Assignment 3

Math 351

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1 Instructions

There are two exercises (which were typeset using the theorem environment).

Exercise 1. *Recreate this entire document.*[1]

Exercise 2. *Create a new document containing a short description of three of your favorite books, papers, or other publications. Be sure to include a bibliography, created using BibTeX.*

An assignment which completes Exercise 2 in an interesting way or makes amusing use of mathematical typesetting will earn the coveted LaTeXer of the week distiction.

1.1 When to turn it in

Please upload the `.tex` and `.bib` source files and the `.pdf` output files to your solutions to Assingment 3 on or before Sunday.

2 Euler was excellent

Euler proved many statements, such as

$$\prod_{m=1}^{\infty} (1 - q^m) = \sum_{n=\infty}^{\infty} (-1)^n \, q^{\left(3n^2 - n\right)/2}. \tag{1}$$

where q is an indeterminate. Equation (1) is known as Euler's pentagonal number theorem. Euler also proved theorem 1 below.

[1]How meta.

Theorem 1 (The Basel Problem.)**.** *We have* $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Euler's original proof of Theorem 1 makes unjustified assumptions that infinite products and sums behave like finite products and sums, but interesting nonetheless and worth displaying.

Proof. Using the power series for sin x, we have

$$\begin{aligned}
\frac{\sin x}{x} &= \frac{1}{x}\left(x - \frac{x^3}{3!} - \frac{x^5}{5!} - \cdots\right) \\
&= 1 - \frac{x^2}{3!} - \frac{x^4}{5!} - \cdots \\
&= \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\cdots \qquad (2)
\end{aligned}$$

where the reasoning[2] behind (2) is that a polynomial can be factored if its roots are known, and the roots of $\sin x/x$ are $\pm\pi, \pm 2\pi, \ldots$ Multiplying each pair of consecutive terms in this product gives

$$\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots \qquad (3)$$

The coefficient of x^2 in (3) is $-\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \cdots = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ the coefficient of x^2 in $\sin x/x$ is $-1/3!$, so equating these two expressions proves the theorem. □

[2]This reasoning is actually true, just needs further justification.

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