MathLab Write Up

Sam Dalton, Cameron Egbert, Ayham Yousef

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1 Introduction

Definition 1.1. Let G be a multi-spoked graph with center vertex v_0 and additional vertices v_i along each corresponding spoke of weight ℓ_i. All edges are of the form (v_0, v_i). Let c_i denote the number of chips placed on any vertex v_i.

Definition 1.2. Chip-firing is the operation in which chips placed on vertices are then transferred to adjacent vertices through the weighted edges. The weight of the edges

Lemma 1.3. *Let G be a spoke graph on n vertices with edges weighted ℓ_i. Let D be a winning divisor for r-gonality with the necessary assumptions. If $\ell_j > k$, then there are at least r chips on v_j.*

Proof. Given k chips for v_m, and weight $\ell_i > k$, it is not possible to fire from v_0 to v_1 without creating new debt of $k - \ell_i$ on v_0. Since chip firing from v_0 to v_i creates debt $\Rightarrow \nexists$ any chip firing operation between $v_0, v_i | c_0, c_i \geq 0$. Since $\nexists$ any valid chip-firing operations, there must be at least r chips on v_i in order to account for the case where r chips are removed from v_i. □

Lemma 1.4. *Let G be a spoke graph on n vertices with edges weighted ℓ_i. Let $A_k = \{\ell_i \mid \ell_i > k\}$.*

$$gon(G) = \min_k \{k + |A_k|\}$$

Proof. □

Lemma 1.5. *Let G be a spoke graph on n vertices with edges weighted ℓ_i. Let $A_{1,k} = \{\ell_i \mid \frac{k}{2} < \ell_i \leq k\}$ and $A_{2,k} = \{\ell_i \mid \ell_i > k\}$.*

$$gon_2(G) = \begin{cases} \min_k\{k + |A_{1,k}| + 2|A_{2,k}|\} & \textit{if } \max\{\ell_i \mid \ell_i \leq \frac{k}{2}\} + \min\{\ell_i \mid \frac{k}{2} < \ell_i \leq k\} > k \\ \min_k\{k + |A_{1,k}| - 1 + 2|A_{2,k}|\} & \textit{if } \max\{\ell_i \mid \ell_i \leq \frac{k}{2}\} + \min\{\ell_i \mid \frac{k}{2} < \ell_i \leq k\} \leq k \end{cases}$$

Proof. □

Lemma 1.6. *Let G be a spoke graph on n vertices with edges weighted ℓ_i. Let D be a winning divisor for r-gonality with the necessary assumptions. If $\ell_j = k - a$ for some $a \in \mathbb{Z}_{\geq 0}$, then there are at least $r - (a+1)$ chips on v_j.*

Proof. □

Lemma 1.7. *Let G be the spoke graph with $n-1$ spokes and $\ell_i = i$ for all $1 \leq i \leq n-1$. Then*

$$gon(G) =$$

Lemma 1.8. *Let G be the spoke graph with $n-1$ spokes and $\ell_i = i$ for all $1 \leq i \leq n-1$. Then*

$$gon_2(G) = \left\lfloor \frac{3}{2} gon(G) \right\rfloor$$

Lemma 1.9. *Let D be a winning divisor. D is rank r if the following hold:*

$$\text{if } \sum_{i \in \mathcal{I}} \ell_i > k \text{ then there exist at least } r - |\mathcal{I}| + 1 \text{ chips among } \ell_{i \in \mathcal{I}}$$

$$\text{if } \sum_{i \in \mathcal{I}} \ell_i = k - a \text{ then there exist at least } r - |\mathcal{I}| - a \text{ chips among } \ell_{i \in \mathcal{I}}$$

Proof. For any arrangement of $r - |\mathcal{I}|$ chips among $v_{i \in \mathcal{I}}$ with c_i chips on v_i, there always exists a patteron of debt such that every v_i is in debt, namely, $c_i + 1$ chips from every vertex. The total debt is $\sum_{i \in \mathcal{I}} (c_i + 1) = r - |\mathcal{I}| + |\mathcal{I}| = r$.
Additionally, for any arrangement of $r - |\mathcal{I}| - a - 1$ chips among $v_{i \in \mathcal{I}}$ with c_i chips on v_i. □