A higher-order adaptive DPG method for compressible flow problems

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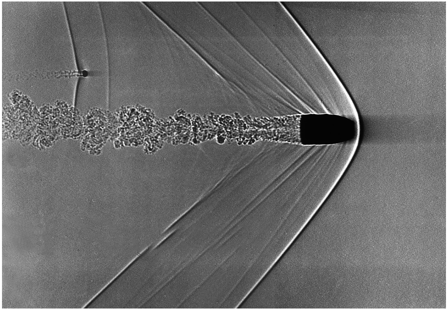
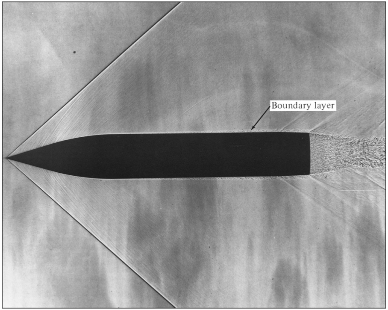
April 19, 2013

Phenomena in compressible flow



(a) Shock wave



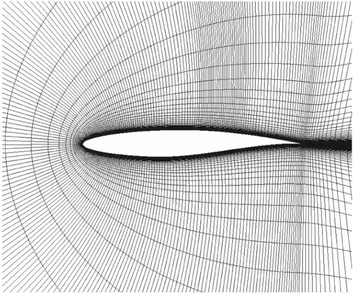
(b) Boundary layer

Compressible flow plays an important role in the aerospace and energy industries - transonic and supersonic aircraft, combustion engines, etc.

Compressible Navier-Stokes equations

Numerical difficulties:

- Resolving solution features (sharp, localized viscous-scale phenomena)
  - Shocks
  - Boundary layers - resolution needed for drag/load
  - Turbulence
- Nonlinear convergence and uniqueness of solutions
- Stability of numerical schemes
  - Coarse/adaptive grids
  - Higher order



Idea: begin first with the model problem of convection-diffusion.

Robustness: convection-diffusion as a model problem

$$\nabla \cdot (\beta u) - \epsilon \Delta u = f, \quad \text{on } \Omega \in \mathbb{R}^3$$

In 1D: $\beta u' - \epsilon u'' = f$. Standard continuous Galerkin variational formulation: solve

$$b(u,v) = \ell(v), \quad u, v \in H_0^1([0,1])$$

where

$$b(u,v) = \int_\Omega -\beta u v' + \epsilon u' v'$$

$$\ell(v) = \int_\Omega f v$$

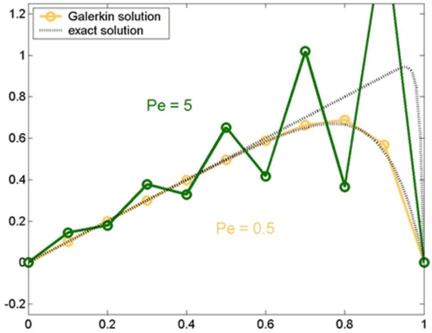


Figure: Solution for $f = 1$. Oscillations in the standard Galerkin method for large Peclet numbers $\mathrm{Pe} := \frac{h}{\epsilon}$.

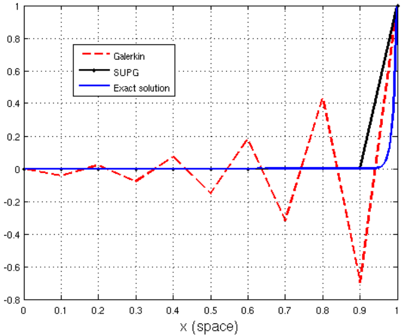
Streamline-upwind Petrov-Galerkin (SUPG)

SUPG solves $b_{\rm SUPG}(u,v) = l_{\rm SUPG}(v)$, where

$$b_{\rm SUPG}(u,v) = b(u,v) + \sum_K \int_K \tau \left(L_{\rm adv} v\right) Lu$$

$$l_{\rm SUPG}(v) = \ell(v) + \sum_K \int_K \tau \left(L_{\rm adv} v\right) f.$$

- $Lu = \nabla \cdot (\beta u) - \epsilon \triangle u$, $L_{\rm adv} u = \nabla \cdot (\beta u)$, and τ is a parameter.
- Restores discrete coercivity.
- Effective for $f \neq 0$.
- Residual-based stabilization.



Can be "interpreted" as a Petrov-Galerkin method,

$$b(u, \tilde{v}_i) = \ell(\tilde{v}_i), \quad \forall i = 1, \ldots, N-1,$$

where the SUPG test function $\tilde{v}_i$ is defined elementwise as[1]

$$\tilde{v}_i = \phi_i(x) + \tau L_{\mathrm{adv}} \phi_i.$$

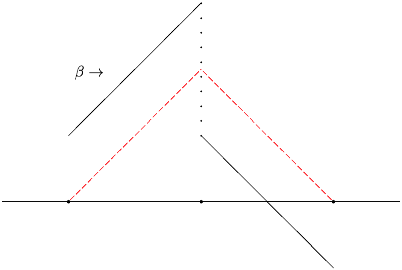


Figure: SUPG test function v_i.

[1] A. Brooks and T. Hughes. Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. *Comp. Meth. Appl. Mech. Engr*, 32:199–259, 1982

DPG: a minimum residual method via optimal testing

Given a trial space U and Hilbert test space V,

$$b(u,v) = \ell(v), \quad u \in U, \quad v \in V$$

This is equivalent to the operator equation posed in V'

$$Bu = \ell$$

if we identify $B : U \to V'$ and $\ell \in V'$ such that

$$\begin{aligned} \langle Bu, v\rangle_V &:= b(u,v), \quad u \in U, v \in V, \\ \langle \ell, v\rangle_V &:= \ell(v), \quad v \in V. \end{aligned}$$

We seek to minimize the dual residual over $U_h \subset U$

$$J(u_h) = \frac{1}{2}\|Bu_h - \ell\|_{V'}^2 := \frac{1}{2} \sup_{v \in V\setminus\{0\}} \frac{|b(u_h,v) - \ell(v)|^2}{\|v\|_V^2}.$$

Let $R_V : V \to V'$ be the isometric Riesz map st.

$$\langle R_V v, \delta v \rangle_V := (v, \delta v)_V, \quad \forall \delta v \in V.$$

Then, our residual $J(u_h)$ is equal to

$$\min_{u_h \in U_h} J(u_h) = \frac{1}{2} \|Bu_h - \ell\|_{V'}^2 = \frac{1}{2} \left\| R_V^{-1}(Bu_h - \ell) \right\|_V^2 .$$

First order optimality: Gâteaux derivative is zero in all directions $\delta u \in U_h$

$$\begin{aligned} \left(R_V^{-1}(Bu_h - \ell), R_V^{-1} B\delta u \right)_V &= 0, \quad \forall \delta u \in U_h. \\ \to \left\langle (Bu_h - \ell), R_V^{-1} B\delta u \right\rangle &= 0, \\ \to b(u_h, R_V^{-1} B\delta u) - \ell(R_V^{-1} B\delta u) &= 0 \end{aligned}$$

Summary: select test functions to minimize residuals

For $\delta u \in U_h$, define optimal test functions $v_{\delta u} := R_V^{-1} B\delta u$ as solutions to

$$\left(v_{\delta u}, \delta v\right)_V = b(\delta u, \delta v), \quad \forall \delta v \in V.$$

The optimal test space $V_{\rm opt}$

$$V_{\rm opt} := \left\{ v_{\delta u} : \delta u \in U_h \right\}.$$

Then, the residual $J(u_h) = \frac{1}{2} \left\| Bu_h - \ell \right\|_{V'}^2$ is minimized by the solution of

$$b\left(u_h, v\right) = \ell\left(v\right), \quad \forall v \in V_{\rm opt}.$$

Practical details of DPG

Computation of $v_{\delta u} := R_V^{-1} B\delta u$ is **global** and **infinite-dimensional**.

- By choosing a broken test space V and localizable norm $\|v\|_V^2 = \sum_K \|v\|_{V(K)}^2$, test functions can be determined locally.
- In practice, we use an enriched space $V_h \subset V$, where $\dim(V_h) > \dim(U_h)$ elementwise, and optimal test functions are approximated by computing $v_{\delta u} := R_{V_h}^{-1} B\delta u \in V_h$ through[2]

$$(v_{\delta u}, \delta v)_V = b(\delta u, \delta v), \quad \delta u \in U_h, \quad \forall \delta v \in V_h$$

Typically, if $U_h = \mathcal{P}^p(\mathbb{R}^n)$, $V_h = \mathcal{P}^{p+\triangle p}(\mathbb{R}^n)$, where $\triangle p \geq n$.[3]

[2] L. Demkowicz and J. Gopalakrishnan. A class of discontinuous Petrov-Galerkin methods. II. Optimal test functions. *Num. Meth. for Partial Diff. Eq*, 27:70–105, 2011

[3] J. Gopalakrishnan and W. Qiu. An analysis of the practical DPG method. Technical report, IMA, 2011. Submitted

Properties of DPG

DPG provides a symmetric positive-definite stiffness matrix. Let $\{\phi_j\}_{j=1}^m$ be a basis for U_h, and $\{v_i\}_{i=1}^n$ a basis for V_h, such that $n > m$. Then, for

$$B_{ij} = b(\phi_j, v_i),$$
$$l_i = \ell(v_i),$$

DPG solves the discrete system for degrees of freedom u

$$\left(B^T R_V^{-1} B\right) u = \left(B^T R_V^{-1}\right) l,$$

where, under a localizable norm and discontinuous test functions, R_V^{-1} is block diagonal.

Properties of DPG

Additional properties of DPG include[4]

- DPG provides the best approximation in the energy norm

$$\|u\|_E = \|Bu\|_{V'} = \sup_{\|v\|_V=1} |b(u,v)| \,.$$

- The energy error is computable through the residual

$$\|u-u_h\|_E = \|B(u-u_h)\|_{V'} = \left\|R_V^{-1}(l - Bu_h)\right\|_V = \|e\|_V$$

  where the error representation function e is defined through $(e, \delta v)_V = \ell(v) - b(u_h, \delta v)$ for all $\delta v \in V$.
- Adaptivity driven by local error indicator $\|e\|^2_{V(K)}$

[4] L. Demkowicz, J. Gopalakrishnan, and A. Niemi. A class of discontinuous Petrov-Galerkin methods. Part III: Adaptivity. *Appl. Numer. Math.*, 62(4):396–427, April 2012

Residual-based stabilization (with a twist)

Can rewrite least-squares minimization as a saddle point problem.[5] Define

$$R_V^{-1}(Bu - l) = e \in V.$$

Undoing the right Riesz operator leads to the Galerkin orthogonality condition

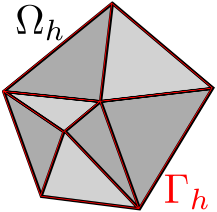
$$\left(e, R_V^{-1} B\delta u\right)_V = \langle e, B\delta u\rangle = b(\delta u, e) = 0, \quad \forall \delta u \in U_h$$

which gives the system for $(u, e) \in U \times V$

$$\begin{aligned} b(u, v) + (e, v)_V &= l(v), \quad \forall v \in V \\ b(\delta u, e) &= 0, \quad \forall \delta u \in U_h \end{aligned}$$

[5] W. Dahmen A. Cohen and G. Welper. Adaptivity and variational stabilization for convection-diffusion equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 46(5):1247–1273, 2012

Ultra-weak formulation



Given a first order system $Au = f$, we identify the **partition** Ω_h and **mesh skeleton** Γ_h.

The ultra-weak formulation for $Au = f$ on Ω_h is

$$b\left((u, \widehat{u}), v\right) := \sum_K \langle \widehat{u}, v \rangle_{\partial K} + (u, A_h^* v)_{\Omega_h} = (f, v)_{\Omega_h}.$$

Under proper assumptions, $\sum_K \langle \widehat{u}, v \rangle_{\partial K} = \langle \widehat{u}, [\![v]\!] \rangle_{\Gamma_h}$, with energy setting

$$u \in L^2(\Omega_h) \equiv L^2(\Omega), \quad v \in V = D(A_h^*), \quad \widehat{u} \in \gamma(D(A)),$$

where $D(A_h^*)$ is the broken graph space of the formal adjoint A_h^*, and $\gamma(D(A))$ the trace space of the graph space of operator A.[6]

[6] L. Demkowicz and J. Gopalakrishnan. Analysis of the DPG method for the Poisson equation. *SIAM J. Numer. Anal.*, 49(5):1788–1809, September 2011

The canonical "graph" test norm

Recall $\|u\|_E := \sup_{v\in V\setminus\{0\}} \frac{b(u,v)}{\|v\|_V}$. Under the ultra-weak formulation, the trial norm

$$\|(u,\widehat{u})\|_U^2 := \|u\|_{L^2(\Omega)}^2 + \|\widehat{u}\|^2$$

generates a test norm *equivalent* to

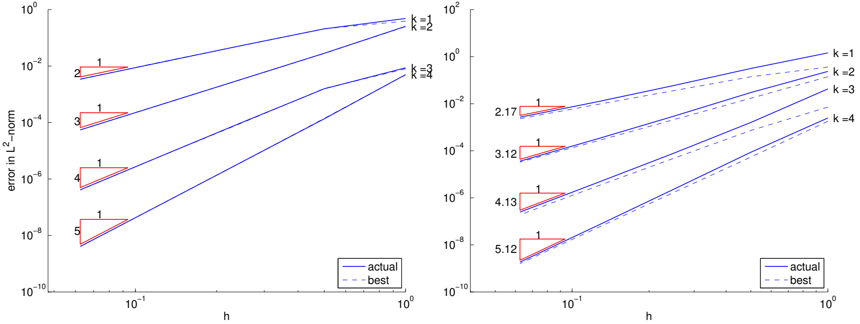
$$\|A_h^* v\|_{L^2(\Omega)}^2 + \left(\sup_{\widehat{u}} \frac{\langle \widehat{u}, [\![v]\!]\rangle_{\Gamma_h}}{\|\widehat{u}\|}\right)^2 .$$

This norm is not localizable, so we instead substitute the jump terms for an L^2 term, giving us the graph norm[7]

$$\|v\|_V^2 := \|A_h^* v\|_{L^2(\Omega)}^2 + \|v\|_{L^2}^2 .$$

[7] T. Bui-Thanh, L. Demkowicz, and O. Ghattas. A unified discontinuous Petrov-Galerkin method and its analysis for Friedrichs' systems. *Submitted to SIAM J. Numer. Anal.*, 2011. Also ICES report 11-34, November 2011

The graph test norm - near L^2 best approx for Stokes.



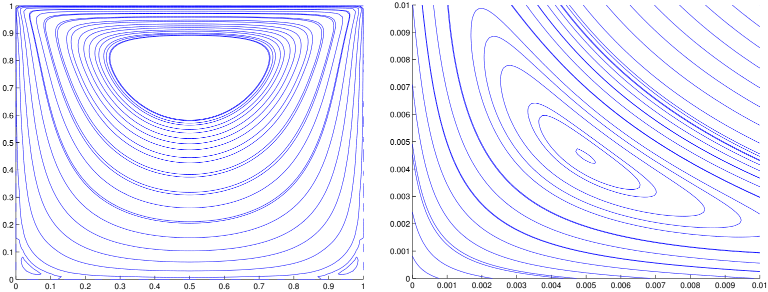
(a) Manufactured solution: u_1 error

(b) Manufactured solution: p error

Figure: Results under the graph norm for Stokes flow.[8]

[8] N. Roberts, T. Bui Thanh, and L. Demkowicz. The DPG method for the Stokes problem. Technical Report 12-22, ICES, June 2012

The graph test norm - Stokes lid-driven cavity flow.



(a) Cavity flow streamlines (b) Second Moffat eddy

Similar results for elasticity, electromagnetics, and wave propagation[9].

[9] J. Zitelli, I. Muga, L. Demkowicz, J. Gopalakrishnan, D. Pardo, and V.M. Calo. A class of discontinuous Petrov–Galerkin methods. Part IV: The optimal test norm and time-harmonic wave propagation in 1D. *Journal of Computational Physics*, 230(7):2406 – 2432, 2011

DPG as a non-conforming DG method on V

For DPG using the ultra-weak variational formulation

$$b\left(\left(u, \widehat{u}\right), v\right) := \langle \widehat{u}, [\![v]\!]\rangle_{\Gamma_h} + (u, A_h^* v)_{\Omega_h} = (f, v)_{\Omega_h},$$

the saddle point problem is

$$\begin{aligned} (e, v)_V &= l(v) - b\left(\left(u, \widehat{u}\right), v\right), \quad \forall v \in V(\Omega_h) \\ (\delta u, A_h^* e)_{L^2(\Omega)} &= 0, \quad \forall (\delta u, 0) \in U_h \\ \left\langle \widehat{\delta u}, [\![e]\!] \right\rangle_{\Gamma_h} &= \int_{\Gamma_h} \widehat{\delta u}\, [\![e]\!] = 0, \quad \forall \left(0, \widehat{\delta u}\right) \in U_h, \end{aligned}$$

This implies $e \in \tilde{V}$, the *weakly conforming* space

$$\tilde{V} = \left\{ v \in V(\Omega_h) : \langle \widehat{u}, [\![v]\!]\rangle_{\Gamma_h} = 0, \quad \forall \left(0, \widehat{\delta u}\right) \in U_h \right\}.$$

Global properties of DPG test spaces

If we define $\tilde{V}_{\rm opt}$ as the optimal test space constructed using $\tilde{V}$

$$\tilde{V}_{\rm opt} = \left\{ R_{\tilde{V}}^{-1} B\delta u, \quad \forall \delta u \in U_h \right\},$$

we have the following theorem[10]

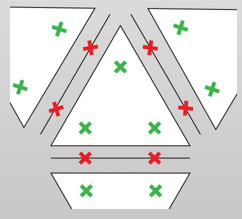
Theorem

$\tilde{V}_{\rm opt} \subset V_{\rm opt}$. Furthermore, the L^2/interior solutions of $b(u,v) = l(v)$ over $V_{\rm opt}$ and $\tilde{V}_{\rm opt}$ coincide.

- The locally determined optimal test functions approximate globally optimal test functions.
- Under a variant of the graph norm, DPG approximates test functions which deliver L^2 optimality.

[10] J. Chan, J. Gopalakrishnan, and L. Demkowicz. Global properties of DPG test spaces for convection-diffusion problems. Technical Report 13-05, ICES, 2013

Ultra-weak formulation for convection-diffusion



The first order convection-diffusion system:

$$A(u,\sigma) := \begin{bmatrix} \nabla\cdot(\beta u - \sigma) \\ \frac{1}{\epsilon}\sigma - \nabla u \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

The variational formulation is

$$b\left(\left(u,\sigma,\widehat{u},\widehat{f}_n\right),(v,\tau)\right) = (u, \nabla_h\cdot\tau - \beta\cdot\nabla_h v)_{\Omega_h} + \left(\sigma, \epsilon^{-1}\tau + \nabla_h v\right)_{\Omega_h} - \langle [\![\tau\cdot n]\!], \widehat{u}\rangle_{\Gamma_h} + \left\langle \widehat{f}_n, [\![v]\!]\right\rangle_{\Gamma_h},$$

where $\widehat{f}_n := \beta_n u - \sigma_n$ and $\left\langle \widehat{f}_n, [\![v]\!]\right\rangle_{\Gamma_h}$ is defined

$$\left\langle \widehat{f}_n, [\![v]\!]\right\rangle_{\Gamma_h} := \sum_K \int_{\partial K} \operatorname{sgn}(\vec{n})\, \widehat{f}_n v.$$

The graph test norm: problems under convection-diffusion

The graph norm[11] for convection-diffusion gives exceptional stability.

$$\|(v,\tau)\|_{V(K)}^2 = \|\nabla\cdot\tau - \beta\cdot\nabla v\|_{L^2(K)}^2 + \|\epsilon^{-1}\tau + \nabla v\|_{L^2(K)}^2 + \|v\|_{L^2(K)}^2.$$

Problem with this test norm: approximability of test functions.

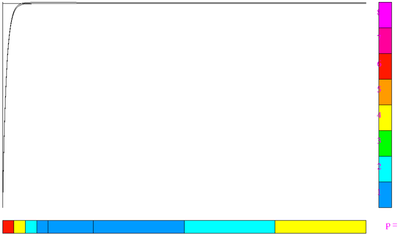


Figure: Components of optimal test functions for flux $\widehat{f}_n$ on the *right-hand* side of a unit element for $\epsilon = 0.01$.

[11] T. Bui-Thanh, L. Demkowicz, and O. Ghattas. A unified discontinuous Petrov-Galerkin method and its analysis for Friedrichs' systems. *Submitted to SIAM J. Numer. Anal.*, 2011. Also ICES report 11-34, November 2011

Determining an alternative test norm

$$b(\mathbf{U},\mathbf{V}) = (u, \nabla\cdot\tau - \beta\cdot\nabla v)_{\Omega_h} + \left(\sigma, \epsilon^{-1}\tau + \nabla v\right)_{\Omega_h} + \text{boundary terms}$$

Recover $\|u\|^2_{L^2(\Omega)}$ with conforming (v,τ) satisfying the *adjoint equations*

$$\begin{array}{ll} \nabla\cdot\tau - \beta\cdot\nabla v & = u \\ \frac{1}{\epsilon}\tau + \nabla v & = 0 \end{array}, \quad \text{boundary terms} = 0$$

"Necessary" conditions for robustness —

$$\|u\|^2_{L^2(\Omega)} = b\left(\mathbf{U},(v,\tau)\right) = \frac{b\left(\mathbf{U},(v,\tau)\right)}{\|(v,\tau)\|_V}\|(v,\tau)\|_V \leq \|\mathbf{U}\|_E\|(v,\tau)\|_V$$

Let $\lesssim$ denote a robust bound - if $\|(v,\tau)\|_V \lesssim \|u\|_{L^2(\Omega)}$, then we have that

$$\|u\|_{L^2(\Omega)} \lesssim \|\mathbf{U}\|_E$$

Main idea: the test norm should measure adjoint solutions robustly.

Choice of inflow boundary condition

We impose the standard outflow wall boundary condition on u. For inflow boundary condition:

- The standard choice of inflow boundary condition: $u = u_0$.
- We impose the non-standard inflow condition: $\widehat{f}_n := \beta_n u - \sigma_n \approx \beta_n u_0$ on $\Gamma_{\rm in}$.

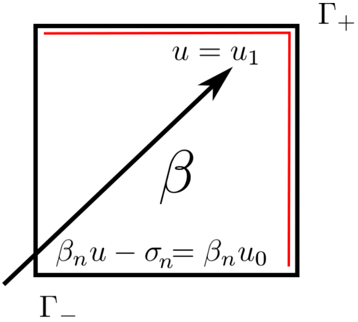
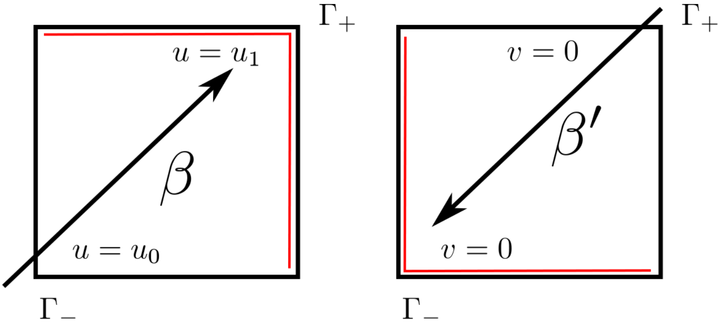


Figure: Non-standard inflow.

When $\sigma_n \approx 0$ near the inflow, condition on $\widehat{f}_n$ approximates condition on u.

Dirichlet inflow condition: issues as $\epsilon \to 0$

Standard choice of boundary condition: $u = u_0$ on inflow boundary $\Gamma_{\rm in}$, induces boundary layers in adjoint problems, $\|\beta \cdot \nabla v\|_{L^2} = O(\epsilon^{-1})$.

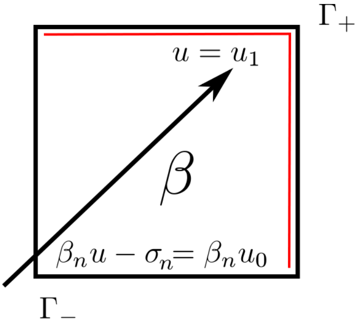


(a) Primal problem (b) Adjoint problem

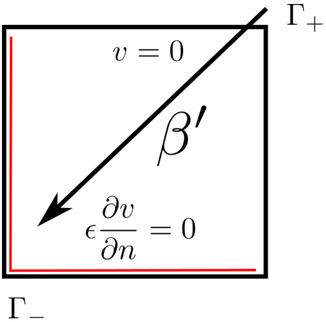
Figure: For the standard Dirichlet inflow condition, the solution to the adjoint problem can develop strong boundary layers at the outflow of the adjoint problem.

Solution: New inflow boundary condition on $\widehat{f}_n$

Non-standard choice of boundary condition: $\widehat{f}_n = \beta_n u_0$ on Γ_{in}, induces smoother adjoint problems, $\|\beta \cdot \nabla v\|_{L^2} = O(1)$.



(a) Primal problem

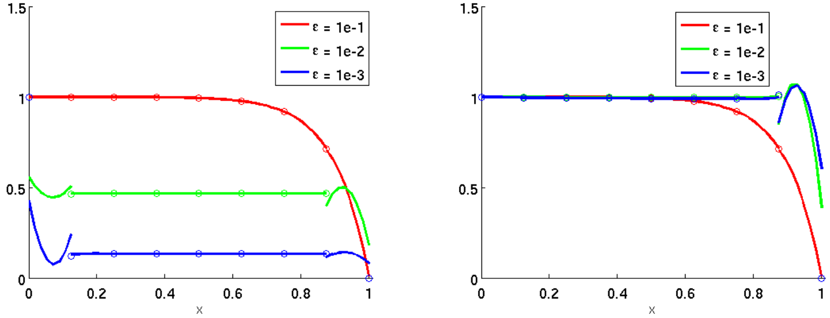


(b) Adjoint problem

Figure: Under the new inflow condition, the wall-stop boundary condition is relaxed to a zero-stress condition at the outflow boundary of the adjoint problem.

Test norms and adjoint solutions

Intuition: the effectiveness of DPG under a test norm is governed by how a **specific test norm** measures the **solutions of the adjoint problem**.



(a) Dirichlet inflow $u = u_0$ (b) “Convection” inflow $u - \epsilon u' \approx u_0$

Figure: DPG solutions to convection-diffusion for both inflow conditions using an H^1 test norm.

Adjoint estimates, or "norm building blocks"

For solutions (v, τ) of the adjoint equations, the following quantities are robustly bounded from above by $\|(u, \sigma)\|_{L^2(\Omega)}$.

$$\|v\|, \sqrt{\epsilon}\|\nabla v\|, \|\beta \cdot \nabla v\|$$
$$\|\nabla \cdot \tau\|, \frac{1}{\sqrt{\epsilon}}\|\tau\|.$$

We will construct a test norm through a combination of the above quantities, such that

- v and τ decoupled (no systems).
- Coefficients are of equal order after transforming to the unit element (no boundary layers).

Mesh-scaled test norms

Our test norm, as defined over a single element K, is now

$$\|(v,\tau)\|_{V,K}^2 = \|v\|^2 + \epsilon\|\nabla v\|^2 + \|\beta\cdot\nabla v\|^2 + \|\nabla\cdot\tau\|^2 + \min\left\{\frac{1}{\epsilon}, \frac{1}{|K|}\right\}\|\tau\|^2.$$

Defining $\|(v,\tau)\|_{V,\Omega_h} = \sum_{K\in\Omega_h}\|(v,\tau)\|_{V,K}$ induces the robust bound[12]

Theorem

Under the test norm $\|(v,\tau)\|_{V,\Omega_h}$, the energy norm for DPG satisfies

$$\|u\|_{L^2(\Omega)} + \|\sigma\|_{L^2(\Omega)} + \epsilon\|\widehat{u}\| + \sqrt{\epsilon}\left\|\widehat{f}_n\right\| \lesssim \left\|\left(u,\sigma,\widehat{u},\widehat{f}_n\right)\right\|_E.$$

[12] J. Chan, N. Heuer, T. Bui Thanh, and L. Demkowicz. Robust DPG method for convection-diffusion problems II: natural inflow conditions. Technical Report 12-21, ICES, June 2012. Submitted

Erikkson-Johnson model problem

On domain $\Omega = [0,1]^2$, with $\beta = (1,0)^T$, $f = 0$ and boundary conditions

$$\beta_n \widehat{u - \sigma_n} = \widehat{f}_n = u_0 - \epsilon \frac{\partial u}{\partial n}, \quad \beta_n \leq 0$$

$$\widehat{u} = 0, \quad \beta_n > 0$$

Separation of variables yields a boundary layer solution.[13]

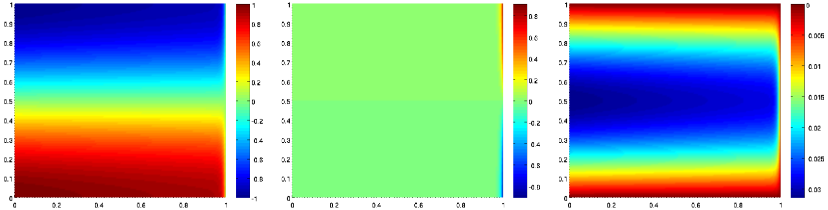


Figure: Exact solution for u, σ_x, and σ_y for $\epsilon = .01$, $C_1 = 1$, $C_n = 0$, $n \neq 1$

[13] L. Demkowicz and N. Heuer. Robust DPG method for convection-dominated diffusion problems. Technical Report 11-33, ICES, 2011

Numerical verification: Erikkson-Johnson problem

Numerical experiments done using the Camellia[14] codebase.

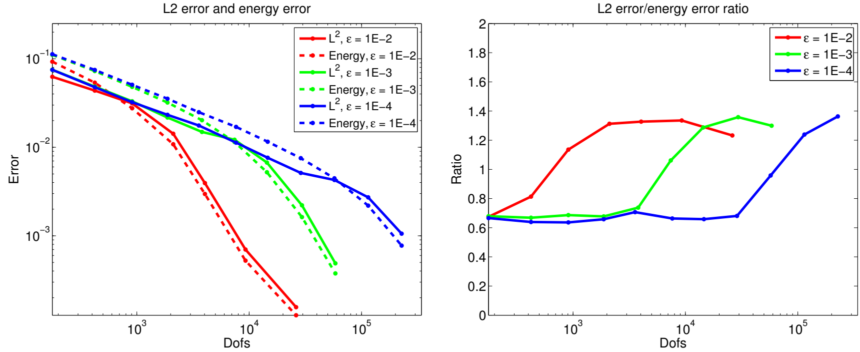


Figure: L^2/energy errors and their ratio for $\epsilon = .01, .001, .0001$.

[14] N. Roberts, D. Ridzal, P. Bochev, and L. Demkowicz. A Toolbox for a Class of Discontinuous Petrov-Galerkin Methods Using Trilinos. Technical Report SAND2011-6678, Sandia National Laboratories, 2011

DPG for nonlinear problems

- DPG applied to linearized problem; equivalent to Gauss-Newton
- Given some linearization (Newton-Raphson, pseudo-timestepping), we measure
  - size of the linearized update Δu

$$\|\Delta u\|_E := \|B_u \Delta u\|_{V'} = \left\|R_V^{-1} B_u \Delta u\right\|_V$$

  - the nonlinear residual

$$\|R(u)\|_E := \|B(u) - \ell\|_{V'} = \left\|R_V^{-1}\left(B(u) - \ell\right)\right\|_V$$

Preliminary experiments were done in 1D[15] and space-time.[16]

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[15] J. Chan, L. Demkowicz, R. Moser, and N. Roberts. A New Discontinuous Petrov-Galerkin Method with Optimal Test Functions. Part V: Solution of 1D Burgers' and Navier-Stokes Equations. Technical Report 10-25, ICES, June 2010

[16] J. Chan, L. Demkowicz, and M. Shashkov. Space-time DPG for shock problems. Technical Report LA-UR 11-05511, LANL, September 2011

2D test case: Burgers equation

$$\frac{\partial\left(u^2/2\right)}{\partial x} + \frac{\partial u}{\partial y} + \epsilon\Delta u = f$$

Burgers equation can be written with $\beta(u) = (u/2, 1)$

$$\nabla\cdot(\beta(u)u - \sigma) = f$$
$$\frac{1}{\epsilon}\sigma - \nabla u = 0.$$

i.e. nonlinear convection-diffusion on domain $[0,1]^2$.

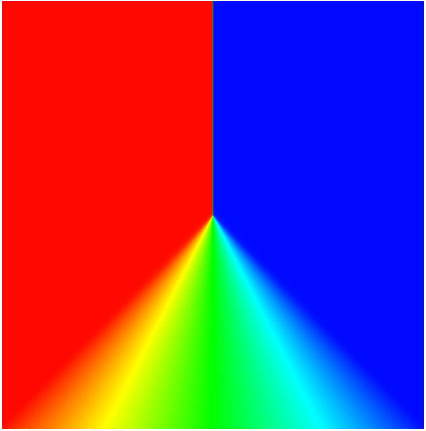


Figure: Shock solution for Burgers' equation, $\epsilon = 1e-4$, using Newton-Raphson.

Adaptivity begins with a cubic 4×4 mesh.

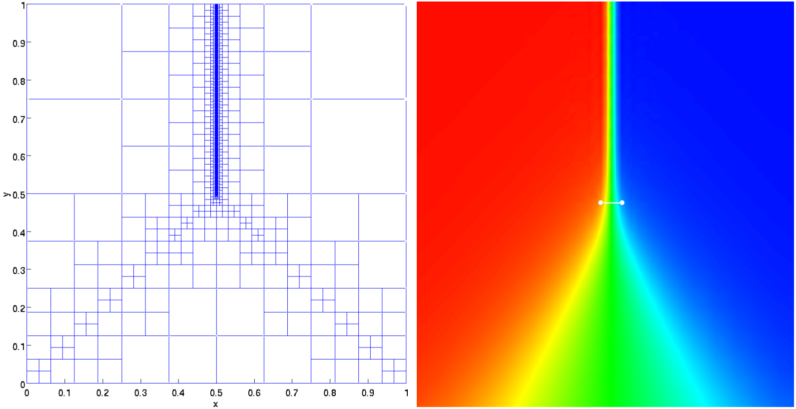


Figure: Adaptive mesh after 9 refinements, and zoom view at point (.5,.5) with shock formation and $1e-3$ width line for reference.

2D Compressible Navier-Stokes equations (ideal gas)

Given density ρ, velocities $\mathbf{u} = (u_1, u_2)$ and temperature T,

$$
\begin{aligned}
\nabla \cdot \begin{bmatrix} \rho u_1 \\ \rho u_2 \end{bmatrix} &= 0, \\
\nabla \cdot \left(\begin{bmatrix} \rho u_1^2 + p \\ \rho u_1 u_2 \end{bmatrix} - \boldsymbol{\sigma}_1 \right) &= 0, \\
\nabla \cdot \left(\begin{bmatrix} \rho u_1 u_2 \\ \rho u_2^2 + p \end{bmatrix} - \boldsymbol{\sigma}_2 \right) &= 0, \\
\nabla \cdot \left(\begin{bmatrix} ((\rho e) + p) u_1 \\ ((\rho e) + p) u_2 \end{bmatrix} - \boldsymbol{\sigma} \mathbf{u} + \vec{q} \right) &= 0, \\
\frac{1}{2\mu} \boldsymbol{\sigma} - \frac{\lambda}{4\mu(\mu + \lambda)} \operatorname{tr}(\boldsymbol{\sigma}) \mathbf{I} &= \nabla \mathbf{u} - \mathrm{Re}\, \mathbf{w}, \\
\kappa^{-1} \vec{q} &= \nabla T,
\end{aligned}
$$

where $\mathbf{w}$ represents the antisymmetric part of ∇u

Extrapolation of test norms

Convection-diffusion:

$$\begin{aligned}\nabla \cdot (\beta u - \sigma) &= f\\ \frac{1}{\epsilon}\sigma - \nabla u &= 0.\end{aligned}$$

Navier-Stokes: defining vector variables $U = \{\rho, u_1, u_2, T\}$ and $\Sigma = \{\boldsymbol{\sigma}, \mathbf{q}, w\}$,

$$\begin{aligned}\nabla \cdot (A_{\rm invisc} U - A_{\rm visc}\Sigma) &= R_{\rm conserv}(U, \Sigma)\\ E_{\rm visc}\Sigma - \nabla U &= R_{\rm constit}(U, \Sigma)\end{aligned}$$

where $R_{\rm conserv}(U, \Sigma)$ and $R_{\rm constit}(U, \Sigma)$ are the conservation/constitutive residuals.

Extrapolation of test norms

Convection-diffusion:

$$\nabla \cdot (\beta u - \sigma) = f$$
$$\frac{1}{\epsilon}\sigma - \nabla u = 0.$$

Navier-Stokes: defining vector variables $U = \{\rho, u_1, u_2, T\}$ and $\Sigma = \{\boldsymbol{\sigma}, \mathbf{q}, w\}$,

$$\nabla \cdot (A_{\text{invisc}} U - A_{\text{visc}} \Sigma) = R_{\text{conserv}}(U, \Sigma)$$
$$E_{\text{visc}} \Sigma - \nabla U = R_{\text{constit}}(U, \Sigma)$$

where $R_{\text{conserv}}(U, \Sigma)$ and $R_{\text{constit}}(U, \Sigma)$ are the conservation/constitutive residuals.

Test norms over one element

Convection-diffusion:

$$\|(v,\tau)\|_{V,K}^2 = \|v\|^2 + \|\beta\cdot\nabla v\|^2 + \epsilon\|\nabla v\|^2 + \|\nabla\cdot\tau\|^2 + \min\left\{\frac{1}{\epsilon},\frac{1}{|K|}\right\}\|\tau\|^2.$$

Navier-Stokes: let V and W be vectors of test functions v_i and τ_i.

$$\|(V,W)\|_{V,K}^2 = \|V\|^2 + \left\|A_{\rm invisc}^T\nabla V\right\|^2 + \frac{1}{\rm Re}\left\|A_{\rm visc}^T\nabla V\right\|^2 + \|\nabla\cdot W\|^2 + \min\left\{1,\frac{1}{\rm Re|K|}\right\}\left\|E_{\rm visc}^T W\right\|^2.$$

Carter's flat plate problem

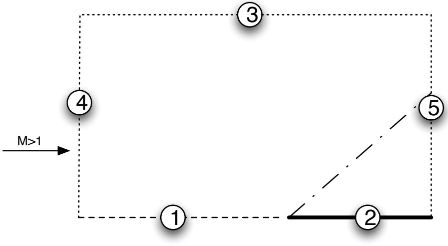
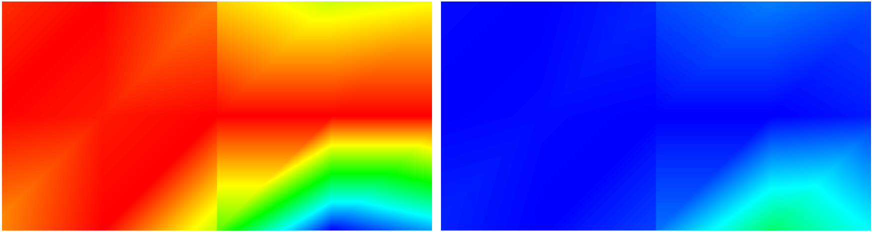


Figure: Carter flat plate problem on domain $[0, 2] \times [0, 1]$. Plate begins at $x = 1$, $\mathrm{Re} = 1000$.

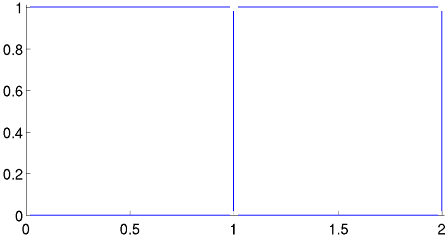
1. Symmetry boundary conditions.
2. Prescribed temperature and wall stagnation conditions.
3. Symmetry boundary conditions.
4. Inflow: conserved quantities specified using far-field values.
5. No outflow condition set.

Stress/heat flux boundary conditions are set in terms of the momentum and energy fluxes.

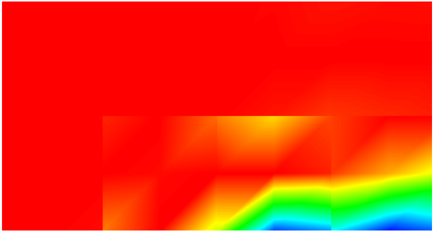
Refinement level 0



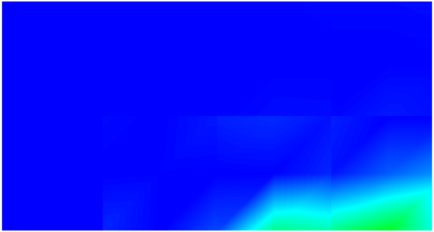
(a) u_1 (b) T



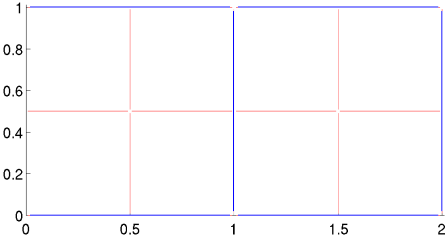
Refinement level 1



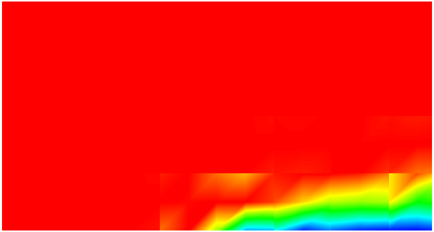
(a) u_1



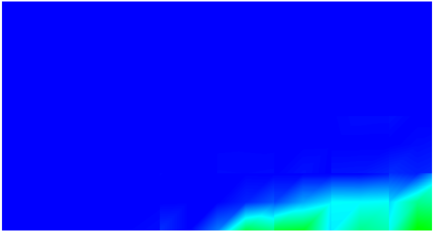
(b) T



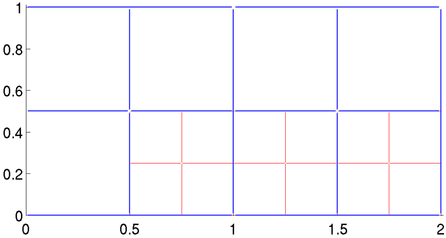
Refinement level 2



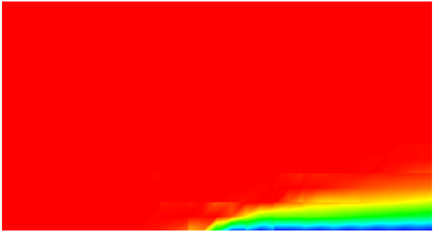
(a) u_1



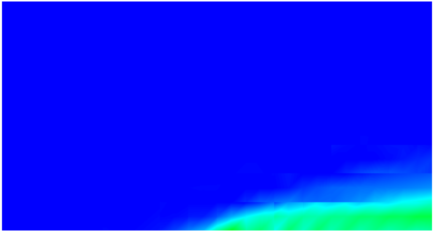
(b) T



Refinement level 3



(a) u_1



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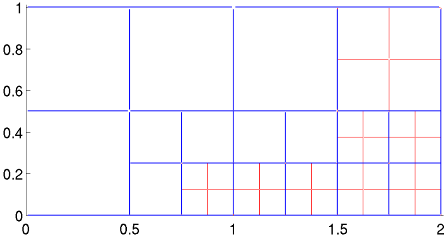
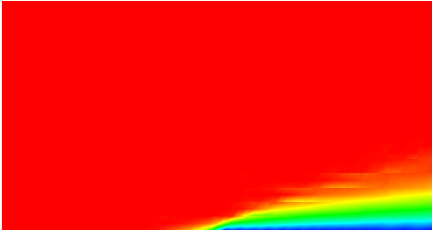
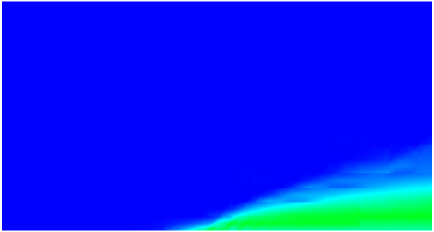


Refinement level 4



(a) u_1



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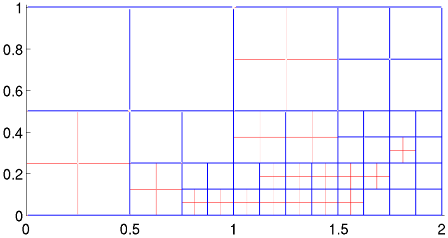
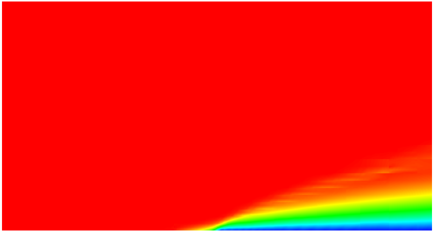
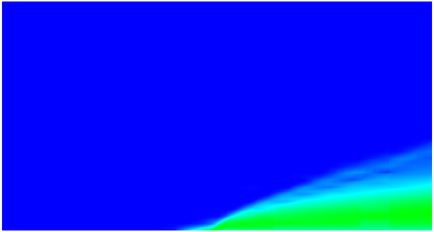


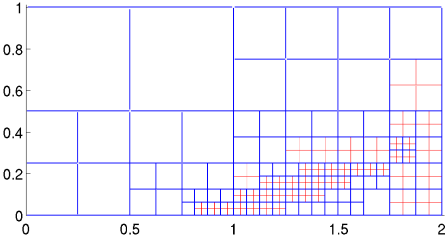
Refinement level 5



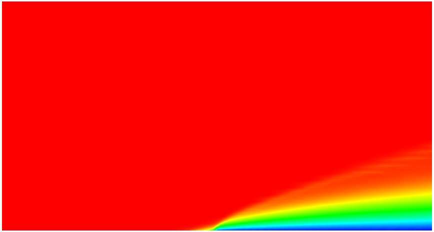
(a) u_1



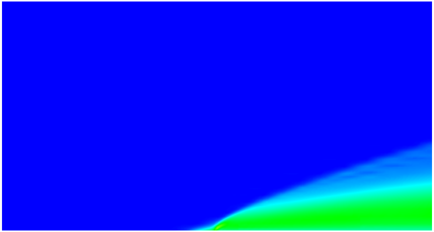
(b) T



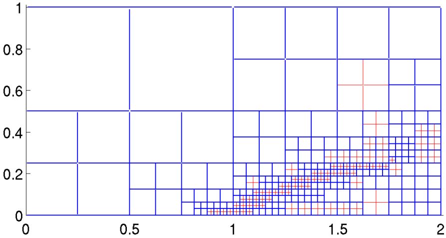
Refinement level 6



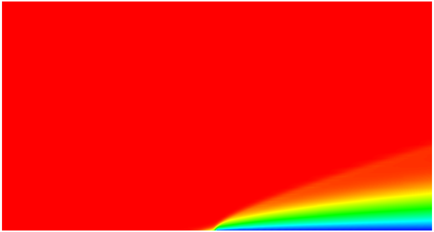
(a) u_1



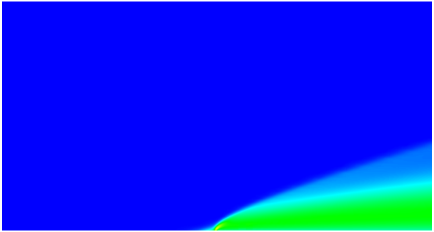
(b) T



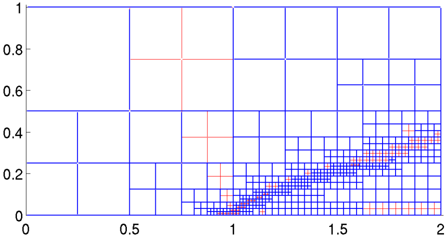
Refinement level 7



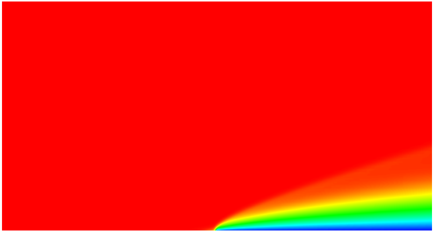
(a) u_1



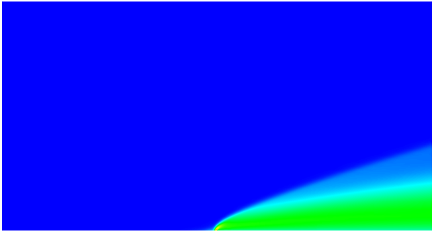
(b) T



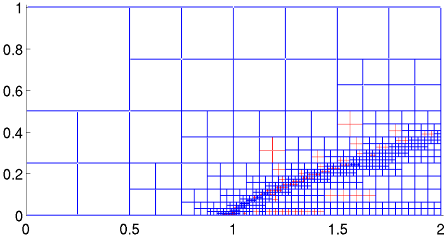
Refinement level 8



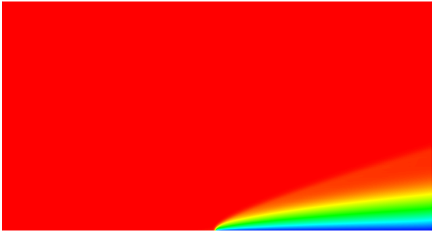
(a) u_1



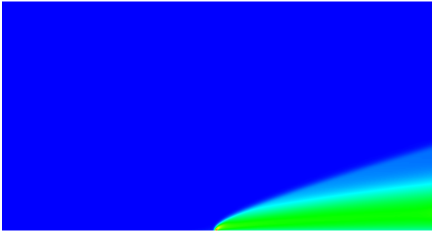
(b) T



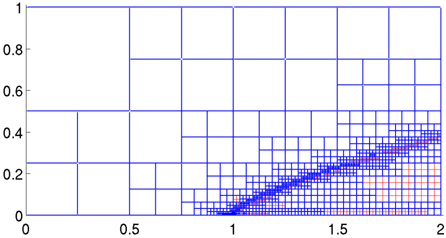
Refinement level 9



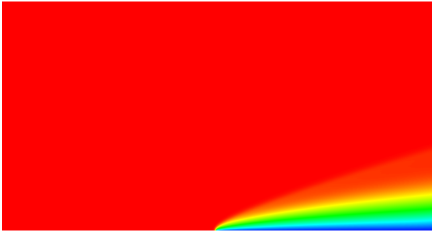
(a) u_1



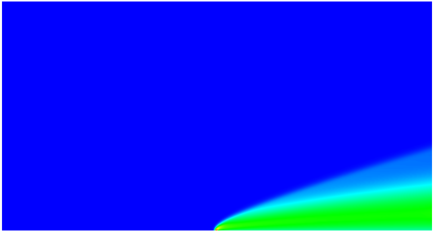
(b) T



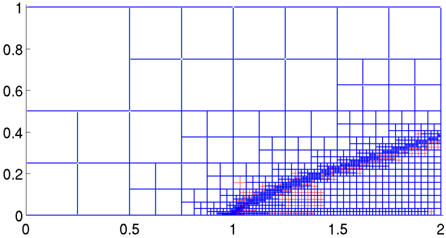
Refinement level 10



(a) u_1

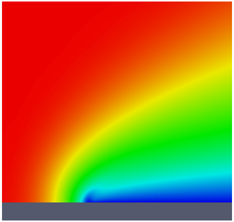


(b) T

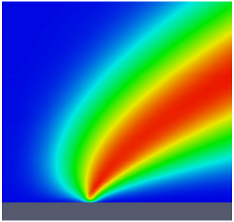


Zoomed solutions at plate/stagnation point

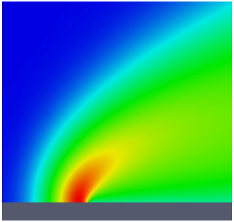
(a) ρ



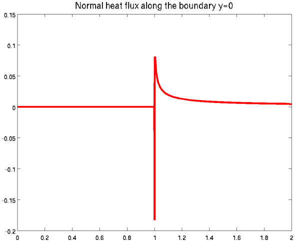
(b) u_1



(c) u_2



(d) T



(e) q_n

Automatic extension to anisotropic/*hp* meshes

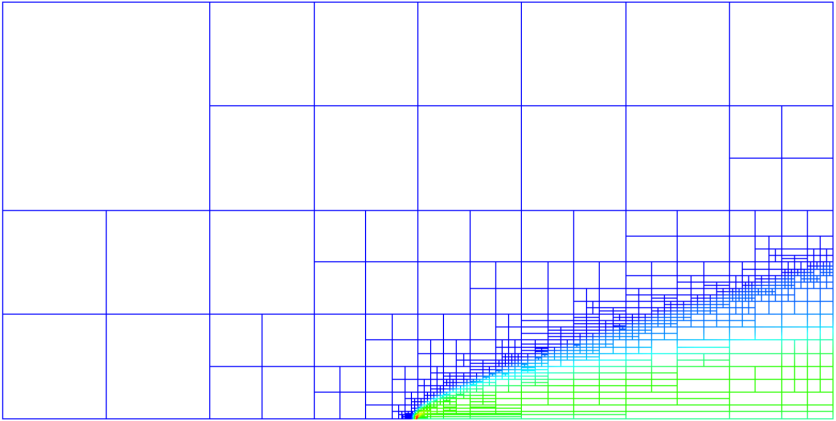
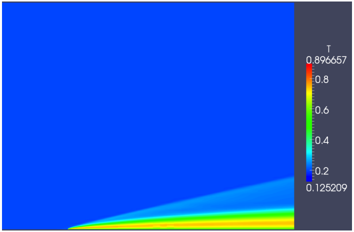
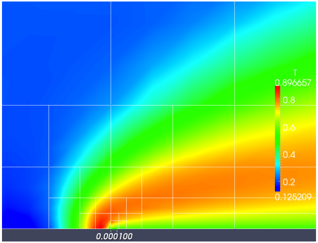


Figure: Trace $\widehat{T}$ for $\mathrm{Re} = 1000$ using an anisotropic refinement scheme.

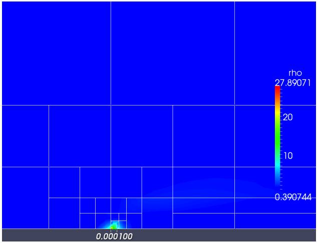
Preliminary results for $\mathrm{Re} = 10000$.



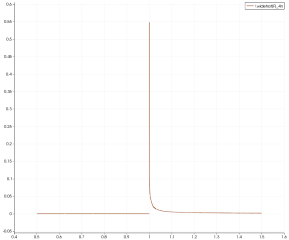
(a) Temperature



(b) T at stagnation



(c) ρ at stagnation



(d) Heat flux at plate

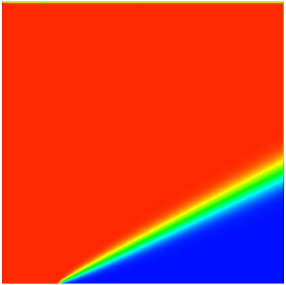
Thank you!

Questions?

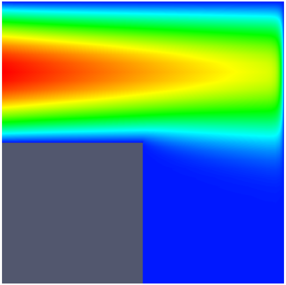
Local conservation

Given stiffness matrix K, can introduce Lagrange multipliers λ to enforce element-wise constraints, leading to the saddle point problem[17]

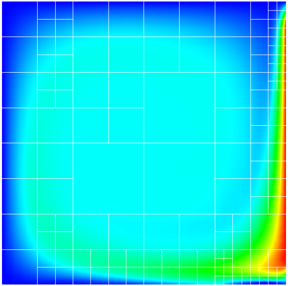
$$\begin{bmatrix} K & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ c \end{bmatrix}.$$



(e) Advection skew



(f) Flow over a step



(g) Double glazing

[17] T. Ellis, J. Chan, N. Roberts, and L. Demkowicz. Element conservation properties in the DPG method. Technical report, ICES. In preparation

Inviscid equations

Issue: consider pure convection, $\nabla \cdot \beta u = f$. The ultra-weak variational formulation is

$$\left\langle \widehat{f}_n, v \right\rangle - (u, \beta \cdot \nabla v) = (f, v),$$

where $\widehat{f}_n := \beta_n u$. When $\beta_n = 0$, v has only a streamline derivative, and $\widehat{f}_n$ becomes an ill-defined trace in the cross-stream direction. For hyperbolic *systems*, this issue manifests as *sonic lines*.

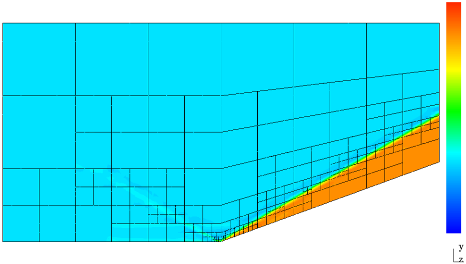


Figure: Sonic lines ($u_n - c = 0$) appear for linearized Euler.

Regularization

For $\beta = (-y, x)^T$ on $\Omega = [-1, 1]^2$. Ill posed in the convection setting. Similar tests have been done with discontinuous data.

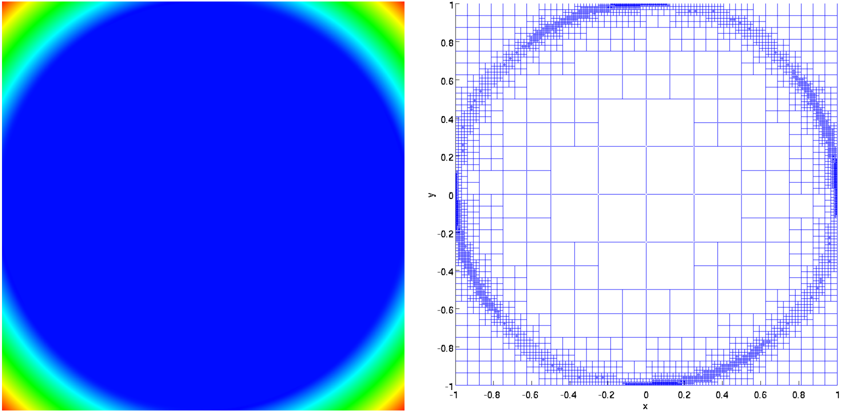


Figure: Steady vortex problem with $\epsilon = 1e - 4$.

Anisotropic refinement scheme

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Adaptivity and variational stabilization for convection-diffusion equations.
ESAIM: Mathematical Modelling and Numerical Analysis, 46(5):1247–1273, 2012.

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Also ICES report 11-34, November 2011.

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A Toolbox for a Class of Discontinuous Petrov-Galerkin Methods Using Trilinos.
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