Assignment 1
MECHTRON 3X03: Scientific Computation

Due at 11:59 PM on Friday, October 6

Fall 2023

Use of Generative AI Policy

If you use it, treat generative AI as you would a search engine: you may use it to answer general queries about scientific computing, but any specific component of a solution or lines of code must be cited (see the syllabus for citation guidelines).

Submission Guidelines

Submit the following files on Avenue:

1. A PDF called `<FIRSTNAME>_<LASTNAME>_assignment1.pdf` containing all answers (no need to include Julia code here). Show all steps in your solutions.
2. A file called `spacing.jl` for Problem 6.
3. A file called `truncation.jl` for Problem 7.
4. A file called `expm1mx.jl` for Problem 8.

The PDF can be any combination of typed/scanned/handwritten, so long as it is legible (you will receive a grade of zero on any section that cannot be understood).

This is an individual assignment. All submitted work must be your own, or appropriately cited from scholarly references. Submitting all or part of someone else's solution is an academic offence.

Problems

There are 9 problems for a total of 30 points.

Analysis

For the problems in this section, you must use mathematical analysis to derive and justify your solution. You can check your answers or compute values with code or a calculator, but you will not be given marks for simply computing the correct values.

Problem 1 (2 points)**:** For small x, the approximation $\sin x \approx x$ is often used. For what range of x is this good to a relative accuracy of $\frac{1}{2}10^{-14}$?

Problem 2 (4 points)**:** Write the Taylor series at x (i.e., as a polynomial in h) for

a) (2 points) e^{x+2h},

b) (2 points) $\sin(x - 3h)$.

Problem 3 (3 points)**:** Suppose you approximate e^x by its truncated Maclaurin series. For $x = 0.5$, derive how many terms of the series are needed to achieve (absolute) accuracy of 10^{-10}.

Problem 4 (2 points)**:** Consider the expression $1 - a^2$. In IEEE 754 double precision, for what values of a does this expression evaluate to 1?

Problem 5 (2 points)**:** Give an example in base-10 floating point arithmetic (with precision t of your choosing) where

a) (1 points) $(a + b) + c \neq a + (b + c)$,

b) (1 points) $(a * b) * c \neq a * (b * c)$.

Programming

The problems in this section (except for Problem 9) contain a programming portion in the Julia language. Include the written parts of each question in your submitted PDF, and be sure to correctly name each Julia file you submit.

Problem 6 (4 points)**:** Suppose you need to generate $n + 1$ equally spaced points in the interval $[a, b]$ with spacing $h = (b - a)/n$, where $n > 1$. You can use either

$$x_0 = a, x_i = x_{i-1} + h, i = 1, \dots, n \text{ or} \tag{1}$$

$$x_i = a + ih, i = 0, \dots, n. \tag{2}$$

Denote by $\tilde{x}_i$ the computed value in (1) and by $\hat{x}_i$ the computed value in (2).

a) (2 points) Which of $|x_i - \tilde{x}_i|$ and $|x_i - \hat{x}_i|$ is more accurate? Explain why.

b) (2 points) Write a Julia program that implements both methods and illustrates the difference between them by plotting (on a log y-axis) their errors relative to an accurate solution using `BigFloat`. Include this plot in your PDF submission (see Problem 8 for an example using `savefig`).

Submit your Julia code to Avenue as a file called `spacing.jl`.

Problem 7 (6 points)**:** Consider the approximation

$$f'(x) \approx \frac{f(x + h) - f(x - h)}{2h}, \tag{3}$$

where $h > 0$. Assume that $f'''(x)$ is continuous on $[x - h, x + h]$.

a) (2 points) If we approximate $f'(x)$ by using Taylor series with Eq. 3, what is the truncation error of this approximation?

b) (2 points) When evaluated on a computer, for what value of h is the error of this approximation the smallest?

c) (2 points) For the function $f(x) = \sin(x)e^{\cos x}$, plot the error

$$\left| f'(x) - \frac{f(x + h) - f(x - h)}{2h} \right| \tag{4}$$

versus h for appropriate values of h. Plot on a log-log scale (`xaxis=:log` and `yaxis=:log`) and include the figure in your PDF submission (once again, see Problem 8 for an example using `savefig`). Submit your code in a file called `truncation.jl`. How does the error match your derivation in the previous part?

Problem 8 (4 points)**:** Consider

$$f(x) = \frac{e^x - x - 1}{x^2}. \tag{5}$$

When evaluated with $|x| < 1$, the relative error can be large.

a) (1 point) Explain why this error can be large.

b) (3 points) Write a Julia function

```
function expm1mx(x)
        # Evaluates (exp(x) - 1 - x)/x^2 for |x| < 1.
end
```

that evaluates $f(x)$ as accurately as possible when $|x| < 1$. You must use only double precision and must not use any of Julia's built-in functions. Store your function in a file called `expm1mx.jl` and submit it to Avenue. Then run the Julia script

```
# Define our function and domain
f(x) = (exp(x) - 1 - x)/x^2
N = -16:1:0
x = 10. .^N

# Compute accurate values in higher precision
accurate = f.(BigFloat.(x))

# Relative error in f(x)
error_f = abs.((f.(x) - accurate)./accurate)

# Relative error in expm1mx(x)
error_app = abs.((expm1mx.(x) - accurate)./accurate)

# Plot and save the output
p = plot(x, error_f, xaxis=:log, yaxis=:log,
         label="Rel. Error in f(x)", color=:blue)
plot!(x, error_app, xaxis=:log, yaxis=:log,
         label="Rel. Error in expm1mx(x)", color=:red)
xaxis!("x")
yaxis!("Relative Error")
savefig(p, "relative_error.png")
```

and submit the saved figure as part of your PDF submission.

Problem 9 (3 points)**:** The following Julia script

```
g(x) = (exp(x)-1-x)/x^2;
h(x) = (exp(x)-x-1)/x^2;
x = 1e-10;
println("x=$(x)")
println("g(x)=$(g(x))")
println("h(x)=$(h(x))")
x = 2^(-33);
println("x=$(x)")
println("g(x)=$(g(x))")
println("h(x)=$(h(x))")
```

produces (on my machine):

```
x=1.0e-10
g(x)=827.4037096265816
h(x)=0.0
x=1.1641532182693481e-10
g(x)=0.0
h(x)=0.0
```

Explain each of the values for $g(x)$ and $h(x)$.