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# MECHTRON 4AX3:A3 Due: Oct/26/23

1) Two sensors $S_1$ $S_2$ measuring quantity $x$.
Independent and $var(S_1) = \sigma_1^2$, $var(S_2) = \sigma_2^2$
Best estimate for $x$ and variance given $S_1$ $S_2$.

Use a linear merge for this where $x = S_{new}$:

$$S_{new} = K S_1 + (1-K) S_2 \quad \text{where} \quad 0 \leq K \leq 1 \qquad (1)$$

$$var(S_{new}) = var(K S_1 + (1-K) S_2)$$

$$= var(K S_1) + var((1-K) S_2)$$

$S_1$ $S_2$ independent

$$= K^2 var(S_1) + (1-K)^2 var(S_2) + \underbrace{\cancel{2 covar(K S_1, (1-K) S_2)}}_{0}$$

$$= K^2 \sigma_1^2 + (1-K)^2 \sigma_2^2 \qquad (2)$$

Also want to minimize the variance of $S_{new}$:

$$\min_K var(S_{new}) = \min_K K^2 \sigma_1^2 + (1-K)^2 \sigma_2^2$$

$$\frac{d}{dK}\left(K^2\sigma_1^2 + (1-K)^2\sigma_2^2\right) = 0$$

$$0 = 2K\sigma_1^2 - 2(1-K)\sigma_2^2$$

$$0 = K\sigma_1^2 - \sigma_2^2 + K\sigma_2^2 \qquad K = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

We can substitute K back into our equations:

1

$$S_{new} = \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right) S_1 + \left(1 - \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right) S_2$$

$$\boxed{S_{new} = \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right) S_1 + \left(\frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right) S_2}$$

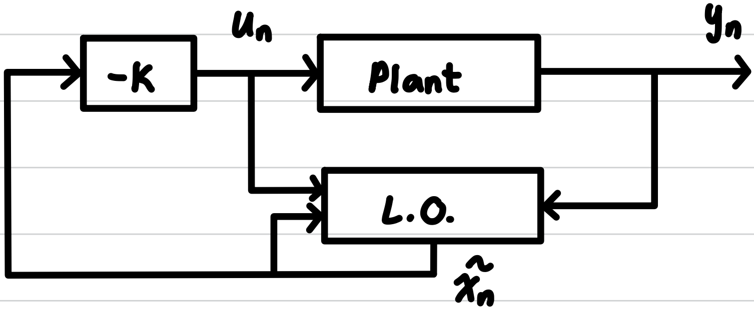
2

$$var(S_{new}) = \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)^2 \sigma_1^2 + \left(\frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right)^2 \sigma_2^2$$

$$\boxed{var(S_{new}) = \frac{\sigma_2^4\sigma_1^2 + \sigma_1^4\sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}}$$

## 2) Lüneberger Observer for L.T.D. System

Show error of observation can be controlled to zero.




$e_{n+1} = x_{n+1} - \tilde{x}_{n+1}$ (control this to zero)

① $x_{n+1} = A_n x_n + B_n u_n \quad y_n = C_n x_n$ (state)

② $\tilde{x}_{n+1} = A_n \tilde{x}_n + B_n u_n + L_n (y_n - \tilde{y}_n)$ (estimation)

① - ②

$e_{n+1} = A_n x_n + B_n u_n - A_n \tilde{x}_n - B_n u_n - L_n (y_n - \tilde{y}_n)$

$e_{n+1} = A_n x_n - A_n \tilde{x}_n - L_n (y_n - \tilde{y}_n) \qquad y_n = C_n x_n$

$e_{n+1} = A_n (x_n - \tilde{x}_n) - L_n C_n (x_n - \tilde{x}_n)$

$$e_{n+1} = (A_n - L_n C_n)(x_n - \tilde{x}_n) \qquad e_n = x_n - \tilde{x}_n$$

$$e_{n+1} = (A_n - L_n C_n)\, e_n$$

To control this error of $e_{n+1}$ to zero, $e_n$ must be multiplied by a factor less than 1, for all $n$. Doing this will shrink the error to zero as time approaches infinity.

$$\boxed{\|A_n - L_n C_n\| < 1 \quad \text{for all } n}$$

3) Iterative learning scheme to estimate $y = \lambda_1 x + \lambda_2$ online as samples $(x_i, y_i)$ arrive. Update process of the form $\lambda_{n+1} = \lambda_n - u f(x_n, y_n)$, give $u$ and $f$.

Can use Stochastic Gradient Descent (SGD)

$$Q(w) = \sum_{i=1}^{n} Q_i(w) = \sum_{i=1}^{n} (y_i - \tilde{y}_i)^2 = \sum_{i=1}^{n} (y_i - \lambda_1 x_i - \lambda_2)^2$$

Find partial derivatives of $\lambda_1$ and $\lambda_2$:

$$\frac{d}{d\lambda_1}(y_i - \lambda_1 x_i - \lambda_2)^2 = -2x_i(y_i - \lambda_1 x_i - \lambda_2)$$

$$\frac{d}{d\lambda_2}(y_i - \lambda_1 x_i - \lambda_2)^2 = -2(y_i - \lambda_1 x_i - \lambda_2)$$

Update functions:

$$\lambda_{n+1} = \lambda_n - u \begin{pmatrix} -2x_i(y_i - \lambda_1 x_i - \lambda_2) \\ -2(y_i - \lambda_1 x_i - \lambda_2) \end{pmatrix}$$

$$f_1(x_i, y_i) = -2x_i(y_i - \lambda_1 x_i - \lambda_2)$$
$$f_2(x_i, y_i) = -2(y_i - \lambda_1 x_i - \lambda_2)$$

u: learning rate, hyperparameter
needs to be properly configured
too large may cause divergent behaviours
too small may never reach the minimum point

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}_{n+1} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}_n - u \begin{pmatrix} -2x_i(y_i - \lambda_1 x_i - \lambda_2) \\ -2(y_i - \lambda_1 x_i - \lambda_2) \end{pmatrix}$$

4) Discrete System ABC and state $x^*$, $x(0) = 0$
Find a control input $u(0), u(1), \dots, u(N-1)$ such that $x(N) = x^*$ (set up matrix and determine solution)

$$x_{n+1} = Ax_n + Bu_n \qquad y_n = Cx_n$$

Setting up the reachability matrix.

$$x_0 = 0$$
$$x_1 = Ax_0 + Bu_0 = Bu_0$$
$$x_2 = Ax_1 + Bu_1 = ABu_0 + Bu_1$$
$$x_3 = Ax_2 + Bu_2 = A^2Bu_0 + ABu_1 + Bu_2$$
$$\vdots$$
$$x_N = Ax_{N-1} + Bu_{N-1} = A^{N-1}Bu_0 + A^{N-2}Bu_1 + \cdots + Bu_{N-1}$$

$$x_N = \underbrace{\begin{pmatrix} A^{N-1}B & A^{N-2}B & \cdots & B \end{pmatrix}}_{\uparrow} \underbrace{\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix}}_{u}$$

$R_N$ matrix for N time steps

To find if an input sequence from $x_0 = 0$ to $x_N = x^*$ we need to solve the following:

$$x^* = R_N u = \begin{pmatrix} A^{N-1}B & A^{N-2}B & \cdots & B \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix}$$

When does a solution exis?

- R is full rank (rank N)
- R is invertible

  Determinant of R is non-zero

A matrix with full rank indicates the system is fully controllable and any state can be achieved with a set of control inputs.

5) 3 points $(x_i, y_i)$ $i = 1, 2, 3$ and model $y = ax + b$, best estimate for $a, b$ and justify

$$y_i = f(x_i) = ax_i + b = f_i$$

$$\min_{a,b} \sum_{i=1}^{3} (y_i - f_i)^2 = \min_{a,b} \sum_{i=1}^{n} v_i^2 \qquad v_i = y_i - f_i$$

$$f_i = ax_i + b = \begin{pmatrix} x_i & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \qquad F = A\theta$$

$$F = \begin{pmatrix} ax_1 + b \\ ax_2 + b \\ ax_3 + b \end{pmatrix} \quad A = \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \end{pmatrix} \quad \theta = \begin{pmatrix} a \\ b \end{pmatrix} \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\min_{a,b} \sum_{i=1}^{3} (y_i - f_i)^2 = \min_{a,b} V^T V \qquad V = Y - F$$

$$= \min_{a,b} (Y - F)^T (Y - F)$$

$$= \min_{\theta} (Y - A\theta)^T (Y - A\theta)$$

$$= \min_{\Theta} (Y^T - (A\Theta)^T)(Y - A\Theta)$$

$$= \min_{\Theta} Y^T Y - Y^T A\Theta - (A\Theta)^T Y - (A\Theta)^T A\Theta$$

$$= \min_{\Theta} Y^T Y - 2Y^T A\Theta - (A\Theta)^T A\Theta$$

Take derivative and set to 0 (minimize):

$$\frac{d}{d\Theta}\left(Y^T Y - 2Y^T A\Theta - (A\Theta)^T A\Theta\right) = 0$$

$$0 = 0 - 2A^T Y - 2A^T A\Theta$$

$$\Theta = (A^T A)^{-1} A^T Y \qquad \text{(solve using MATLAB)}$$

$$\boxed{\Theta = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \dfrac{y_1(2x_1 - x_2 - x_3) + y_2(2x_2 - x_1 - x_3) + y_3(2x_3 - x_1 - x_2)}{2x_1^2 - 2x_1x_2 - 2x_1x_3 + 2x_2^2 + 2x_3^2 - 2x_2x_3} \\ \dfrac{y_1(-x_1x_2 - x_1x_3 + x_2^2 + x_3^2) + y_2(-x_1x_2 - x_2x_3 + x_1^2 + x_3^2) + y_3(-x_1x_3 - x_2x_3 + x_1^2 + x_2^2)}{2x_1^2 - 2x_1x_2 - 2x_1x_3 + 2x_2^2 + 2x_3^2 - 2x_2x_3} \end{pmatrix}}$$

(Solution 1)

6) State Space Equation of G, define new system $\tilde{A}\ \tilde{B}\ \tilde{C}$ that has an integrator in the input path $1/s\, G(s)$.

$$A = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Integrator state denoted by z

$\dot{z} = u$    u: input to system

$v = z$    v: output of integrator

Augmented System:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B \\ 1 \end{pmatrix} u \qquad y = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}$$

$$\boxed{\tilde{A} = \begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \tilde{C} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}}$$

(Solution 2)

6) State Space Equation of G, define new system $\check{A}\ \tilde{B}\ \hat{C}$ that has an integrator in the input path $1/s\, G(s)$.

$$A = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Laplace Transform:

$$sX(s) = AX(s) + BU(s) \qquad Y(s) = CX(s)$$

$$(sI - A)X(s) = BU(s)$$

$$X(s) = (sI - A)^{-1}BU(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{C(sI-A)^{-1}BU(s)}{U(s)} = C(sI-A)^{-1}$$

$$sI-A = \begin{pmatrix} s-1 & 0.5 \\ 0 & s-1 \end{pmatrix}$$

$$(sI-A)^{-1} = \frac{1}{(s-1)^2} \begin{pmatrix} s-1 & -0.5 \\ 0 & s-1 \end{pmatrix}$$

$$C(sI-A)^{-1} = \begin{pmatrix} 0 & \frac{s-1}{(s-1)^2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{s-1} \end{pmatrix}$$

$C(sI-A)^{-1} = \frac{1}{s-1} \cdot G(s)$ (add $1/s$ integrator)

$$Y(s) = G(s)U(s)$$

$$s(s-1)\,Y(s) = U(s)$$

$$s^2 Y(s) - sY(s) = U(s)$$

$\ddot{y} - \dot{y} = u$ and $\ddot{y} = \dot{y} + u$

Let $x_1 = y$ and $x_2 = \dot{y}$

$$\dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}}$$

7) Variance of $AX+BY$

$A, B$ matricies $X, Y$ variables

$\operatorname{var}(AX+BY)$

$= \operatorname{var}(AX) + \operatorname{var}(BY) + 2\operatorname{covar}(AX, BY)$

Proof of $\operatorname{var}(AX) = A\operatorname{var}(X)A^T$:

$= E[(AX - E(AX))(AX - E(AX))^T]$

$= E[(AX - AE(X))(AX - AE(X))^T$

$= E[A(X - E(X))(X - E(X))^T A^T]$

$= AE[(X - E(X))(X - E(X))^T]A^T$

$= A\operatorname{var}(X)A^T$

Proof of $covar(AX, BY) = A\,covar(X,Y)B^T$

$$= E[(AX - E(AX))(BY - E(BY))^T]$$

$$= E[(AX - AE(X))(BY - BE(Y))^T]$$

$$= E[A(X - E(X))(Y - E(Y))^T B^T]$$

$$= AE[(X - E(X))(Y - E(Y))^T]B^T$$

$$= A\,covar(X,Y)B^T$$

Subbing in value of proofs, we get:

$$\boxed{= A\,var(X)A^T + B\,var(Y)B^T + 2A\,covar(X,Y)B^T}$$

Note: Dimensions of A, B should conform with both X, Y vectors and covariance matricies.