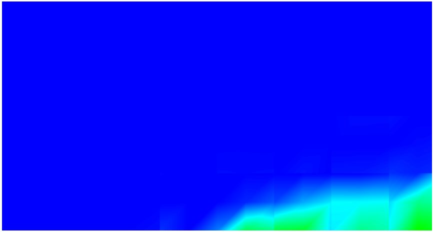
(b) T



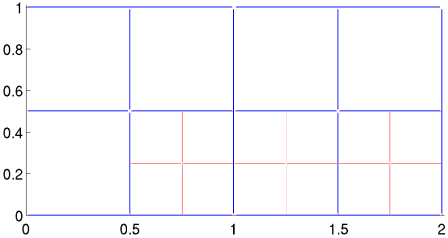
Refinement level 2



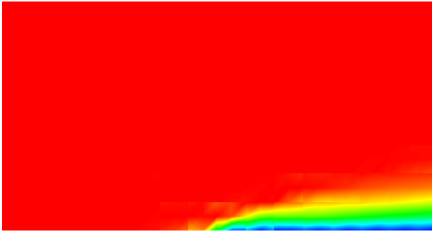
(a) u_1



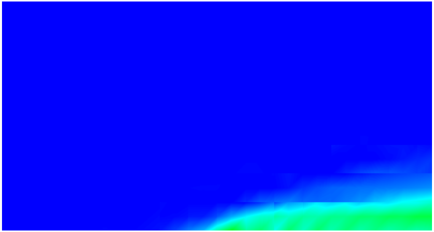
(b) T



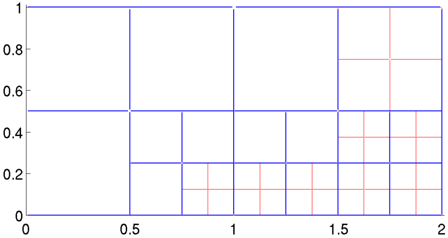
Refinement level 3



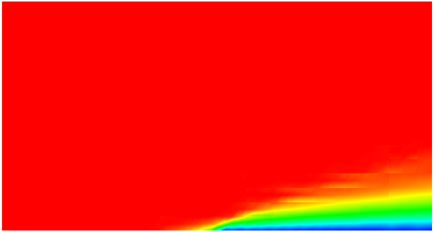
(a) u_1



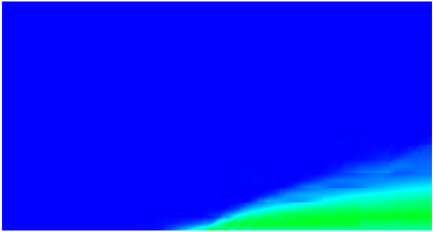
(b) T



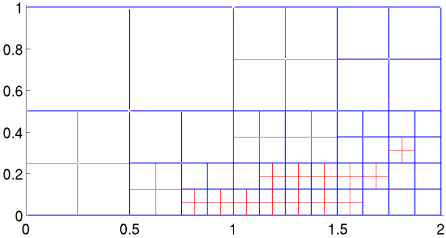
Refinement level 4



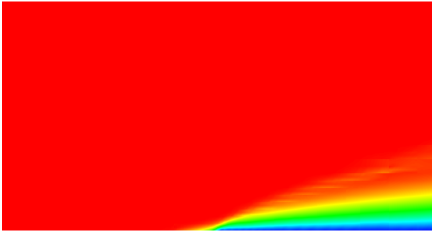
(a) u_1



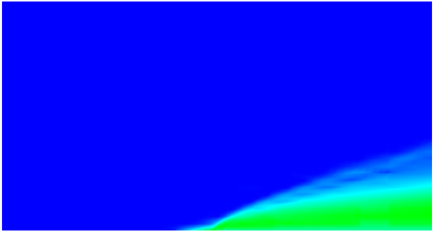
(b) T



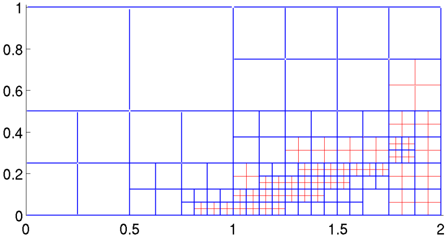
Refinement level 5



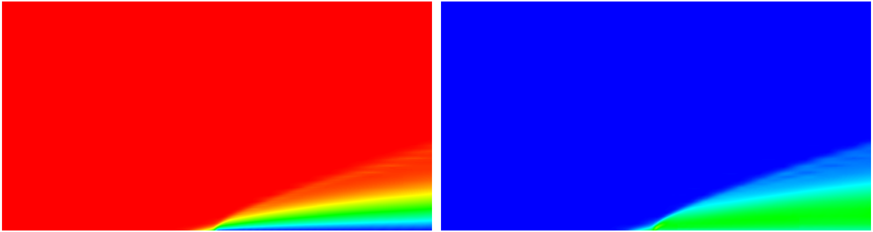
(a) u_1



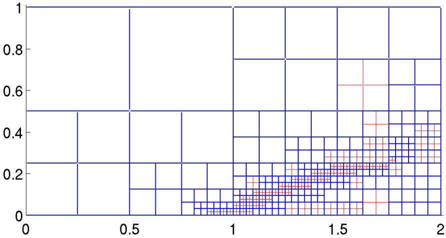
(b) T



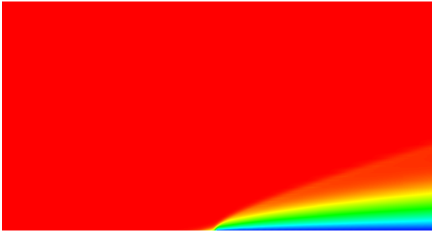
Refinement level 6



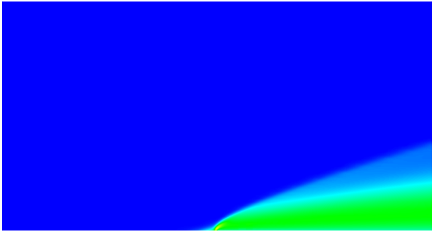
(a) u_1 (b) T



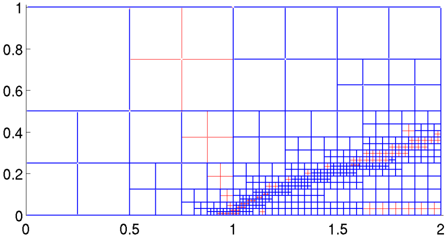
Refinement level 7



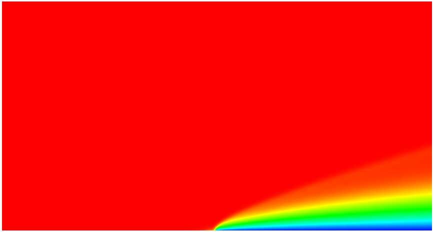
(a) u_1



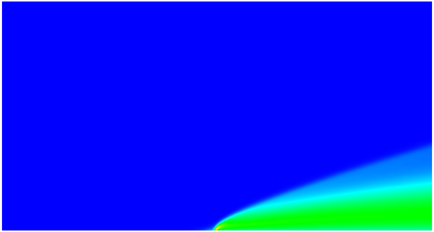
(b) T



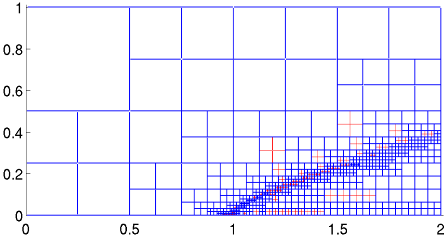
Refinement level 8



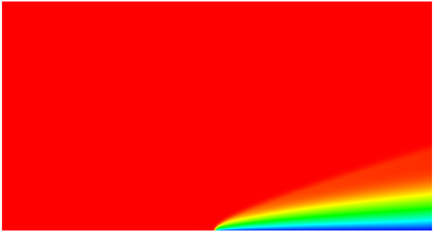
(a) u_1



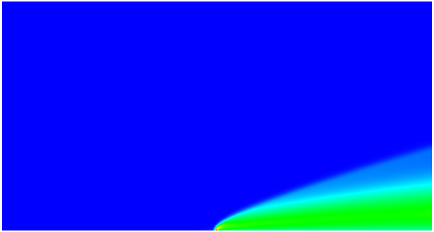
(b) T



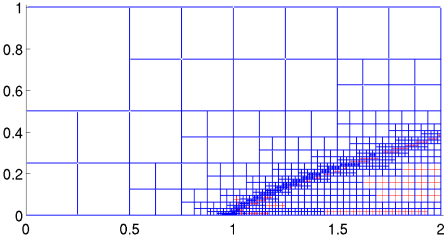
Refinement level 9



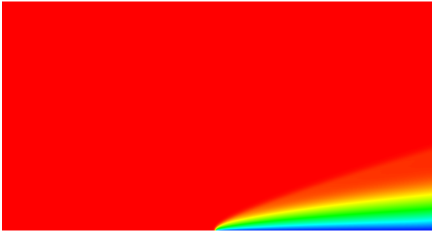
(a) u_1



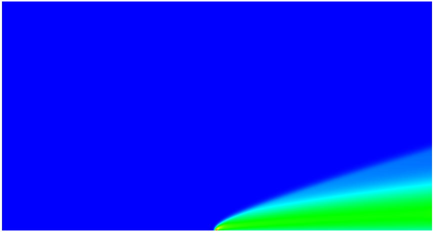
(b) T



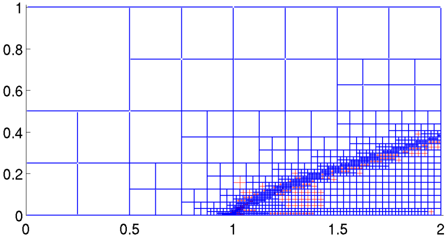
Refinement level 10



(a) u_1

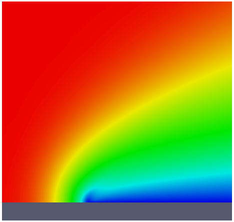


(b) T

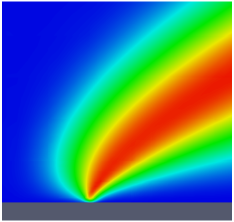


Zoomed solutions at plate/stagnation point

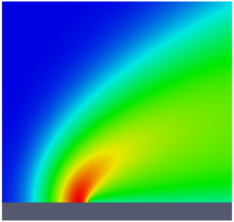
(a) ρ



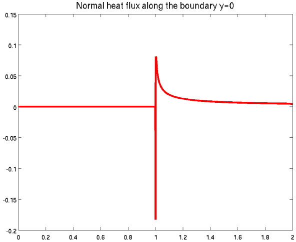
(b) u_1



(c) u_2



(d) T



(e) q_n

Proposed work: Area A

- **Completed: Prove robustness of DPG method for the scalar convection-diffusion problem.**
  We have introduced a test norm under which the DPG method robustly bounds the L^2 error in the field variables u and the scaled stress σ. Numerical results confirm the theoretical bounds given.
- **Proposed: Attempt analysis of the linearized Navier-Stokes system.**
  We hope to analyze the linearized Navier-Stokes equations to determine an optimal extrapolation of the test norm for the scalar convection-diffusion problem to systems.

Proposed work: Area B

- **Completed: Collaborative work with Nathan Roberts on the higher order parallel adaptive DPG code Camellia.**
  Numerical experiments use the higher-order adaptive codebase Camellia, built by Nathan Roberts upon the Trilinos library.
- **Proposed: Anisotropic refinements and *hp*-schemes.**
  The error representation function drives refinement effectively, and we hope to generalize its use to anisotropic adaptive schemes.

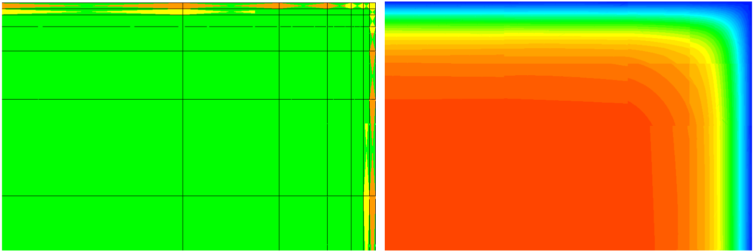


Figure: Anisotropic mesh for a convection-diffusion boundary layer.

Proposed: Distributed iterative static condensation.

$$Ku = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} u_{\text{flux}} \\ u_{\text{field}} \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} = l$$

where D has a block-diagonal structure. The system can be reduced to yield the condensed system

$$\left(A - BD^{-1}B^T\right) u_{\text{flux}} = f - BD^{-1}g$$

where D^{-1} can be inverted block-wise. For FE stiffness matrices under the Laplace equation, the Schur complement has reduced condition number of $O(h^{-1})$ as opposed to $O(h^{-2})$.[11]

[11] L. Mansfield. On the Conjugate Gradient Solution of the Schur Complement System Obtained from Domain Decomposition. *SIAM Journal on Numerical Analysis*, 27(6):pp. 1612–1620, 1990

Proposed: a Nonlinear Hessian-based DPG method. Given a nonlinear variational problem $b(u,v) = \ell(v)$, linear in v but not in u, beginning with the *nonlinear* dual residual

$$J(u_h) = \frac{1}{2}\|B(u_h) - \ell\|_{V'}^2 := \frac{1}{2} \sup_{v \in V\setminus\{0\}} \frac{|b(u_h,v) - \ell(v)|^2}{\|v\|_V^2}.$$

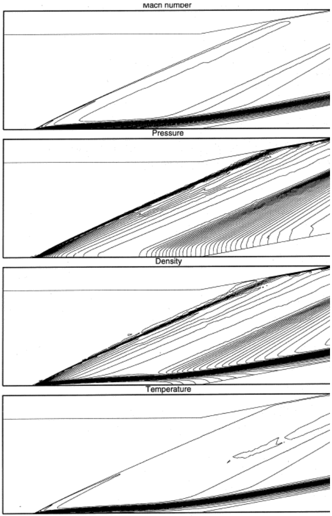
produces a Hessian-based DPG method, which solves

$$b_u(\Delta u, v) + b''(\Delta u, \delta u, v_{R(u)}) = \ell(v) - b(u,v) = r(u,v),$$

and aims to minimize the nonlinear dual residual instead of the linearized problem residual.

Proposed work: Area C

- **Completed: convection-dominated diffusion, Burgers, and a model problem for Navier-Stokes.**
- **Proposed: Range of Reynolds/Mach numbers.**
- **Proposed: Flow over a bump, Holden's ramp problem (shock-boundary layer interaction).**
- **Proposed (time permitting): airfoils, regularized Euler.**



Thank you!

Questions?

Extended literature review

- Finite difference/volume methods, upwind stencils, monotonicity, ENO/WENO.
- Artificial diffusion, flux limiters/discontinuity-capturing operators.
- DG fluxes can be interpreted as residual-based stabilizations similar to SUPG.[12]
- Hybridized DG (HDG) methods introduce independent unknowns for numerical traces. Fluxes are defined in terms of DG fluxes and the numerical trace.[13]

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[12] F. Brezzi, B. Cockburn, L.D. Marini, and E. Süli. Stabilization mechanisms in discontinuous Galerkin finite element methods. *Computer Methods in Applied Mechanics and Engineering*, 195(25–28):3293 – 3310, 2006

[13] B. Cockburn, J. Gopalakrishnan, and R. Lazarov. Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems. *SIAM J. Numer. Anal.*, 47(2):1319–1365, February 2009

Specifics of the ultra-weak formulation

Given a first order system $Au = f$, multiply by test function v and integrate

$$(Au, v) = \langle \gamma (Au), v \rangle + (u, A_h^* v) = (f, v)$$

If $u \in L^2(\Omega)$, the trace of u is undefined, so we identify boundary terms $\langle \gamma (Au), v \rangle_{\Gamma_h} = \langle \widehat{u}, v \rangle_{\Gamma_h}$ as unknowns $\widehat{u}$ on Γ_h. For convection-diffusion,

$$\begin{aligned} b\left(\left(u, \sigma, \widehat{u}, \widehat{f}_n\right), (v, \tau)\right) = (u, \nabla_h \cdot \tau - \beta \cdot \nabla_h v)_{\Omega_h} + \left(\sigma, \epsilon^{-1}\tau + \nabla_h v\right)_{\Omega_h} \\ - \langle [\![\tau \cdot n]\!], \widehat{u} \rangle_{\Gamma_h} + \left\langle \widehat{f}_n, [\![v]\!] \right\rangle_{\Gamma_h}, \end{aligned}$$

where

$$\widehat{f}_n := \beta_n u - \sigma_n \in H^{-1/2}(\Gamma_h), \quad \widehat{u} \in H^{1/2}(\Gamma_h)$$

with the minimum energy extension norm on Γ_h. We note that $H^{\pm 1/2}(\Gamma_h)$ are *closed* subspaces of $\prod_K H^{\pm 1/2}(\partial K)$.

Inviscid equations

Issue: consider pure convection, $\nabla \cdot \beta u = f$. The ultra-weak variational formulation is

$$\left\langle \widehat{f}_n, v \right\rangle - (u, \beta \cdot \nabla v) = (f, v),$$

where $\widehat{f}_n := \beta_n u$. When $\beta_n = 0$, v has only a streamline derivative, and $\widehat{f}_n$ becomes an ill-defined trace in the cross-stream direction. For hyperbolic *systems*, this issue manifests as *sonic lines*.

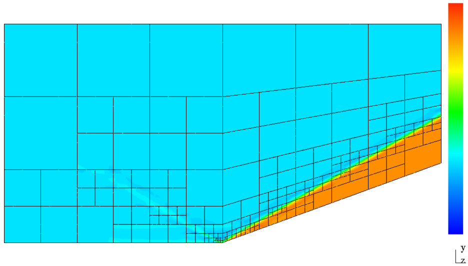

Figure: Sonic lines ($u_n - c = 0$) appear for linearized Euler.

F. Brezzi, B. Cockburn, L.D. Marini, and E. Süli.
Stabilization mechanisms in discontinuous Galerkin finite element methods.
Computer Methods in Applied Mechanics and Engineering, 195(25–28):3293 – 3310, 2006.

A. Brooks and T. Hughes.
Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations.
Comp. Meth. Appl. Mech. Engr, 32:199–259, 1982.

T. Bui-Thanh, L. Demkowicz, and O. Ghattas.
A unified discontinuous Petrov-Galerkin method and its analysis for Friedrichs' systems.
Submitted to SIAM J. Numer. Anal., 2011.
Also ICES report 11-34, November 2011.

J. Chan, L. Demkowicz, R. Moser, and N. Roberts.
A New Discontinuous Petrov-Galerkin Method with Optimal Test Functions. Part V: Solution of 1D Burgers' and Navier-Stokes Equations.
Technical Report 10-25, ICES, June 2010.

J. Chan, L. Demkowicz, and M. Shashkov.
Space-time DPG for shock problems.
Technical Report LA-UR 11-05511, LANL, September 2011.

J. Chan, N. Heuer, T. Bui Thanh, and L. Demkowicz.
Robust DPG method for convection-diffusion problems II: natural inflow conditions.
Technical Report 12-21, ICES, June 2012.
Submitted.

B. Cockburn, J. Gopalakrishnan, and R. Lazarov.
Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems.
SIAM J. Numer. Anal., 47(2):1319–1365, February 2009.

L. Demkowicz and J. Gopalakrishnan.
Analysis of the DPG method for the Poisson equation.
SIAM J. Numer. Anal., 49(5):1788–1809, September 2011.

L. Demkowicz and J. Gopalakrishnan.
A class of discontinuous Petrov-Galerkin methods. II. Optimal test functions.
Num. Meth. for Partial Diff. Eq, 27:70–105, 2011.

L. Demkowicz, J. Gopalakrishnan, and A. Niemi.
A class of discontinuous Petrov-Galerkin methods. Part III: Adaptivity.
Appl. Numer. Math., 62(4):396–427, April 2012.

J. Gopalakrishnan and W. Qiu.
An analysis of the practical DPG method.
Technical report, IMA, 2011.
Submitted.

L. Mansfield.
On the Conjugate Gradient Solution of the Schur Complement System Obtained from Domain Decomposition.
SIAM Journal on Numerical Analysis, 27(6):pp. 1612–1620, 1990.

N. Roberts, D. Ridzal, P. Bochev, and L. Demkowicz.

A Toolbox for a Class of Discontinuous Petrov-Galerkin Methods Using Trilinos.
Technical Report SAND2011-6678, Sandia National Laboratories, 2011.