# Burrows-Wheeler Transform and FM Index

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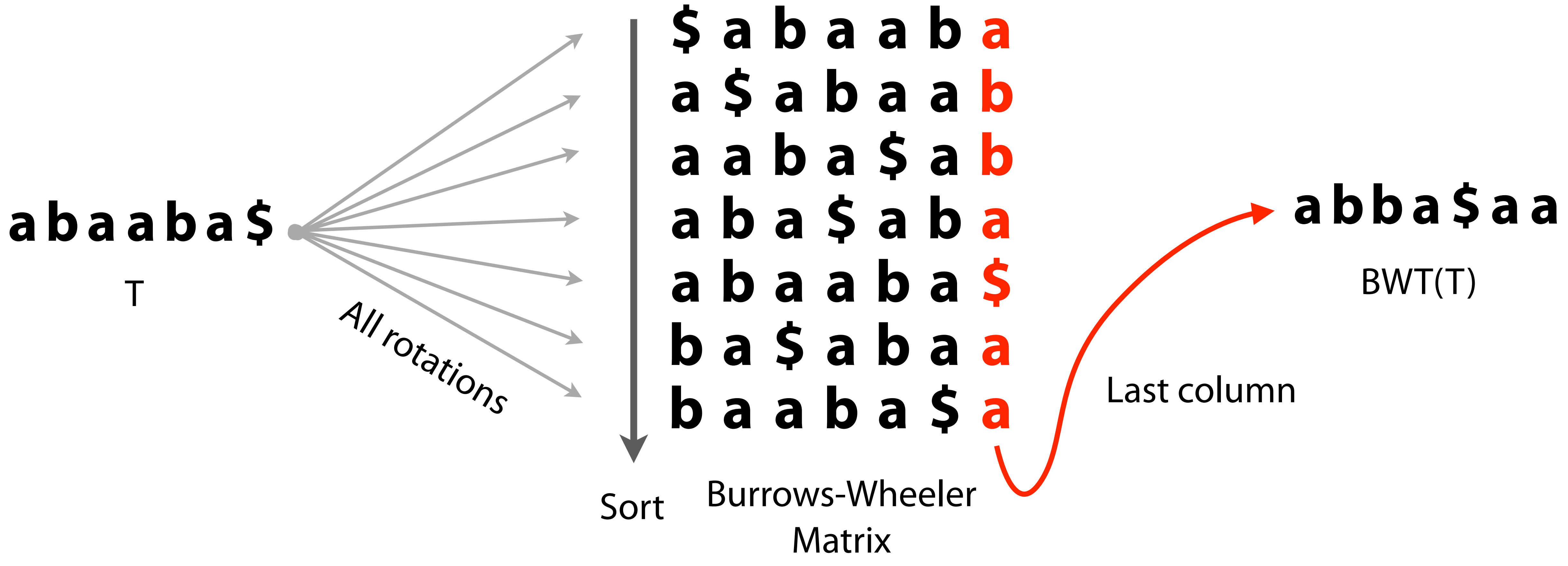
# Burrows-Wheeler Transform

*Reversible permutation* of the characters of a string, used originally for compression



How is it useful for compression? How is it reversible? How is it an index?

Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm. *Digital Equipment Corporation, Palo Alto, CA* 1994, Technical Report 124; 1994

# Burrows-Wheeler Transform

```
def rotations(t):
    """ Return list of rotations of input string t """
    tt = t * 2
    return [ tt[i:i+len(t)] for i in xrange(0, len(t)) ]

def bwm(t):
    """ Return lexicographically sorted list of t’s rotations """
    return sorted(rotations(t))

def bwtViaBwm(t):
    """ Given T, returns BWT(T) by way of the BWM """
    return ''.join(map(lambda x: x[-1], bwm(t)))
```

Make list of all rotations

Sort them

Take last column

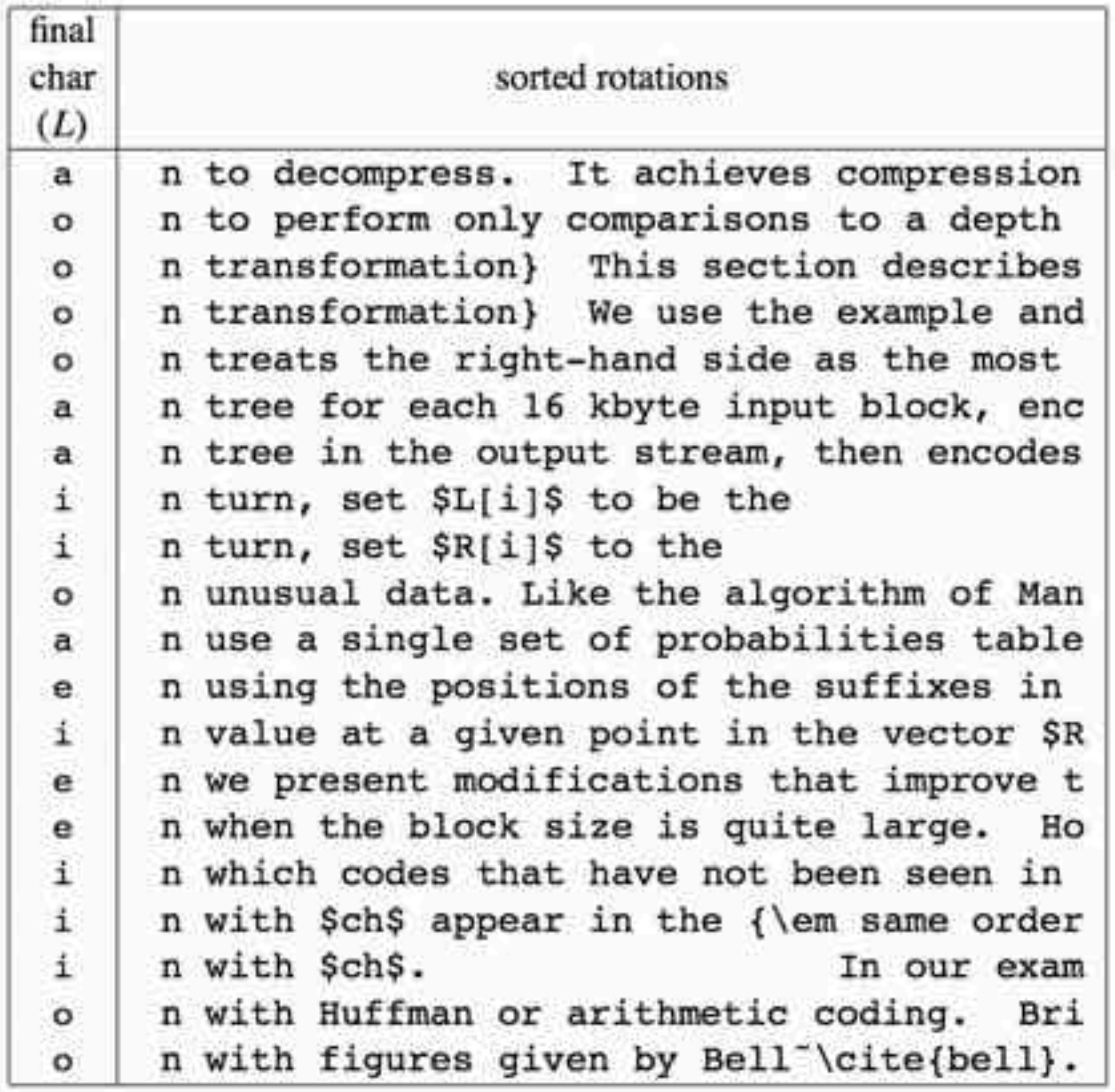
```
>>> bwtViaBwm("Tomorrow_and_tomorrow_and_tomorrow$")
'w$wwdd__nnoooaattTmmmrrrrrrooo__ooo'

>>> bwtViaBwm("It_was_the_best_of_times_it_was_the_worst_of_times$")
's$esttssfftteww_hhmmbootttt_ii__woeeaaressIi_______'

>>> bwtViaBwm('in_the_jingle_jangle_morning_Ill_come_following_you$')
'u_gleeeengj_mlhl_nnnnt$nwj__lggIolo_iiiiarfcmylo_oo_'
```

Python example: http://nbviewer.ipython.org/6798379

# Burrows-Wheeler Transform

Characters of the BWT are sorted by their *right-context*

This lends additional structure to BWT(T), tending to make it more compressible



| final char ($L$) | sorted rotations |
|---|---|
| a | n to decompress.  It achieves compression |
| o | n to perform only comparisons to a depth |
| o | n transformation}  This section describes |
| o | n transformation}  We use the example and |
| o | n treats the right-hand side as the most |
| a | n tree for each 16 kbyte input block, enc |
| a | n tree in the output stream, then encodes |
| i | n turn, set $L[i]$ to be the |
| i | n turn, set $R[i]$ to the |
| o | n unusual data. Like the algorithm of Man |
| a | n use a single set of probabilities table |
| e | n using the positions of the suffixes in |
| i | n value at a given point in the vector $R |
| e | n we present modifications that improve t |
| e | n when the block size is quite large.  Ho |
| i | n which codes that have not been seen in |
| i | n with $ch$ appear in the {\em same order |
| i | n with $ch$.                    In our exam |
| o | n with Huffman or arithmetic coding.  Bri |
| o | n with figures given by Bell~\cite{bell}. |

Figure 1: Example of sorted rotations. Twenty consecutive rotations from the sorted list of rotations of a version of this paper are shown, together with the final character of each rotation.

Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm. *Digital Equipment Corporation, Palo Alto, CA* 1994, Technical Report 124; 1994

# Burrows-Wheeler Transform

BWM bears a resemblance to the suffix array

```
$ a b a a b a
a $ a b a a b
a a b a $ a b
a b a $ a b a
a b a a b a $
b a $ a b a a
b a a b a $ a
```

BWM(T)

| | |
|---|---|
| 6 | $ |
| 5 | a $ |
| 2 | a a b a $ |
| 3 | a b a $ |
| 0 | a b a a b a $ |
| 4 | b a $ |
| 1 | b a a b a $ |

SA(T)

Sort order is the same whether rows are rotations or suffixes

# Burrows-Wheeler Transform

In fact, this gives us a new definition / way to construct BWT(T):

$$BWT[i] = \begin{cases} T[SA[i]-1] & \text{if } SA[i] > 0 \\ \$ & \text{if } SA[i] = 0 \end{cases}$$

"BWT = characters just to the left of the suffixes in the suffix array"

```
$ a b a a b a
a $ a b a a b
a a b a $ a b
a b a $ a b a
a b a a b a $
b a $ a b a a
b a a b a $ a
```

BWM(T)

| | |
|---|---|
| 6 | $ |
| 5 | a $ |
| 2 | a a b a $ |
| 3 | a b a $ |
| 0 | a b a a b a $ |
| 4 | b a $ |
| 1 | b a a b a $ |

SA(T)

# Burrows-Wheeler Transform

```
def suffixArray(s):
    """ Given T return suffix array SA(T).  We use Python's sorted
        function here for simplicity, but we can do better. """
    satups = sorted([(s[i:], i) for i in xrange(0, len(s))])
    # Extract and return just the offsets
    return map(lambda x: x[1], satups)
```

Make suffix array

```
def bwtViaSa(t):
    """ Given T, returns BWT(T) by way of the suffix array. """
    bw = []
    for si in suffixArray(t):
        if si == 0: bw.append('$')
        else: bw.append(t[si-1])
    return ''.join(bw) # return string-ized version of list bw
```

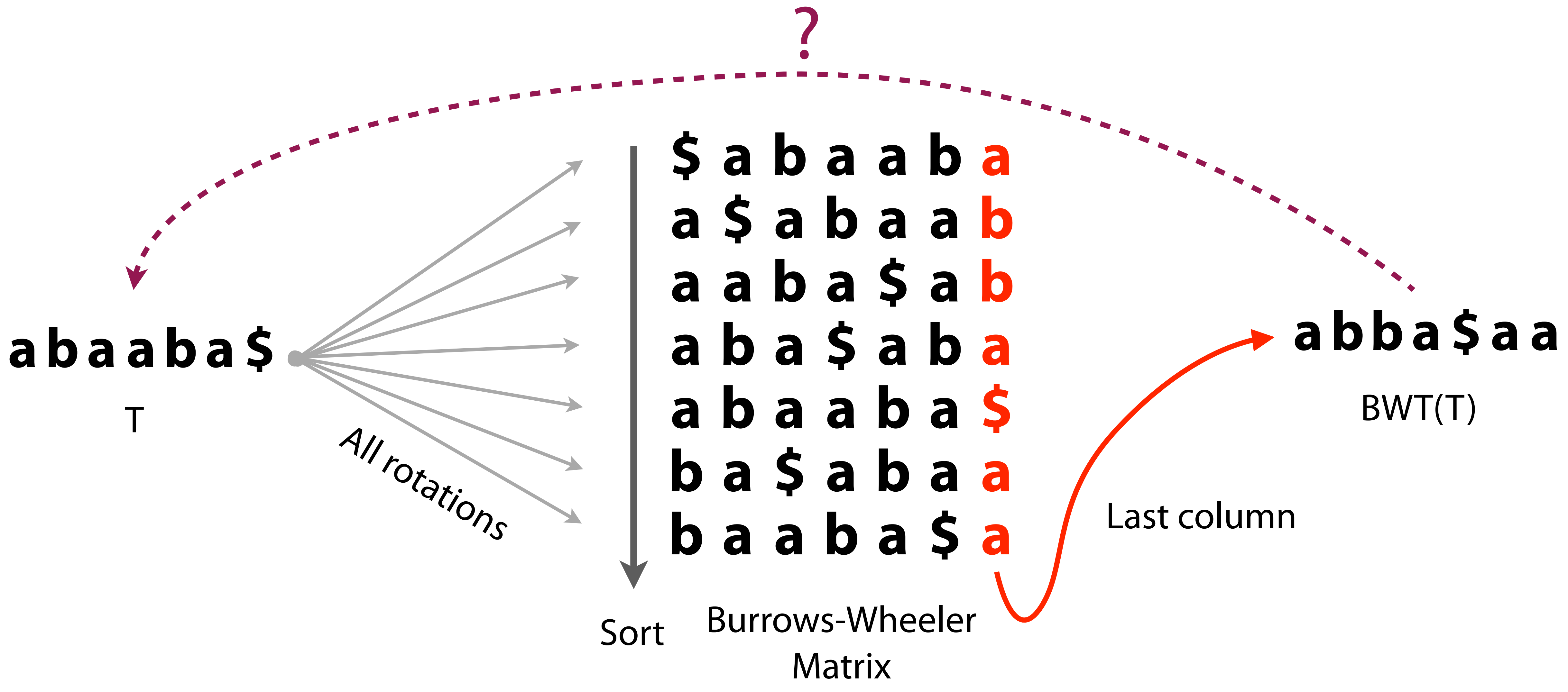
Take characters just to the left of the sorted suffixes

```
>>> bwtViaSa("Tomorrow_and_tomorrow_and_tomorrow$")
'w$wwdd__nnoooaattTmmmrrrrrrooo__ooo'

>>> bwtViaSa("It_was_the_best_of_times_it_was_the_worst_of_times$")
's$esttssfftteww_hhmmbootttt_ii__woeeaaressIi_______'

>>> bwtViaSa('in_the_jingle_jangle_morning_Ill_come_following_you$')
'u_gleeeengj_mlhl_nnnnt$nwj__lggIolo_iiiiarfcmylo_oo_'
```

Python example: http://nbviewer.ipython.org/6798379

# Burrows-Wheeler Transform

How to reverse the BWT?



BWM has a key property called the *LF Mapping*...

# Burrows-Wheeler Transform: T-ranking

Give each character in *T* a rank, equal to # times the character occurred previously in *T*. Call this the *T-ranking*.

$$\mathbf{a}_0\ \mathbf{b}_0\ \mathbf{a}_1\ \mathbf{a}_2\ \mathbf{b}_1\ \mathbf{a}_3\ \$$$

Now let's re-write the BWM including ranks...

# Burrows-Wheeler Transform

BWM with T-ranking:

$$\begin{array}{ccccccc}
F & & & & & & L \\
\$ & a_0 & b_0 & a_1 & a_2 & b_1 & \mathbf{a_3} \\
\mathbf{a_3} & \$ & a_0 & b_0 & a_1 & a_2 & b_1 \\
\mathbf{a_1} & a_2 & b_1 & a_3 & \$ & a_0 & b_0 \\
\mathbf{a_2} & b_1 & a_3 & \$ & a_0 & b_0 & \mathbf{a_1} \\
\mathbf{a_0} & b_0 & a_1 & a_2 & b_1 & a_3 & \$ \\
b_1 & a_3 & \$ & a_0 & b_0 & a_1 & \mathbf{a_2} \\
b_0 & a_1 & a_2 & b_1 & a_3 & \$ & \mathbf{a_0}
\end{array}$$

Look at first and last columns, called *F* and *L*

And look at just the **a**s

**a**s occur in the same order in *F* and *L*. As we look down columns, in both cases we see: $\mathbf{a_3}$, $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_0}$

# Burrows-Wheeler Transform

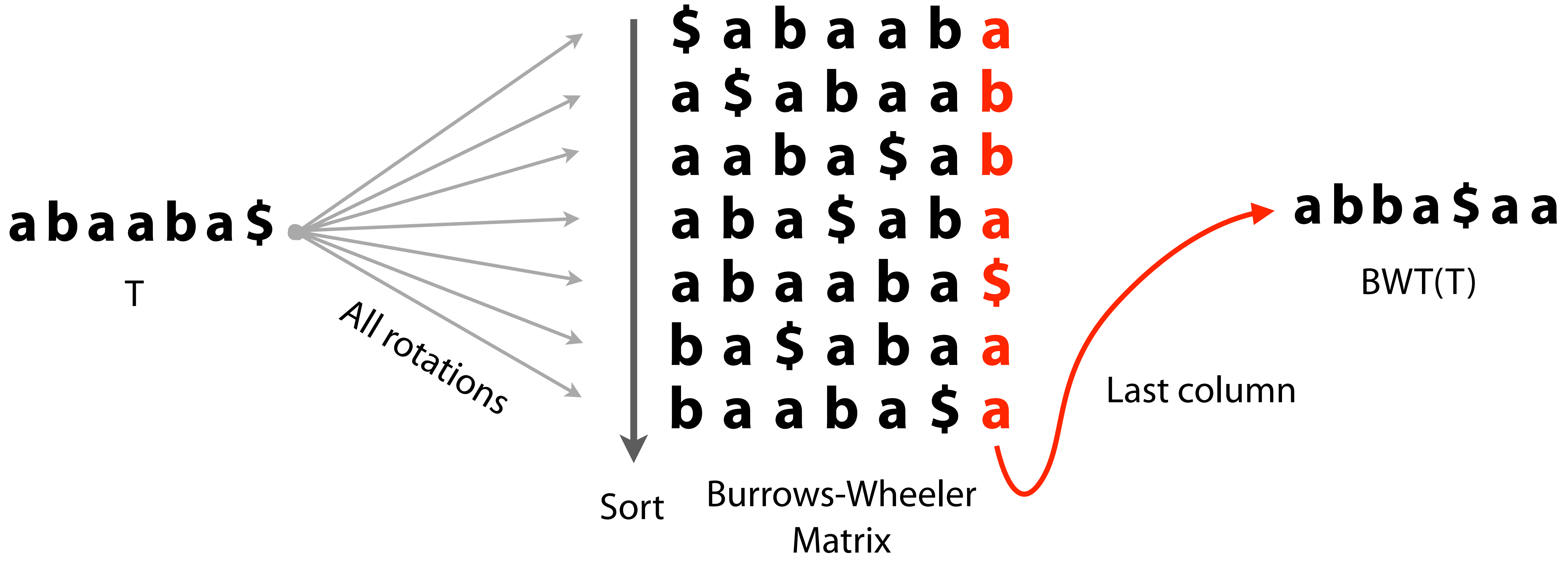
BWM with T-ranking:

$$\begin{array}{ccccccc}
F & & & & & & L \\
\$ & a_0 & b_0 & a_1 & a_2 & b_1 & a_3 \\
a_3 & \$ & a_0 & b_0 & a_1 & a_2 & \mathbf{b}_1 \\
a_1 & a_2 & b_1 & a_3 & \$ & a_0 & \mathbf{b}_0 \\
a_2 & b_1 & a_3 & \$ & a_0 & b_0 & a_1 \\
a_0 & b_0 & a_1 & a_2 & b_1 & a_3 & \$ \\
\mathbf{b}_1 & a_3 & \$ & a_0 & b_0 & a_1 & a_2 \\
\mathbf{b}_0 & a_1 & a_2 & b_1 & a_3 & \$ & a_0
\end{array}$$

Same with **b**s: $\mathbf{b}_1$, $\mathbf{b}_0$

# Burrows-Wheeler Transform

*Reversible permutation* of the characters of a string, used originally for compression



How is it useful for compression? How is it reversible? How is it an index?

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# Burrows-Wheeler Transform: LF Mapping

BWM with T-ranking:

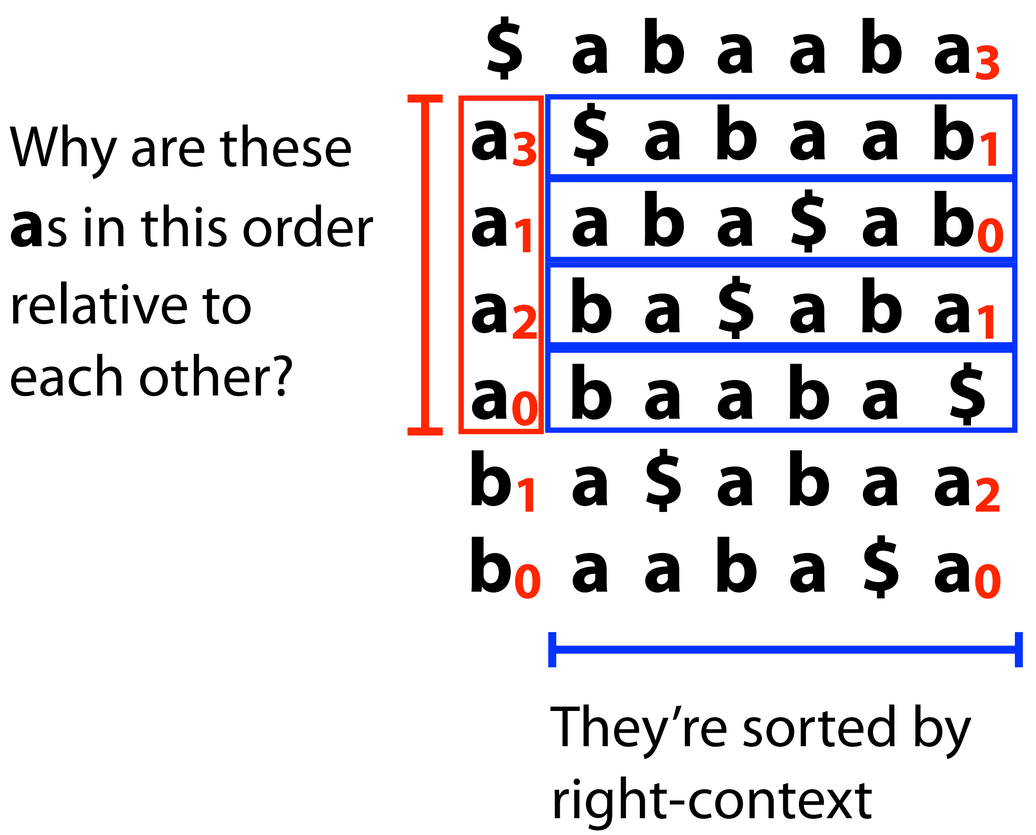
$$
\begin{array}{ccccccc}
F & & & & & & L \\
\$ & a_0 & b_0 & a_1 & a_2 & b_1 & a_3 \\
a_3 & \$ & a_0 & b_0 & a_1 & a_2 & b_1 \\
a_1 & a_2 & b_1 & a_3 & \$ & a_0 & b_0 \\
a_2 & b_1 & a_3 & \$ & a_0 & b_0 & a_1 \\
a_0 & b_0 & a_1 & a_2 & b_1 & a_3 & \$ \\
b_1 & a_3 & \$ & a_0 & b_0 & a_1 & a_2 \\
b_0 & a_1 & a_2 & b_1 & a_3 & \$ & a_0
\end{array}
$$

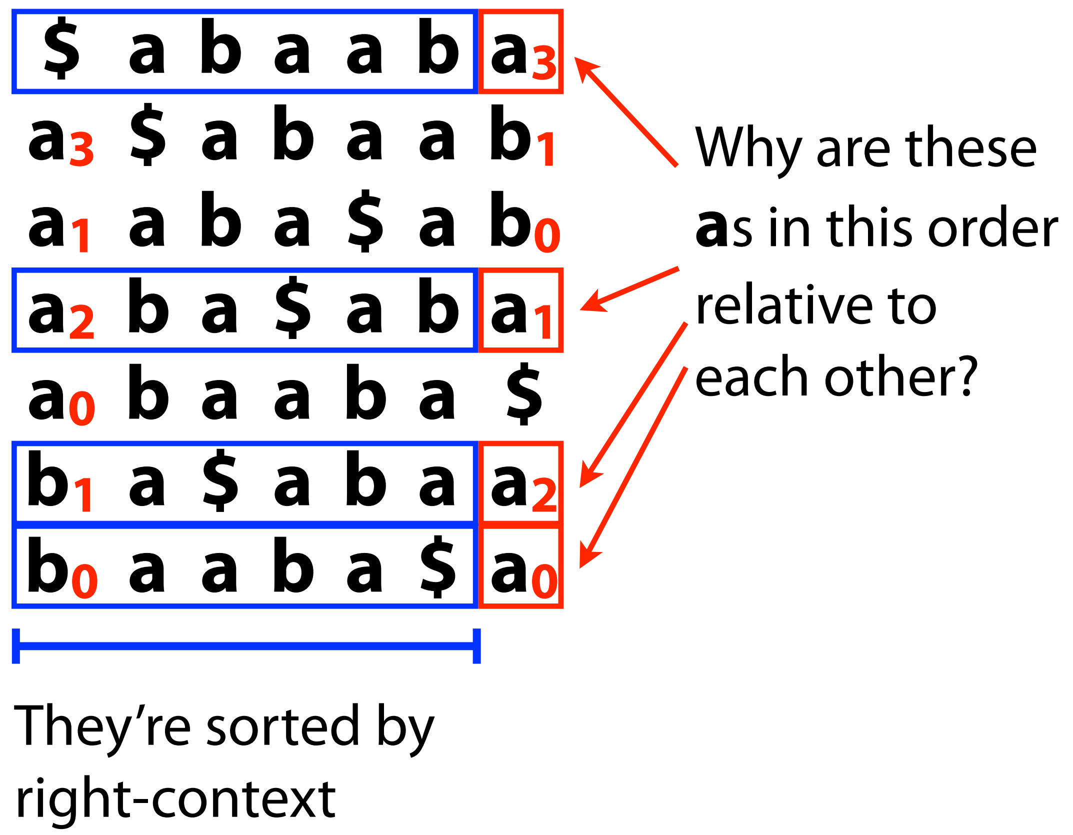
LF Mapping: The *i*th occurrence of a character *c* in *L* and the *i*th occurrence of *c* in *F* correspond to the *same* occurrence in *T*

However we rank occurrences of *c*, ranks appear in the same order in *F* and *L*

# Burrows-Wheeler Transform: LF Mapping

Why does the LF Mapping hold?





Occurrences of *c* in *F* are sorted by right-context. Same for *L*!

Whatever ranking we give to characters in *T*, rank orders in *F* and *L* will match

# Burrows-Wheeler Transform: LF Mapping

BWM with T-ranking:

$$\begin{array}{ccccccc}
F & & & & & & L \\
\$ & a_0 & b_0 & a_1 & a_2 & b_1 & a_3 \\
a_3 & \$ & a_0 & b_0 & a_1 & a_2 & b_1 \\
a_1 & a_2 & b_1 & a_3 & \$ & a_0 & b_0 \\
a_2 & b_1 & a_3 & \$ & a_0 & b_0 & a_1 \\
a_0 & b_0 & a_1 & a_2 & b_1 & a_3 & \$ \\
b_1 & a_3 & \$ & a_0 & b_0 & a_1 & a_2 \\
b_0 & a_1 & a_2 & b_1 & a_3 & \$ & a_0
\end{array}$$

We'd like a different ranking so that for a given character, ranks are in ascending order as we look down the F / L columns...

# Burrows-Wheeler Transform: LF Mapping

BWM with B-ranking:

$$
\begin{array}{c|ccccccc|l}
 & F & & & & & & L & \\
\downarrow & \$ & a_3 & b_1 & a_1 & a_2 & b_0 & a_0 & \downarrow \\
 & a_0 & \$ & a_3 & b_1 & a_1 & a_2 & b_0 & \\
 & a_1 & a_2 & b_0 & a_3 & \$ & a_3 & b_1 & \\
 & a_2 & b_0 & a_0 & \$ & a_3 & b_1 & a_1 & \text{Ascending rank} \\
 & a_3 & b_1 & a_1 & a_2 & b_0 & a_0 & \$ & \\
 & b_0 & a_0 & \$ & a_3 & b_1 & a_1 & a_2 & \\
 & b_1 & a_1 & a_2 & b_0 & a_0 & \$ & a_3 & \\
\end{array}
$$

*F* now has very simple structure: a **$**, a block of **a**s *with ascending ranks*, a block of **b**s *with ascending ranks*

## Burrows-Wheeler Transform

|  | *F* | *L* |  |
|---|---|---|---|
|  | \$ | $a_0$ |  |
|  | $a_0$ | $b_0$ |  |
|  | $a_1$ | $b_1$ | ⟵ Which BWM row *begins* with $b_1$? |
|  | $a_2$ | $a_1$ |  |
|  | $a_3$ | \$ |  |
|  | $b_0$ | $a_2$ |  |
| row 6 ⟶ | $b_1$ | $a_3$ |  |

Skip row starting with **\$** (1 row)

Skip rows starting with **a** (4 rows)

Skip row starting with $b_0$ (1 row)

Answer: row 6

# Burrows-Wheeler Transform

Say *T* has 300 **A**s, 400 **C**s, 250 **G**s and 700 **T**s and **\$** < **A** < **C** < **G** < **T**

Which BWM row (0-based) begins with $\mathbf{G}_{100}$? (Ranks are B-ranks.)

Skip row starting with **\$** (1 row)

Skip rows starting with **A** (300 rows)

Skip rows starting with **C** (400 rows)

Skip first 100 rows starting with **G** (100 rows)

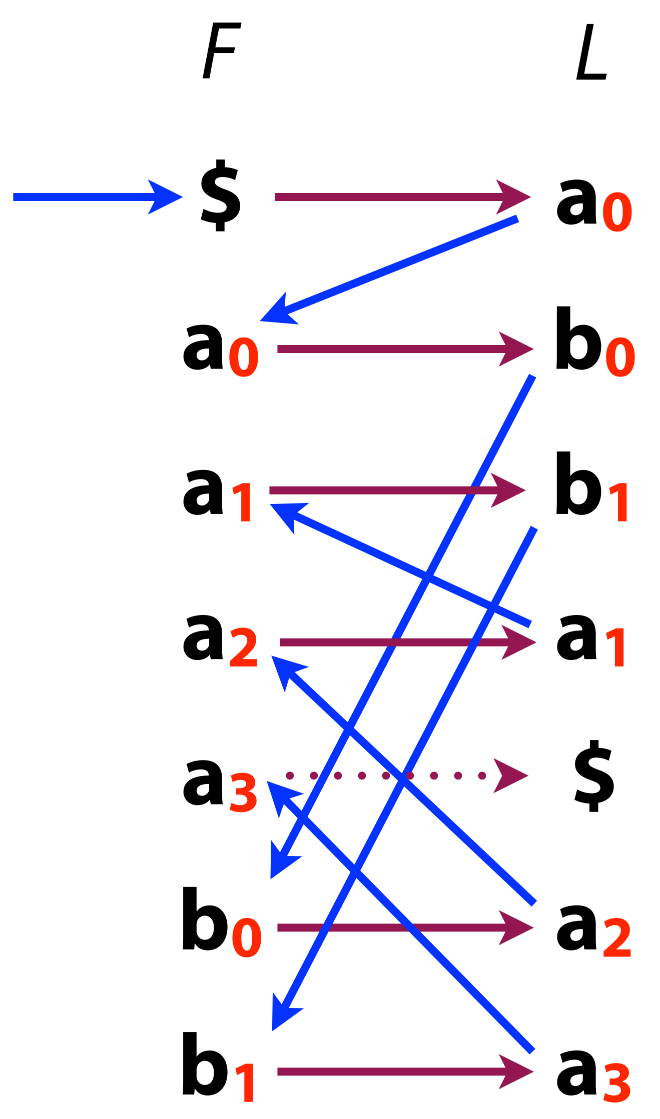
Answer: row 1 + 300 + 400 + 100 = **row 801**

# Burrows-Wheeler Transform: reversing

Reverse BWT(T) starting at right-hand-side of *T* and moving left

Start in first row. *F* must have **$**. *L* contains character just prior to **$**: $\mathbf{a}_0$

$\mathbf{a}_0$: LF Mapping says this is same occurrence of **a** as first **a** in *F*. Jump to row *beginning* with $\mathbf{a}_0$. *L* contains character just prior to $\mathbf{a}_0$: $\mathbf{b}_0$.



Repeat for $\mathbf{b}_0$, get $\mathbf{a}_2$

Repeat for $\mathbf{a}_2$, get $\mathbf{a}_1$

Repeat for $\mathbf{a}_1$, get $\mathbf{b}_1$

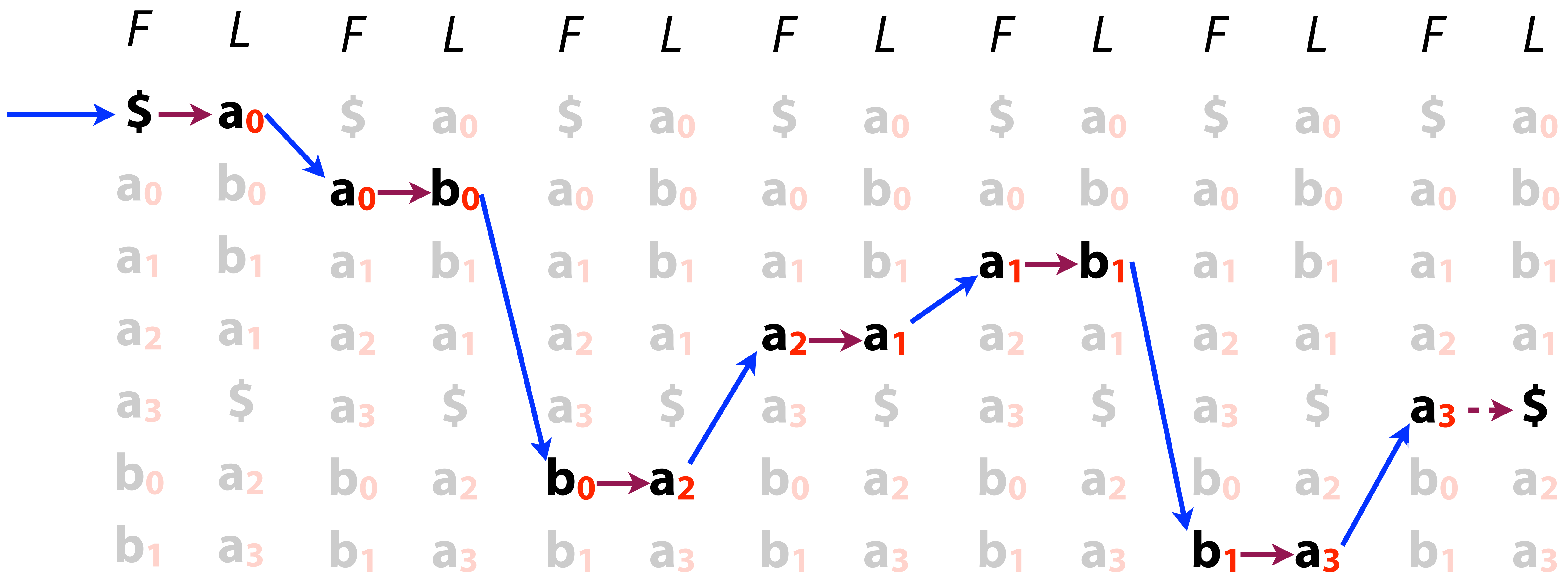
Repeat for $\mathbf{b}_1$, get $\mathbf{a}_3$

Repeat for $\mathbf{a}_3$, get **$**, done

Reverse of chars we visited = $\mathbf{a}_3\ \mathbf{b}_1\ \mathbf{a}_1\ \mathbf{a}_2\ \mathbf{b}_0\ \mathbf{a}_0\ \$ = T$

# Burrows-Wheeler Transform: reversing

Another way to visualize reversing BWT(T):



*T:* $a_3\, b_1\, a_1\, a_2\, b_0\, a_0\, \$$

# Burrows-Wheeler Transform: reversing

```
def rankBwt(bw):
    ''' Given BWT string bw, return parallel list of B-ranks.  Also
        returns tots: map from character to # times it appears. '''
    tots = dict()
    ranks = []
    for c in bw:
        if c not in tots: tots[c] = 0
        ranks.append(tots[c])
        tots[c] += 1
    return ranks, tots
```

Calculate B-ranks and count occurrences of each char

```
def firstCol(tots):
    ''' Return map from character to the range of rows prefixed by
        the character. '''
    first = {}
    totc = 0
    for c, count in sorted(tots.iteritems()):
        first[c] = (totc, totc + count)
        totc += count
    return first
```

Make concise representation of first BWM column

```
def reverseBwt(bw):
    ''' Make T from BWT(T) '''
    ranks, tots = rankBwt(bw)
    first = firstCol(tots)
    rowi = 0 # start in first row
    t = '$' # start with rightmost character
    while bw[rowi] != '$':
        c = bw[rowi]
        t = c + t # prepend to answer
        # jump to row that starts with c of same rank
        rowi = first[c][0] + ranks[rowi]
    return t
```

Do reversal

Python example: http://nbviewer.ipython.org/6860491

# Burrows-Wheeler Transform: reversing

```
>>> reverseBwt("w$wwdd__nnoooaattTmmmrrrrrrooo__ooo")
'Tomorrow_and_tomorrow_and_tomorrow$'

>>> reverseBwt("s$esttssfftteww_hhmmbootttt_ii__woeeaaressIi_______")
'It_was_the_best_of_times_it_was_the_worst_of_times$'

>>> reverseBwt("u_gleeeengj_mlhl_nnnnt$nwj__lggIolo_iiiiarfcmylo_oo_")
'in_the_jingle_jangle_morning_Ill_come_following_you$'
```

ranks list is *m* integers long! We'll fix later. →

```
def reverseBwt(bw):
    ''' Make T from BWT(T) '''
    ranks, tots = rankBwt(bw)
    first = firstCol(tots)
    rowi = 0 # start in first row
    t = '$' # start with rightmost character
    while bw[rowi] != '$':
        c = bw[rowi]
        t = c + t # prepend to answer
        # jump to row that starts with c of same rank
        rowi = first[c][0] + ranks[rowi]
    return t
```

# Burrows-Wheeler Transform

We've seen how BWT is useful for compression:

- Sorts characters by right-context, making a more compressible string

And how it's reversible:

- Repeated applications of LF Mapping, recreating *T* from right to left

How is it used as an index?

# FM Index

FM Index: an index combining the BWT *with a few small auxilliary data structures*

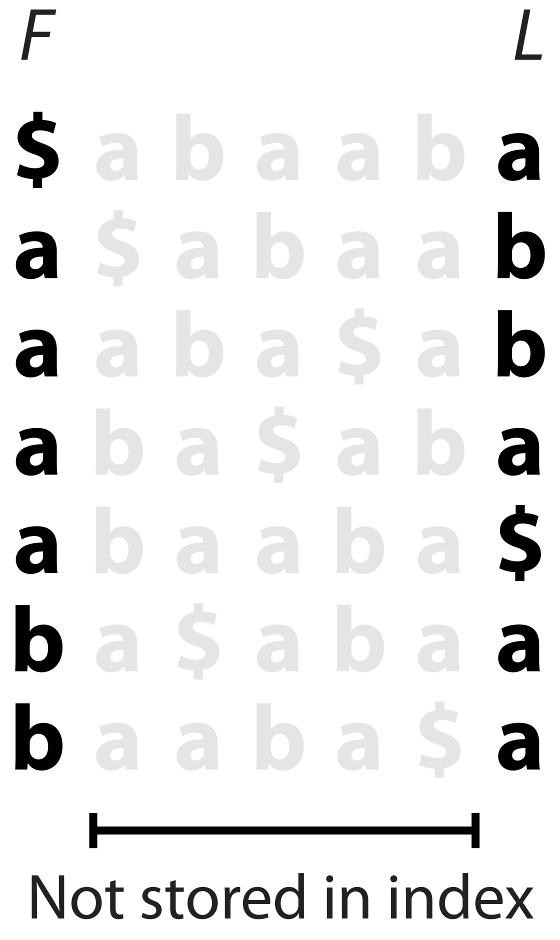
“FM” supposedly stands for “Full-text Minute-space.” (But inventors are named Ferragina and Manzini)

Core of index consists of *F* and *L* from BWM:

*F* can be represented very simply (1 integer per alphabet character)

And *L* is compressible

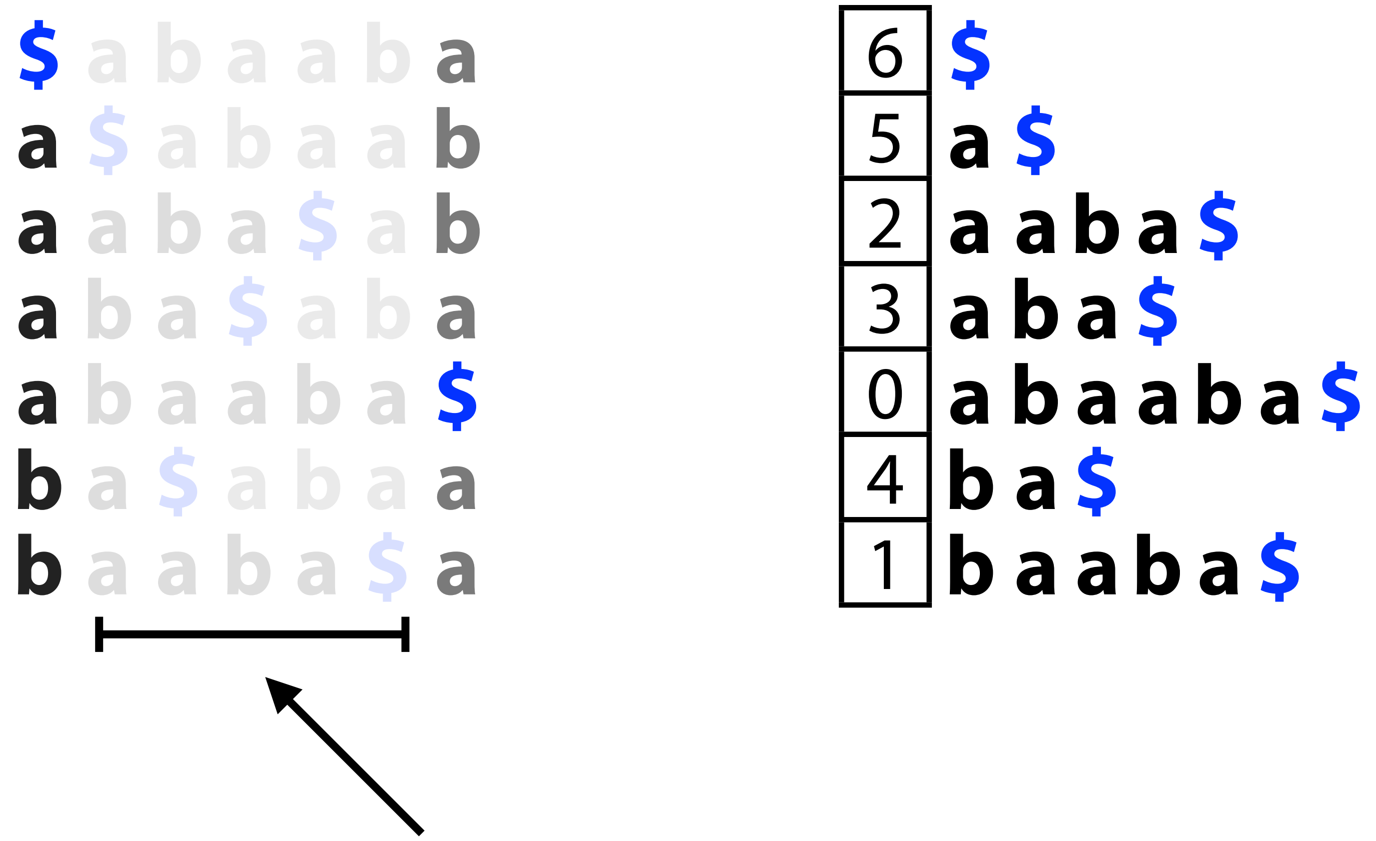
Potentially very space-economical!



Paolo Ferragina, and Giovanni Manzini. "Opportunistic data structures with applications." *Foundations of Computer Science, 2000. Proceedings. 41st Annual Symposium on*. IEEE, 2000.

# FM Index: querying

Though BWM is related to suffix array, we can't query it the same way



We don't have these columns; binary search isn't possible

# FM Index: querying

Look for range of rows of BWM(T) with *P* as prefix

Do this for *P*'s shortest suffix, then extend to successively longer suffixes until range becomes empty or we've exhausted *P*

$P =$ **aba**

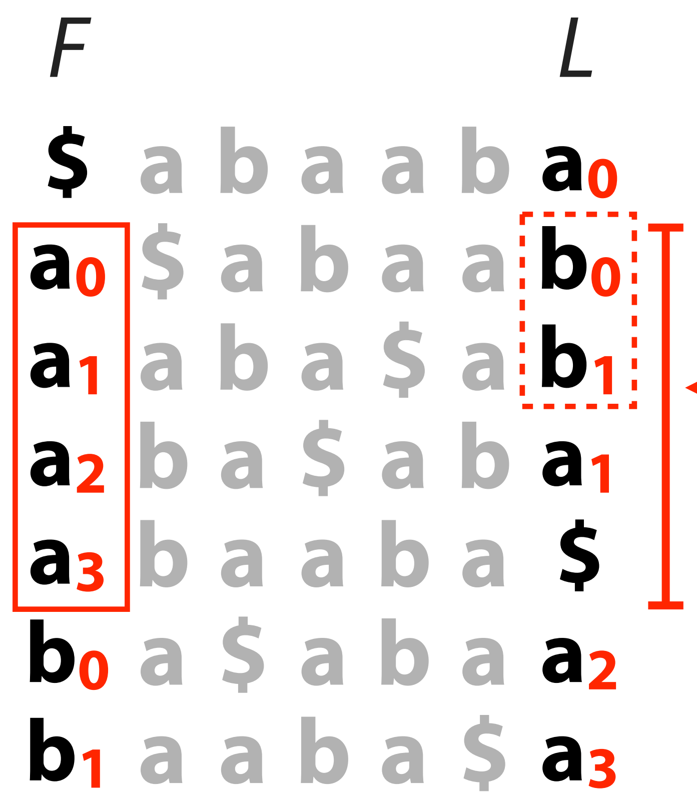
Easy to find all the rows beginning with **a**, thanks to *F*'s simple structure

$$
\begin{array}{ccccccc}
F & & & & & & L \\
\$ & a & b & a & a & b & a_3 \\
a_0 & \$ & a & b & a & a & b_1 \\
a_1 & a & b & a & \$ & a & b_0 \\
a_2 & b & a & \$ & a & b & a_1 \\
a_3 & b & a & a & b & a & \$ \\
b_0 & a & \$ & a & b & a & a_2 \\
b_1 & a & a & b & a & \$ & a_0
\end{array}
$$

# FM Index: querying

We have rows beginning with **a**, now we seek rows beginning with **ba**

*P* = **aba**

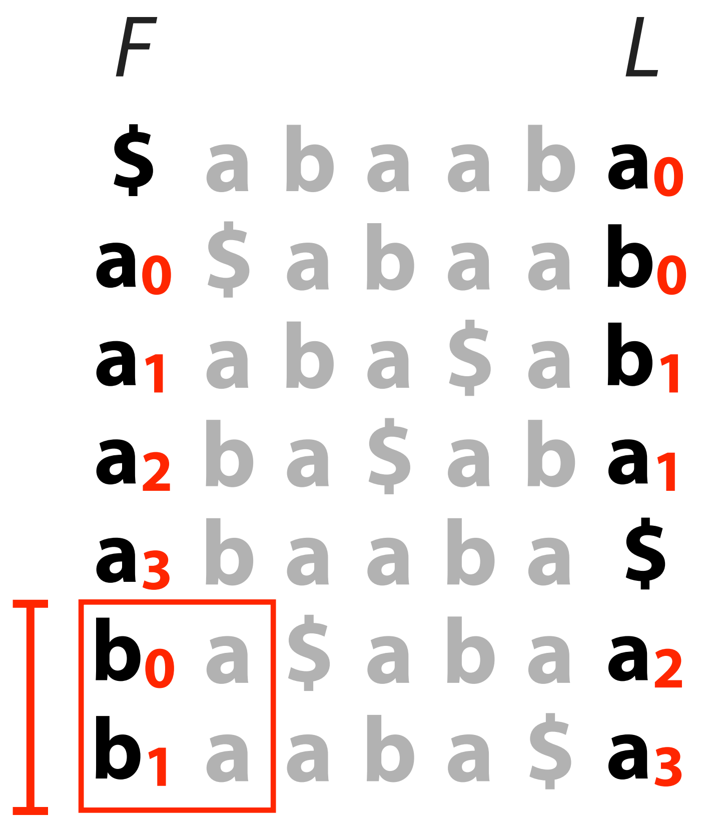


Look at those rows in *L*.

**b**0, **b**1 are **b**s occuring just to left.

*P* = **aba**

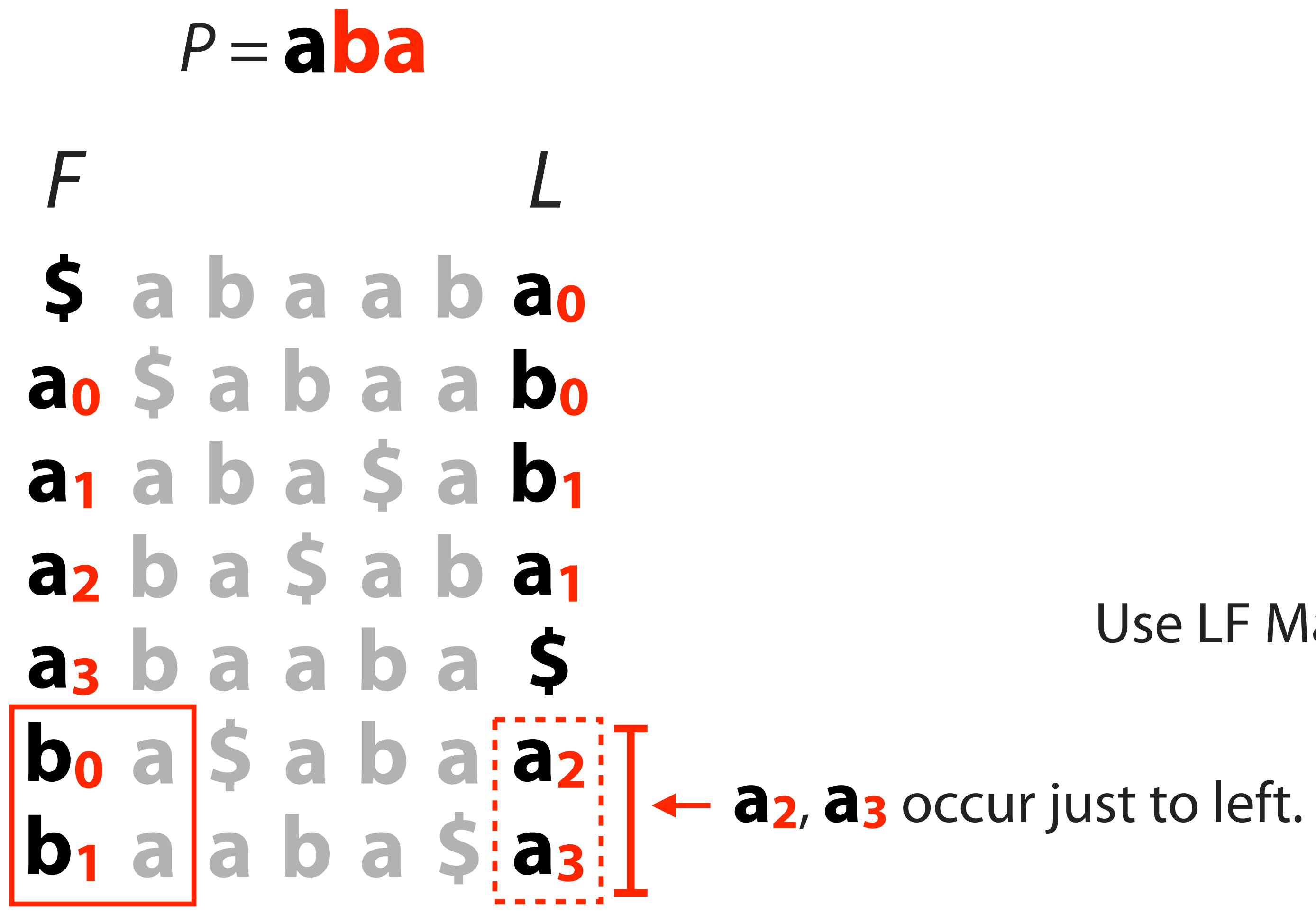
Use LF Mapping. Let new range delimit those **b**s

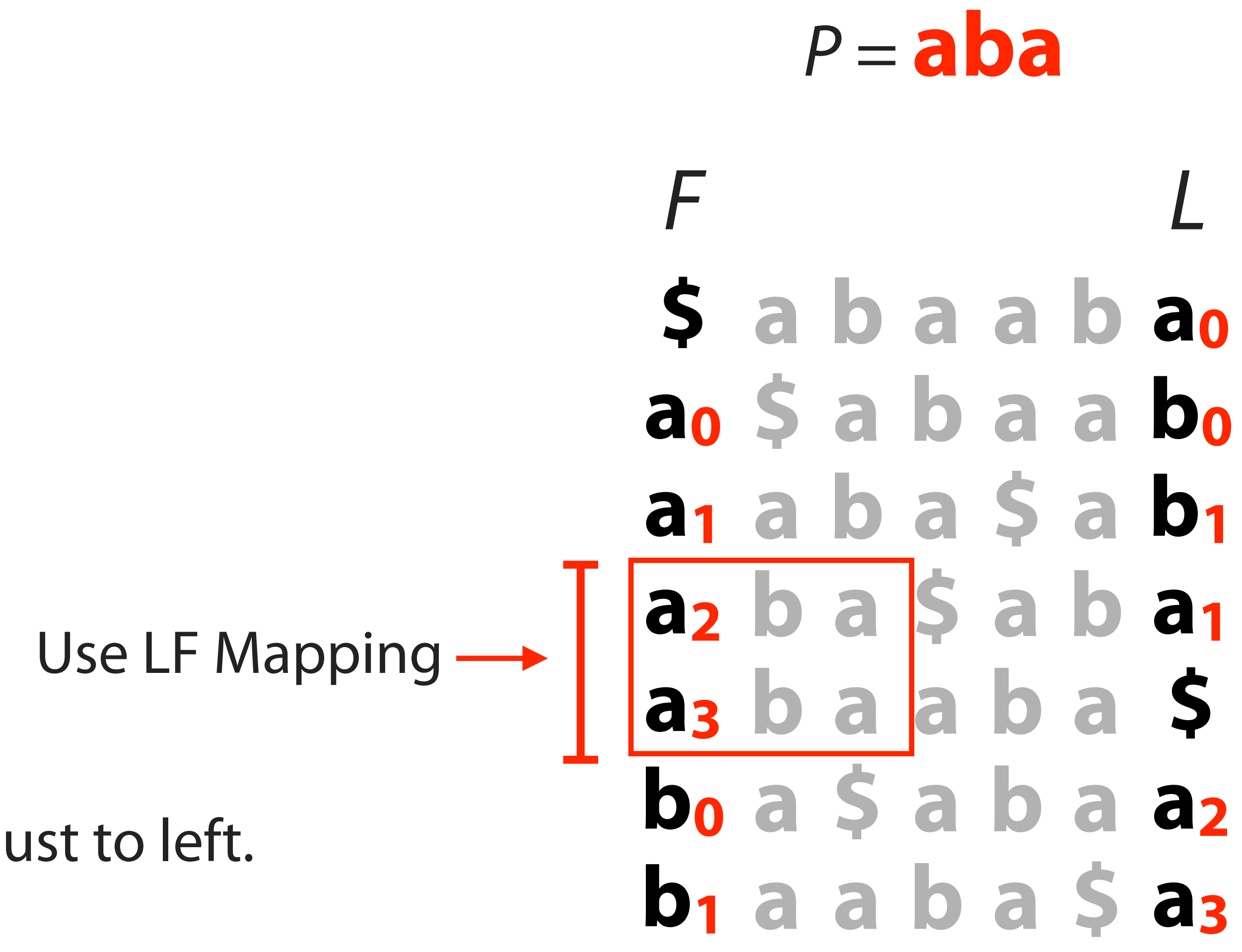


Now we have the rows with prefix **ba**

# FM Index: querying

We have rows beginning with **ba**, now we seek rows beginning with **aba**



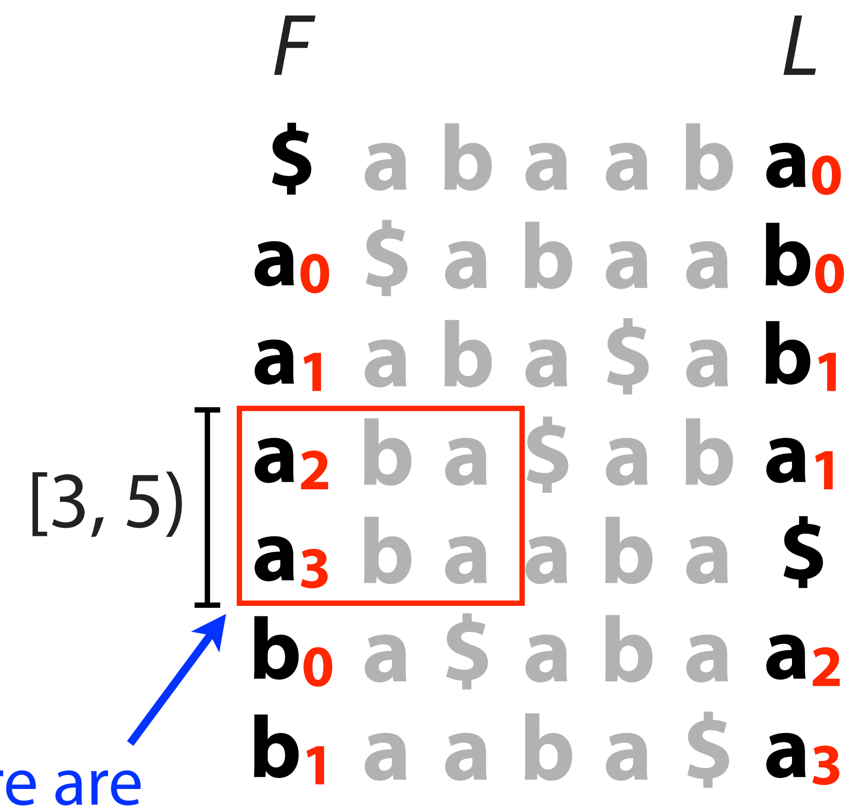


Now we have the rows with prefix **aba**

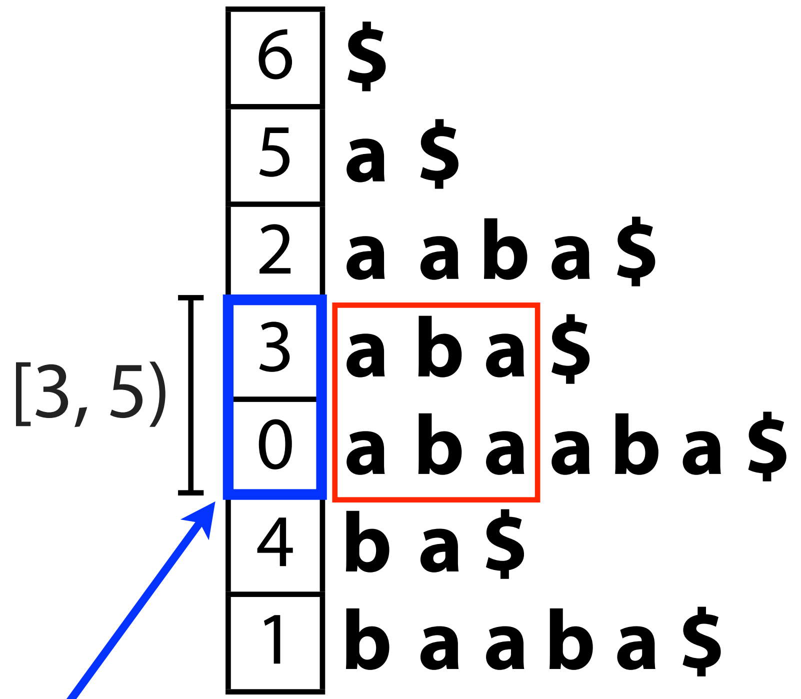
# FM Index: querying

$P =$ **aba**

Now we have the same range, [3, 5), we would have got from querying suffix array



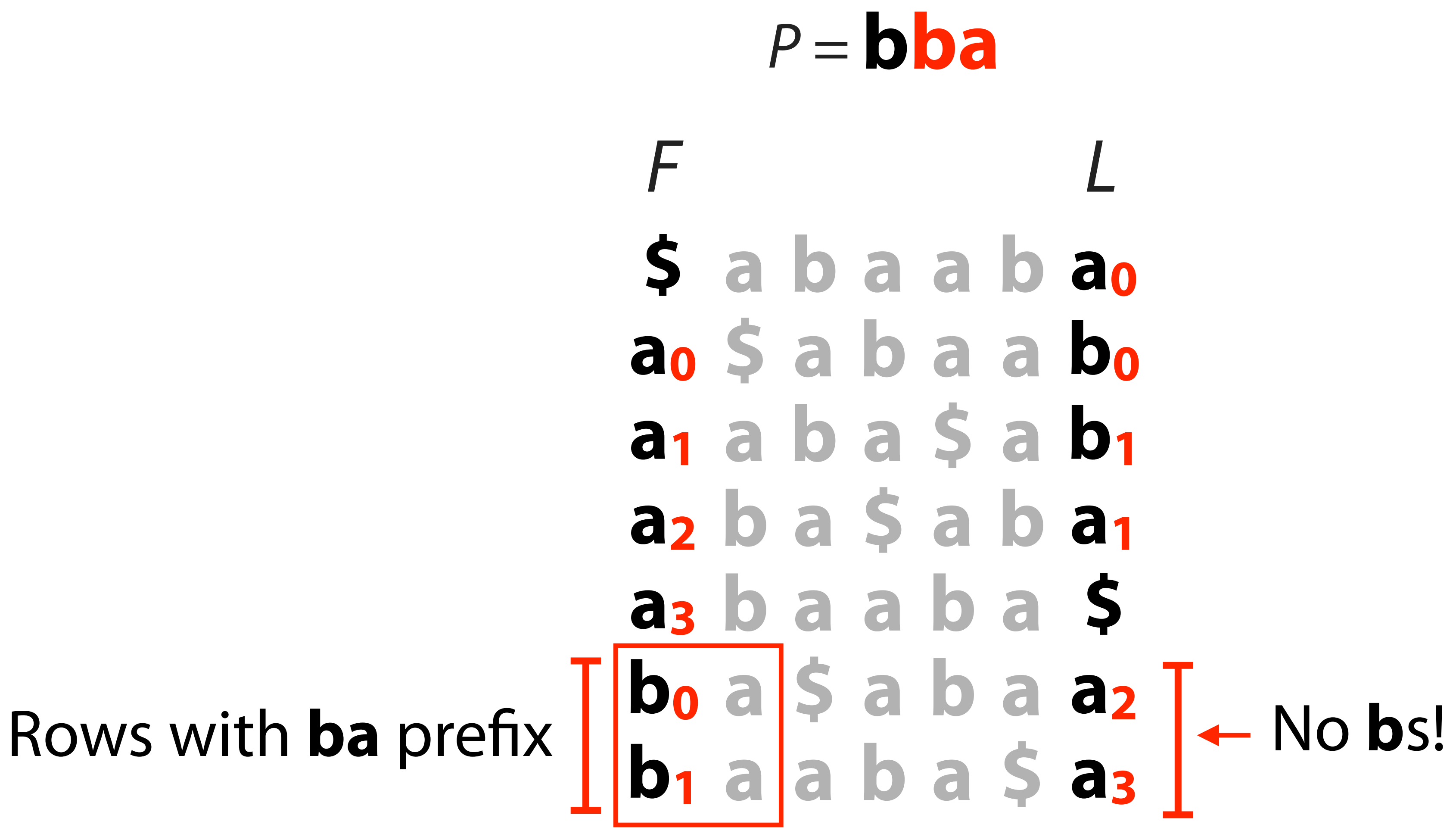
Where are these?



Unlike suffix array, we don't immediately know *where* the matches are in T...

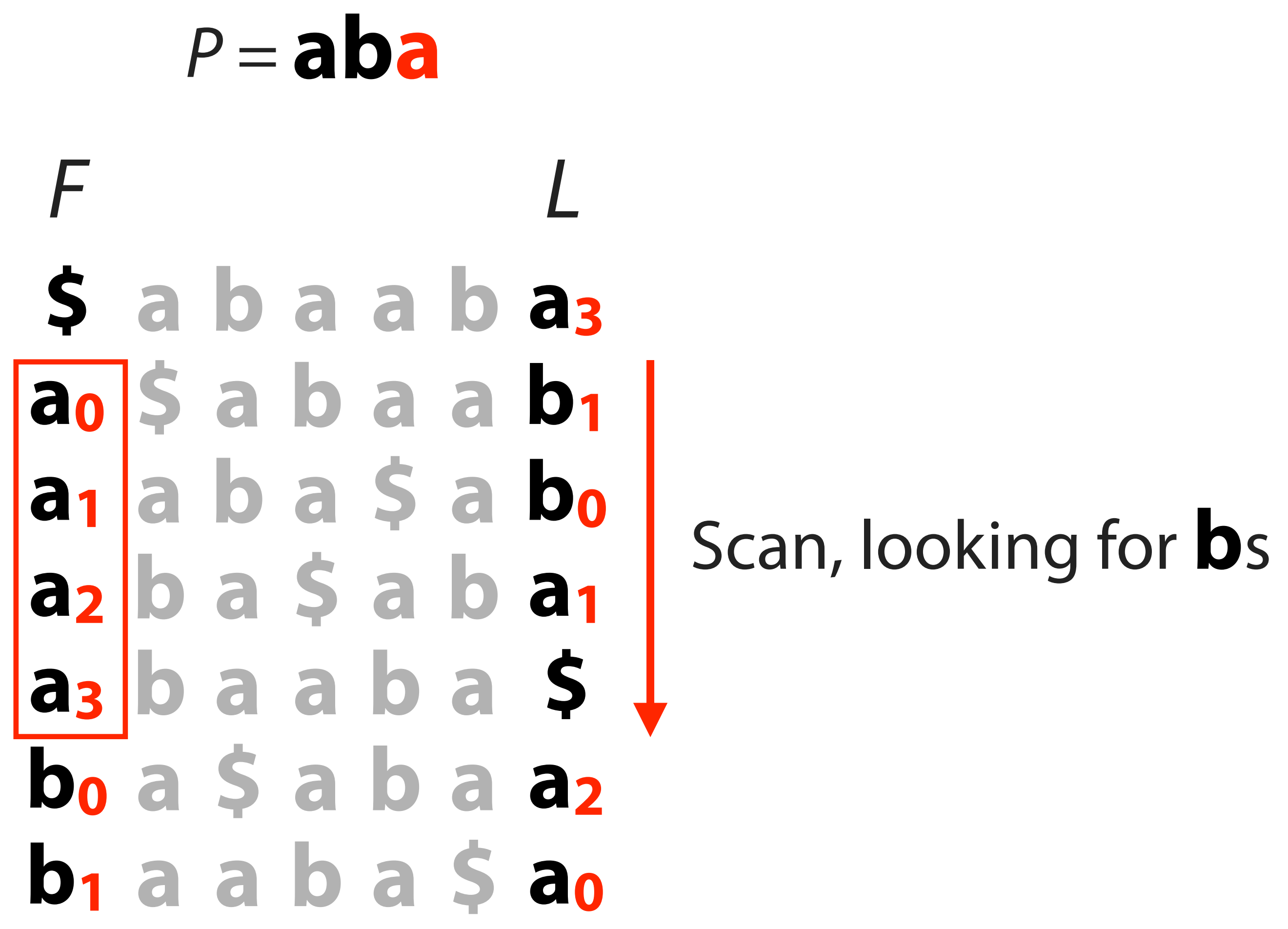
# FM Index: querying

When *P* does not occur in *T*, we will eventually fail to find the next character in *L*:



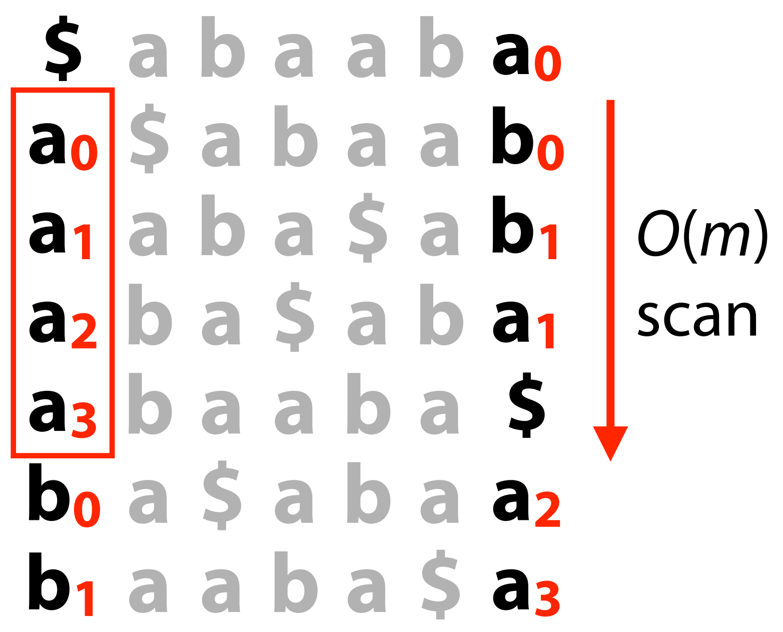
# FM Index: querying

If we *scan* characters in the last column, that can be very slow, *O*(*m*)



# FM Index: lingering issues

(1) Scanning for preceding character is slow

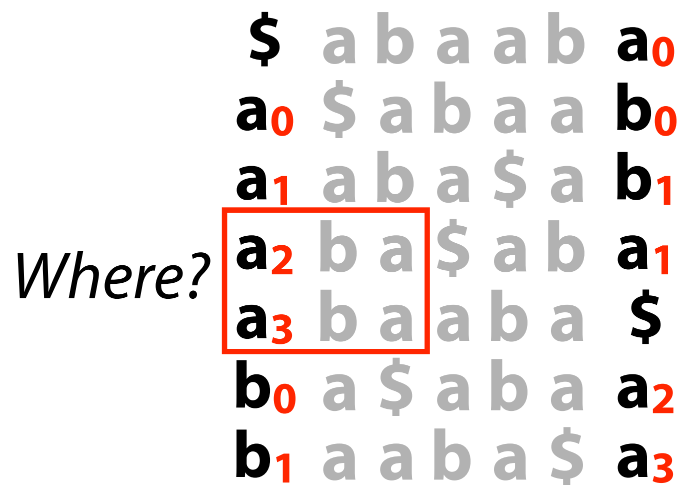


(2) Storing ranks takes too much space

*m* integers

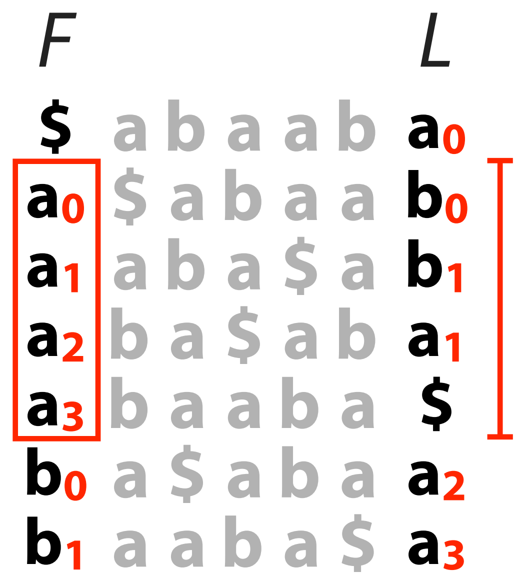
```
def reverseBwt(bw):
    """ Make T from BWT(T) """
    ranks, tots = rankBwt(bw)
    first = firstCol(tots)
    rowi = 0
    t = "$"
    while bw[rowi] != '$':
        c = bw[rowi]
        t = c + t
        rowi = first[c][0] + ranks[rowi]
    return t
```

(3) Need way to find where matches occur in *T*:



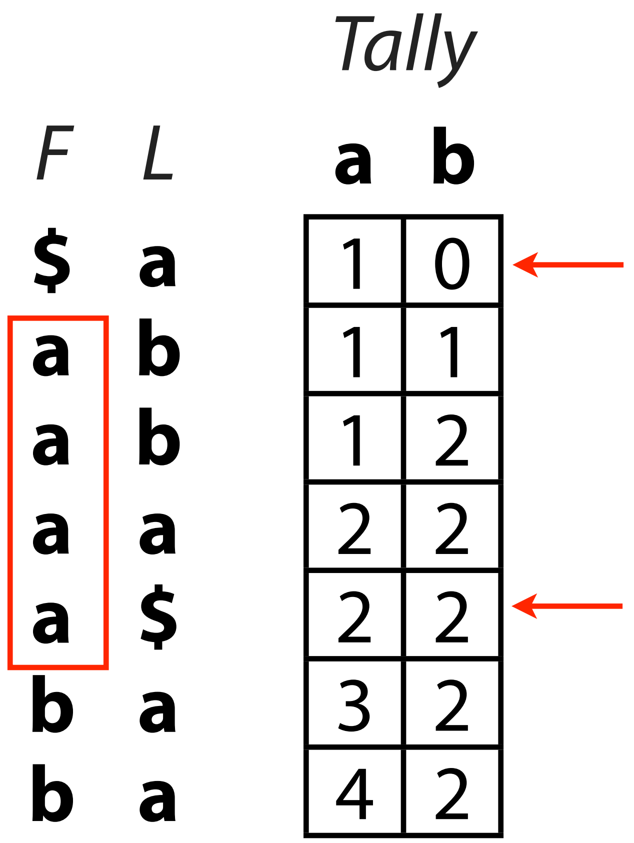
# FM Index: fast rank calculations

Is there an O(1) way to determine which **b**s precede the **a**s in our range?



| *F* | | | | | | *L* |
|---|---|---|---|---|---|---|
| $ | a | b | a | a | b | $a_0$ |
| $a_0$ | $ | a | b | a | a | $b_0$ |
| $a_1$ | a | b | a | $ | a | $b_1$ |
| $a_2$ | b | a | $ | a | b | $a_1$ |
| $a_3$ | b | a | a | b | a | $ |
| $b_0$ | a | $ | a | b | a | $a_2$ |
| $b_1$ | a | a | b | a | $ | $a_3$ |

Idea: pre-calculate # **a**s, **b**s in *L* up to every row:



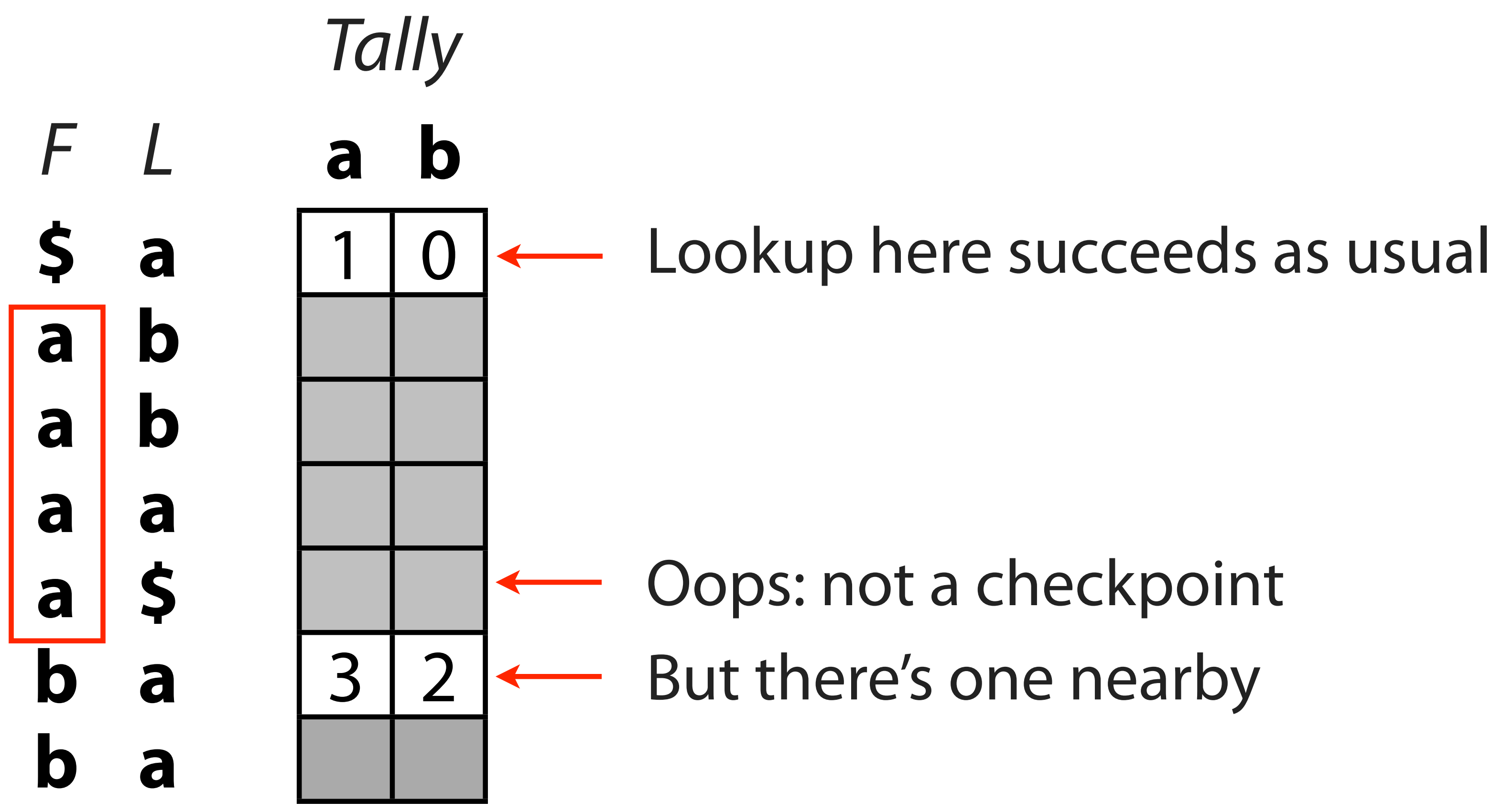
| *F* | *L* | *Tally* **a** | *Tally* **b** |
|---|---|---|---|
| $ | a | 1 | 0 |
| a | b | 1 | 1 |
| a | b | 1 | 2 |
| a | a | 2 | 2 |
| a | $ | 2 | 2 |
| b | a | 3 | 2 |
| b | a | 4 | 2 |

We infer $\mathbf{b}_0$ and $\mathbf{b}_1$ appear in *L* in this range

O(1) time, but requires $m \times |\Sigma|$ integers

# FM Index: fast rank calculations

Another idea: pre-calculate # **a**s, **b**s in *L* up to *some* rows, e.g. every 5th row. Call pre-calculated rows *checkpoints*.



To resolve a lookup for character *c* in non-checkpoint row, scan along *L* until we get to nearest checkpoint. Use tally at the checkpoint, *adjusted for # of cs we saw along the way*.

# FM Index: fast rank calculations

What's my rank?

$482 + 2 - 1 = 483$

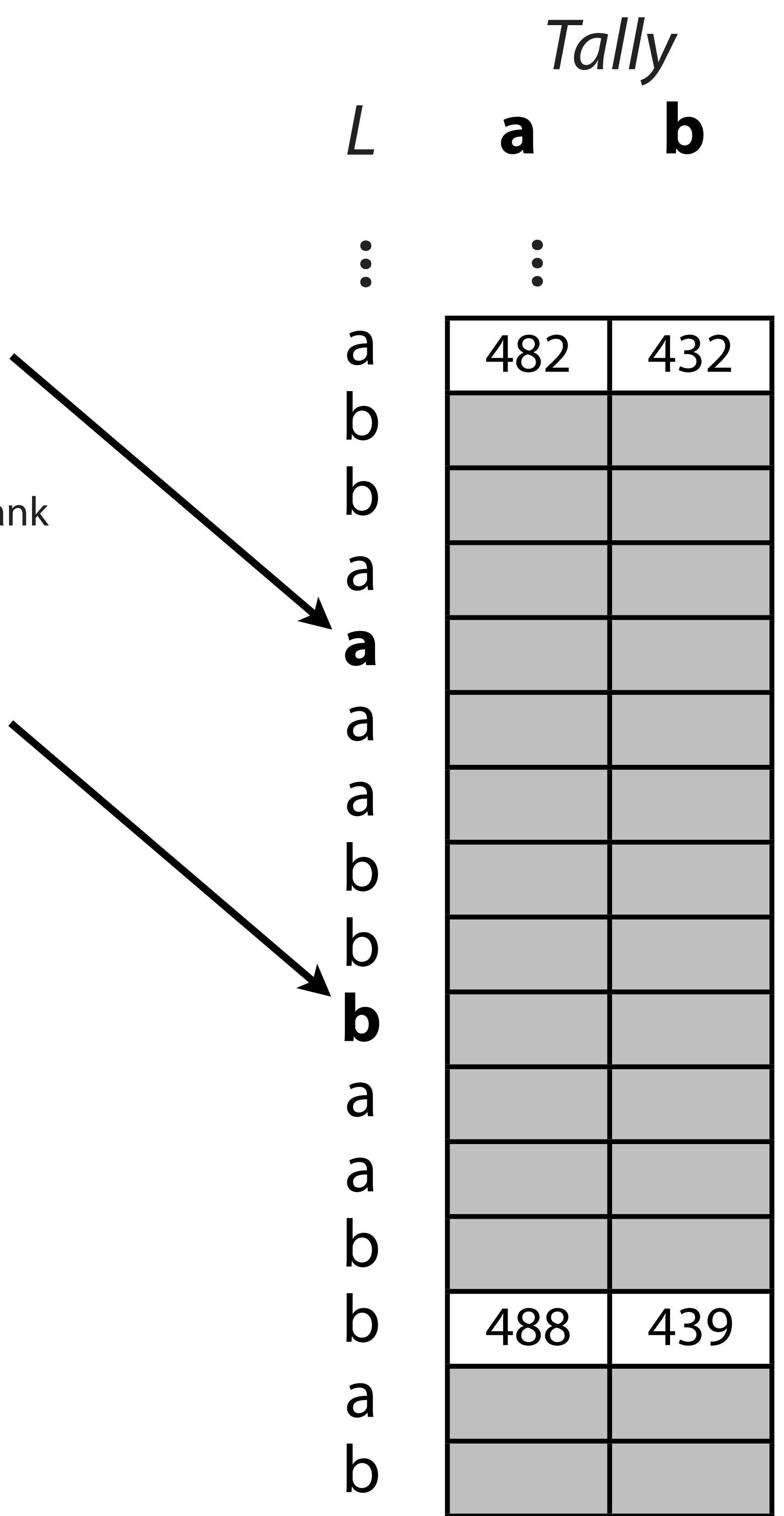
Checkpoint

**a**s along the way

tally -> rank

What's my rank?

$439 - 2 - 1 = 436$

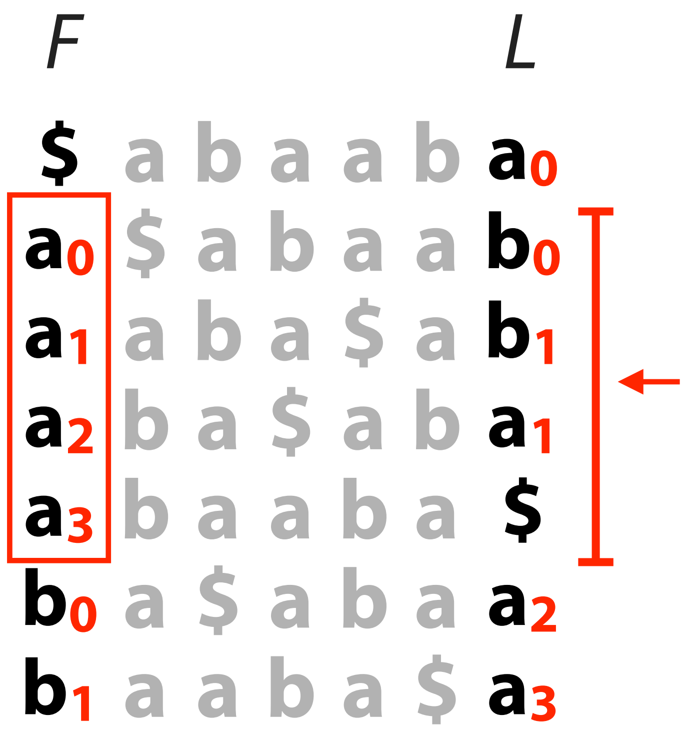


Assuming checkpoints are spaced *O*(1) distance apart, lookups are *O*(1)

# FM Index: a few problems

Solved! At the expense of adding checkpoints ($O(m)$ integers) to index.

**(1)**



This scan is $O(m)$ work

**With checkpoints it's $O(1)$**

**(2)** Ranking takes too much space

*m* integers

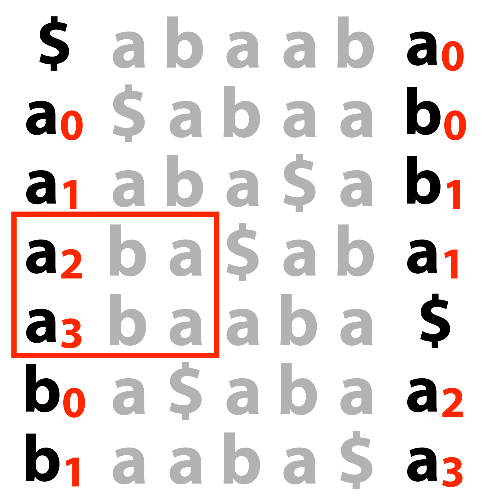
```
def reverseBwt(bw):
    """ Make T from BWT(T) """
    ranks, tots = rankBwt(bw)
    first = firstCol(tots)
    rowi = 0
    t = "$"
    while bw[rowi] != '$':
        c = bw[rowi]
        t = c + t
        rowi = first[c][0] + ranks[rowi]
    return t
```

**With checkpoints, we greatly reduce # integers needed for ranks**

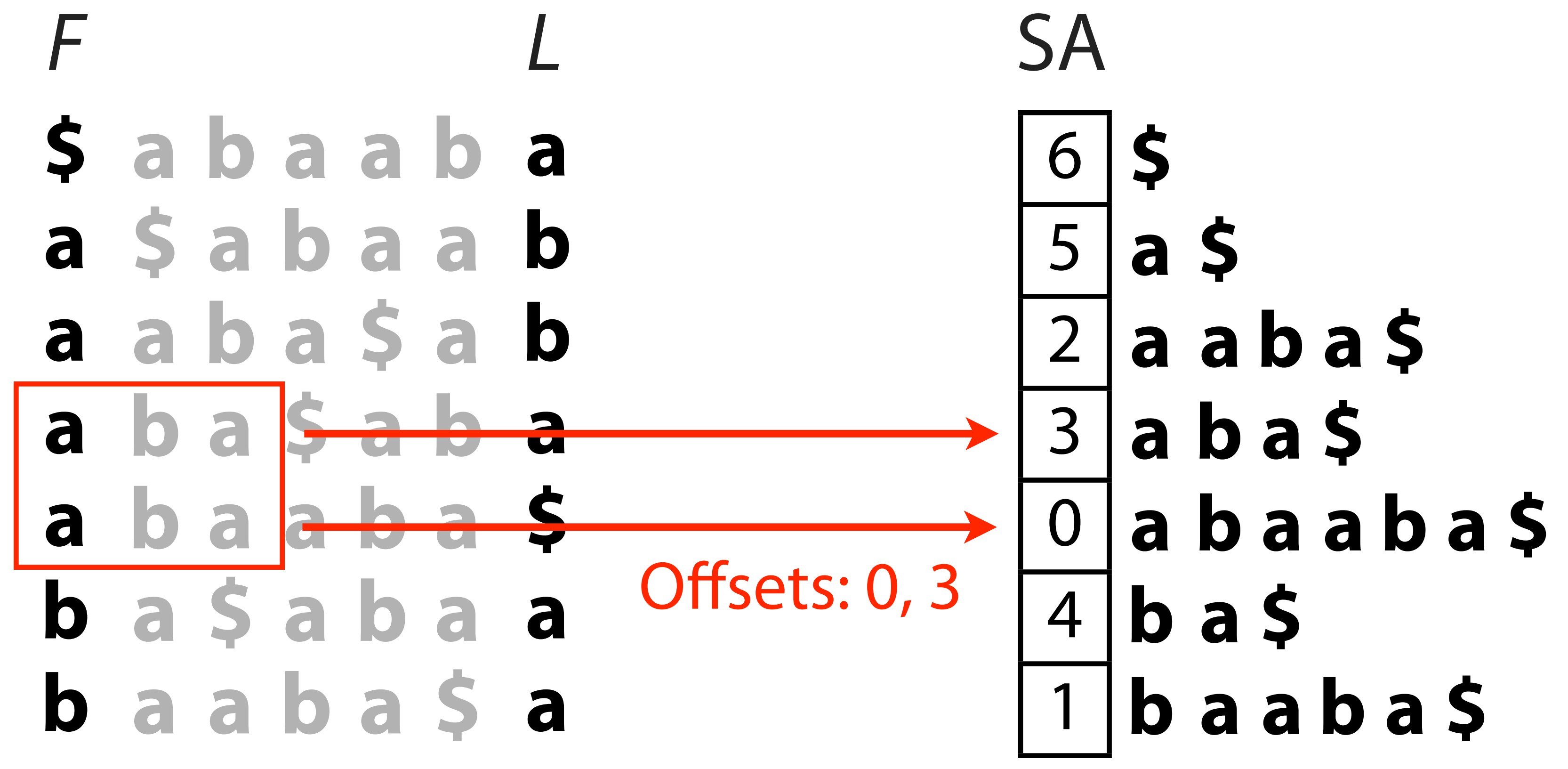
But it's still $O(m)$ space - there's literature on how to improve this space bound

# FM Index: a few problems

Not yet solved: **(3)** Need a way to find where these occurrences are in *T*:



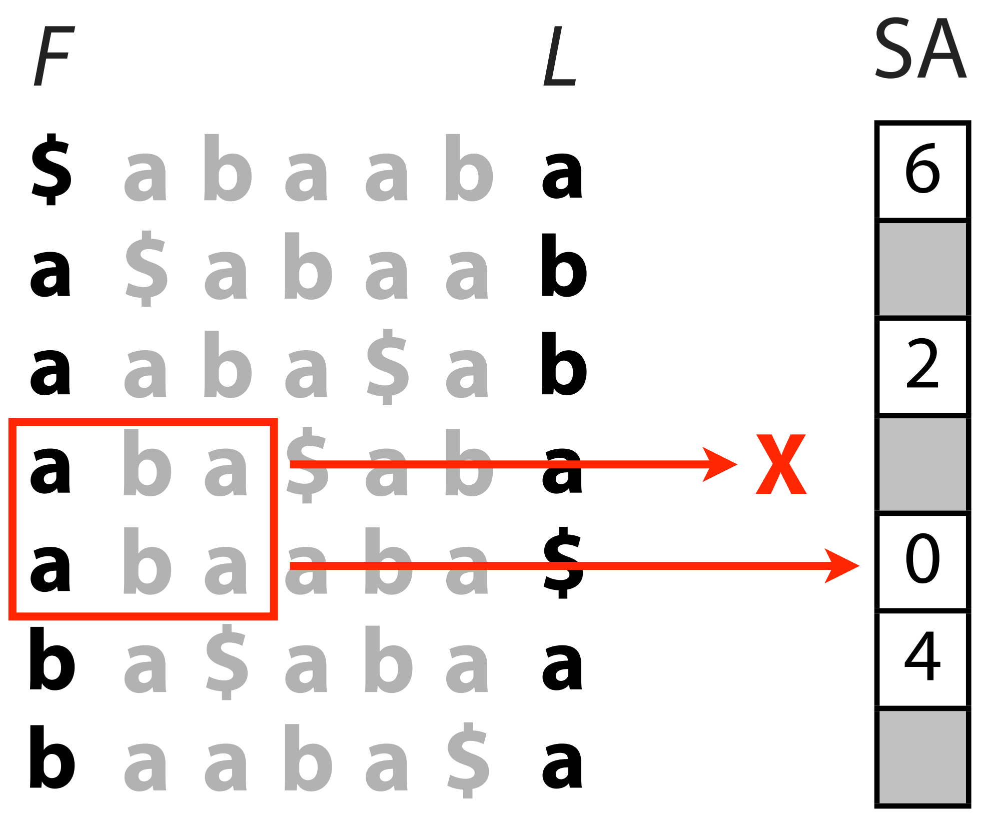
If suffix array were part of index, we could simply look up the offsets



But SA requires *m* integers

# FM Index: resolving offsets

Idea: store some, but not all, entries of the suffix array



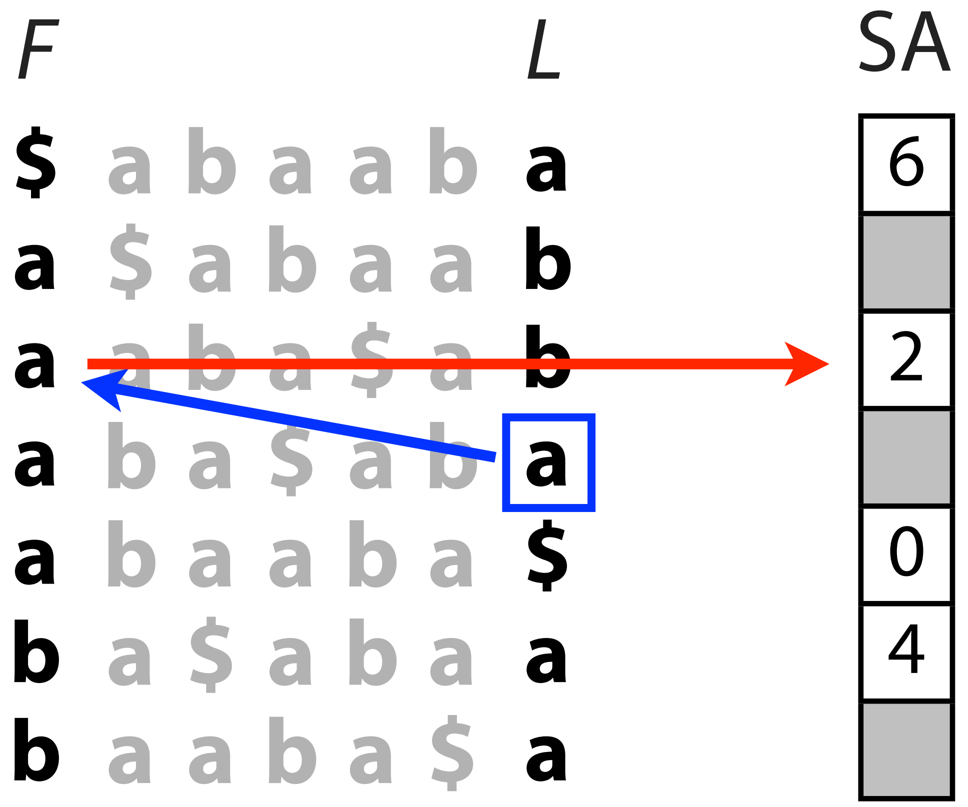
Lookup for row 4 succeeds - we kept that entry of SA

Lookup for row 3 fails - we discarded that entry of SA

## FM Index: resolving offsets

But LF Mapping tells us that the **a** at the end of row 3 corresponds to...

...the **a** at the begining of row 2



And row 2 has a suffix array value = 2

So row 3 has suffix array value = 3 = 2 (row 2's SA val) + 1 (# steps to row 2)

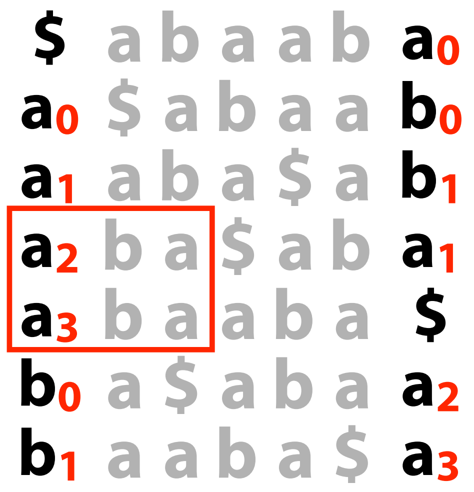
If saved SA values are O(1) positions apart in *T*, resolving offset is O(1) time

# FM Index: problems solved

Solved! At the expense of adding some SA values ($O(m)$ integers) to index

Call this the "SA sample"

**(3)** Need a way to find where these occurrences are in *T*:



**With SA sample we can do this in *O*(1) time per occurrence**

# FM Index: small memory footprint

Components of the FM Index:

| | |
|---|---|
| First column (*F*): | ~ $\lvert\Sigma\rvert$ integers |
| Last column (*L*): | *m* characters |
| SA sample: | $m \cdot a$ integers, where *a* is fraction of rows kept |
| Checkpoints: | $m \times \lvert\Sigma\rvert \cdot b$ integers, where *b* is fraction of rows checkpointed |

Example: DNA alphabet (2 bits per nucleotide), $T$ = human genome, $a = 1/32$, $b = 1/128$

| | |
|---|---|
| First column (*F*): | 16 bytes |
| Last column (*L*): | 2 bits * 3 billion chars = 750 MB |
| SA sample: | 3 billion chars * 4 bytes/char / 32 = ~ 400 MB |
| Checkpoints: | *3 billion* * 4 bytes/char / *128* = ~ 100 MB |

Total < 1.5 GB