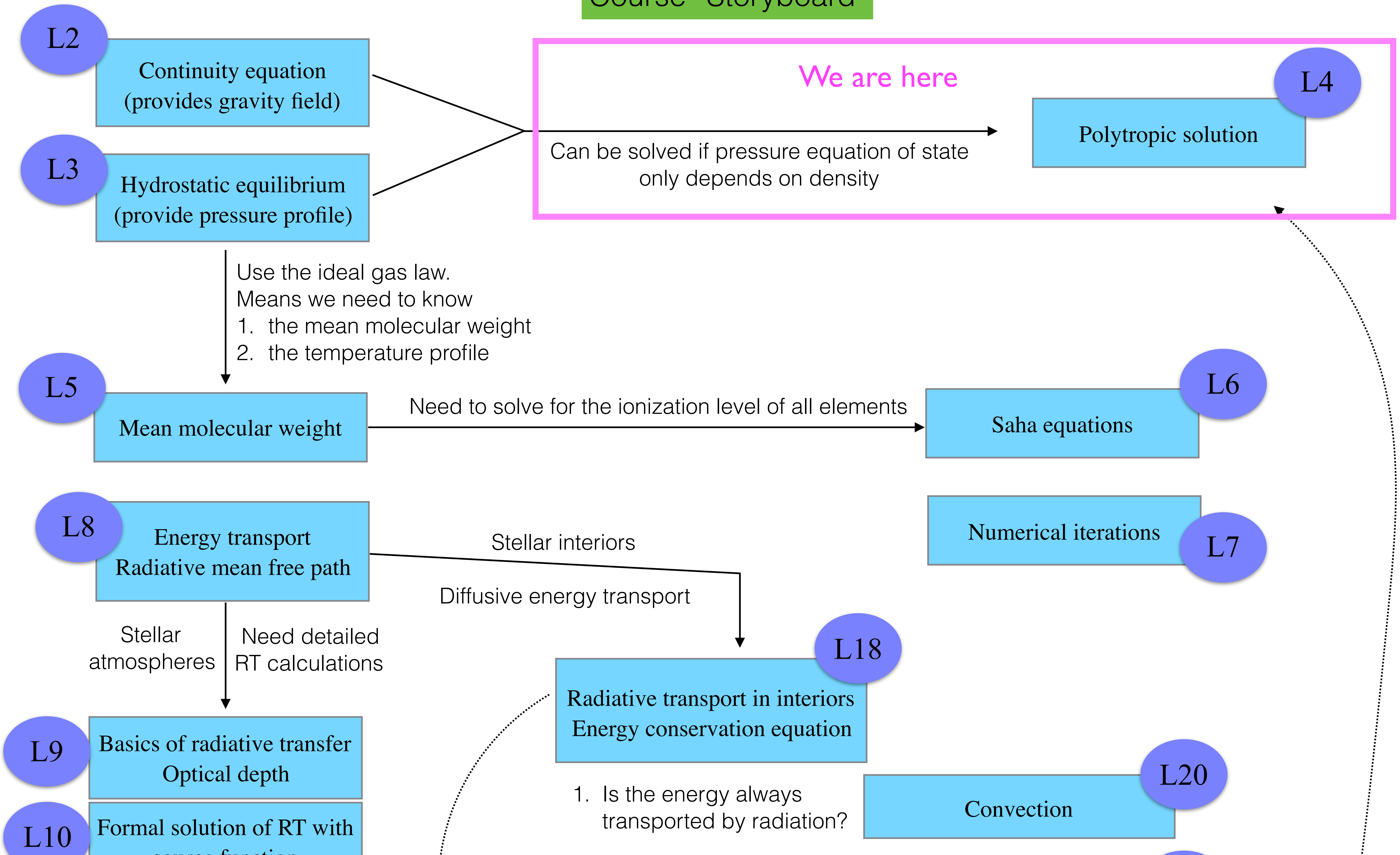
Week 2 Thursday
L-4
Polytropes

Course “Storyboard”



Differential equations:

Continuity equation

$$\frac{dM_r(r)}{dr} = 4\pi r^2 \rho(r)$$

Hydrostatic equilibrium

$$\frac{dP(r)}{dr} = -\rho(r)\overbrace{\frac{GM_r(r)}{r^2}}^{g(r)}$$

Two equations, three unknowns (plus R)

We need a relationship between $P(r)$ and $\rho(r)$

Variables

$M_r(r)$
$\rho(r)$
$P(r)$

Boundary conditions

$M_r(r = 0) = 0$
$P(r = R) = 0$

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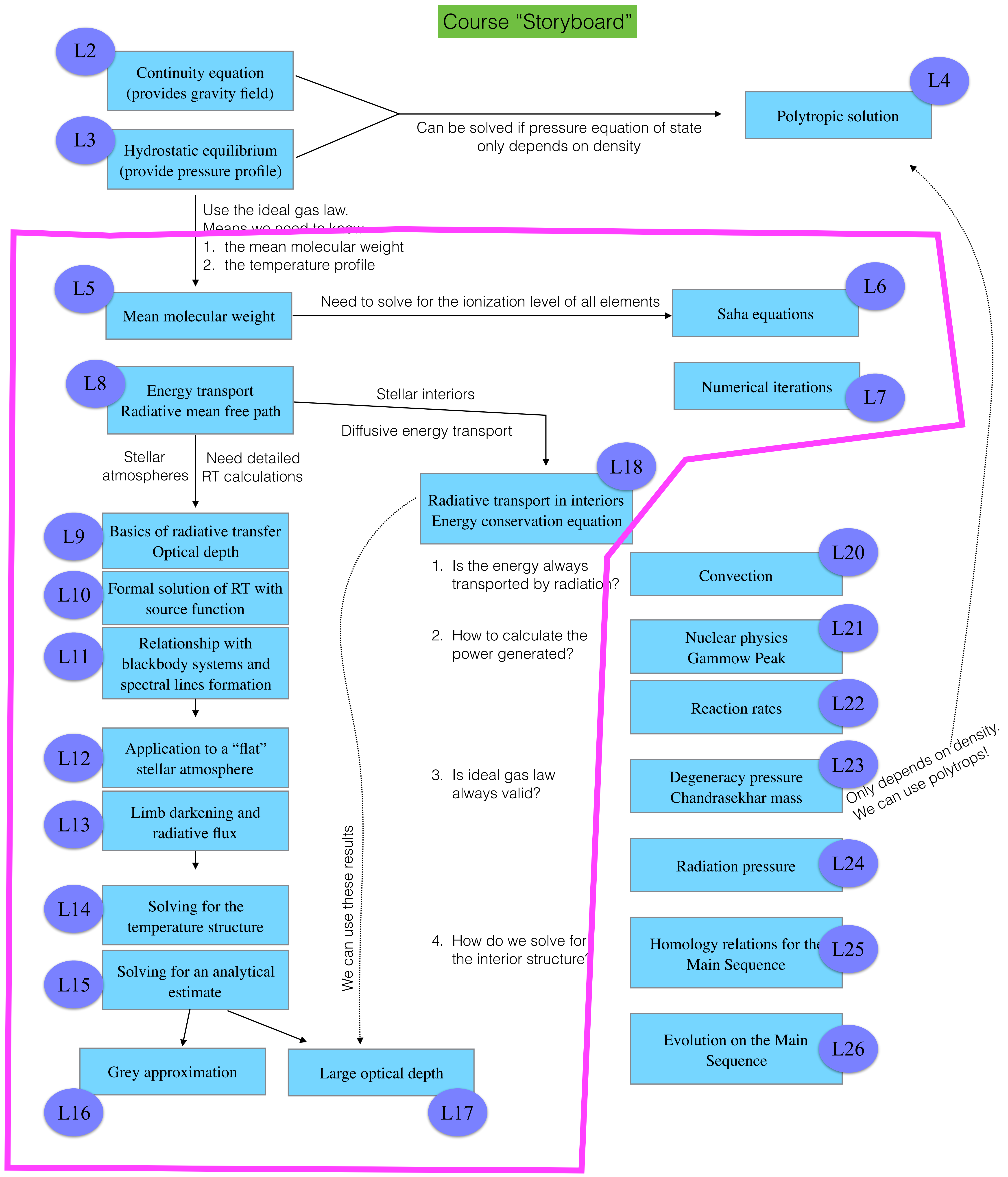
We need a relationship between $P(r)$ and $\rho(r)$ (An "equation of state")

$P = nkT$?

If we consider ideal gas, OK… But we don't know T (yet).
And to relate n to ρ, we need to know the composition…

All of this is basically to get $T(r)$… yikes!



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We need a relationship between $P(r)$ and $\rho(r)$ (An "equation of state")

But if P is only dependent on ρ, we could in principle solve the problem!

$$P(r) = K\rho^{\frac{n+1}{n}}$$

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Polytropic equation of state

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Hydrostatic equilibrium

$$\frac{dP(r)}{dr} = -\rho(r)\frac{GM_r(r)}{r^2}$$

$$\frac{d}{dr}\quad \frac{r^2}{\rho(r)}\frac{dP(r)}{dr} = -G\frac{d}{dr}M_r(r)$$

$$\frac{d}{dr}\left[\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right] = -G\,4\pi r^2\rho(r)$$

Polytropic equation of state

$$P(r) = K\rho(r)^{\frac{n+1}{n}}$$

Put all the *r*s on one side: $$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right] = -G\,4\pi\rho(r)$$

Differential equations:

Continuity + hydrostatic equations

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right] = -\,G\,4\pi\rho(r)$$

Polytropic equation of state

$$P(r) = K\rho(r)^{\frac{n+1}{n}}$$

1. We would like to solve for $\rho(r)$.
2. But, we like things to be unit-less

   (can you think why?)

$$\rho(r) = \boxed{\rho_o}\;\theta^n(r) \qquad \epsilon = \frac{r}{\alpha}$$

A scale factor (undefined yet, but see later), that will scale the radial coordinate to be unit-less (Why not use $R_\star$ here?)

Differential equations:

Continuity + hydrostatic equations

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right] = -G\,4\pi\rho(r)$$

1. We will to solve for $\theta(\epsilon)$.

$$\rho(r) = \rho_o\ \theta^n(r) \qquad \epsilon = \frac{r}{\alpha}$$

(But to go back to $\rho(r)$, we will need to also get the value of ρ_o in the process….)

Polytropic equation of state

$$P(r) = K\rho(r)^{\frac{n+1}{n}}$$

$$P(r) = K\left[\rho_o\theta^n(r)\right]^{\frac{n+1}{n}} = K\rho_o^{\frac{n+1}{n}}\ \theta^{n+1}(r) = P_o\ \theta^{n+1}(r)$$

$$P(r=0) = P_o = K\rho_o^{\frac{n+1}{n}}$$

Differential equations:

Continuity + hydrostatic equations

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right] = -G\,4\pi\rho(r)$$

$$\underbrace{\frac{(n+1)P_o}{4\pi G\rho_o^2}}_{\alpha^2}\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\theta(r)}{dr}\right] = -\theta^n(r)$$

This whole thing has units of length^2 !

α^2

$$\rho(r) = \rho_o\,\theta^n(r)$$

$$\epsilon = \frac{r}{\alpha}$$

$$P(r) = P_o\,\theta^{n+1}(r)$$

1. Replace $P(r)$ and $\rho(r)$.
2. Do the inner derivative
$$\frac{d\theta^{n+1}}{dr} = (n+1)\theta^n\frac{d\theta}{dr}$$
3. Get all of the constants out of the derivatives and on the left-side

Differential equations:

Continuity + hydrostatic equations

$$\rho(r) = \boxed{\rho_o}\ \theta^n(r) \qquad \epsilon = \frac{r}{\alpha}$$

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$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right] = -G\,4\pi\rho(r)$$

1. Replace $P(r)$ and $\rho(r)$.
2. Do the inner derivative
   $$\frac{d\theta^{n+1}}{dr} = (n+1)\theta^n\frac{d\theta}{dr}$$
3. Get all of the constants out of the derivatives and on the left-side

$$\frac{(n+1)\boxed{P_o}}{4\pi G\boxed{\rho_o^2}}\ \frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\theta(r)}{dr}\right] = -\theta^n(r)$$

$$\frac{\alpha^2}{r^2}\frac{d}{dr}\left[r^2\frac{d\theta(r)}{dr}\right] = -\theta^n(r)$$

4. Make a change of variable
   $r = \alpha\epsilon,\ dr = \alpha d\epsilon$

$$\frac{1}{\epsilon^2}\frac{d}{d\epsilon}\left[\epsilon^2\frac{d\theta(\epsilon)}{d\epsilon}\right] = -\theta^n(\epsilon)$$

The “Lane-Emden” equation

$$\frac{1}{\epsilon^2}\frac{d}{d\epsilon}\left[\epsilon^2\frac{d\theta(\epsilon)}{d\epsilon}\right] = -\theta^n(\epsilon)$$

The "Lane-Emden" equation

$$\rho(r) = \rho_o \; \theta^n(r)$$

$$\epsilon = \frac{r}{\alpha}$$

Second order differential equation

Need boundary conditions

At the center:

$\epsilon = 0$ (why?)

$\theta(\epsilon = 0) = 1$ (why?)

$\left.\frac{d\theta}{d\epsilon}\right|_{\epsilon=0} = 0$ (why?)

$$\frac{1}{\epsilon^2}\frac{d}{d\epsilon}\left[\epsilon^2\frac{d\theta(\epsilon)}{d\epsilon}\right] = -\theta^n(\epsilon)$$

The "Lane-Emden" equation

Second order differential equation

There is an analytical solution for only 3 values of "n"

(For other values, need to use a numerical method)

n = 0 $$\theta(\epsilon) = 1 - \frac{\epsilon^2}{6}$$

n = 1 $$\theta(\epsilon) = \frac{\sin\epsilon}{\epsilon}$$

n = 5 $$\theta(\epsilon) = \frac{1.0}{(1.0+\epsilon^2/3)^{1/2}}$$

In notebook: let's graph these solutions

* How to define functions
* How to make a 'loop'

=> From now on, you are responsible for your axis labels

```
ax.set_xlabel('your label)
ax.set_ylabel('your label)
```

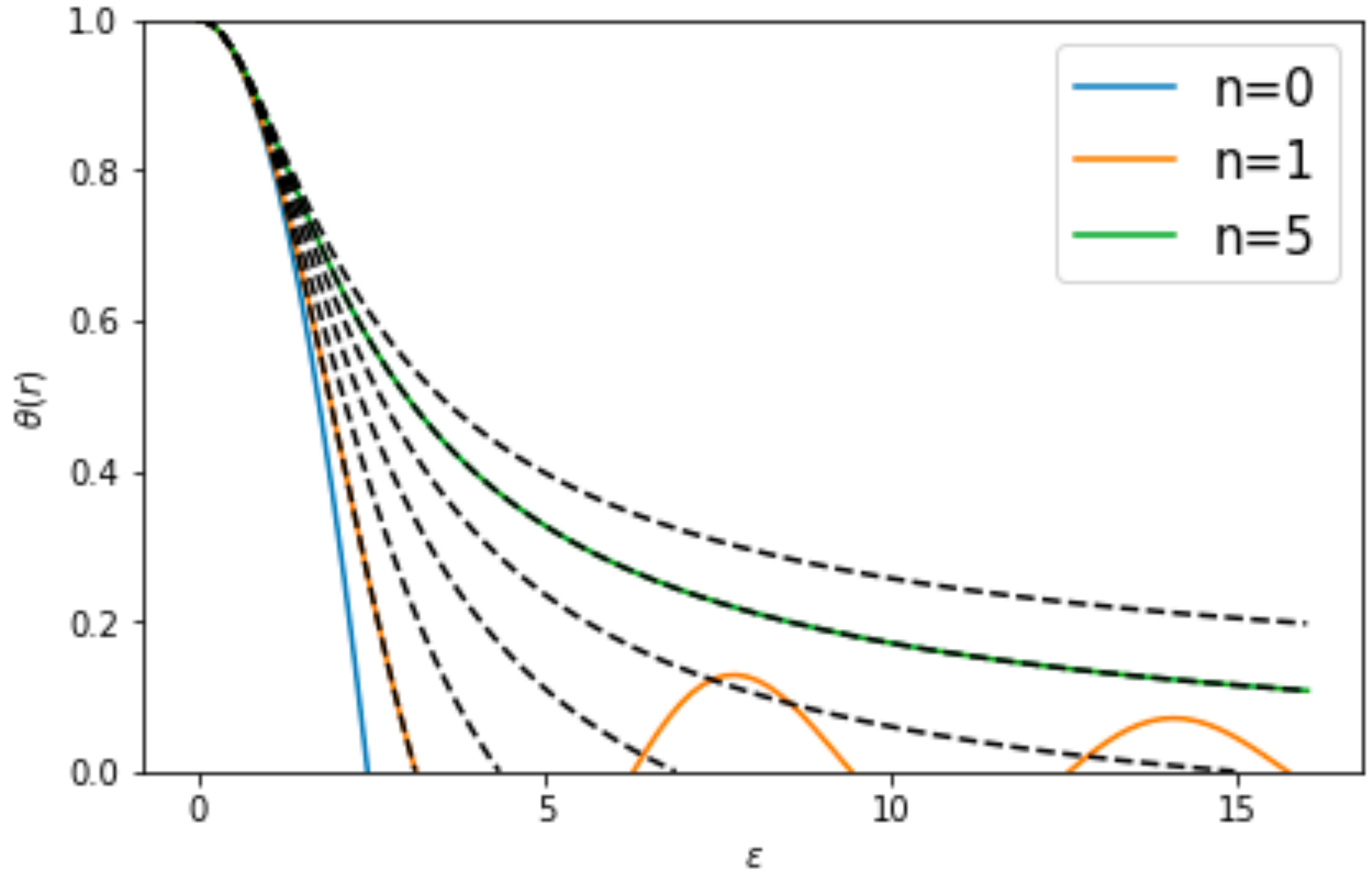
For math symbols:

```
ax.set_xlabel(r'$\alpha$ and $\beta')
```

Now that we have $\theta(\epsilon)$ (for a given n), how do we go back to $\rho(r)$?

$$\rho(r) = \boxed{\rho_o} \quad \theta^n(r)$$

$$\epsilon = \frac{r}{\alpha}$$



Where is the ‘surface’? ϵ_1

$\theta(\epsilon = \epsilon_1) = 0$

In real units:

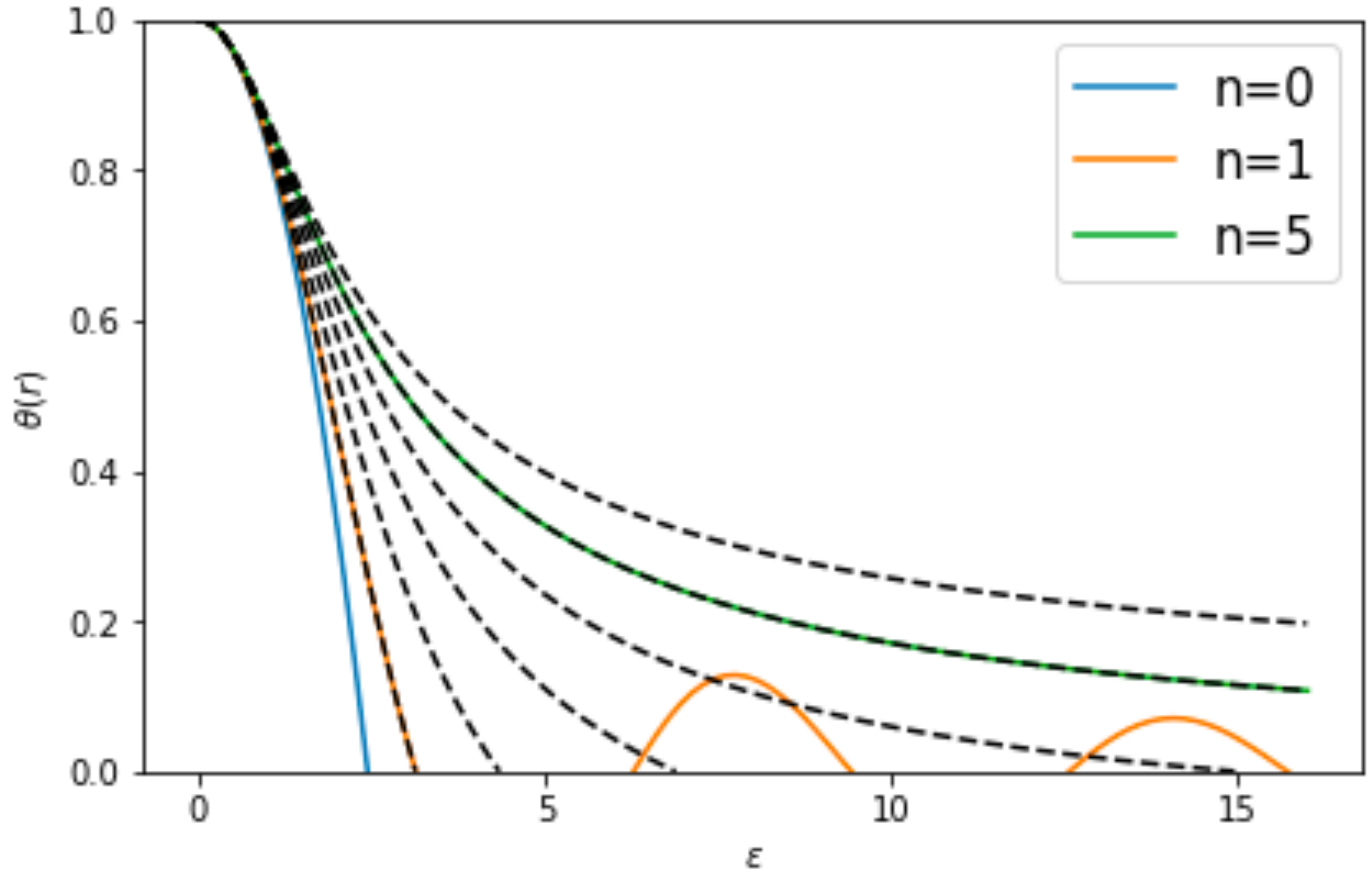
$R_\star = \alpha\epsilon_1$ $\quad \Rightarrow \quad$ $\dfrac{r}{R_\star} = \dfrac{\epsilon}{\epsilon_1}$

In notebook:
* let’s graph these solutions transferred to $\rho(r)/\rho_o$ versus $r/R_\star$ and compare with the sun!

Now that we have $\theta(\epsilon)$ (for a given n), how do we go back to $\rho(r)$?

$$\rho(r) = \rho_o \quad \theta^n(r)$$

$$\epsilon = \frac{r}{\alpha}$$



Where is the 'surface'? $\quad \epsilon_1$

$$\theta(\epsilon = \epsilon_1) = 0$$

In real units:

$$R_\star = \alpha\epsilon_1 \quad \Rightarrow \quad \frac{r}{R_\star} = \frac{\epsilon}{\epsilon_1}$$

or

$$R_\star = \left[\frac{(n+1)P_o}{4\pi G\rho_o^2}\right]^{1/2} \epsilon_1$$

Now that we have $\theta(\epsilon)$ (for a given n), how do we go back to $\rho(r)$?

$$\frac{r}{R_\star} = \frac{\epsilon}{\epsilon_1}$$

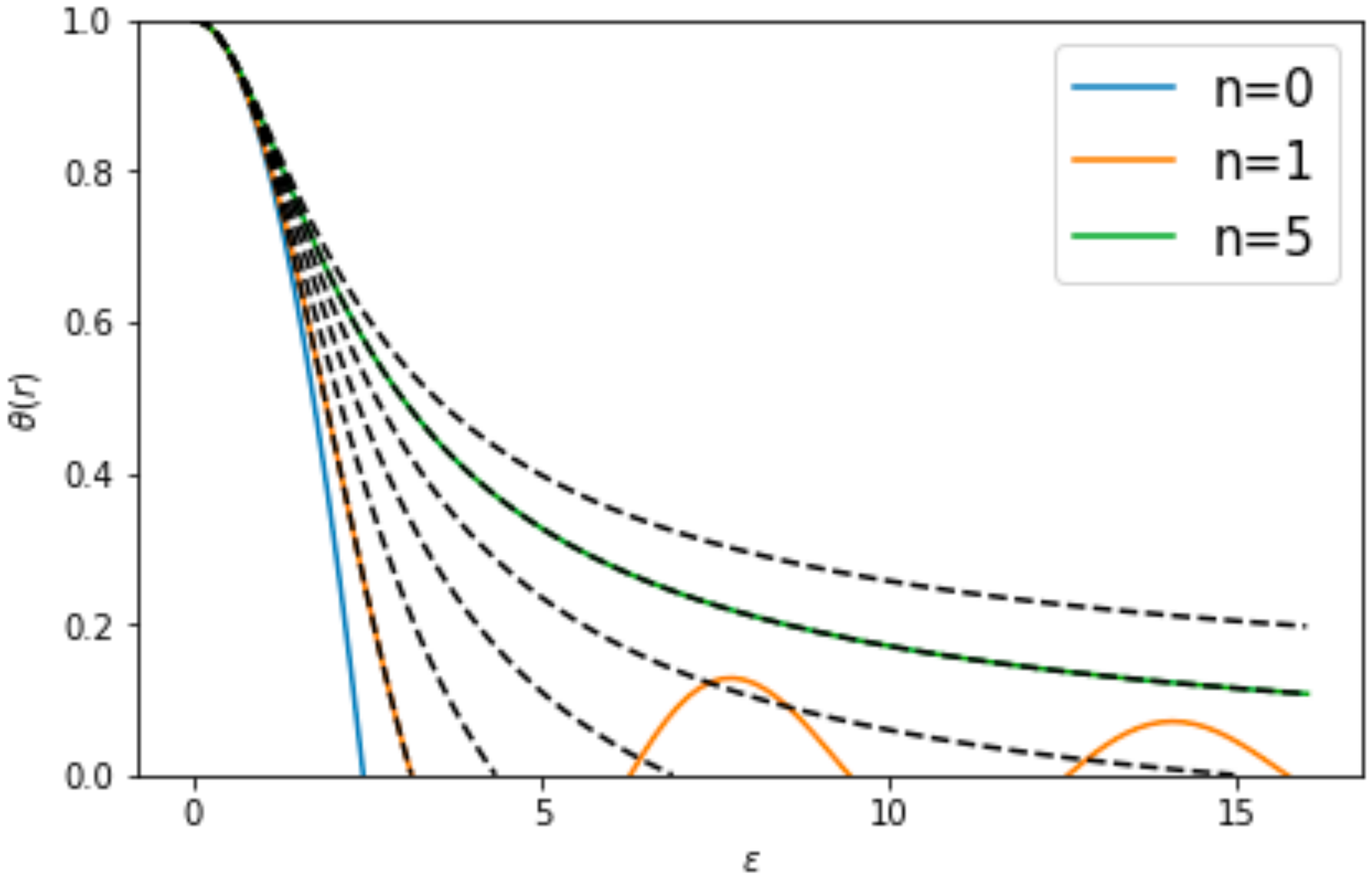
$$\rho(r) = \rho_o \quad \theta^n(r)$$

$$\epsilon = \frac{r}{\alpha}$$

1 $$R_\star = \left[\frac{(n+1)P_o}{4\pi G \rho_o^2}\right]^{1/2} \epsilon_1$$

2 $$P_o = K \rho_o^{\frac{n+1}{n}}$$

3



Let's go back to the continuity equation for a moment

$$M_\star = M_r(r = R_\star) = 4\pi \int_0^{R_\star} r^2 \rho(r) dr$$

$r = \alpha\epsilon \qquad dr = \alpha d\epsilon \qquad R_\star = \alpha\epsilon_1 \qquad \rho(r) = \rho_o \;\; \theta^n(r)$

$$M_\star = 4\pi\alpha^3\rho_o \boxed{\int_0^{\epsilon_1} \epsilon^2 \theta^n(\epsilon) d\epsilon}$$

yah! Unit-less integral!
And if we have a numerical vector for $\theta(\epsilon)$,
we can numerically get that number

But, in textbooks they go a bit further analytically

$$M_\star = 4\pi\alpha^3\rho_c \int_0^{\epsilon_1} \epsilon^2 \theta(\epsilon)^n d\epsilon$$

$$\frac{1}{\epsilon^2}\frac{d}{d\epsilon}\left(\epsilon^2\frac{d\theta(\epsilon)}{d\epsilon}\right) = -\theta^n(\epsilon)$$

1. Put the whole Lane-Emden equation back in there

$$M_\star = 4\pi\alpha^3\rho_c \int_0^{\epsilon_1} \cancel{\epsilon^2} \frac{-1}{\cancel{\epsilon^2}}\frac{d}{\cancel{d\epsilon}}\left(\epsilon^2\frac{d\theta(\epsilon)}{d\epsilon}\right) \cancel{d\epsilon}$$

2. Cancel out a bunch of terms

$$M_\star = -4\pi\alpha^3\rho_c \int_0^{\epsilon_1} d\left(\epsilon^2\frac{d\theta(\epsilon)}{d\epsilon}\right)$$

3. Integrate this puppy

$$M_\star = -4\pi\alpha^3\rho_c \left[\epsilon^2\frac{d\theta(\epsilon)}{d\epsilon}\right]_0^{\epsilon_1}$$

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1. Evaluate the term in bracket at the bounds

$$M_\star = -4\pi\alpha^3\rho_c \left[\epsilon_1^2 \frac{d\theta(\epsilon)}{d\epsilon}\Big|_{\epsilon_1} - 0^2 \frac{d\theta(\epsilon)}{d\epsilon}\Big|_{\epsilon=0}\right]$$

Zero from boundary condition

2. Replace α with its definition

$$M_\star = -\frac{1}{(4\pi)^{1/2}} \left(\frac{n+1}{G}\right)^{3/2} \frac{P_c^{3/2}}{\rho_c^2}\ \epsilon_1^2 \theta'(\epsilon_1)$$

Slope of $\theta(\epsilon)$ at ϵ_1

Now that we have $\theta(\epsilon)$ (for a given n),
how do we go back to $\rho(r)$?

$$\rho(r) = \rho_o \quad \theta^n(r)$$

For a given n we know $\theta(\epsilon), \epsilon_1, \theta'(\epsilon_1)$

1 $$R_\star = \left[\frac{(n+1)P_o}{4\pi G\rho_o^2}\right]^{1/2} \epsilon_1$$

2 $$P_o = K\rho_o^{\frac{n+1}{n}}$$

3 $$M_\star = -\frac{1}{\sqrt{4\pi}}\left(\frac{n+1}{G}\right)^{3/2} \frac{P_o^{3/2}}{\rho_o^2} \epsilon_1^2 \, \theta'(\epsilon_1)$$

K

$R_\star$ P_o

$M_\star$ ρ_o

5 quantities, 3 equations
If we specify 2 quantities, the other 3 are constrained

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For a given n we know $\theta(\epsilon), \epsilon_1, \theta'(\epsilon_1)$

K

$R_\star$ P_o

$M_\star$ ρ_o

For example:

Degenerate matter (white dwarfs and neutron stars) equation of state IS a polytrop!

(K and n are known)

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K

$R_\star$ P_o

$M_\star$ ρ_o

Another example:

Normal stars (just an approximation — no constrains on what K (or n, really) is.

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Notebook: K

If $= 1R_\odot \longleftarrow R_\star$ $\quad P_o$

If $= 1M_\odot \longleftarrow M_\star$ $\quad \rho_o \longrightarrow$ What is ?

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