Probability Basics

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June 2, 2021

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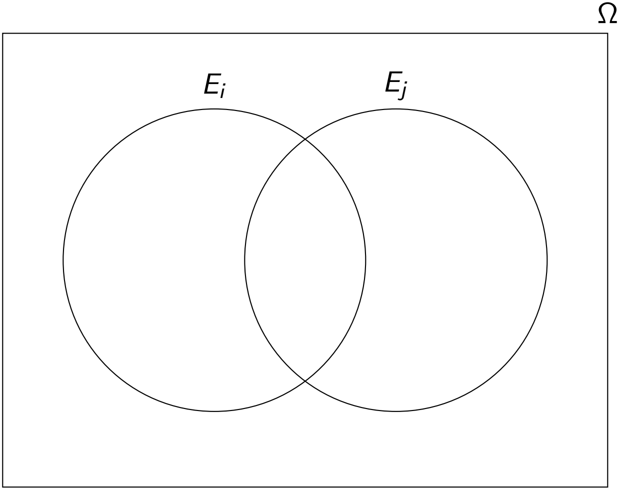
  4. NOTE: Notation and terminology can vary. E.g. conjunctions of events (joint probabilities) might be represented by $\cap$, $\wedge$, &, or commas.

Venn Diagram Representations



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- Screening Off: $P(A|B \cap C) = P(A|B)$ implies $A \perp\!\!\!\perp C$.
- Bayes' Theorem: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$, $P(B) \neq 0$

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Law of Large Numbers

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- Or, a limiting ratio as more samples are observed. (Fisher)

Fisher Quote

To such a [man gambling on a single die throw] the information supplied by a familiar mathematical statement such as: "If a aces are thrown in n trials, the probability that the difference in absolute value between a/n *and* $1/6$ *shall exceed any positive value* ϵ*, however small, shall tend to zero as the number n is increased indefinitely", will seem not merely remote, but also incomplete and lacking in definiteness in its application to the particular throw in which he is interested. Indeed, by itself it says nothing about that throw. It is obvious, moreover, that many subsets of future throws, which may include his own, can be shown to give probabilities, in this sense, either greater or less than* $1/6$.

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- Price: $Pr(E) :=$ price an agent willing to pay for a ticket that pays 1 in the case that E occurs, and 0 otherwise.

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- Instead, he preferred what is called proper scoring rules.
  - Belief calibration should occur against one's own lights: it is not fundamentally about what you can win on the open market, but about not losing what you already have.

Proper Scoring

An individual is asked to provide their DoB or subjective estimation π of an event E, with the knowledge that he or she will be scored according to whether E obtains or not—and whether π corresponds with their actual degree of belief p. A penalization is calculated using the quadratic formula $(E-\pi)^2$, where E is 1 or 0. The prevision or expectation of this penalization is $(E-\pi)^2(1-p)+(E^c-\pi)^2(p)$, or the sum of the assessments in the case that E obtains or does not obtain (E^c). When $E=1$ (and $E^c=0$) this simplifies to $\pi^2+p-2\pi p$. In the case that one does not state one's *actual* degree of belief $(\pi \neq p)$, the prevision of penalization increases by $(\pi-p)^2$. This algebraic fact is the basis of why proper scores are seen to be an effective elicitor of subjective DOB.

Principal Principle (Lewis)

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- That is, $P_b(E)$ should equal $P_f(E)$

Propensities (Popper)

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- Objective vs. Subjective