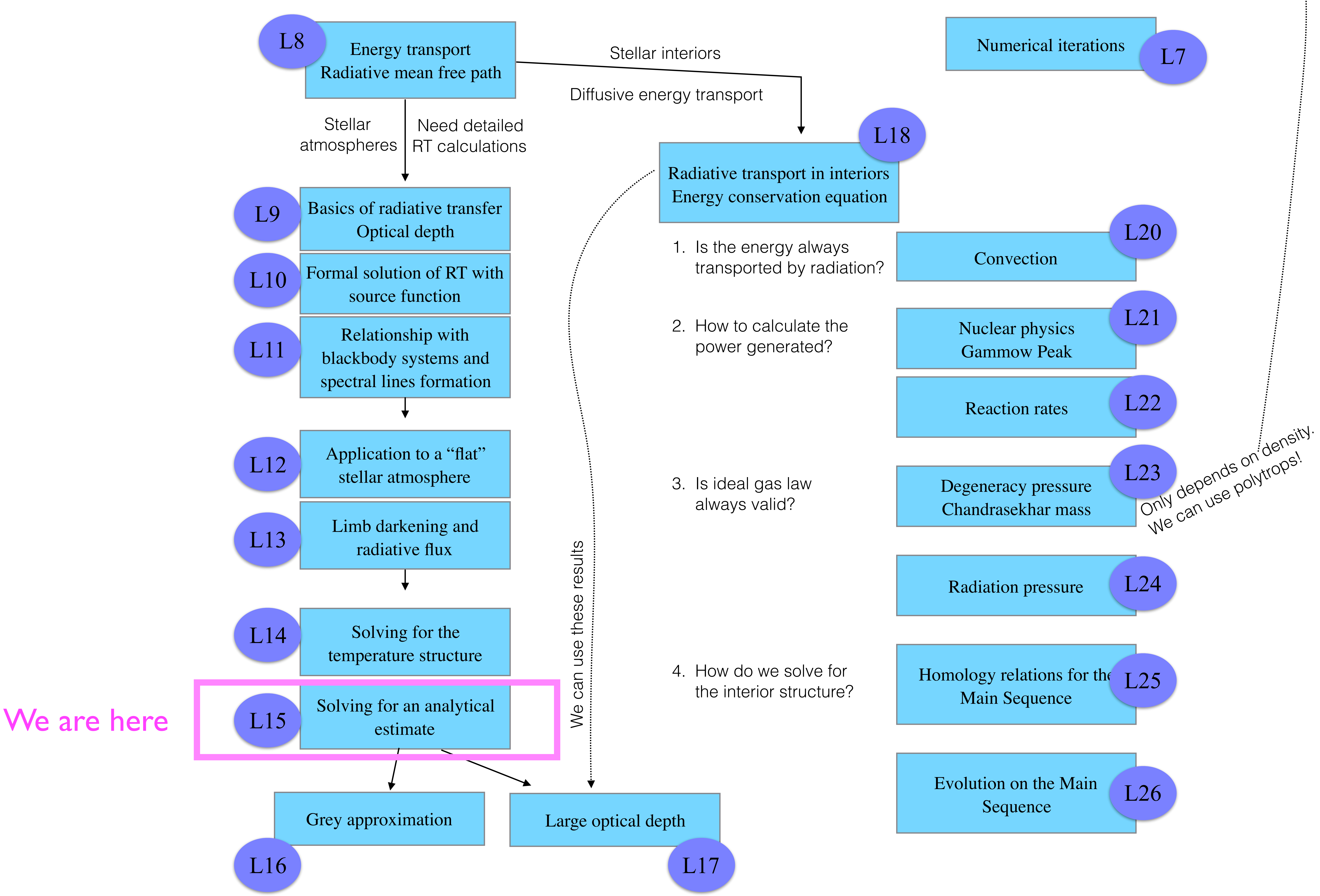
# Week 9 Tuesday

# L-15

# Analytic estimate


L8
Energy transport
Radiative mean free path
Stellar interiors
Diffusive energy transport
Numerical iterations
L7
Stellar atmospheres
Need detailed RT calculations
L18
Radiative transport in interiors
Energy conservation equation
L9
Basics of radiative transfer
Optical depth
L10
Formal solution of RT with source function
L11
Relationship with blackbody systems and spectral lines formation
1. Is the energy always transported by radiation?
L20
Convection
2. How to calculate the power generated?
L21
Nuclear physics
Gammow Peak
L22
Reaction rates
L12
Application to a "flat" stellar atmosphere
L13
Limb darkening and radiative flux
3. Is ideal gas law always valid?
L23
Degeneracy pressure
Chandrasekhar mass
Only depends on density.
We can use polytrops!
L24
Radiation pressure
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Solving for the temperature structure
We can use these results
4. How do we solve for the interior structure?
L25
Homology relations for the Main Sequence
We are here
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Evolution on the Main Sequence
Grey approximation
Large optical depth
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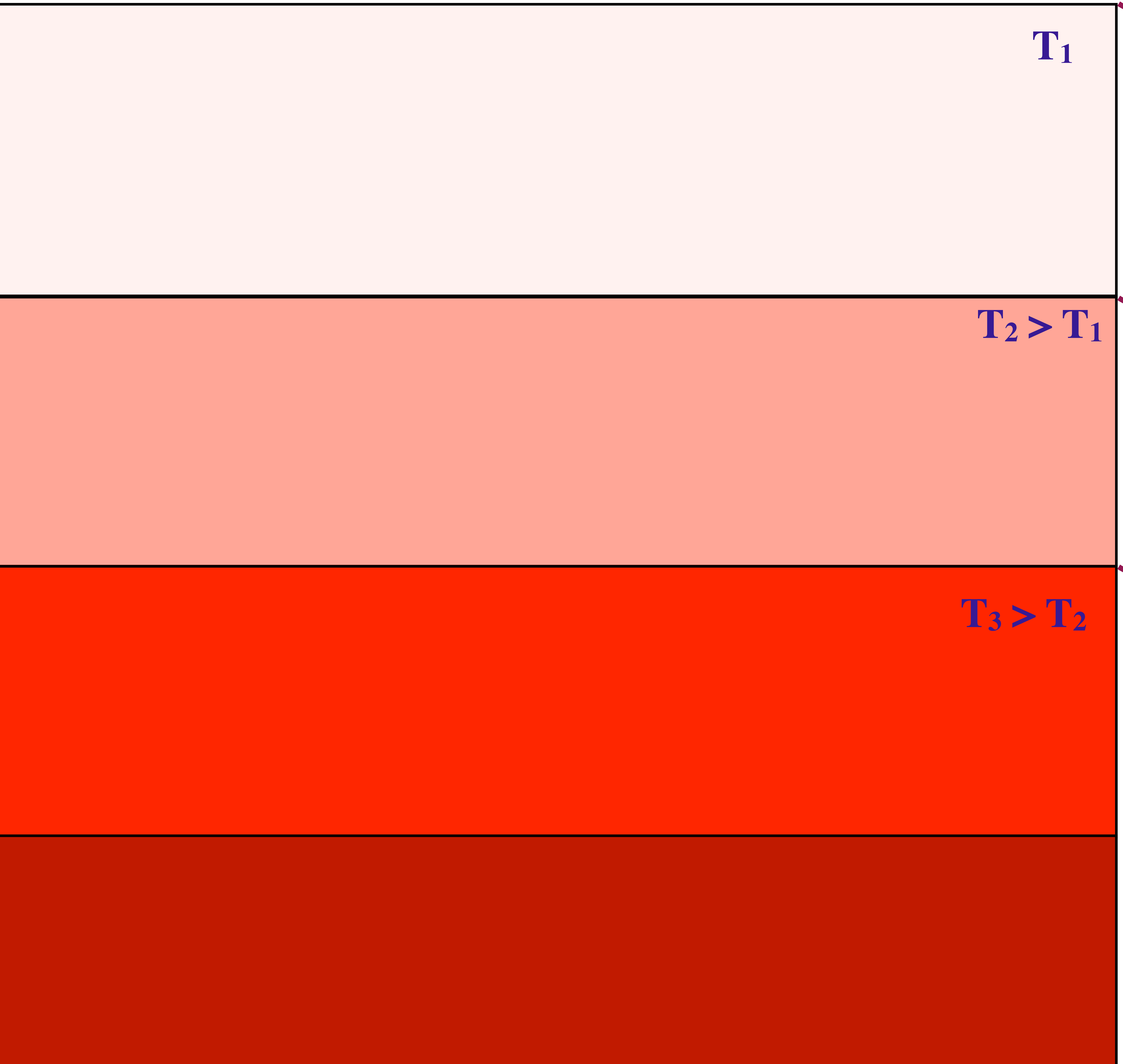
# Key points from the last few lectures

$$\frac{dF}{dz} = 0 \qquad F(z) = \sigma T_{\mathrm{eff}}^4$$

Reminder




Flux through that surface = $\sigma T_{eff}^4$

**(For the Sun: 63 MegaWatt / m$^2$ upward)**

Flux through that surface also

Flux through that surface also

# So we need to relate the flux to source function.

Reminder

**General**

**Spherical, with azimuthal symmetry (u = cos θ)**

**Flux** $$F_\lambda = \int I_\lambda \cos\theta \, d\Omega$$

$$\frac{F_\lambda}{4\pi} = \frac{1}{2}\int_{-1}^{+1} I_\lambda(u)\; u\; du$$

Put the solution for $I(\tau_z, u)$ in the flux equation. On the board

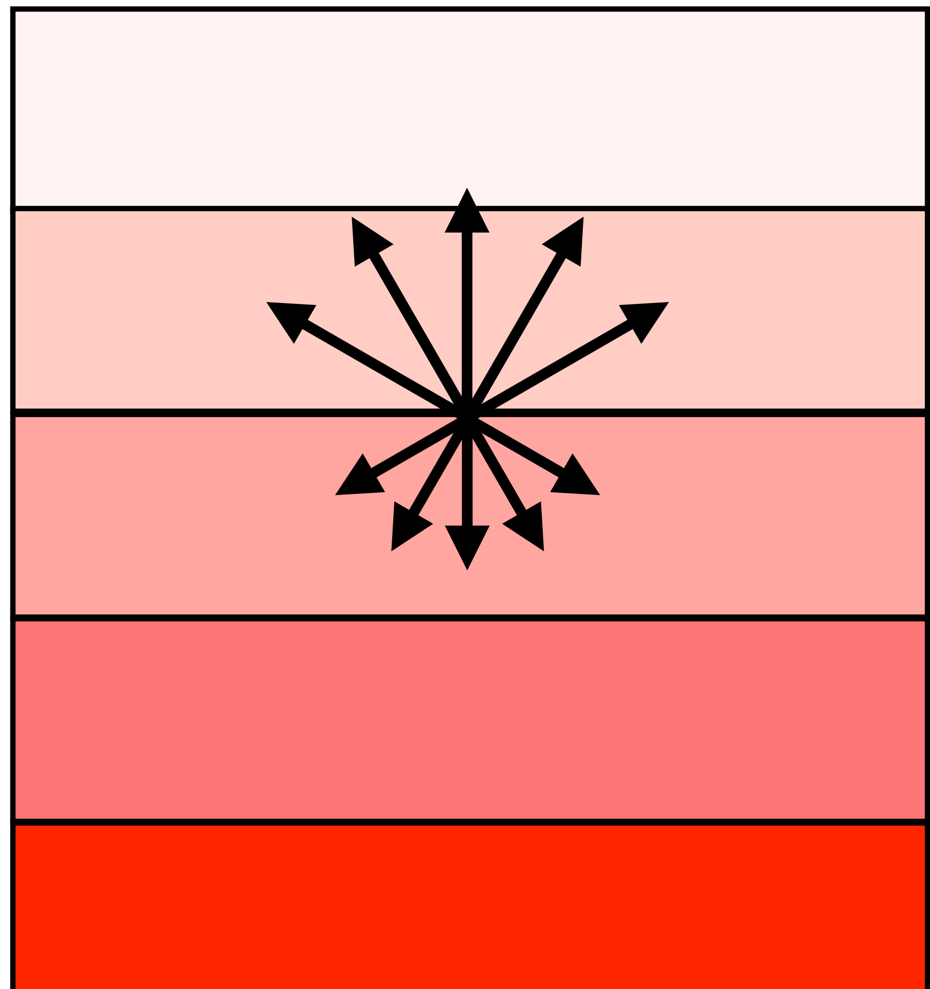
Solution for a flat, semi-infinite atmosphere:

out

$$I(\tau_z, u > 0) = \int_{\tau_z'=\tau_z}^{\tau_z'=\infty} S(\tau_z')\; e^{\frac{\tau_z - \tau_z'}{u}}\; \frac{d\tau_z}{u}$$

in

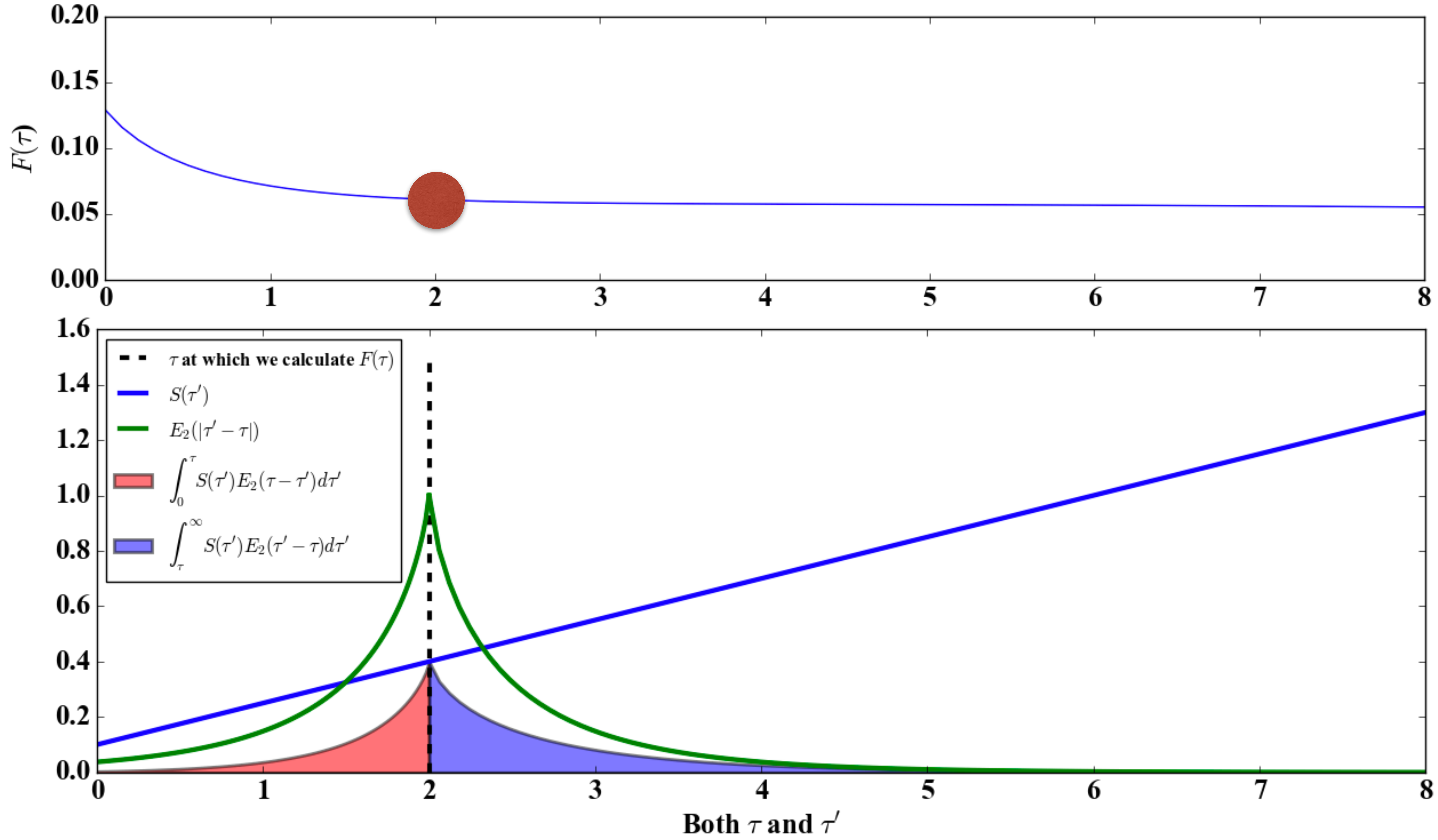
$$I(\tau_z, u < 0) = \int_{\tau_z'=\tau_z}^{\tau_z'=0} S(\tau_z')\; e^{\frac{\tau_z - \tau_z'}{u}}\; \frac{d\tau_z}{u}$$



# Flux for a flat, semi-infinite atmosphere

Reminder

$$\frac{F_\lambda(\tau)}{4\pi} = -\frac{1}{2}\int_{\tau'=0}^{\tau} S_\lambda(\tau')E_2(\tau-\tau')d\tau' + \frac{1}{2}\int_{\tau'=\tau}^{\infty} S_\lambda(\tau')E_2(\tau'-\tau)d\tau'$$



Reminder

1 $$\int_0^\infty F(\lambda, \tau_\lambda)\ d\lambda = \sigma T_{\rm eff}^4$$

2 $$F(\lambda, \tau_\lambda) = 2\pi \int_{\tau'=\tau}^{\tau'=\infty} S(\lambda, \tau'_\lambda) E_2(\tau'_\lambda - \tau_\lambda) d\tau'_\lambda - 2\pi \int_{\tau'=0}^{\tau'=\tau} S(\lambda, \tau'_\lambda) E_2(\tau_\lambda - \tau'_\lambda) d\tau'_\lambda$$

3 $$S(\lambda, \tau_\lambda) = B(\ \lambda, T(z(\tau_\lambda))\ )$$

4 $$d\tau_\lambda(z) = -\kappa_\lambda(z)\rho(z)dz$$

5 $$\kappa_\lambda(z) = f(\text{composition}, \text{T(z)})$$

6 $$P(z) = \frac{\rho(z)kT(z)}{\mu(z)m_H}$$

7 $$\mu(z) = f(\text{composition}, \text{T(z)}, P(z))$$

8 $$\frac{dP(z)}{dz} = -g(z)\rho(z)$$

9 $$g(z) \simeq g_\star$$

$F(\lambda, \tau_\lambda)$ $T_{\rm eff}$

$S(\lambda, \tau_\lambda)$

$z(\tau_\lambda)$ $T(z)$

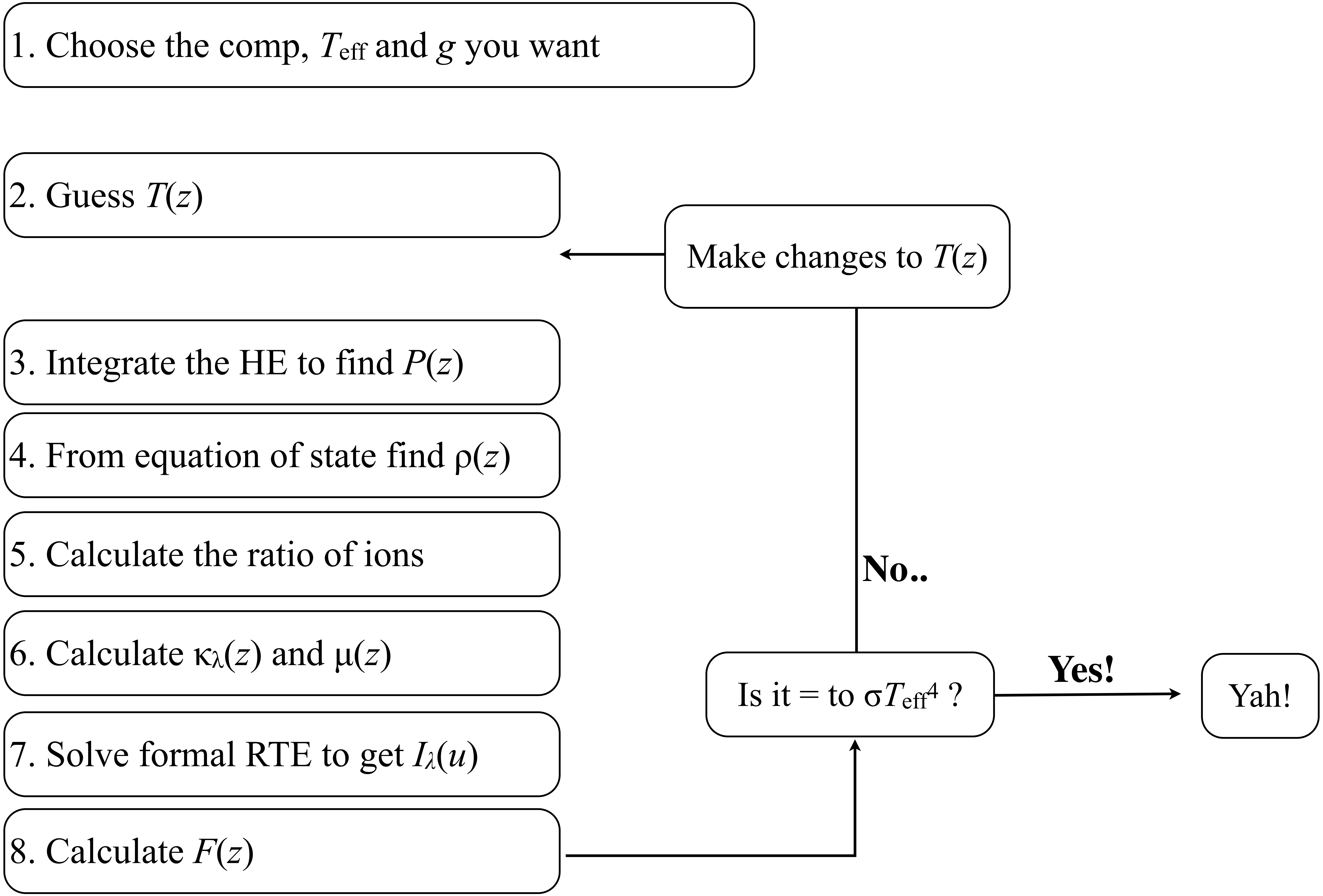
$\kappa_\lambda(z)$ $\rho(z)$

composition

$P(z)$ $\mu(z)$

$g(z)$

$g_\star$


1. Choose the comp, $T_{\mathrm{eff}}$ and $g$ you want
2. Guess $T(z)$
3. Integrate the HE to find $P(z)$
4. From equation of state find $\rho(z)$
5. Calculate the ratio of ions
6. Calculate $\kappa_\lambda(z)$ and $\mu(z)$
7. Solve formal RTE to get $I_\lambda(u)$
8. Calculate $F(z)$
Is it = to $\sigma T_{\mathrm{eff}}^4$ ?
Yes!
Yah!
No..
Make changes to $T(z)$

1 $$\int_0^\infty F(\lambda, \tau_\lambda)\, d\lambda = \sigma T_{\rm eff}^4$$ $F(\lambda, \tau_\lambda)$ $T_{\rm eff}$

2 $$F(\lambda, \tau_\lambda) = 2\pi \int_{\tau'=\tau}^{\tau'=\infty} S(\lambda, \tau'_\lambda) E_2(\tau'_\lambda - \tau_\lambda) d\tau'_\lambda - 2\pi \int_{\tau'=0}^{\tau'=\tau} S(\lambda, \tau'_\lambda) E_2(\tau_\lambda - \tau'_\lambda) d\tau'_\lambda$$ $S(\lambda, \tau_\lambda)$

3 $$S(\lambda, \tau_\lambda) = B(\ \lambda, T(z(\tau_\lambda))\ )$$ $z(\tau_\lambda)$ $T(z)$

4 $$d\tau_\lambda(z) = -\kappa_\lambda(z)\rho(z)dz$$ $\kappa_\lambda(z)$ $\rho(z)$

5 $$\kappa_\lambda(z) = f(\text{composition}, \text{T(z)})$$ composition

6 $$P(z) = \frac{\rho(z)kT(z)}{\mu(z)m_H}$$ $P(z)$ $\mu(z)$

7 $$\mu(z) = f(\text{composition}, \text{T(z)}, P(z))$$

8 $$\frac{dP(z)}{dz} = -g(z)\rho(z)$$ $g(z)$

9 $$g(z) \simeq g_\star$$ $g_\star$

We can make further approximations for an analytical estimate

# Schwarzschild-Milnes equations

The 1st, 2nd, and 3rd "moment"

$$J_\lambda = \frac{1}{2}\int_{-1}^{1} I_\lambda \quad du$$

$$\frac{F_\lambda}{4\pi} = \frac{1}{2}\int_{-1}^{1} I_\lambda \quad u du$$

$$K_\lambda = \frac{1}{2}\int_{-1}^{1} I_\lambda \quad u^2 du$$

What these moments look like when we plug in the intensity solutions for flat atmosphere

$$J_\lambda(\tau) = +\frac{1}{2}\int_{\tau'=0}^{\tau} S_\lambda(\tau')E_1(\tau-\tau')d\tau' + \frac{1}{2}\int_{\tau'=\tau}^{\infty} S_\lambda(\tau')E_1(\tau'-\tau)d\tau'$$

$$\frac{F_\lambda(\tau)}{4\pi} = -\frac{1}{2}\int_{\tau'=0}^{\tau} S_\lambda(\tau')E_2(\tau-\tau')d\tau' + \frac{1}{2}\int_{\tau'=\tau}^{\infty} S_\lambda(\tau')E_2(\tau'-\tau)d\tau'$$

$$K_\lambda(\tau) = +\frac{1}{2}\int_{\tau'=0}^{\tau} S_\lambda(\tau')E_3(\tau-\tau')d\tau' + \frac{1}{2}\int_{\tau'=\tau}^{\infty} S_\lambda(\tau')E_3(\tau'-\tau)d\tau'$$

1 $$\int_0^\infty F(\lambda, \tau_\lambda)\, d\lambda = \sigma T_{\rm eff}^4$$ $F(\lambda, \tau_\lambda)$ $T_{\rm eff}$

2 $$F(\lambda, \tau_\lambda) = 2\pi \int_{\tau'=\tau}^{\tau'=\infty} S(\lambda, \tau'_\lambda) E_2(\tau'_\lambda - \tau_\lambda) d\tau'_\lambda - 2\pi \int_{\tau'=0}^{\tau'=\tau} S(\lambda, \tau'_\lambda) E_2(\tau_\lambda - \tau'_\lambda) d\tau'_\lambda$$ $S(\lambda, \tau_\lambda)$

$\boxed{z(\tau_\lambda) \quad T(z)} \rightarrow T(\tau_\lambda)$

3 $$S(\lambda, \tau_\lambda) = B(\ \lambda, T(z(\tau_\lambda))\ )$$

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4 $$d\tau_\lambda(z) = -\kappa_\lambda(z) \rho(z) dz$$ $\kappa_\lambda(z)$ $\rho(z)$

5 $$\kappa_\lambda(z) = f(\text{composition}, \mathrm{T(z)})$$ composition

6 $$P(z) = \frac{\rho(z) k T(z)}{\mu(z) m_H}$$ $P(z)$ $\mu(z)$

7 $$\mu(z) = f(\text{composition}, \mathrm{T(z)}, P(z))$$

8 $$\frac{dP(z)}{dz} = -g(z)\rho(z)$$ $g(z)$

9 $$g(z) \simeq g_\star$$ $g_\star$

1 $$\int_0^\infty F(\lambda, \tau_\lambda)\, d\lambda = \sigma T_{\text{eff}}^4$$

2 $$F(\lambda, \tau_\lambda) = 2\pi \int_{\tau'=\tau}^{\tau'=\infty} S(\lambda, \tau'_\lambda) E_2(\tau'_\lambda - \tau_\lambda) d\tau'_\lambda - 2\pi \int_{\tau'=0}^{\tau'=\tau} S(\lambda, \tau'_\lambda) E_2(\tau_\lambda - \tau'_\lambda) d\tau'_\lambda$$

3 $$S(\lambda, \tau_\lambda) = B(\ \lambda, T(z(\tau_\lambda))\ )$$

$F(\lambda, \tau_\lambda)$ $T_{\text{eff}}$

$S(\lambda, \tau_\lambda)$

$\boxed{z(\tau_\lambda) \quad T(z)} \rightarrow T(\tau_\lambda)$

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A flat atmosphere, with "grey" opacity

$$\frac{d\left[\int F_\lambda d\lambda\right]}{dz} = 0$$

$$\kappa \neq \kappa_\lambda$$

$$-u\frac{dI_\lambda}{dz} = \kappa_\lambda\rho \quad I_\lambda - \kappa_\lambda\rho \quad S_\lambda$$

$$-\int_u u\frac{dI_\lambda}{dz} = \kappa_\lambda\rho\underbrace{\int_u I_\lambda}_{2J_\lambda} - \kappa_\lambda\rho\underbrace{\int_u S_\lambda}_{2S_\lambda}$$

$$-\frac{d}{dz}\left[\underbrace{\int_u uI_\lambda}_{F_\lambda/2\pi}\right] = \kappa_\lambda\rho\; 2J_\lambda - \kappa_\lambda\rho\; 2S_\lambda$$

$$-\underbrace{\frac{d}{dz}\int_\lambda F_\lambda}_{0} = 4\pi\rho\int_\lambda(\kappa_\lambda J_\lambda - \kappa_\lambda S_\lambda)$$

$$-u^2\frac{dI_\lambda}{dz} = \kappa_\lambda\rho\; uI_\lambda - \kappa_\lambda\rho\; uS_\lambda$$

$$-\int_u u^2\frac{dI_\lambda}{dz} = \kappa_\lambda\rho\underbrace{\int_u uI_\lambda}_{F_\lambda/2\pi} - \kappa_\lambda\rho\underbrace{\int_u uS_\lambda}_{0}$$

$$-\frac{d}{dz}\left[\underbrace{\int_u u^2 I_\lambda}_{2K_\lambda}\right] = \kappa_\lambda\rho\,\frac{F_\lambda}{2\pi} - 0$$

$$-\int_\lambda \frac{d}{\kappa\rho dz}\quad K_\lambda = \underbrace{\int_\lambda \frac{F_\lambda}{4\pi}}_{\frac{\sigma T_{\rm eff}^4}{4\pi}}$$

$$-\underbrace{\frac{d}{dz}\int_\lambda}_{0} F_\lambda = 4\pi\rho\int_\lambda(\kappa_\lambda J_\lambda - \kappa_\lambda S_\lambda)$$

$$0 = \int_0^\infty \kappa_\lambda\,(J_\lambda - S_\lambda)d\lambda$$

$$\int_0^\infty \kappa_\lambda\, J_\lambda\, d\lambda = \int_0^\infty \overset{j_\lambda}{\boxed{\kappa_\lambda\, S_\lambda}}\, d\lambda$$

Total integrated energy absorbed = Total integrated energy emitted

If the opacity is “grey” $\kappa \neq \kappa_\lambda$:

$$0 = \kappa\int_0^\infty (J_\lambda - S_\lambda)d\lambda$$

$$\int_0^\infty J_\lambda\, d\lambda = \int_0^\infty S_\lambda\, d\lambda$$

$$\tilde{J}(\tau_z) = \tilde{S}(\tau_z)$$

But if $S_\lambda \simeq B_\lambda$, we can also write:

$$\tilde{J}(\tau_z) = \tilde{B}(T(\tau_z)) = \frac{\sigma T^4(\tau_z)}{\pi}$$

$$-\int_\lambda \frac{d}{\kappa\rho dz}\,K_\lambda = \underbrace{\int_\lambda \frac{F_\lambda}{4\pi}}_{\frac{\sigma T_{\mathrm{eff}}^4}{4\pi}}$$

If the opacity is “grey” $\kappa \neq \kappa_\lambda$:

$$-\frac{d}{\kappa\rho dz}\int_0^\infty K_\lambda d\lambda = \frac{\sigma T_{\mathrm{eff}}^4}{4\pi}$$

$$\frac{d\tilde{K}(\tau_z)}{d\tau_z} = \frac{\sigma T_{\mathrm{eff}}^4}{4\pi}$$

# The grey case (opacity is not a function of wavelength)

1 $$\tilde{J}(\tau_z) = \frac{\sigma \boxed{T^4(\tau_z)}}{\pi}$$

2 $$\frac{d\tilde{K}(\tau_z)}{d\tau_z} = \frac{\sigma T_{\rm eff}^4}{4\pi}$$

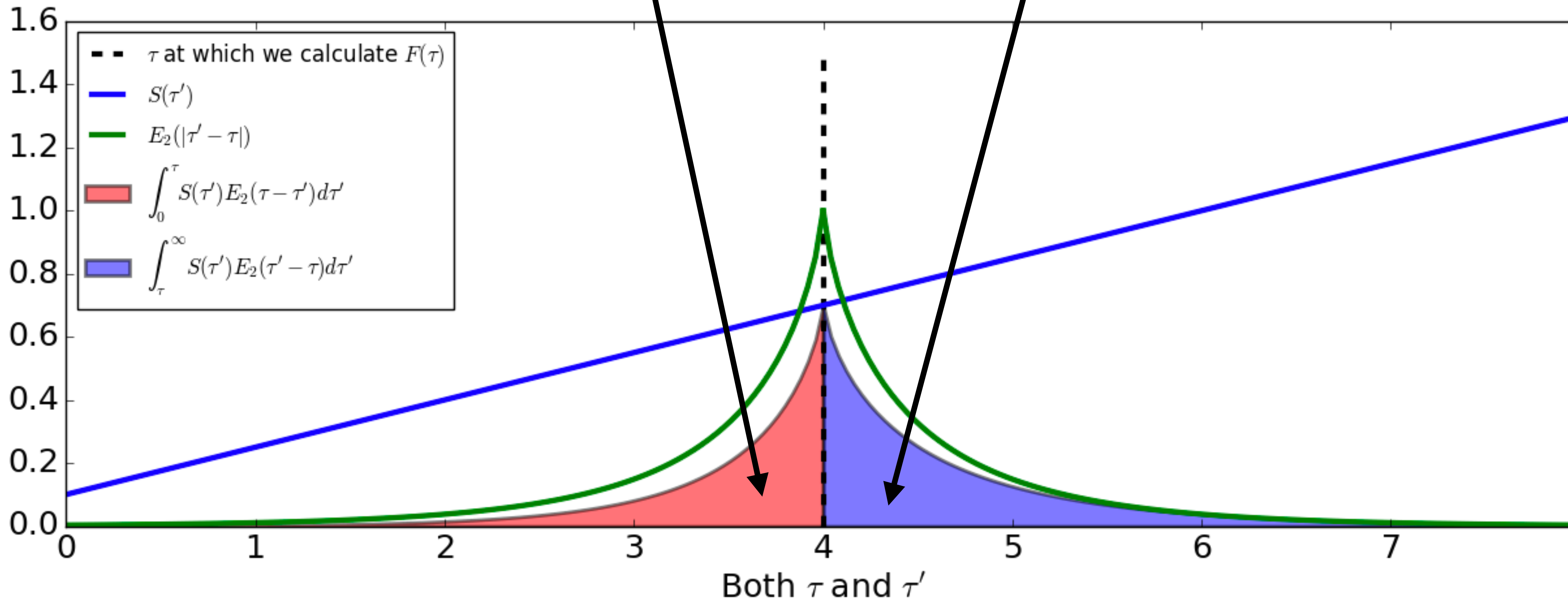
If we could find a (simple) relation between $\tilde{J}$ and $\tilde{K}$, we are in business

$\downarrow$

A further approximation: <u>large optical depth</u>

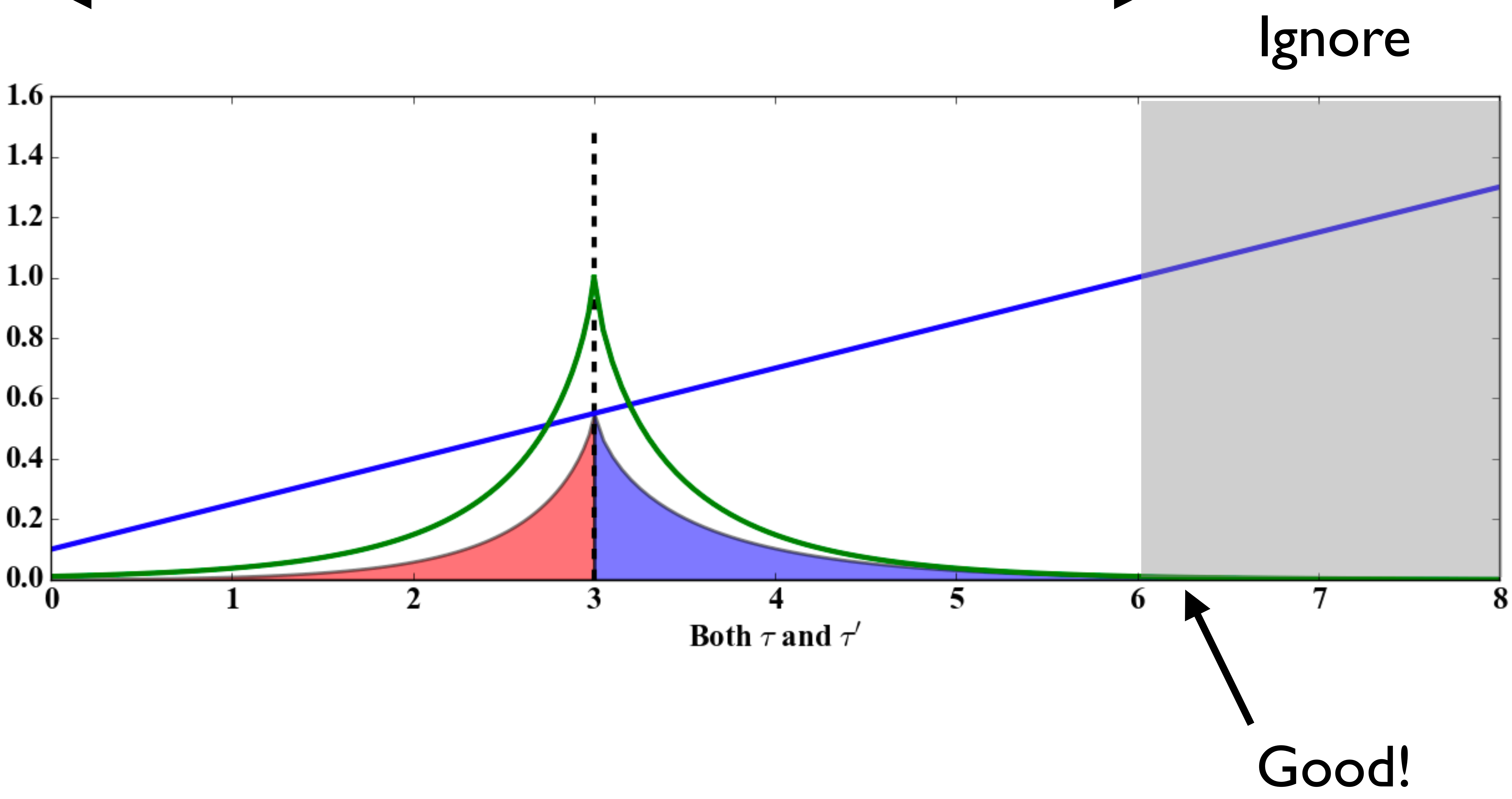
Q: why should this make you cringe a little?

$$\frac{F_\lambda(\tau)}{4\pi} = -\frac{1}{2}\int_{\tau'=0}^{\tau} S_\lambda(\tau')E_2(\tau-\tau')d\tau' + \frac{1}{2}\int_{\tau'=\tau}^{\infty} S_\lambda(\tau')E_2(\tau'-\tau)d\tau'$$



$$\frac{F_\lambda(3)}{4\pi} = -\frac{1}{2}\int_{\tau'=0}^{3} S_\lambda(\tau')E_2(3-\tau')d\tau' + \frac{1}{2}\int_{\tau'=3}^{\infty} S_\lambda(\tau')E_2(\tau'-3)d\tau'$$

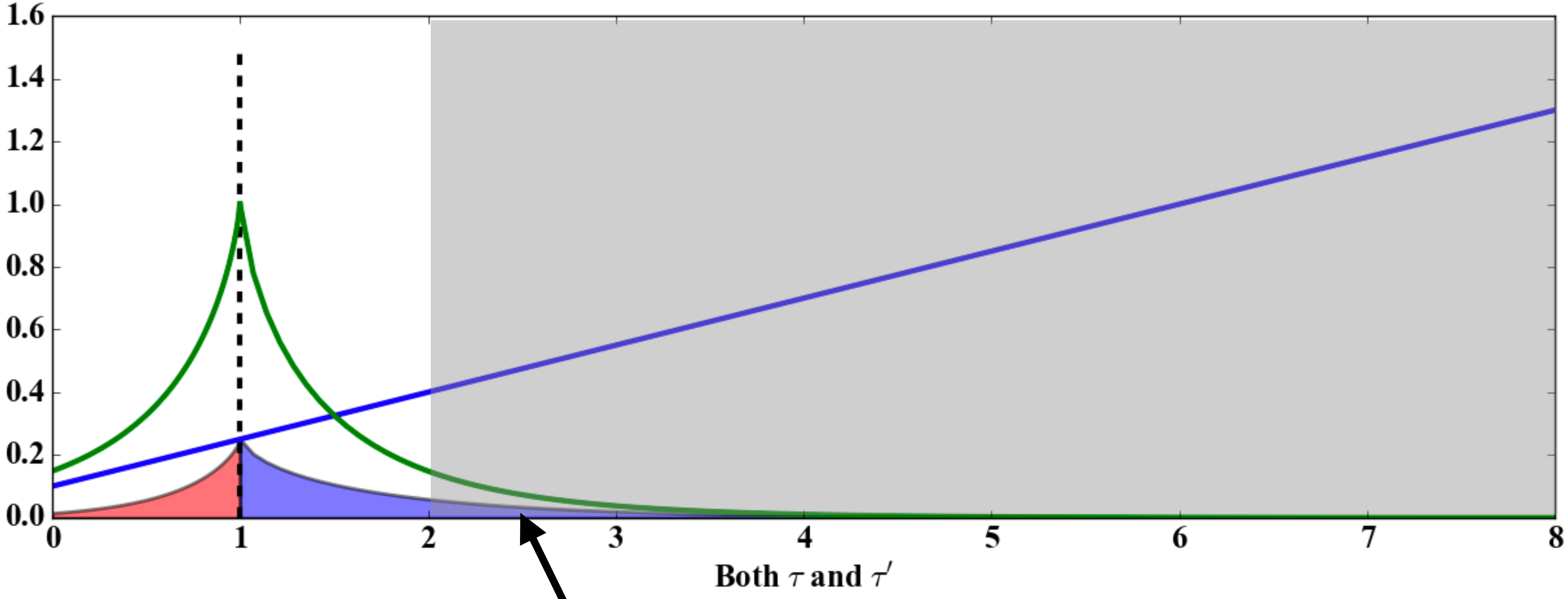
$\tau - \tau' = 3$ $\qquad$ $\tau' - \tau = 3$



$$\frac{F_\lambda(1)}{4\pi} = -\frac{1}{2}\int_{\tau'=0}^{1} S_\lambda(\tau')E_2(1-\tau')d\tau' + \frac{1}{2}\int_{\tau'=1}^{\infty} S_\lambda(\tau')E_2(\tau'-1)d\tau'$$

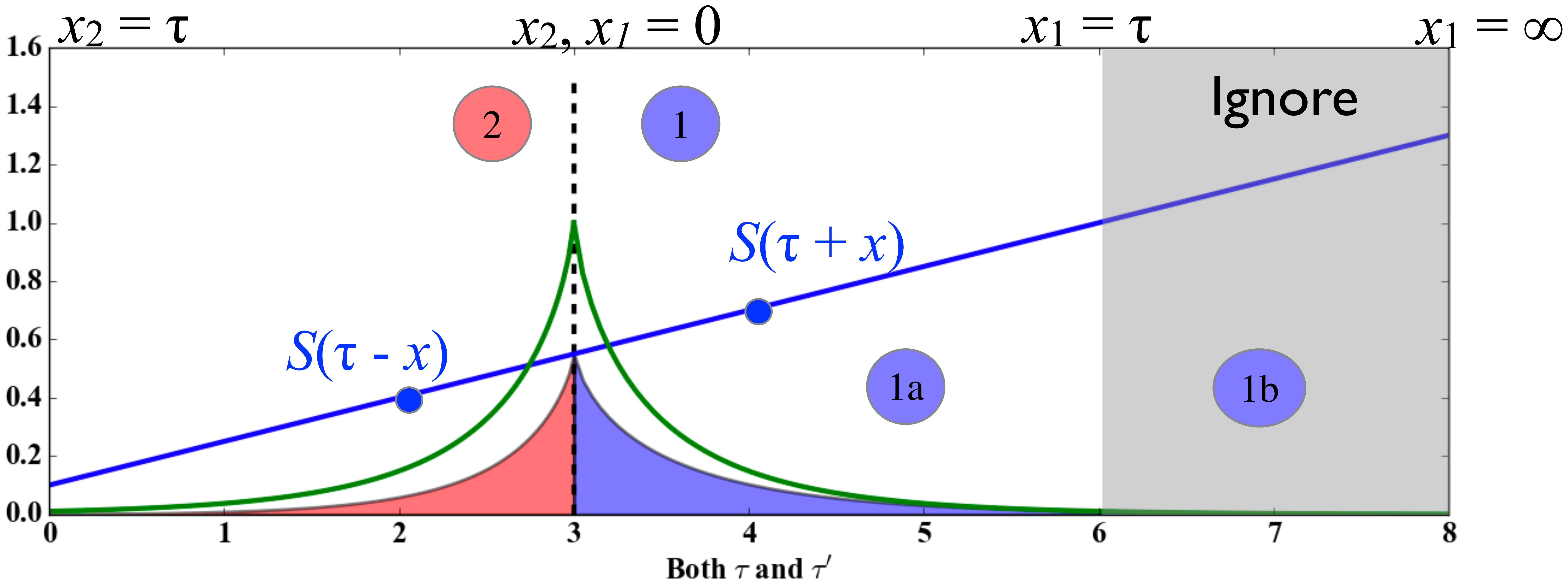
$\tau - \tau' = 1$ $\qquad$ $\tau' - \tau = 1$

Ignore



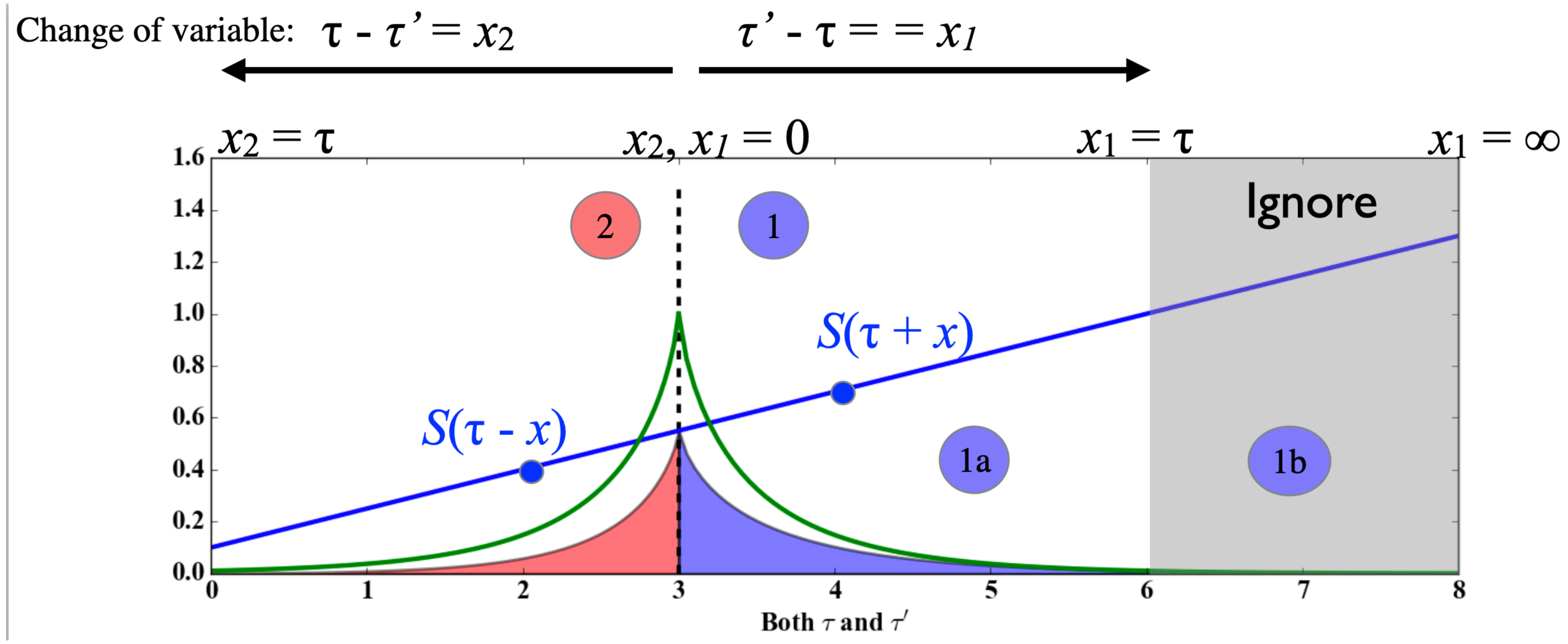

Not so good....

Change of variable: $\tau - \tau' = x_2$ $\tau' - \tau = = x_1$




$$\frac{F_\lambda(\tau)}{4\pi} = -\frac{1}{2}\int_{\tau'=0}^{\tau} S_\lambda(\tau')E_2(\tau-\tau')d\tau' + \frac{1}{2}\int_{\tau'=\tau}^{\infty} S_\lambda(\tau')E_2(\tau'-\tau)d\tau'$$

$$\frac{F_\lambda(\tau)}{4\pi} = -\frac{1}{2}\int_{x_2=0}^{\tau} S_\lambda(\tau-x_2)E_2(x_2)dx_2 + \frac{1}{2}\int_{x_1=0}^{\infty} S_\lambda(\tau+x_1)E_2(x_1)dx_1$$




$$\frac{F_\lambda(\tau)}{4\pi} = -\frac{1}{2}\int_{x_2=0}^{\tau} S_\lambda(\tau - x_2)E_2(x_2)dx_2 + \frac{1}{2}\int_{x_1=0}^{\infty} S_\lambda(\tau + x_1)E_2(x_1)dx_1$$

$$\frac{F_\lambda(\tau)}{4\pi} = -\frac{1}{2}\int_{x_2=0}^{\tau} S_\lambda(\tau - x_2)E_2(x_2)dx_2 + \frac{1}{2}\int_{x_1=0}^{\tau} S_\lambda(\tau + x_1)E_2(x_1)dx_1 + \frac{1}{2}\int_{x_1=\tau}^{\infty} S_\lambda(\tau + x_1)E_2(x_1)dx_1$$

$$\frac{F_\lambda(\tau)}{4\pi} = \frac{1}{2}\int_{x=0}^{\tau} \left[ -S_\lambda(\tau - x)E_2(x) + S_\lambda(\tau + x)E_2(x) \right] dx$$

$$\frac{F_\lambda(\tau)}{4\pi} = \frac{1}{2}\int_{x=0}^{\tau} \left[ S_\lambda(\tau + x) - S_\lambda(\tau - x) \right] E_2(x)dx$$

$$\frac{F_\lambda(\tau)}{4\pi} = \frac{1}{2}\int_{x=0}^{\tau} \left[S_\lambda(\tau+x) - S_\lambda(\tau-x)\right] \; E_2(x)dx$$

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(a)}{m!}(x-a)^m$$

$$f(a+\Delta a) = \sum_{m=0}^{\infty} \frac{f^{(m)}(a)}{m!}(\Delta a)^m$$

$$\left[S_\lambda(\tau+x) - S_\lambda(\tau-x)\right] = \sum_{m=1}^{\infty} \left(1+(-1)^{1+m}\right) \frac{x^m}{m!} \left.\frac{d^m S}{d\tau^m}\right|_\tau$$

$$\frac{F_\lambda(\tau)}{4\pi} = \sum_{m=1}^{\infty} \left(1+(-1)^{1+m}\right) \frac{1}{m!} \left.\frac{d^m S}{d\tau^m}\right|_\tau \frac{1}{2} \underbrace{\boxed{\int_{x=0}^{\tau} x^m \; E_2(x)dx}}_{\frac{m!}{2+m}}$$

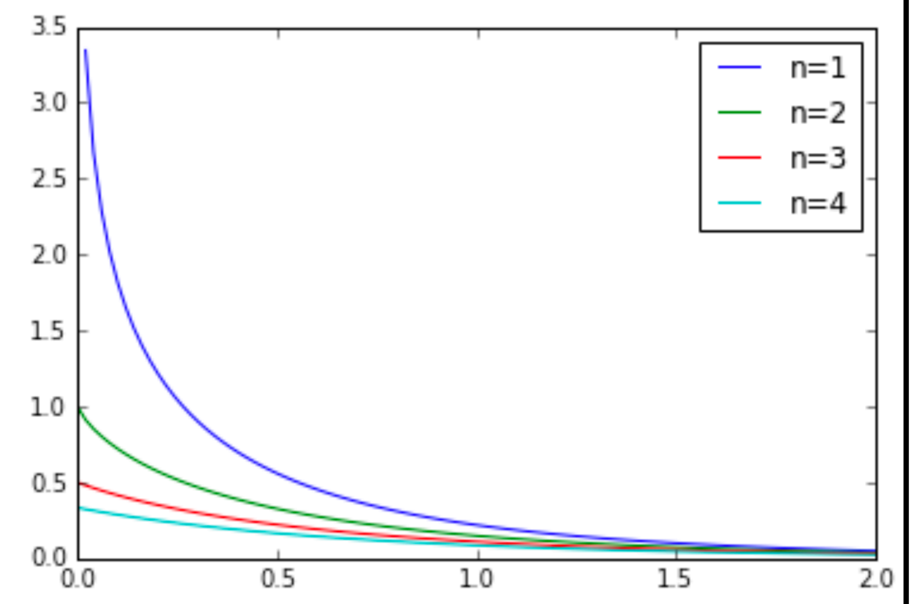
Exponential functions [python: scipy.special.expn(n, a)]

$$E_n(a) = \int_1^\infty \frac{\mathrm{e}^{-ax}}{x^n}dx$$

Chain of integrations by parts using:

$$E_n(a=0) = \frac{1}{n-1}$$

$$\frac{dE_n(a)}{da} = -E_{n-1}(a)$$




$$nE_{n+1}(a) = \mathrm{e}^{-a} - aE_n(a)$$

$$\frac{F_\lambda(\tau)}{4\pi} \simeq \frac{1}{3}\frac{dS_\lambda}{d\tau_\lambda} + \frac{1}{5}\frac{d^3 S_\lambda}{d\tau_\lambda^2} + \ldots$$

We can do the same procedure (approximation for large $\tau$) for all of the 'moment' equations, and also for the intensity solution.

$$\boxed{I(\tau, u) = S(\tau) + u \left.\frac{dS(\tau')}{d\tau'}\right|_{\tau}} + 2!\; u^2 \left.\frac{d^2S(\tau')}{d\tau'^2}\right|_{\tau} + \ldots$$

$$\boxed{J(\tau) = S(\tau)} + \frac{1}{3}\left.\frac{d^2S(\tau')}{d\tau'^2}\right|_{\tau} + \ldots$$

$$\boxed{\frac{F}{4\pi} = \frac{1}{3}\left.\frac{dS(\tau')}{d\tau'}\right|_{\tau}} + \frac{1}{5}\left.\frac{d^3S(\tau')}{d\tau'^3}\right|_{\tau} + \ldots$$

$$\boxed{K(\tau) = \frac{1}{3}S(\tau)} + \frac{1}{5}\left.\frac{d^2S(\tau')}{d\tau'^2}\right|_{\tau} + \ldots$$

# The grey case (opacity is not a function of wavelength)

1 $$\tilde{J}(\tau_z) = \frac{\sigma \boxed{T^4(\tau_z)}}{\pi}$$

2 $$\frac{d\tilde{K}(\tau_z)}{d\tau_z} = \frac{\sigma T_{\text{eff}}^4}{4\pi}$$

If we could find a (simple) relation between $\tilde{J}$ and $\tilde{K}$, we are in business

$\downarrow$

A further approximation: <u>large optical depth</u>

$$J_\lambda(\tau_z) = 3K_\lambda(\tau_z)$$

$$\tilde{J}(\tau_z) = 3\tilde{K}(\tau_z)$$

# The grey case (opacity is not a function of wavelength) + large $\tau$

1 $$\tilde{J}(\tau_z) = \frac{\sigma \boxed{T^4(\tau_z)}}{\pi}$$

2 $$\frac{d\tilde{K}(\tau_z)}{d\tau_z} = \frac{\sigma T_{\text{eff}}^4}{4\pi}$$

$\rightarrow$ Step 1: Integrate this $\tilde{K}(\tau_z) = \frac{\sigma T_{\text{eff}}^4}{4\pi}(\tau_z + \text{const})$

3 $$\tilde{J}(\tau_z) = 3\tilde{K}(\tau_z)$$

$\rightarrow$ Step 2: replace $\tilde{K}$ in here $\tilde{J}(\tau_z) = \frac{3\sigma T_{\text{eff}}^4}{4\pi}(\tau_z + \text{const})$

Step 3: replace $\tilde{J}$ in here $\frac{\sigma \boxed{T^4(\tau_z)}}{\pi} = \frac{3\sigma T_{\text{eff}}^4}{4\pi}(\tau_z + \text{const})$

$$T^4(\tau_z) = \frac{3T_{\text{eff}}^4}{4}(\tau_z + q(\tau_z))$$

# Schwarzschild-Milnes equations

The 1st, 2nd, and 3rd “moment”

$$J_\lambda = \frac{1}{2}\int_{-1}^{1} I_\lambda \; du$$

$$\frac{F_\lambda}{4\pi} = \frac{1}{2}\int_{-1}^{1} I_\lambda \; u du$$

$$K_\lambda = \frac{1}{2}\int_{-1}^{1} I_\lambda \; u^2 du$$

What these moments look like when we plug in the intensity solutions for flat atmosphere

$$J_\lambda(\tau) = +\frac{1}{2}\int_{\tau'=0}^{\tau} S_\lambda(\tau')E_1(\tau-\tau')d\tau' + \frac{1}{2}\int_{\tau'=\tau}^{\infty} S_\lambda(\tau')E_1(\tau'-\tau)d\tau'$$

$$\frac{F_\lambda(\tau)}{4\pi} = -\frac{1}{2}\int_{\tau'=0}^{\tau} S_\lambda(\tau')E_2(\tau-\tau')d\tau' + \frac{1}{2}\int_{\tau'=\tau}^{\infty} S_\lambda(\tau')E_2(\tau'-\tau)d\tau'$$

$$K_\lambda(\tau) = +\frac{1}{2}\int_{\tau'=0}^{\tau} S_\lambda(\tau')E_3(\tau-\tau')d\tau' + \frac{1}{2}\int_{\tau'=\tau}^{\infty} S_\lambda(\tau')E_3(\tau'-\tau)d\tau'$$

# Schwarzschild-Milnes equations

The 1st, 2nd, and 3rd "moment"

What these moments look like when we plug in the intensity solutions for flat atmosphere

$$J_\lambda = \frac{1}{2}\int_{-1}^{1} I_\lambda \;\; du$$

$$J_\lambda(\tau) = +\frac{1}{2}\int_{\tau'=0}^{\tau} S_\lambda(\tau')E_1(\tau-\tau')d\tau' + \frac{1}{2}\int_{\tau'=\tau}^{\infty} S_\lambda(\tau')E_1(\tau'-\tau)d\tau'$$

$$\frac{F_\lambda}{4\pi} = \frac{1}{2}\int_{-1}^{1} I_\lambda \;\; udu$$

$$\frac{F_\lambda(\tau)}{4\pi} = -\frac{1}{2}\int_{\tau'=0}^{\tau} S_\lambda(\tau')E_2(\tau-\tau')d\tau' + \frac{1}{2}\int_{\tau'=\tau}^{\infty} S_\lambda(\tau')E_2(\tau'-\tau)d\tau'$$

$$K_\lambda = \frac{1}{2}\int_{-1}^{1} I_\lambda \;\; u^2du$$

$$K_\lambda(\tau) = +\frac{1}{2}\int_{\tau'=0}^{\tau} S_\lambda(\tau')E_3(\tau-\tau')d\tau' + \frac{1}{2}\int_{\tau'=\tau}^{\infty} S_\lambda(\tau')E_3(\tau'-\tau)d\tau'$$

# The grey case (opacity is not a function of wavelength) + large $\tau$

1 $$\tilde{J}(\tau_z) = \frac{\sigma T^4(\tau_z)}{\pi}$$

$\rightarrow$Remember, this came for out result that $\tilde{J} = \tilde{S}$

If the opacity is "grey" $\kappa \neq \kappa_\lambda$:

$$0 = \kappa \int_0^\infty (J_\lambda - S_\lambda) d\lambda$$

$$\int_0^\infty J_\lambda \, d\lambda = \int_0^\infty S_\lambda \, d\lambda$$

$$\tilde{J}(\tau_z) = \tilde{S}(\tau_z)$$

But if $S_\lambda \simeq B_\lambda$, we can also write:

$$\tilde{J}(\tau_z) = \tilde{B}(T(\tau_z)) = \frac{\sigma T^4(\tau_z)}{\pi}$$

2 $$\frac{d\tilde{K}(\tau_z)}{d\tau_z} = \frac{\sigma T_{\text{eff}}^4}{4\pi}$$

3 $$\tilde{J}(\tau_z) = 3\tilde{K}(\tau_z)$$

$\rightarrow$ Step 2: replace $\tilde{K}$ in here $\tilde{J}(\tau_z) = \frac{3\sigma T_{\text{eff}}^4}{4\pi}(\tau_z + \text{const})$

$$J_\lambda(\tau) = +\frac{1}{2}\int_{\tau'=0}^{\tau} S_\lambda(\tau')E_1(\tau - \tau')d\tau' + \frac{1}{2}\int_{\tau'=\tau}^{\infty} S_\lambda(\tau')E_1(\tau' - \tau)d\tau'$$

$$\tilde{J}(\tau) = +\frac{1}{2}\int_{\tau'=0}^{\tau} \tilde{S}(\tau')E_1(\tau - \tau')d\tau' + \frac{1}{2}\int_{\tau'=\tau}^{\infty} \tilde{S}(\tau')E_1(\tau' - \tau)d\tau'$$

$$\frac{3\sigma T_{\text{eff}}^4}{4\pi}(\tau_z + \boxed{q(\tau_z)}) \qquad \frac{3\sigma T_{\text{eff}}^4}{4\pi}(\tau_z + \boxed{q(\tau_z)}) \qquad \frac{3\sigma T_{\text{eff}}^4}{4\pi}(\tau_z + \boxed{q(\tau_z)})$$