# MT5846: Advanced Computational Techniques

## Project Three

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**Abstract**

This paper investigates the use of numerical method to solve two advection PDEs. The first part of the paper discusses the Von-Neuman stability analysis for the Lax-Wendroff and MacCormack Methods. It then investigates the implementation of the First Upwind Difference, Lax-Wendroff and Euler BTCS method to solve an advection equation. It does this for a variety of temporal spacing values and investigates their impact on the results. It finds that the explicit methods have their best results when the courant number $c = 1$. It also shows that the implicit methods struggles to have accurate results due to both a phase error and a reduction in amplitude. Having done this it proposes an improvement to the Euler BTCS to improve the accuracy of the method. This improvement results in a method that has an improvement in accuracy but the explicit methods still remain the most accurate when $c = 1$, in addition this improvement came at an increased computational cost. The final section of the paper discusses the implementation of the Lax and MacCormack methods to solve the Burgers' equation again for a variety of temporal spacing values. It finds that both suffer from instability when the CFL condition is not met and that the MacCormack method appears to result in smoother solutions than the Lax method.

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# 1 Introduction

In this project, we will be investigating two advection equations as described below.

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} \tag{1}$$

and the Inviscid Burgers' Equation:

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} \tag{2}$$

For this problem the quantity $u(x,t)$ describes the velocity at a point $x$ on a one dimensional domain at a time $t$. For the investigations concerning (1) we will be using a domain of $0 \le x \le 300$ with $a = 300m/s$ and the condition outlined below:

- Initial Conditions

$$\begin{aligned} u(x,0) &= 0 && 0 \le x < 50 \\ u(x,0) &= 100\sin\left[\pi\left(\frac{x-50}{60}\right)\right] && 50 \le x < 110 \\ u(x,0) &= 0 && 110 \le x \le 300 \end{aligned}$$

- Boundary Conditions

$$\begin{aligned} u(0,t) &= 0 \\ u(300,t) &= 0 \end{aligned}$$

For investigations concerning the Inviscid Burgers' equation (2) we will be using the domain $0 \le x \le 4$, with the conditions outlined below:

- Initial Conditions

$$\begin{aligned} u(x,0) &= 1 && 0 \le x < 2 \\ u(x,0) &= 0 && 2 \le x \le 4 \end{aligned}$$

- Boundary Conditions

$$\begin{aligned} u(0,t) &= 1 \\ u(4,t) &= 0 \end{aligned}$$

To solve these PDE numerous methods have been employed these have been outlined below. These all being valid for $1 < i < im$, and where, $c := \frac{a\Delta t}{\Delta x}$:

The Lax-Wendroff Method:

$$u_i^{n+1} = u_i^n - \frac{c}{2}(u_{i+1}^n - u_{i-1}^n) + \frac{c^2}{2}(u_{i-1}^n - 2u_i^n + u_{i+1}^n) \tag{3}$$

First Upwind Difference Method:

$$u_i^{n+1} = u_i^n - c(u_i^n - u_{i-1}^n) \tag{4}$$

Euler BTCS Method:

$$\frac{c}{2}u_{i-1}^{n+1} - u_i^{n+1} - \frac{c}{2}u_{i+1}^{n+1} = -u_i^n \tag{5}$$

MacCormack Method:

$$\begin{aligned} \text{Predictor Step : } u_i^* &= u_i^n - \frac{\Delta t}{\Delta x}(E_{i+1}^n - E_{i-1}^n), \\ \text{Corrector Step : } u_i^{n+1} &= \frac{1}{2}\left[(u_i^n + u_i^*) - \frac{\Delta t}{\Delta x}(E_{i-1}^*)\right], \end{aligned} \tag{6}$$

where $E_i^n = \frac{(u_i^n)^2}{2}$ and $E_i^* = \frac{(u_i^*)^2}{2}$

**Analytical Solution** : For Equation (1) we will be comparing the results of the numerical methods with the analytical soltion given by $u(x,t) = u_0(x - at)$ with $u_o(x) = u(x,0)$ and $a = 300m/s$

Although it is not mentioned in the project outline to discuss it is important to note here that all the method discussed in this project are **consistent** to their respective equations, and as a result when their respective stability conditions are satisfied the methods do **converge** to their correct solutions.

## 2 Part 1: Stability Analysis

The first section of this report is dedicated to the Von-Neumann stability analysis for Lax-Wendroff method (3) and the MacCormack method (6)

It can be shown that the round-off error $\epsilon^n$ evolves in time like the finite difference method. We set $u_i^n \to \epsilon_i^n := \xi^n e^{I\theta i}$ where $I$ is the imaginary unit and $\theta \in [-\pi, \pi]$

### 2.1 Lax-Wendroff Method Stability

Substituting the values above into Equation (1) we obtain:

$$\xi^{n+1} e^{I\theta i} = \xi^n e^{I\theta i} - \frac{c}{2}(\xi^n e^{I\theta(i+1)} - \xi^n e^{I\theta(i-1)}) + \frac{c^2}{2}(\xi^n e^{I\theta(i-1)} - 2\xi^n e^{I\theta(i)} + \xi^n e^{I\theta(i+1)})$$

dividing through by $\xi^n e^{I\theta i}$ we obtain:

$$G = \frac{\xi^{n+1}}{\xi^n} = 1 - \frac{c}{2}(e^{I\theta} - e^{-I\theta}) + \frac{c^2}{2}(e^{I\theta} + e^{-I\theta}) - c^2$$

where $G$ is the **Amplification Factor**, $\xi^n$ is not growing when $|G| \leq 1$. Using the identity $2\cos\theta = e^{I\theta} + e^{-I\theta}$ and $2i\sin\theta = e^{I\theta} - e^{-I\theta}$ we obtain:

$$G = \underbrace{1 + c^2(\cos\theta) - c^2}_{\text{Real Part}} - \underbrace{c(\sin\theta)}_{\text{Imaginary Part}} \; i$$

From this we clearly observe $G = a + bi \in \mathbb{C}$ hence for $|G| \leq 1$ we require $\sqrt{a^2 + b^2} \leq 1$:

$$\sqrt{(1 + c^2(\cos\theta - 1))^2 + c^2\sin^2\theta} \leq 1 \iff (1 + c^2(\cos\theta - 1))^2 + c^2\sin^2\theta \leq 1$$

Expanding the brackets and simplifying terms we obtain:

$$2c^2(\cos\theta - 1) + c^4(\cos\theta - 1)^2 + c^2(\sin^2\theta) \leq 0$$

assuming $c \neq 0$, we divide by $c^2$ and use the identity $\sin^2\theta + \cos^2\theta = 1$ to obtain :

$$2(\cos\theta - 1) + c^2(\cos\theta - 1)^2 + 1 - \cos^2\theta \leq 0$$

$$\iff c^2(\cos\theta - 1)^2 \leq \cos^2\theta - 2\cos\theta + 1 = (\cos\theta - 1)^2$$

since we are attempting to show that the stability condition is met for all values of theta a small caveat should be briefly discussed here. For values of $\theta$ that result in $\cos\theta = 1$ the stability condition collapses to the trivially true condition that $0 \leq 0$, so we continue having accounted for this value of theta by dividing by $(\cos\theta - 1)$

$$c^2 \leq 1 \iff c \leq 1 \iff \frac{a\Delta t}{\Delta x} \leq 1$$

This concludes the stability analysis for the Lax-Wendroff Method.

### 2.2 MacCormack Method Stability

Since this method contains two steps but a stability condition is required for the whole method the predictor step has been substitued into the equation of the corrector step for the purpose of the stability analysis. This obtains:

$$u_i^{n+1} = \frac{1}{2}\left[(2u_i^n - c(u_{i+1}^n - u_i^n)) - c(u_i^n - c(u_{i+1}^n - u_i^n) - u_{i-1}^n + c(u_i^n - u_{i-1}^n))\right]$$

Now making the same substitutions as before we obtain:

$$\begin{aligned}\xi^{n+1}e^{I\theta i} = \frac{1}{2}[2\xi^n e^{I\theta i} - c(\xi^n e^{I\theta(i+1)} - \xi^n e^{I\theta i})\\ - c(\xi^n e^{I\theta i} - c(\xi^n e^{I\theta(i+1)} - \xi^n e^{I\theta}) - \xi^n e^{I\theta(i-1)} + c(\xi^n e^{I\theta i} - \xi^n e^{I\theta(i-1)}))]\end{aligned} \quad (7)$$

Dividing through by $\xi^n e^{I\theta i}$ and simplifying we obtain:

$$\frac{\xi^{n+1}}{\xi^n} = G = \frac{1}{2}\left[2 - c(e^{I\theta} - 1) - c(1 - c(e^{I\theta} - 1) - e^{-I\theta} + c(1 - e^{-I\theta}))\right]$$

$$= \frac{1}{2}\left[2 + c^2(e^{I\theta} + e^{-I\theta}) - 2c^2 - c(e^{I\theta} - e^{-I\theta}))\right]$$

making the same substitutions as before for $\cos\theta$ and $\sin\theta$ and simplifying the expression we obtain:

$$G = \underbrace{1 + c^2(\cos\theta) - c^2}_{\text{Real Part}} - \underbrace{c(\sin\theta)}_{\text{Imaginary Part}} \; i$$

This is ideal since its exactly the same condition we had previously hence we follow the same method to obtain the stability condition for $|G| \leq 1$:

$$(1 + c^2(\cos\theta - 1))^2 + c^2\sin^2\theta \leq 1$$

$$\sqrt{(1 + c^2(\cos\theta - 1))^2 + c^2\sin^2\theta} \leq 1 \iff (1 + c^2(\cos\theta - 1))^2 + c^2\sin^2\theta \leq 1$$

dividing by $c^2$ assuming $c \neq 0$ and using identity for $\sin^2\theta = 1 - \cos^2\theta$

$$2(\cos\theta - 1) + c^2(\cos\theta - 1)^2 + 1 - \cos^2\theta \leq 0$$

$$\iff c^2(\cos\theta - 1)^2 \leq \cos^2\theta - 2\cos\theta + 1 = (\cos\theta - 1)^2$$

Using the same reasoning as before to exclude the case $\cos\theta = 1$, we divide by $(\cos\theta - 1)$ to obtain:

$$c^2 \leq 1 \iff c \leq 1 \iff \frac{a\Delta t}{\Delta x} \leq 1$$

# 3 Part 2: Advection Equation (Linear Problem)

The next section will be dedicated to the advection equation (1) mentioned in the introduction of this report. Specifically investigations will be conducted to show how the numeric results of the methods changes as the $\Delta t$ value is changed keeping $\Delta x = 5$. Having done this further investigation is conducted for the Euler BTCS method to see how its parameters can be changed to achieve an improvement in its accuracy.

Before investigating all of the numerical results for the different methods it is important to obtain a holistic view of what the result of the problem was attempting to achieve.

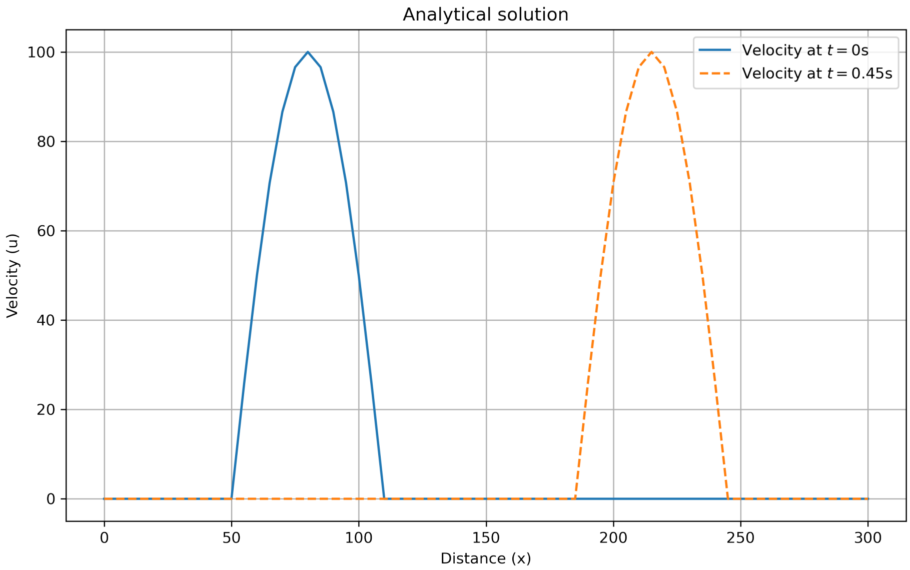


Figure 1: Velocity Profile

To that affect figure 1 shows how the analytical solution of the advection equation being investigated changes from its initial condition to its solution at $t = 0.45$s. We can see from this, the physical interpretation of what the advection equation is doing. We can see that as time passes the shape of the non-zero values of the initial conditions are moving from left to right. Specifically it is moving at a speed of $a = 300m/s$.

## 3.1 Numerical Methods Results

This section will discuss the results of implementing the following methods: First Upwind difference, Lax-Wendroff and the Euler BTCS, to solve the advection equation and evaluate its performance at the time $t = 0.45$s.

To break up the results for this section we will discuss the plots for all 3 methods for each $\Delta t \in \{0.018, 0.01666, 0.0075\}$ separately. Additionally, a table has been added for each $\Delta t$ value to summarise the quantitative results for the errors for the three methods under both the $L_1$ and $L_\infty$ norms, accompanied with a discussion.

### 3.1.1 Velocity profile when $\Delta t = 0.018$

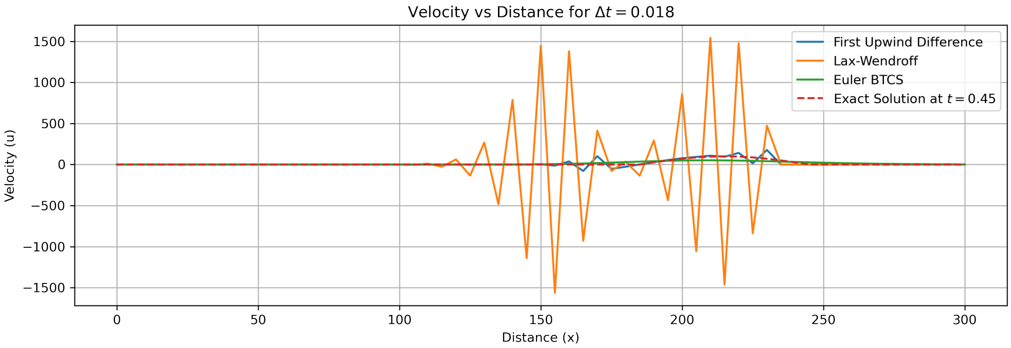


Figure 2: Velocity Profile when $\Delta t = 0.018$

Table 1: Error results when $\Delta t = 0.018$

| | $L_1$ | $L_{\text{inf}}$ |
|---|---|---|
| First Upwind Difference | 653.89 | 106.541 |
| Lax-Wendroff | 17358.831 | 1561.952 |
| Euler BTCS | 590.105 | 49.754 |

From figure 2 we can clearly see that the Lax-Wendroff method has become unstable and diverged from the analytical solution. This is expected since based on the parameters used for this simulation $c = \frac{300\times0.018}{5} = 1.08 > 1$ violates the stability condition of $c \leq 1$. Interestingly if we compare where the instability is at its greatest which is near $x \approx 160, 240$. To the analytical solution we notice that this lies close to where the initial condition transitions from a zeros to non-zero value, more specifically the discontinuities of the initial condition. This makes intuitive sense since the the upwind method is calculating numerical gradients as part of its numerical method and these discontinuities cause a big change in that calculation. This connects to the wider issue that explicit methods experience of numerical diffusion which causes oscillatory behaviour near gradient discontinuities, this will be seen and explored in further analysis.

Before we continue since it is not possible to see how other 2 methods have done at approximating the analytical solution we plot again with $\Delta t = 0.018$ removing the Lax-Wendroff method.

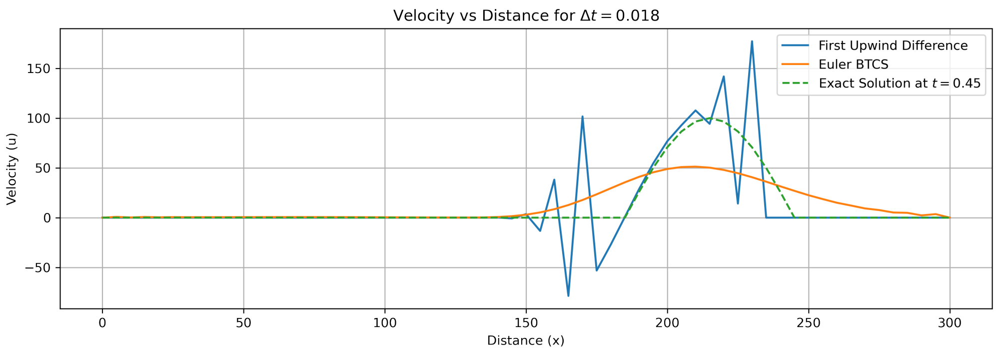


Figure 3: Velocity Profile when $\Delta t = 0.018$ excluding Lax-Wendroff Method

From figure 3 we can see how the remaining two methods have approximated their solution compared to the analytical solution at $t = 0.45$. When we look at the first upwind difference method we again see that it has also become unstable, this again is expected since as discussed in lecture the stability condition is again $c \leq 1$ which has as with the Lax-Wendroff method has been violated. Note we again see a similar trend that the error is at its largest near the gradient discontinuities. What we see more from this method is that the larger error appear to the left of both of these discontinuities. This again makes intuitive sense since the wave is moving from left to right so the errors are accumulating from the left and moving to the right, behind the gradient discontinuity points of the moving wave.

Finally we see that, although the Euler BTCS from lectures is unconditionally stable, it appears to be far from from the analytical solution, specifically its amplitude appears decreased and phase seems to be slightly out from the analytical solution. Possible reasons for this may be that the grid size of $\Delta x = 5$ and $\Delta t = 0.018$ are too large for accurate results or more likely the method is experiencing numerical dissipation and numerical dispersion. These are both symptoms of the implicit nature of the method.

Specifically the phase error is caused by dispersive errors resulting from the BTCS method. This is caused by the central difference used in the method which results in errors in the phase of the wave propagation. Secondly, the reduced amplitude is caused by the smoothing affect that occurs because the method attempts to average future states leading to a smoother solution.

### 3.1.2 Velocity profile when $\Delta t = 0.01666$

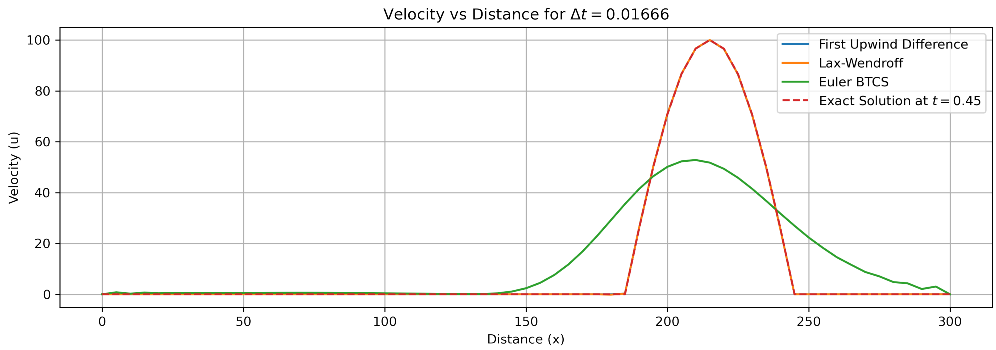


Figure 4: Velocity Profile when $\Delta t = 0.01666$

Table 2: Error results when $\Delta t = 0.01666$

| | $L_1$ | $L_{\text{inf}}$ |
|---|---|---|
| First Upwind Difference | 2.16 | 0.279 |
| Lax-Wendroff | 2.474 | 0.416 |
| Euler BTCS | 569.98 | 48.218 |

We now move on to the results when $\Delta t = 0.01666$ keeping $\Delta x = 5$. We note here that with this updated value of $\Delta t$ the value of $c = \frac{300 \times 0.01666}{5} = 0.9996 < 1$ which implies that we expect all of the three methods to now be stable.

As we can see from figure 4 all three methods appear to be stable and both the First Upwind Difference and Lax-Wendroff methods have significantly improved to the point where they appear to overlap the analytical solution. Similar to when $\Delta t = 0.018$ the Euler BTCS methods still appears to experience the same effects of numerical dissipation (reduction of amplitude) and numerical dispersion (phase errors). However it should be noted here that reducing the $\Delta t$ value has improved the results achieved from this method. This can be quantified by a reduction in error in both the $L_{\text{inf}}$ and $L_2$ norms slightly when comparing table 2 to table 1.

An aside is taken here as to consider why this value of $\Delta t = 0.01666$ was selected. Although it appears as a random choice at first for the time step size this specific value is specific to the setup of the problem being discussed. Firstly to ensure stability, we ensure that the condition $c \leq 1$ is satisfied. To this affect we hence calculate the value of $\Delta t$ when $c = 1$. This being $1 = \frac{300 \times \Delta t}{5}$, which when rearranged is $\frac{5}{300} \approx 0.01666$. This is very important as not only has the stability condition been satisfied but also this value aligns the spatial grid and the velocity of the travelling wave perfectly. This has the impact that for each $\Delta t$ time step taken in the simulation the travelling wave solution moves exactly one grid space of $\Delta x = 5$ along. This alignment causes the minimisation of numerical dispersion and provides a more accurate representation of the wave's speed and direction. This results in greatly reduced errors which is quantified by a reduction in both the $L_1$ and $L_\infty$ norms.

### 3.1.3 Velocity profile when $\Delta t = 0.0075$

The next simulation we will investigate is when $\Delta t = 0.0075$. Again as in the previous results with this value of $\Delta t$ the stability condition for all three methods has been satisfied.

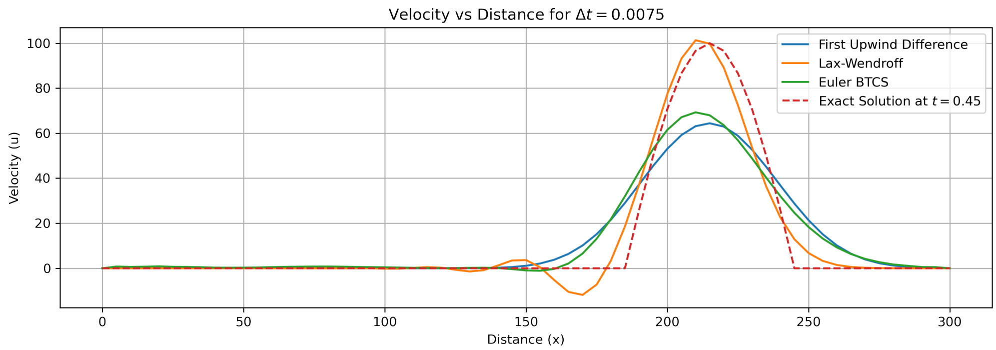


Figure 5: Velocity Profile when $\Delta t = 0.0075$

Table 3: Error results when $\Delta t = 0.0075$

| | $L_1$ | $L_{\text{inf}}$ |
|---|---|---|
| First Upwind Difference | 405.913 | 35.583 |
| Lax-Wendroff | 187.876 | 18.569 |
| Euler BTCS | 382.581 | 33.019 |

We can see from figure 5 that these results appear less accurate to the analytical solution than figure 4 which is surprising considering that the value of $\Delta t$ has been reduced which usually results in more accurate numerical approximations. We first discuss the Euler BTCS methods which has actually reduced in both norms, showing that it has benefited from the smaller value of $\Delta t$ compared to the other 2 methods.

We next discuss the results of the First Upwind Difference Method and Lax-Wendroff method in tandem. As discussed in the results of the previous section the value of $\Delta t = 0.01666$ was optimal as it meant that $c \approx 1$ resulting in the alignment of the spatial grid with the travelling wave. What we see here is what happens when this alignment is not present. Although the time-step for this simulation is smaller it has resulted in this alignment to no longer be present. A physical interpretation of this would be that for every time step that has passed the traveling wave solution has not moved by one spatial grid point $\Delta x = 5$ and as a result an error is introduced to the approximation of the numerical solution. For the Lax-Wendroff method this has the effect of a phase error accumulating which is why the peak of its wave appears to be behind the analytical solution. In addition to this due to the numerical diffusion oscillatory errors have emerged near the gradient discontinuity which is again caused by the numerical solution attempting to deal with this jump from the zero value to non-zero of the travelling wave solution.

For the First Upwind Difference method it appears as though its amplitude has been greatly reduced which is caused by the method not sufficiently capturing the sharp features of the exact speed of the wave. This has caused the wave to experience a smearing or damping effect as it attempts to smooth out the numerical solution for the problem.

## 3.2 Improved Euler BTCS Method

As we have seen from the results of the previous simulations, the reason the Euler BTCS method struggled at providing accurate solutions for the analytical solutions is because it experienced numerical dissipation and numerical dispersion, due to it implicit nature. It was shown thought that the error of the results reduced as the $\Delta t$ value was reduced. To address both of these points an improvement was devised by adjusting the spatial grid and varying the $\Delta t$ values at much smaller values to help the numerical method improve its solution by reducing its error. The new spatial grid had $\Delta x = 0.1$ and the time steps were updated to $\Delta t \in \{0.000333, 0.0001, 0.00001\}$. The choice for this selection was one to show that as $\Delta t$ the method results continued to improve, the second was to include $\Delta t = 0.000333$ which corresponded to when when $c \approx 1$ to show that for the implicit method this did not significantly improve the method, like was the case with the two explicit method. This is likely due to the types of errors that are exhibited in each respective methods, and that implicit methods do not calculate the next time step in the same way as explicit methods. The value of $\Delta t$ was not reduced further as after this point the computational cost became too much considering the gain was minimal.

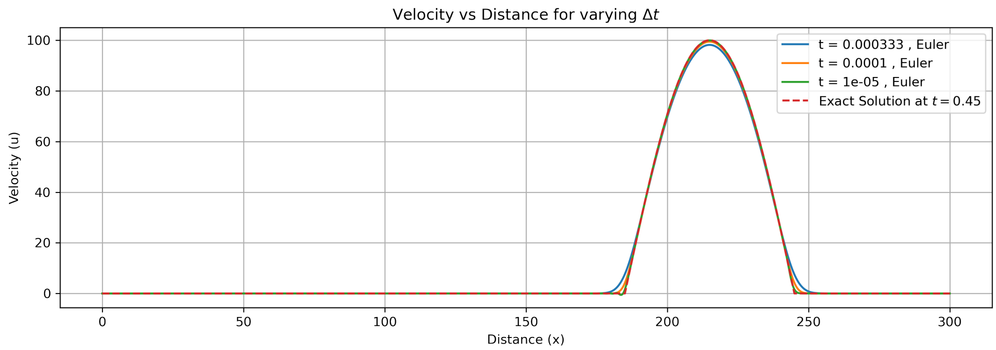


Figure 6: Velocity Profile varying $\Delta t$

Table 4: Error results when varying $\Delta t$

| | $L_1$ | $L_{\text{inf}}$ |
|---|---|---|
| $t = 0.000333$ | 1311.74 | 7.672 |
| $t = 0.0001$ | 411.65 | 4.194 |
| $t = 0.00001$ | 71.139 | 1.611 |

Table 4 shows how as the $\Delta t$ value was reduced the overall errors of the method decreased both in the $L_1$ and $L_\infty$ norms. Additionally, when at the smallest $\Delta t$ value the error is the smallest across all simulations for Euler BTCS, this is more impressive for the $L_1$ norm since by reducing the $\Delta x$ value, more evaluation points for the error had been introduced which usually meant that the overall error of $L_1$ appeared bigger even though it visually was a better fit. As seen in table 4 for the largest $\Delta t$ value which is the largest $L_1$ norm across any simulation for the Euler BTCS method.

The impact of these results shows that the Euler BTCS method could be improved significantly by adjusting the parameters, but the impact of this was that the computaitonal cost increased proportionally. Also even when these parameters were changed they still didn't result in as accurate results compared to the explicit methods when $c = 1$. The conclusion from this is the explicit methods were better methods for investigating this advection equation.

# 4 Part 3: Inviscid Burgers' Equation (Non-Linear Problem)

In this section of the report we will be discussing investigations into equation (2) mentioned in the introduction. To analyse this PDE we will be using two numerical methods, these being the MacCormack Method (6) and the Lax method which for the Burgers' equation becomes:

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{\Delta t}{2\Delta x}(E_{i+1}^n - E_{i-1}^n) \tag{8}$$

where $E_i^n = \frac{(u_i^n)^2}{2}$ and $E_i^* = \frac{(u_i^*)^2}{2}$

## 4.1 Numerical Method Results

For the simulations of the PDE we will be using $\Delta x = 0.1$ and vary $\Delta t \in \{0.14, 0.1, 0.05\}$ for each of these $\Delta t$ values we will show the solution of the numerical results at $T \in \{1.2, 1.8, 2.4\}$.

### 4.1.1 Velocity profile when $\Delta t = 0.14$

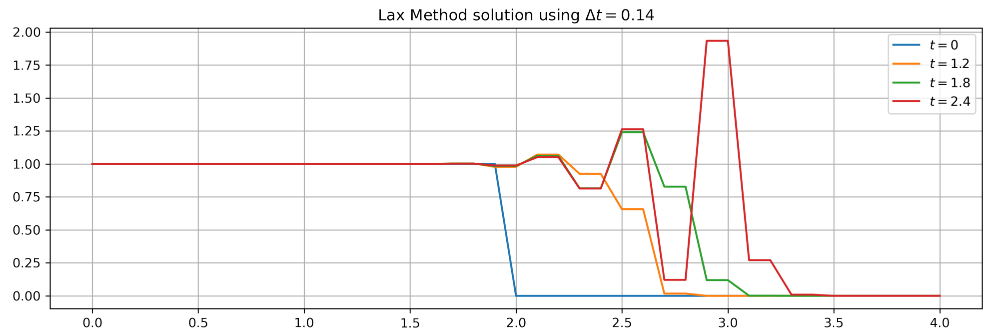


Figure 7: Velocity Profile when $\Delta t = 0.14$

As we can see in figure 7 the solution wave seems to be increasing in amplitude as the time of simulation increases. The initial condition at $t = 0$ has been included to help demonstrate the progression of the wave as time passes. The behaviour of this system appears unstable and this can be verified by the fact the for this system the CFL condition for stability of the Lax method is $|\frac{u\Delta t}{\Delta x}| \leq 1$, as we can see from the initial condition the maximum value of $u(x,t) = 1$, hence plugging this into the stability condition, $\frac{1 \times 0.14}{0.1} = 1.4 > 1$ so the stability condition in violated, this explains why as the simulation progresses the instability appears, in addition to this the oscillatory behaviour exhibited is typical of an explicit method showing numerical dissipation caused by the steep gradient of the initial condition.

From figure 8 we can clearly see that the simulation has become unstable since even at the first time step of $t = 1.2$s the solution has exploded and after this point the later time steps run into overflow errors when being calculated. This is again expected since similarly to the Lax method the stability of

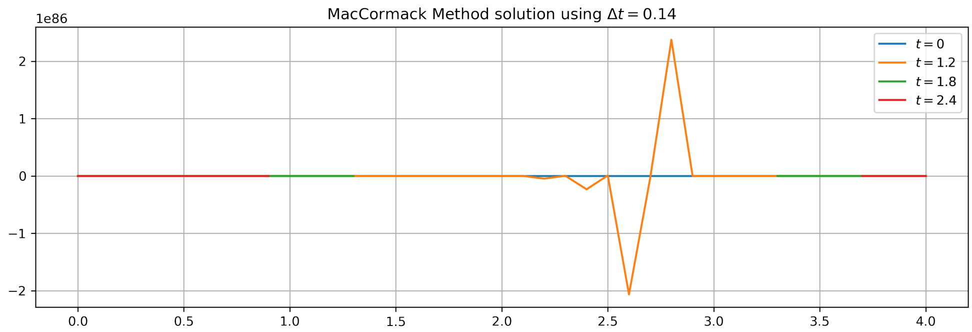


Figure 8: Velocity Profile when $\Delta t = 0.14$

$|\frac{u\Delta t}{\Delta x}| \leq 1$, has again been violated. A few reasons as to why the solution for this method appears to diverge faster than the Lax method are that it introduces more numerical dissipation because it averages the solution between time steps. Due to the predictor-corrector nature of the method once the method becomes unstable errors from the predictor step can be amplified if the corrector step. After this time step the information for the subsequent 2 time steps has lost its value due to the overflow and hence will not be investigated further.

#### 4.1.2 Velocity profile when $\Delta t = 0.1$

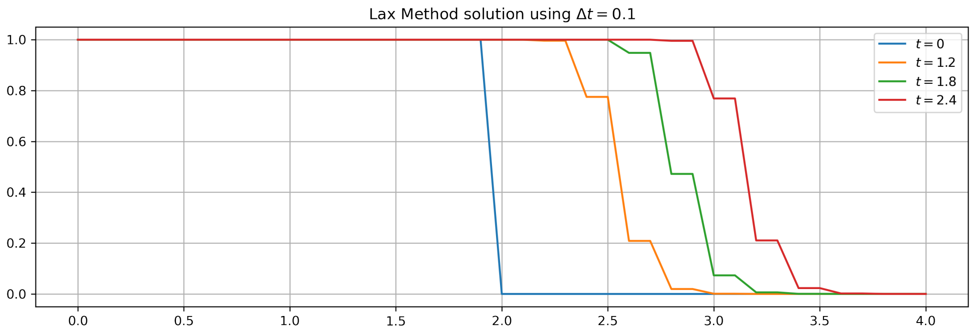


Figure 9: Velocity Profile when $\Delta t = 0.1$

We can see from figure 9 that the method now appears stable which agrees with our analysis since now $\frac{1\times 0.1}{0.1} = 1$. What we notice from this graph is that solution exhibits a travelling wave solution, however the solution to become less smooth as the simulation progresses. This is likely due to the nature of the Burgers equation itself as it creates a steepening of waveforms as time passes. This steepening occurs because faster-moving characteristics catch up to slower ones, leading to gradients becoming increasingly steep over time

From figure 10 we notice that the MacCormack method also appears stable now for the same reasons mentioned previously. We notice however a key difference between the two method here being that the solution curves appear much smoother than for the Lax method. This could be attributed to the second order nature of the method, even though both methods are second order, the predictor-corrector nature of the MacCormack method can often provide a better balance of numerical dissipation than the Lax approach. In addition the predictor-corrector structure allows for a more adaptive response to sharp gradients and discontinuities, which is what is seen in the initial condition.

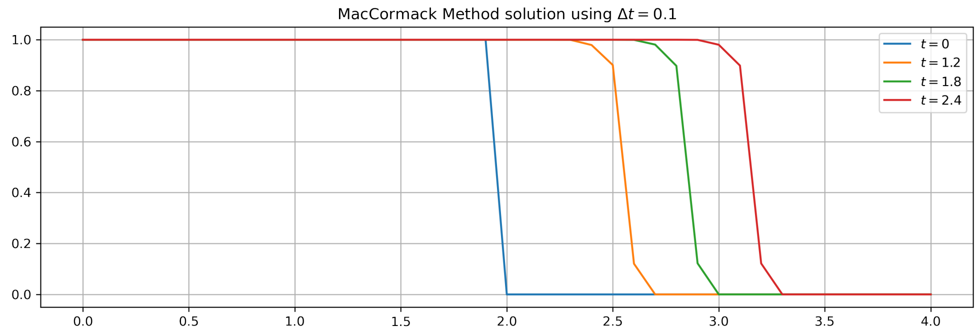


Figure 10: Velocity Profile when $\Delta t = 0.1$

#### 4.1.3 Velocity profile when $\Delta t = 0.05$

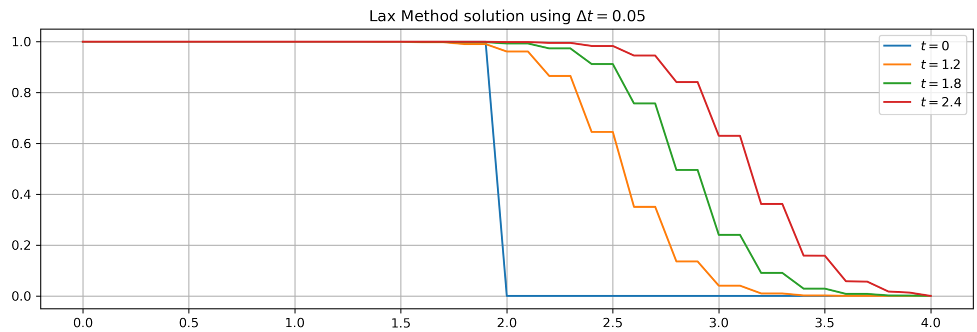


Figure 11: Velocity Profile when $\Delta t = 0.05$

From the figure 11 we can see the nature of the solution has changed slightly than from previous time steps. We notice that again a less smooth solution. What is unique about this simulation is that it appears that instead of a constant shape travelling wave solution it appears to change shape as time progresses. It appears more elongated, showing a less sharp change in $u(x, t)$ compared to the initial condition, again this could be connected with the nature of the Burgers equation itself showing different points on the wave moving at different velocities. It is difficult to interpret the results further since there is no analytical solution to compare them against. However, based on previous simulations it appears as though this is the outlier and hence is likely not exhibiting the true nature of what the results of the simulation should look like.

In this final figure 12 we again see a much smoother solution as expected due to the predictor-corrector type of method. In addition, we notice oscillations forming at the top of the wave front for each solution. This could be caused by numerical dispersion or even predictor-corrector steps overcompensating and then correcting this overcompensation.

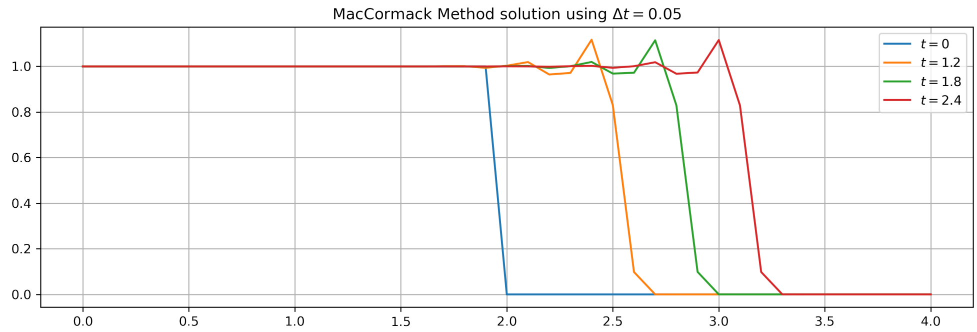


Figure 12: Velocity Profile when $\Delta t = 0.05$

# 5 Discussion and Conclusion

From the analysis above we can see how the different numerical methods employed affect how the numerical solution converges to the solution of this PDE. We discuss the results of each part chronologically.

In part 1, the Von-Neuman stability analysis was conducted for both the Lax-Wendroff and MacCormack method. It was found that both method had the requirement that $c = \frac{a\Delta t}{\Delta x} \leq 1$. As breifely mentioned in that section in addition both the method were also consistent meaning that if the stability condition was satisfied then this ensured convergence for the method.

In part 2, the First Upwind Difference, Lax-Wendroff and Euler BTCS method were employed to solve the advection equation posed. We showed that when $\Delta t = 0.018$ the First Upwind Difference and Lax-Wendroff methods were unstable as expected and the Euler BTCS experiences a phase error and amplitude reduction. When $\Delta t = 0.01666$ the best results were achieved for the First Upwind Difference and Lax-Wendroff methods, this being attributed to the fact that this is when $c = 1$ and the spatial grid aligned with the speed of the travelling wave solution minimising error. The Euler BTCS also improved slightly. Finally for $\Delta t = 0.0075$ the first two methods appeared slightly less effective and this was attributed to the fact that the spatial grid no longer coincided with the speed of the travelling wave solution causes the propagation of errors. However the Euler BTCS method did improve again, showing that a reduction in $\Delta t$ improved its accuracy for each simulation.

The last part of the section 2 discussed and implemented how the Euler BTCS method could be improved. It was shown that if the spatial and temporal grid size was reduced significantly then the results became much more accurate. It also showed that unlike the explicit method it did not benefit as much from the alignment of the spatial/temporal grid and the speed of the wave. The final results for this simulation resulted in errors far lower than any of the previous simulations for the Euler BTCS, showing the trade off that could be accomplished of computational cost and accuracy. However, they still were larger than those of the explict method when $\Delta t = 0.01666$.

Part 3 discussed the implementation of the Lax and MacCormack methods in solving the non-linear Burgers equation. It showed again how the stability condition for both these methods affected the results at $\Delta t = 0.14$. For the remaining two $\Delta t$ values it was shown how each of the methods exhibited different types of solutions and a discussion was provided as to the rational behind why they appeared as they did. Unlike part 2, no analytical solution was provided so it is difficult to say which method performed better as it was not possible to deduce errors, but it appeared as though each method had its benefits depending on the needs of the solution.