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Hawking Radiation: A Review and a Calculator

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ABSTRACT

This papers explores the intriguing mechanism of Hawking radiation. This is mainly a review of the formulation of Hawking radiation, how it came to be, the historical context needed, such as the formulation of the Theory of General Relativity, a solution provided by Schwarzschild, and its implications on the existence of black holes. After presenting the historical context needed to understand the formulation of the theoretical model of black holes, we will then explore Stephen Hawking's formulation of Hawking radiation, describing its mechanism in a descriptive manner, and implications with other fields of physics. Quantum mechanics specifically is one of those fields as a paradox was formulated thanks to the creation of Hawking Radiation and its conflicts with the core tenant of said field. This is known as Quantum Information Paradox, and we will briefly explore solutions to this paradox. Finally, we present a calculator to calculate 12 Hawking radiation related parameters by giving a brief rundown of the derivation of the lifetime of a black hole. In this paper we present a calculator for 12 parameters related to Hawking radiation in SI units. The user input is a mass slider that computes the corresponding Schwarzschild radius, surface area, effective mass density, surface gravity, surface tides, freefall time, entropy, temperature, peak photon wavelength, nominal luminosity, and lifetime. The formulation of Hawking radiation, Schwarzschild solution, and the Quantum Information Paradox are discussed in a review format and serves as background for the calculator. The Cosmic Microwave Background temperature correction is implemented in order to compute a more accurate lifetime.

Keywords: Black Hole, Hawking Radiation, Thermodynamics, Relativity, Information Paradox

1. INTRODUCTION

After the confirmation of the existence of black holes Akiyama et al. (2019), General Relativity passed another test. This observation not only reaffirms the theory more, but also reaffirms the works of Schwarzschild and many others that contributed to the theoretical frame work of black holes. This is especially true for Stephen Hawking.

Schwarzchild introduced the implications of the existence of black holes with his exact solution to Einstein's Field equation, and Hawking contributed by learning more about the mechanism of the black hole with his discovery of Hawking radiation. His work gave the world a new perspective on black holes, but opened up a lot more questions and also a paradox. Hawking radiation is described as black holes radiating away tiny amount of particles very slowly, resulting in lifespan not even imaginable for peoples brain to wrap around.

This paper is organized as follows: we begin by setting the historical and theoretical backdrop, where we discussed the inception of black holes thanks to the works of Albert Einstein and Karl Schwarzschild. This is followed by a descriptive review of the models that describe Hawking Radiation by first discussing its formulation, and the mechanism. We then discuss its implications for black hole thermodynamics and the ongoing debate around the black hole information paradox. Finally, we conclude with potential future directions this research might take, emphasizing the interdisciplinary nature of this inquiry. We also provided how we would use the equations associated with Hawking radiatio effects on black holes to make a Hawking radiation calculator.

2. HISTORICAL CONTEXT

2.1. *Inception*

When Newton's theoretical model of gravity took shape, Newton himself still didn't believe that this wasn't the final words on the nature of gravity. Despite this, it has been widely accepted as a valid description of the gravitational force between masses during and some time after Newton's time. Two-hundred years later, Albert Einstein came along to give a whole new perspective. His eight years long work after his development of

Special Relativity led him to his Theory of General Relativity, where now gravity takes place into his theory of special relativity. One of the biggest results that came out of Einstein's theory is his field equations. These equation were non-linear and difficult to solve, even for an acclaimed genius like Einstein. Fortunately, his theoretical work somehow went to the hands of someone who's volunteering for World War I, Karl Schwarzschild.

To get an exact solution to Einstein's Field Equations, Schwarzschild imagine the simplest possible scenario: a static universe, with a spherically symmetric point mass M with no electric charge, and angular momentum Schwarzschild (1916). He measured everything using spherical coordinates relative to this mass, and he chose a time coordinate where he measures the time from someone that is far way from the mass where spacetime is essentially flat. All this resulted in the Schwarzschild metric.

$$ds^2 = \left(1 - \frac{r_s}{r}\right) c^2\, dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\, dr^2 + r^2\, d\Omega^2 \quad (1)$$

This metric describe how spacetime curves outside of the mass. Despite the success of this derivation, there is a problem. At the center of the mass, where $r = 0$, the first term blows up to infinity, thus the equation breaks down and can no longer describe what's physically happening. This is known as a singularity. The second term is not safe either. At a special distance from the center, known as the Schwarzschild radius r_s, if $r = r_s$, the second term also blows up to infinity, therefore a second singularity. After some analysis of the metric made by Arthur Eddington in 1924, he commented 2 years later that "a star of 250 million km radius could not possibly have so high a density as the Sun. Firstly, the force of gravitation would be so great that light would be unable to escape from it, the rays falling back to the star like a stone to the earth. Secondly, the red shift of the spectral lines would be so great that the spectrum would be shifted out of existence. Thirdly, the mass would produce so much curvature of the spacetime metric that space would close up around the star, leaving us outside." Eddington (1920). All of this, created the seed for the creation of a Black hole. Despite Einstein himself trying to disprove the existence of black holes, Robert Oppenheimer and his student predicted the creation of Black holes based on Einsteins own theory of General Relativity Oppenheimer & Snyder (1939).

2.2. *A Looming Paradox*

The second term of the Schawrzsschild metric blowing up to infinity where $r = r_s$ has a special name for that region. It is known as the event horizon. This named was coined by Wolfgang Rindler, but the one who discovered that the metric that produces this second term blowing up corresponds to a region in space from which nothing escapes is David Finklestein Finkelstein (1958). This also means that he defined a local black hole event horizon as a boundary beyond which events of any kind cannot affect an outside observer.

After some more research of black holes come and go, theoretical physicist and cosmologist Stephen Hawking steps into the picture and introduces a new mechanism of black holes: they evaporate. Hawking predicted that black holes are not entirely black but emit small amounts of thermal radiation at a temperature Hawking (1974). This is known as Hawking Radiation and this implies that black holes has a such a lifetime that is hundreds of orders of magnitude larger than anyone can fathom. By applying quantum field theory to a static black hole background, he determined that a black hole should emit particles that display a perfect black body spectrum. He also argued that the detailed form of the radiation would be independent of the initial state of the black hole, and depend only on its mass, electric charge and angular momentum Hawking (1976). However this mechanism introduced a conflicting truth with quantum mechanics. Quantum mechanics states that information is always retained, no matter what physical properties it goes through, but since Hawking radiation allows the black hole to evaporate, and a property of a black hole is to not let anything, not even light to escape, this would effectively mean that information will cease to be. This creates a paradox between one of the theoretical mechanisms of a black hole and quantum mechanics known as the Black Hole Information Paradox. Later on in the discussion section, there will be more details about this paradox and even the generally accepted solution to it, but for now much of the discussion is about Hawking Radiation.

3. DISCUSSION

Much of the discussion from here on will mostly be about the formulation of Hawking Radiation, its mechanism, and its implications with other fields of physics such as quantum mechanics and the paradox it arises. The calculator will have its own section.

3.1. *Formulation*

Hawking's approach to black hole radiation stemmed from his interests in the implications of black holes on the laws of classical physics, particularly those governing thermodynamic processes. Hawking's breakthrough came when he considered the behavior of quantum fields in the curved space-time around black holes.

Quantum Field Theory (QFT) posits that particle-antiparticle pairs are constantly being created and annihilated in the vacuum. Normally, these virtual particles have no observable effects. However, Hawking proposed that if such a pair were to form near the event horizon of a black hole, the intense gravitational pull might separate them before they could annihilate each other. This led to the radical idea that black holes might not be entirely black but could emit radiation due to quantum mechanical effects Hawking (1974).

Hawking applied the mathematical formalism of QFT in curved spacetime to calculate the effects near a black hole's event horizon. His calculations showed that if a particle-antiparticle pair forms and one particle is captured by the black hole while the other escapes, the escaping particle would appear as radiation emitted by the black hole. The particle falling into the black hole, having negative energy, effectively reduces the mass and energy of the black hole—an observation leading to the prediction that black holes could evaporate over time.

3.2. *Mechanism*

One of the more interesting areas in black hole research is black hole thermodynamics. In 1972, Jacob Bekenstein developed a theory about black holes where he showed that black holes have entropy Bekenstein (1972). He suggested that the entropy of a black hole is linked to the area of its event horizon, not its volume like in more usual systems. This connected the shape and size properties of black holes, as understood through general relativity, to thermodynamic qualities.

Hawking later discovered black hole radiation, which helped explain black hole entropy through quantum field theory. When black holes emit radiation, they lose mass, which causes their surface area to shrink. This leads to a decrease in their entropy since entropy is tied to the surface area. However, the entropy that the emitted Hawking radiation carries away makes up for this loss. Overall, this means the total entropy of the universe goes up, supporting the second law of thermodynamics, which states that the total entropy of an isolated system can't decrease. This study of black hole thermodynamics not only increases our knowledge of black holes but also supports key principles of thermodynamics in the universe.

The temperature of this thermal spectrum (Hawking temperature) is proportional to the surface gravity of the black hole, which, for a Schwarzschild black hole, is inversely proportional to the mass. Hence, large black holes emit less radiation than small black holes. One of the conditions for black holes to actually evaporate is for the Hawking temperature to be greater than the cosmic microwave background radiation (which is 2.7 K) Sivaram (2001). At the present time, for a black hole to start evaporating, they would need a mass less than the Moon. There are no black holes that are that size, so the black wholes around the whole universe would need to wait for the CMB radiation to cool down enough to be lower than the Hawking temperature of the black hole. As an example, once the temperature of the cosmic microwave background drops below that of the black hole, a supermassive black hole with a mass of $10^{11} M_{\odot}$ will evaporate in around 2×10^{100} years, however these were based on outdated assumptions about neutrinos where neutrinos have no mass and that only two neutrino flavors exist Page (1976). Much of the equations relating to black hole properties related to Hawking radiation will be discussed in the calculator section of this paper, but the formulation of Hawking radiation led to a looming paradox.

3.3. *Information Paradox*

The black hole information paradox emerges when one contemplates the sequence of events where a black hole is initially formed by a conventional physical process and subsequently vanishes entirely due to Hawking radiation. According to Hawking's calculation, the end state of the radiation only preserves data about the aggregate mass, electric charge, and angular momentum of the originating state. Given that numerous initial states can share identical values for mass, charge, and angular momentum, it implies that various initial physical conditions could progress into the same ultimate state. As a result, details about the original state would be irretrievably lost. This concept fundamentally challenges a key principle shared by both classical and quantum physics—that the condition of any system at a specific moment should theoretically be able to predict its condition at any other time Hossenfelder (2019).

More precisely, in the realm of quantum mechanics, the state of any system is represented by its wave function, which is evolved through time by a unitary operator. The principle of unitarity ensures that the wave function at any given moment can be utilized to reconstruct the wave function at any prior or subsequent time. In 1993, Don Page proposed that if a black hole originates in a pure quantum state and fully evaporates through a unitary mechanism, the von Neumann entropy of the emitted Hawking radiation would initially rise but then diminish to zero once the black hole has entirely dissipated. This theoretical result is known as the Page curve Page (1993), which describes the entropy trajectory of black holes and seeks to reconcile the principles of quantum mechanics with the phenomena of black hole evaporation.

This paradox has reached some major progress in resolving it. We come back to the Page curve which suggest that if the entanglement entropy follows this curve, it indicates that information escapes the black hole rather than being destroyed. This led to a shift from philosophical debates to more quantitative calculations to verify if the Page curve accurately represents the behavior of black holes. Further advances came with the application of the AdS/CFT correspondence, a theoretical framework that relates gravitational physics in a "bulk" space (which can include black holes) to a quantum field theory on the boundary of that space. This model has been used to argue that information about what falls into a black hole is preserved at the boundary, in a scrambled form, suggesting that information is not lost but rather transformed and stored in the quantum correlations of the Hawking radiation Musser (2023). More recently, studies using more detailed quantum mechanical models have suggested practical ways to potentially recover information from black hole radiation, implying that black holes could act like quantum information scramblers rather than information destroyers. This involves complex quantum operations that might theoretically allow for the reconstruction of incoming information from the outgoing radiation, though at an impractically long timescale Wood (2023).

3.4. *The way forward*

Hawking's work sparked a new perspective of the mechanism of black hole, and even a famous paradox involving the violation of a core tenant of quantum mechanics. Although much theoretical work has been done revolving around Hawking radiation, this is still mostly theoretical. Experimental observations seems to be unfeasible due to the Hawking radiation effect being too small to be observed directly even with experimentally achievable conditions for gravitational system. For now, it might be far into the future before any experimental observations can confirm the existence of Hawking radiation, but understanding even more about this mechanism can unlock even more secrets the black hole is hiding.

4. CALCULATOR

Arguably this section can be a part of section 3 as much of the equations for the calculator will be discussed here, but it would make more sense on what the project approach is about by presenting the equations in conjunction on how this calculator would work. The calculator idea is inspired by Jim Wisniewski, whose page from 2006 appears to be no longer available.

There are 12 parameters that can be adjusted in calculating any aspect of the effects of Hawking Radiation: Mass, Schawrzschild radius, Surface area, Effective density, Surface gravity, Surface tides, Time to singularity, Entropy, Temperature, Peak photons, Nominal luminosity, and finally, the lifetime. Adjusting any of these parameter will affect every other parameter. Let's begin introducing the equations.

We can see how most of the equations contribute to the liftetime of a black hole thanks to Hawking's work. Beginning with the standard formula for the Schwarzschild radius of a mass M

$$R = \frac{2GM}{c^2} \tag{2}$$

From Hawking (1974), the thermodynamic temperature of such a black hole is

$$T = \frac{\kappa}{2\pi} = \frac{\hbar c^3}{8\pi k_B G}\frac{1}{M} \tag{3}$$

where κ is the surface gravity. The surface gravity discussed here refers to its Newtonian approximation. At the event horizon, however, the actual surface gravity becomes effectively infinite. This means that to hold an object stationary right at the event horizon would require a rocket of infinite power. The surface area is

$$A = 4\pi R^2 = \frac{16\pi G^2}{c^4}M^2 \tag{4}$$

This makes the Hawking radiation luminosity to become

$$L = A\sigma T^4 = \frac{\hbar c^6}{15360\pi G^2}\frac{1}{M^2} \tag{5}$$

Peak photons in Hawking radiation relate to the characteristics of the emitted radiation, particularly its spectral properties. In other words, this represent the peak of the black-body radiation curve per unit logarithm (of wavelength or frequency) that corresponds to the black hole temperature. Peak photons are calculated using Planck's radiation law

$$B_\nu(\nu, T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/k_B T} - 1} \tag{6}$$

where ν is the frequency. We alternatively can express this as a function of wavelength $\lambda = c/\nu$

$$B_\lambda(\lambda, T) = B_\nu(\nu, T)\frac{d\nu}{d\lambda} = \frac{2hc^2}{\lambda}\frac{1}{e^{hc/\lambda k_B T} - 1} \tag{7}$$

We reexpress B in terms of $\log\lambda$ to obtain a solution this derivative

$$\lambda_{\text{logpeak}} = \frac{hc}{k_B T[W(-4e^{-4}) + 4]} \tag{8}$$

The aforementioned Bekenstein entropy that was discussed earlier is also named the Bekenstein-Hawking entropy, and this dimensionless parameter is defined as

$$S = \frac{c^3 A}{4G\hbar} = M^2 \frac{4\pi G}{\hbar c} \tag{9}$$

We multiply this value by the Boltzmann constant to get the entropy a conventional unit. A black hole with mass M at a distance r has an incident radiation flux of the following form

$$\Phi = \frac{L}{4\pi r^2} = \frac{\hbar c^6}{614440\pi^2 G^2 r^2} \frac{1}{M^2} \tag{10}$$

the equation above isn't related to the calculator but just a little fact. The free-fall time t_S from horizon to singularity can be calculated

$$t_S = \frac{1}{c} \int \frac{1}{\sqrt{\frac{2GM}{c^2 r} - 1}} \, dr = \frac{\pi G M}{c^3} \tag{11}$$

The expression for L makes it possible to calculate the lifetime of a black hole of given initial mass M_0, assuming no mass input. Luminosity means energy output, thus

$$-\frac{dE}{dt} = \frac{\hbar c^6}{15360\pi G^2} \frac{1}{M^2} \tag{12}$$

Since $dE = dMc^2$

$$-\frac{dM}{dt} = \frac{\hbar c^4}{15360\pi G^2} \frac{1}{M^2} \tag{13}$$

Separating variables and integrating, we get

$$t = \frac{5120\pi G^2}{1.8083\hbar c^4} M^3 \tag{14}$$

where the 1.8083 factor comes from the calculations of Page (2005) which arises from a combination of effects. The surface tide and effective density, were all mixed into the derivation above. The intended website that will demonstrate these equations will allow you free reign on adjusting any of the parameters relating to Hawking Radiation's effects on black holes.

5. CONCLUSIONS

Given the relevant historical context needed to understand the how black holes were even formulated, especially thanks to the work of Schwarzschild, we explored one particular black hole mechanism that help us shaped our understanding on a black hole's lifetime. As they nothing last forever and Hawking radiation allows for black holes to have a lifespan. Even though the effects of Hawking radiation is very slow, resulting in an unfathomable long lifespan, it is still another piece of the universe that we hopefully solve (as we still haven't experimental observed this). A whole new sub-discipline of black hole research known as black hole thermodynamics was created which led the creation of Hawking radiation thanks to the work of Bekenstein discovering a entropy in black holes. After the formulation of the Hawking radiation, this arises a complex paradox that contradicts the inner workings of quantum mechanics which has been mostly resolved due to recent works of Page and others. Finally, we demonstrated a way to calculated different parameters of hawking radiation effects on black hole such as the lifetime of the black hole (which is the most important one). Although our species might never witness the spectacles of our galaxy colliding with Andromeda and what not, it is fascinating how far we gone to predict the trajectory of the universe, and Hawking radiation gave is just one of those predictors of the universe future that we stumbled upon.

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