Week 13 Tuesday
L-23
Homology

$$\frac{dM_r(r)}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{dP(r)}{dr} = -\rho(r)\frac{GM_r(r)}{r^2}$$

$$\frac{dT(r)}{dr} = -\frac{3\kappa_R(r)\rho(r)}{64\pi\sigma}\frac{L_r(r)}{r^2 T(r)^3}$$

$$\frac{dL_r(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r)$$

$$\boxed{\begin{matrix} M_r(r) & P(r) & L_r(r) & T(r) \\ \rho(r) & \mu(r) & \epsilon_{\rm nuc}(r) & \kappa_R(r) \end{matrix}}$$

$$P(r) = \frac{\rho(r)}{\mu(r)}kT(r)$$

$$\mu(r) = f(\text{comp}, T(r), P(r))$$

$$\kappa_R(r) = f(\text{comp}, T(r), P(r))$$

$$\epsilon_{\rm nuc}(r) = f(\text{comp}, T(r), P(r))$$

Other energy transport?

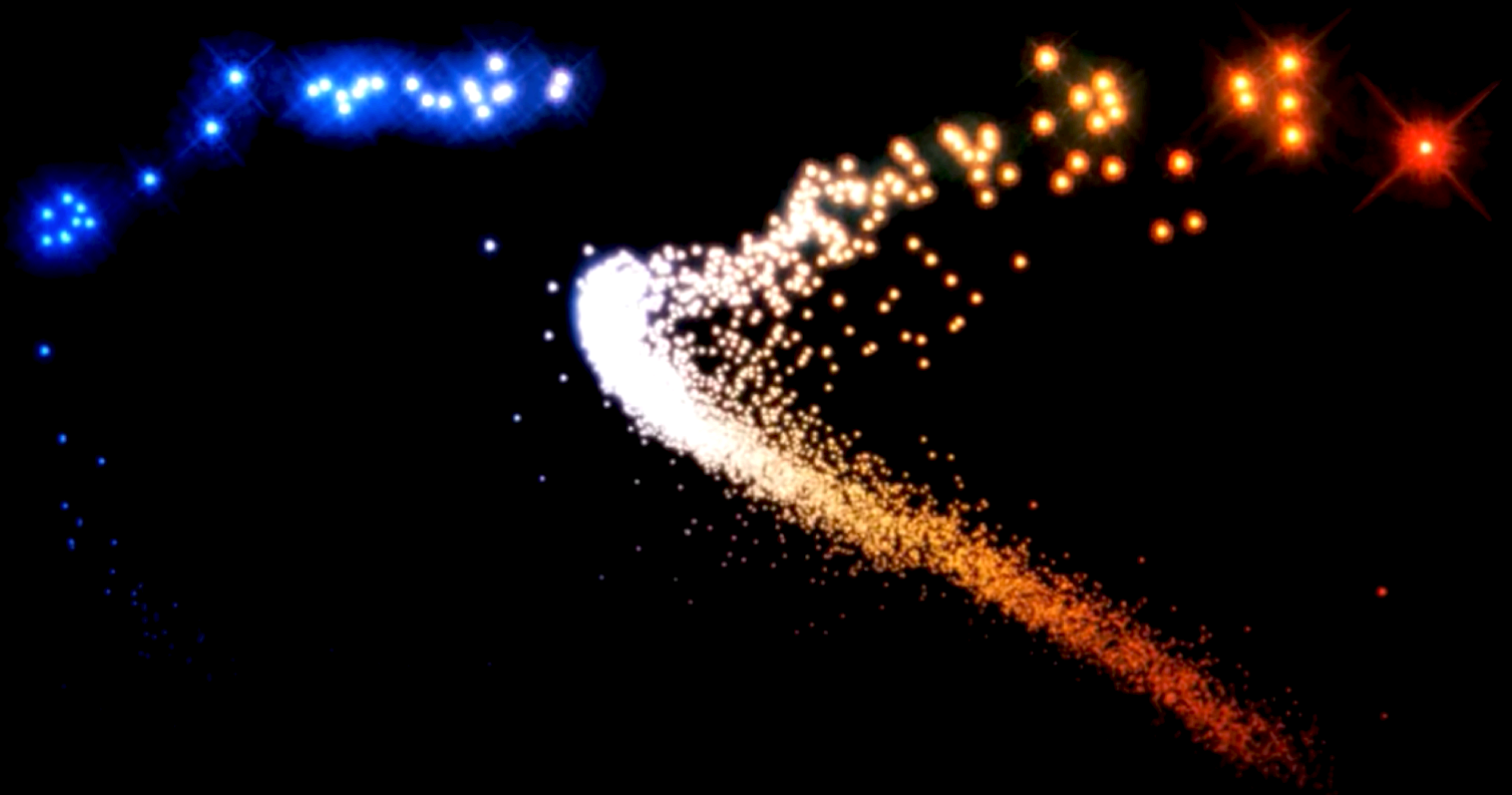
Ideal gas always valid?

Nuclear mechanism?

How to solve?

What changes with time?


Omega Centauri Globular Cluster
APOD 2016 April 27 . Roberto Colombari

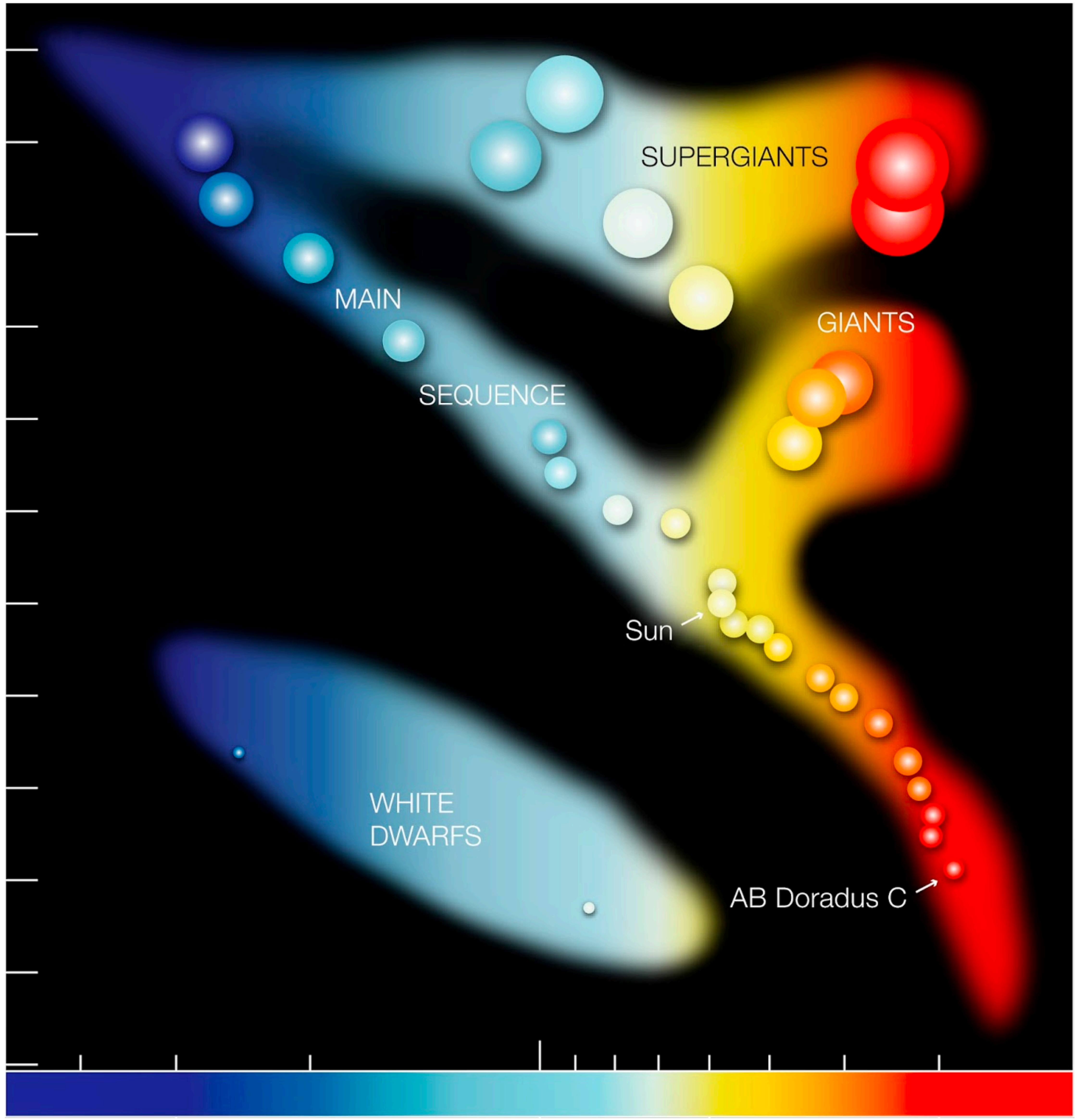


NASA, ESA, J. Anderson and R. van der Marel (STScI)

The Hertzsprung-Russell (HR) Diagram is a family portrait of stars

Power

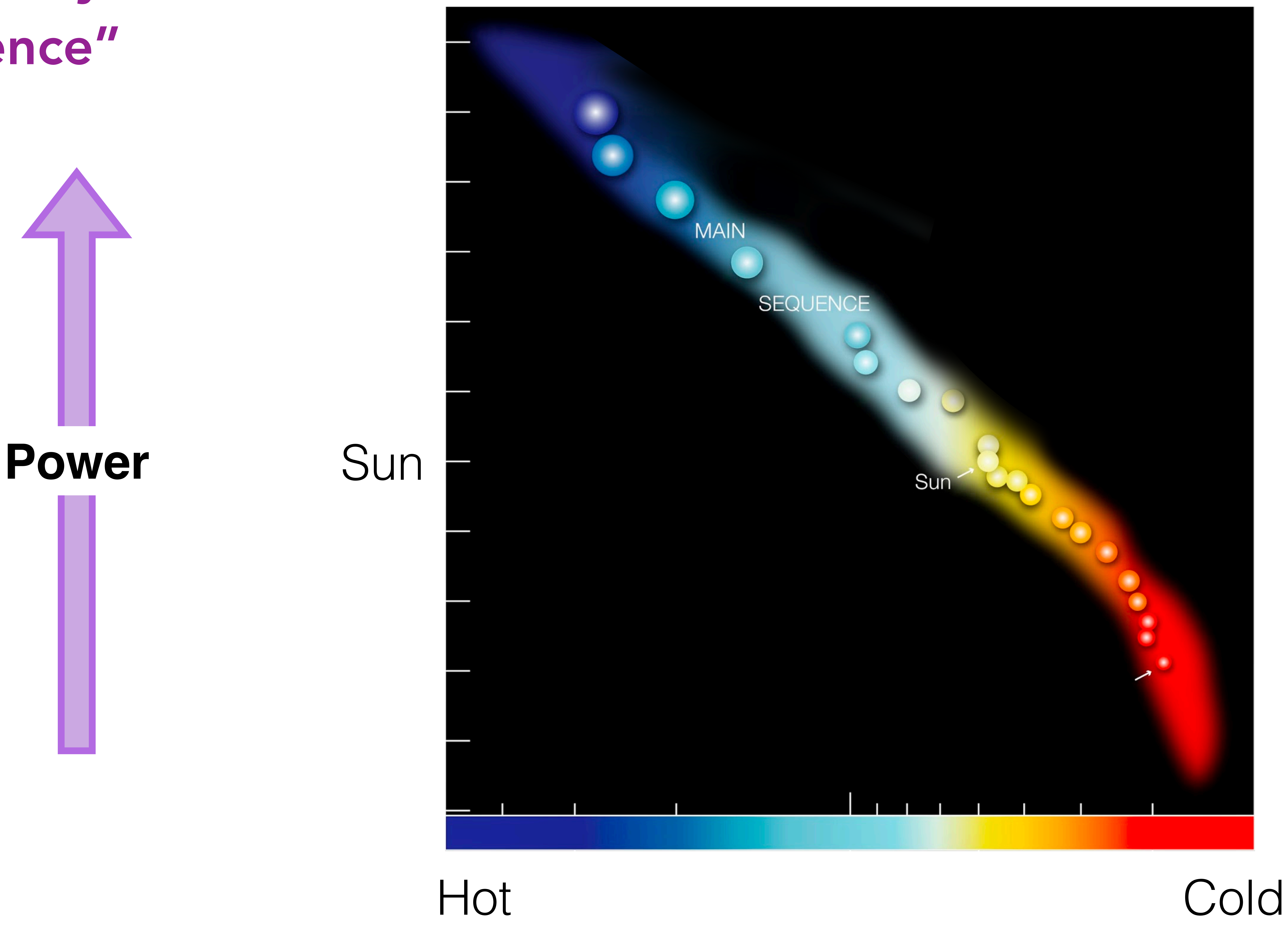
Sun



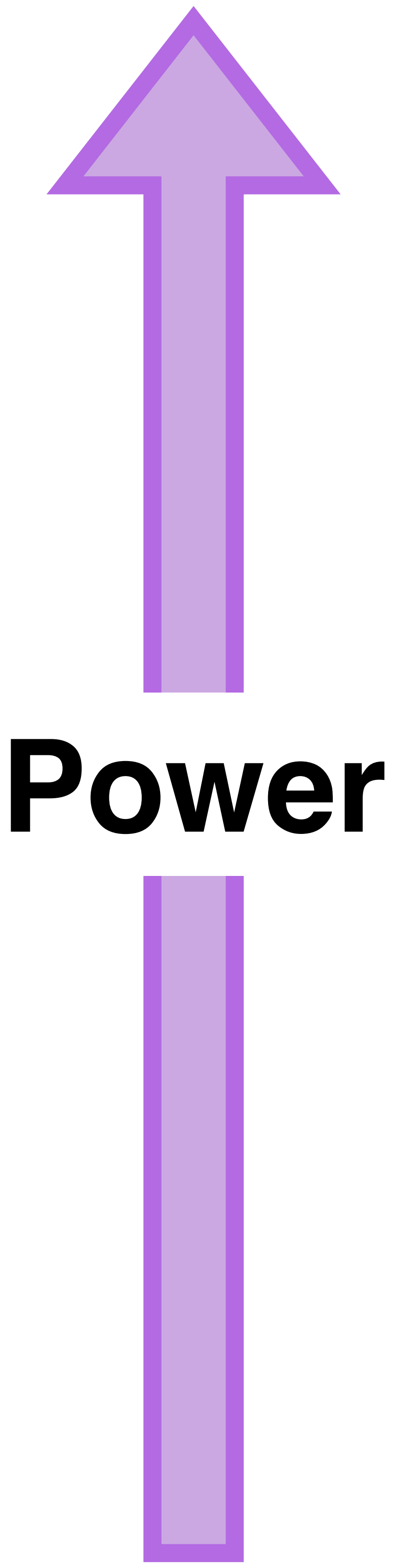
Hot

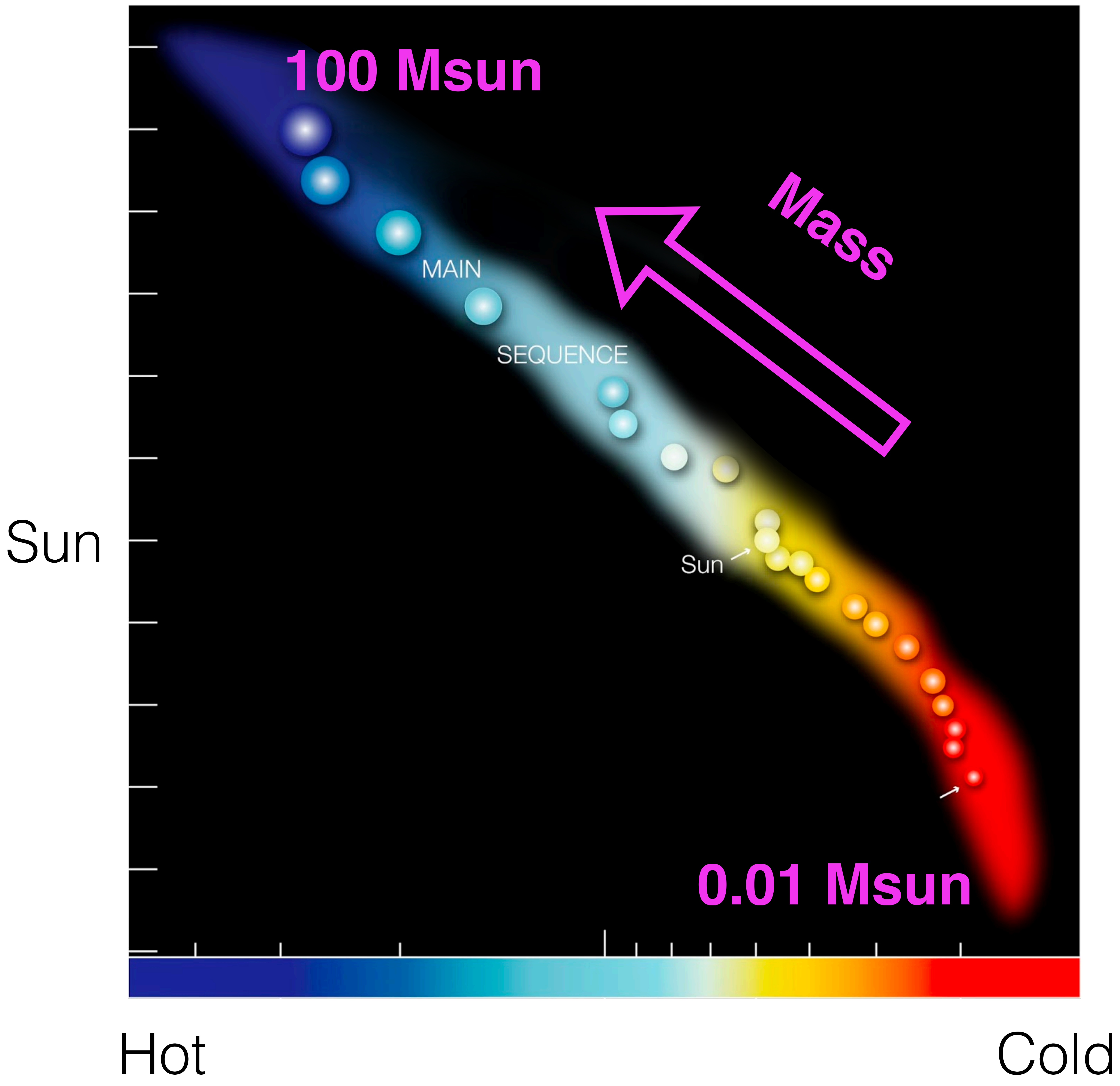
Cold

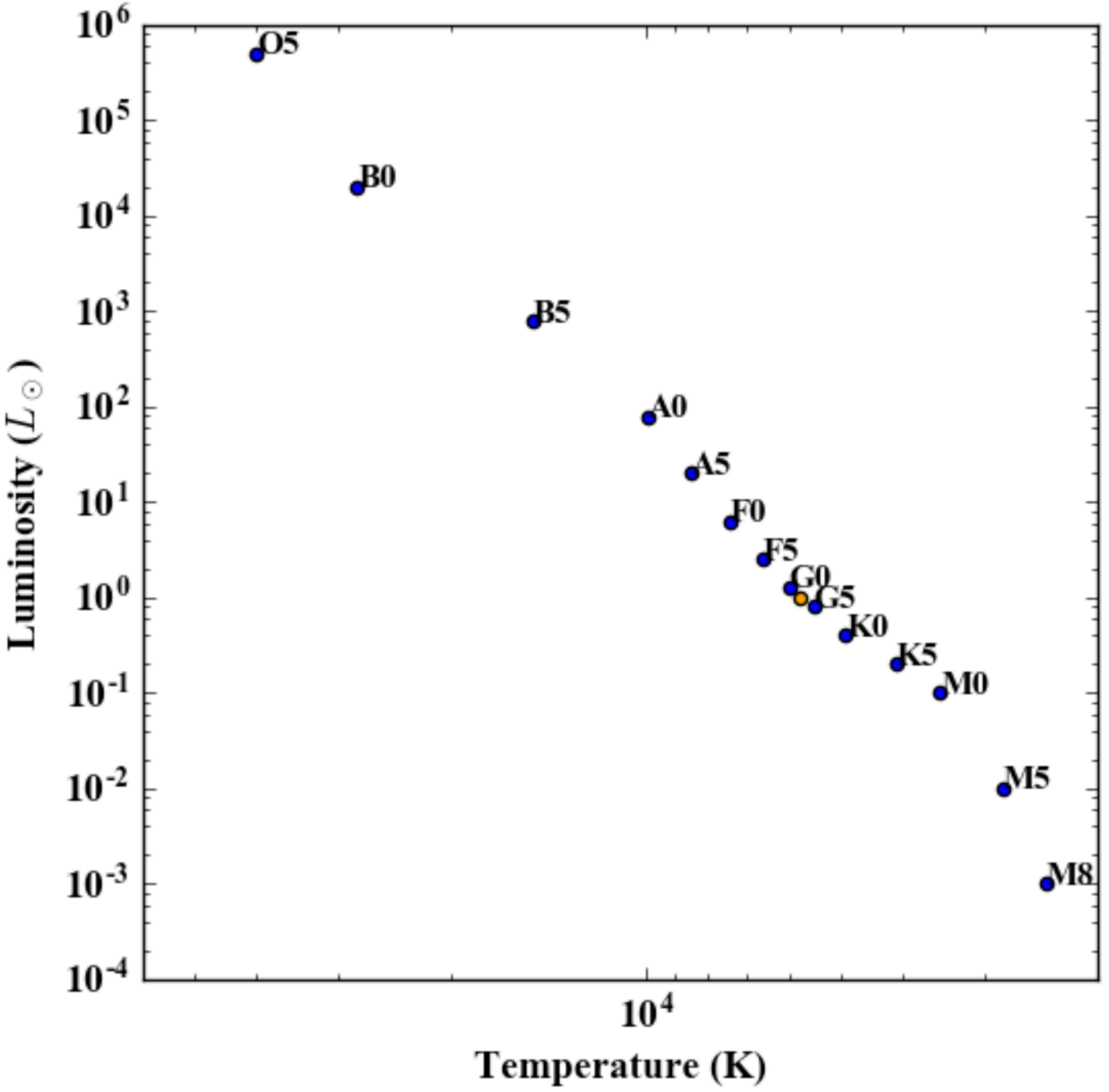
Most stars in our galaxy are on the "Main Sequence"



On this "Main Sequence", stars are also ordered by their mass



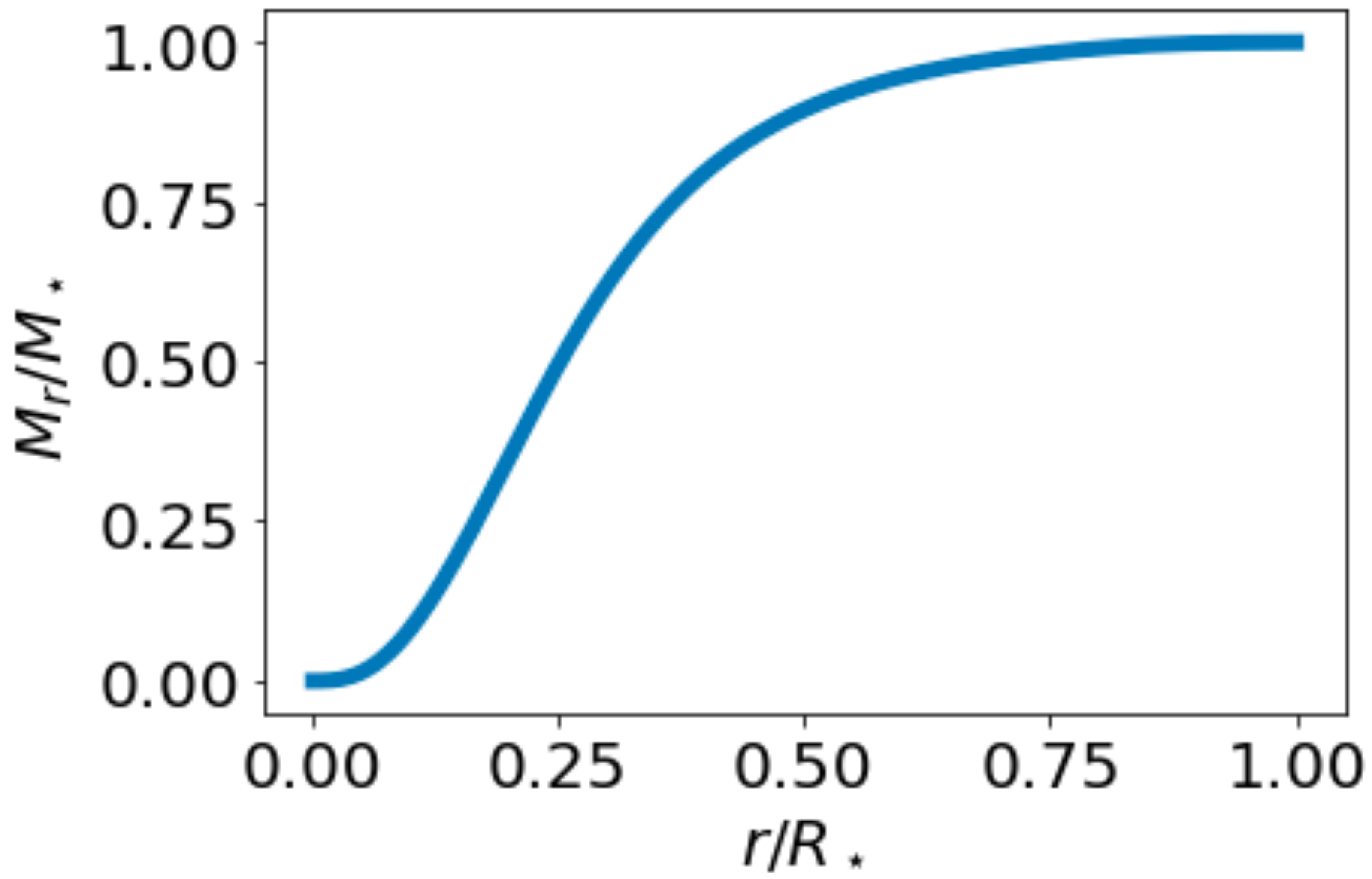


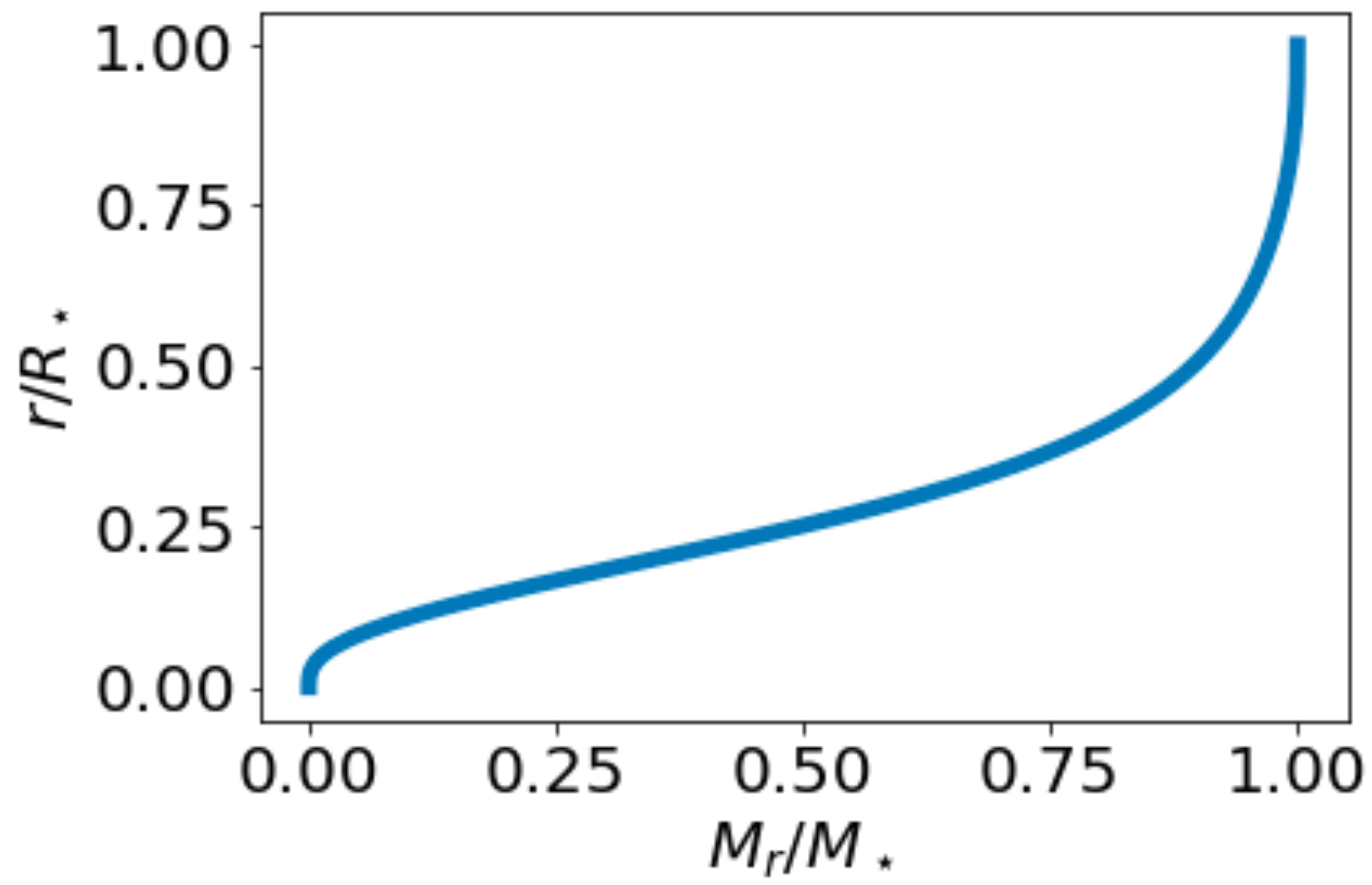


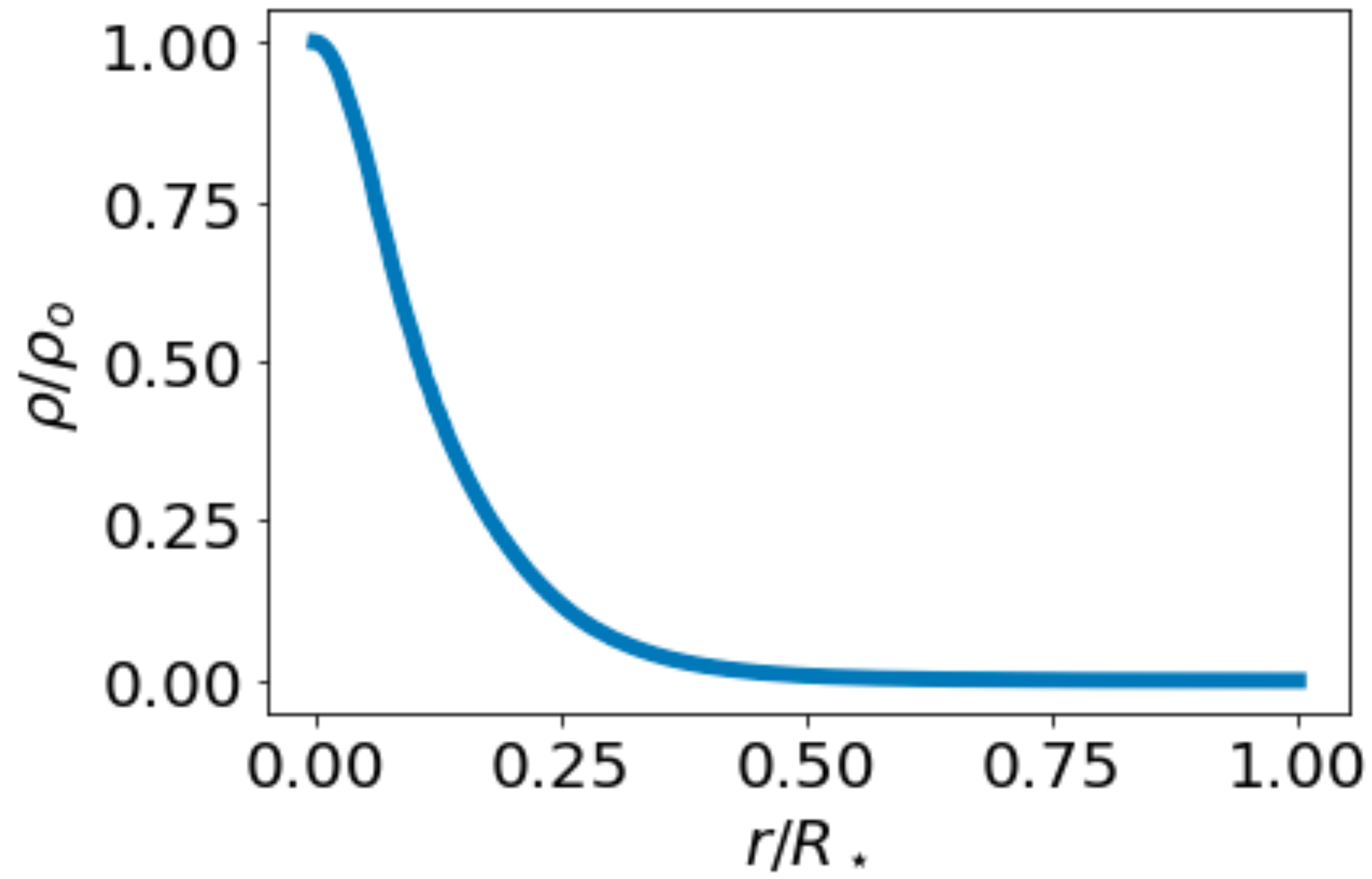
Luminosity ($L_\odot$)
10^6
10^5
10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
O5
B0
B5
A0
A5
F0
F5
G0
G5
K0
K5
M0
M5
M8
10^4
Temperature (K)

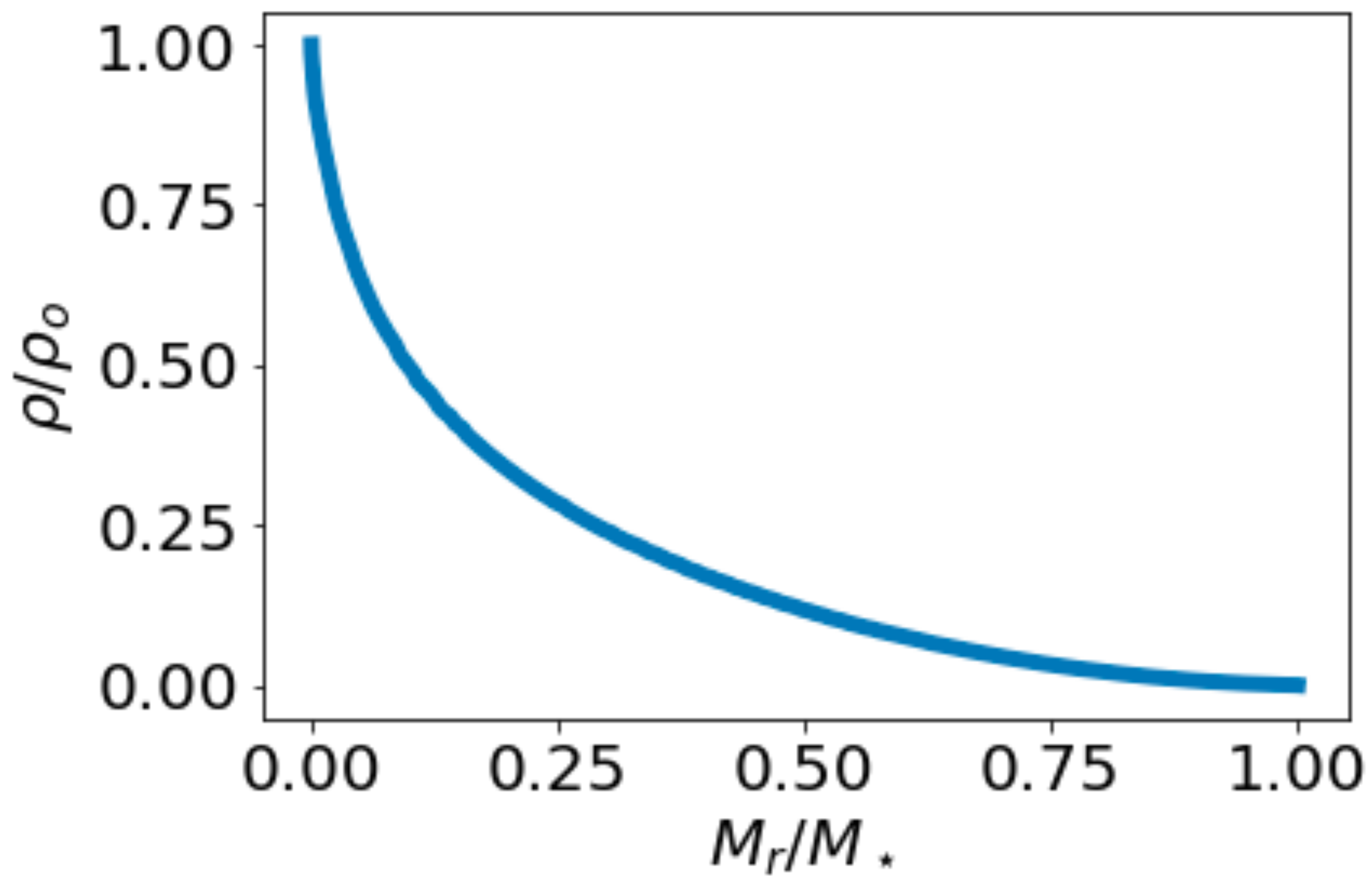
Radius coordinates

Enclosed mass coordinates









Radius coordinates

$$\frac{dM_r(r)}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{dP(r)}{dr} = -\rho(r)\frac{GM_r(r)}{r^2}$$

$$\frac{dL_r(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r)$$

$$\frac{dT(r)}{dr} = -\frac{3\kappa_R(r)\rho(r)}{64\pi\sigma}\frac{L_r(r)}{r^2 T(r)^3}$$

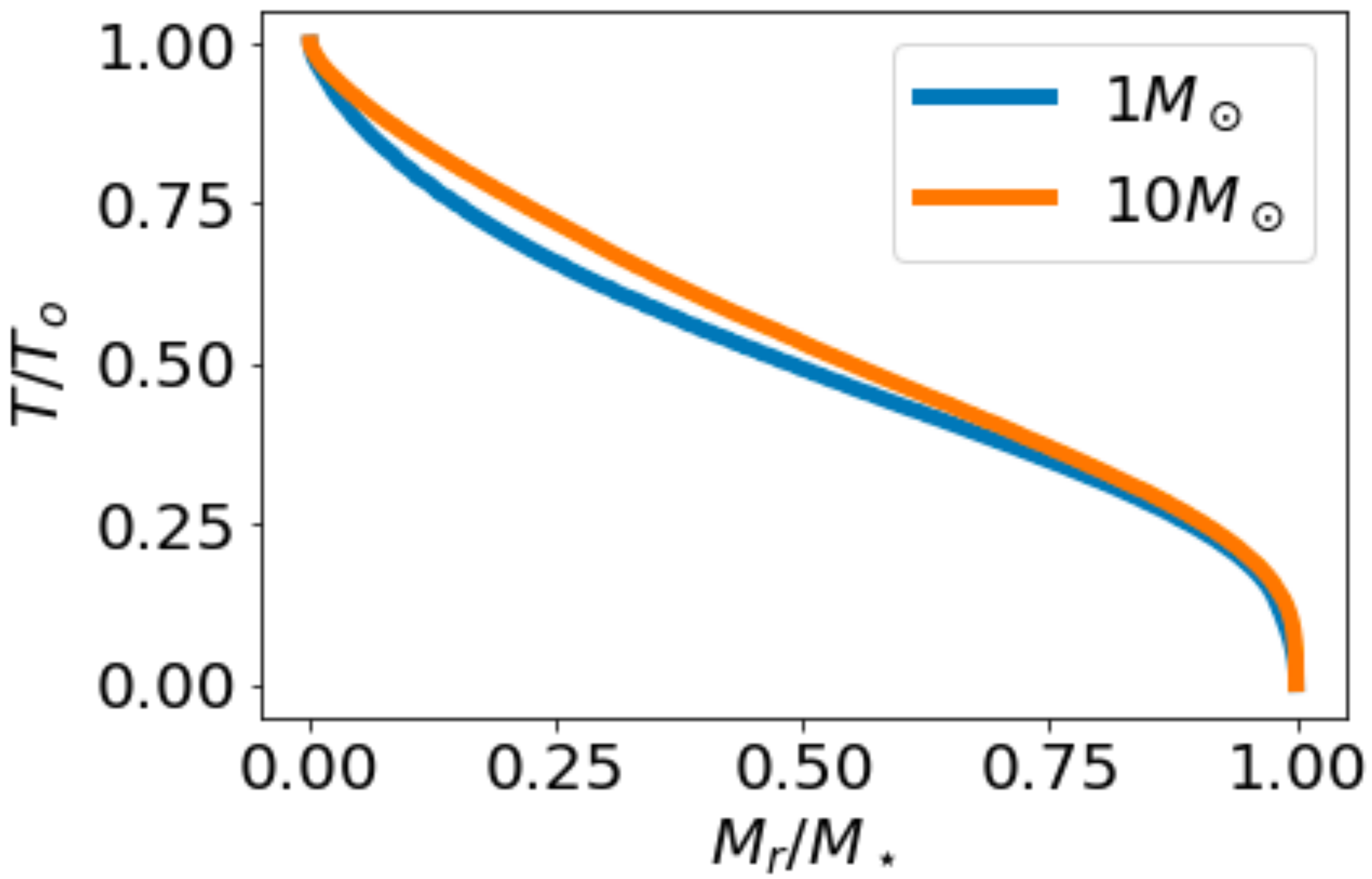
Enclosed mass coordinates

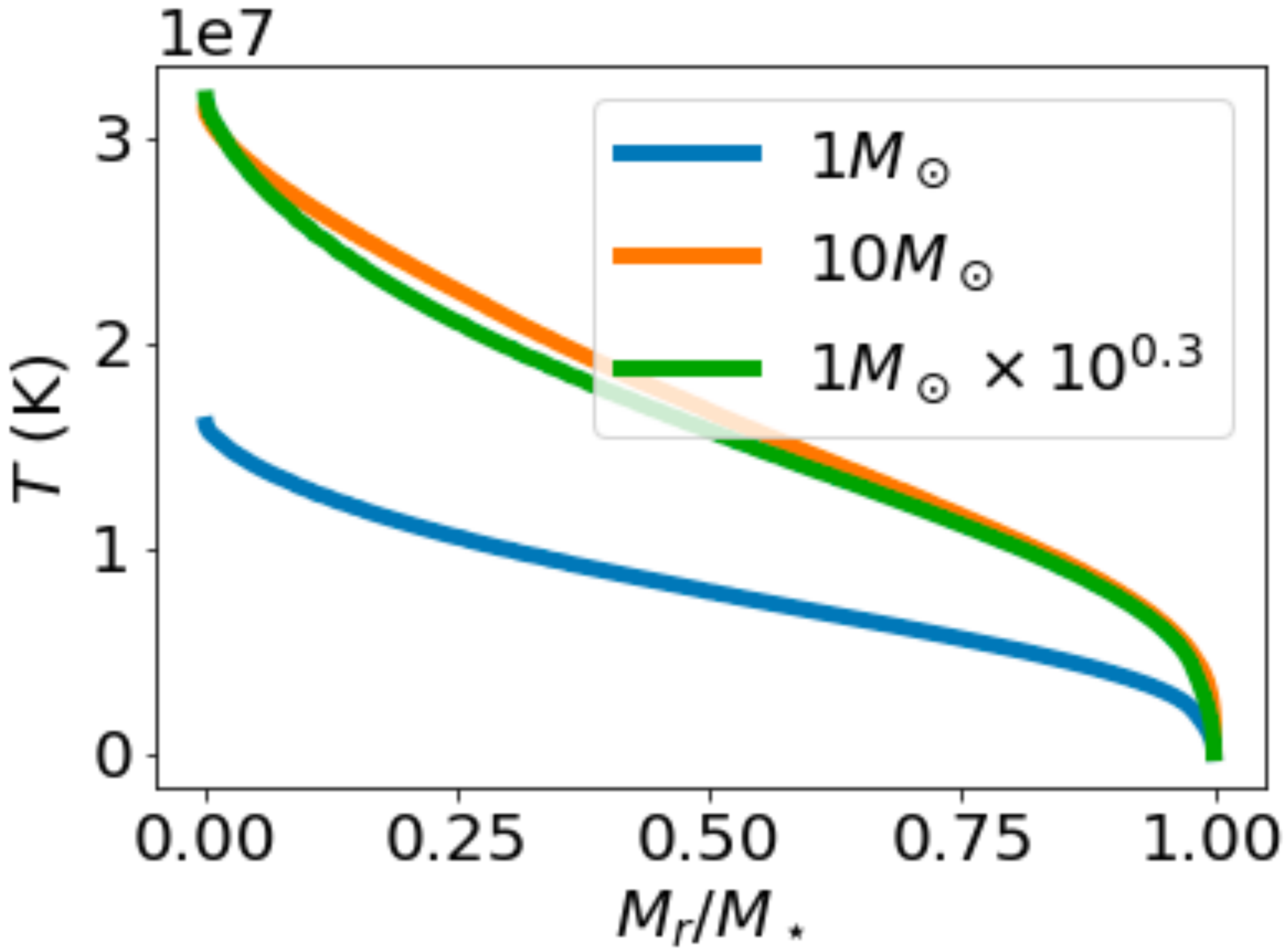
$$\frac{dr(M_r)}{dM_r} = \frac{1}{4\pi\ r^2(M_r)\ \rho(M_r)}$$

$$\frac{dP(M_r)}{dM_r} = -\frac{GM_r}{4\pi\ r^4(M_r)}$$

$$\frac{dL_r(M_r)}{dM_r} = \epsilon(M_r)$$

$$\frac{dT(M_r)}{dM_r} = -\frac{3\kappa(M_r)\ L_r(M_r)}{64\pi^2 ac\ r^4(M_r)\ T^3(M_r)}$$





Homology relations

$$q = \frac{M_r}{M_\star}$$

$$r = M_\star^{a_r} \; \tilde{r}(q)$$

$$P = M_\star^{a_P} \; \tilde{P}(q)$$

$$L_r = M_\star^{a_L} \; \tilde{L}_r(q)$$

$$T = M_\star^{a_T} \; \tilde{T}(q)$$

Enclosed mass coordinates

$$\frac{dr(M_r)}{dM_r} = \frac{1}{4\pi\ r^2(M_r)\ \rho(M_r)}$$

$$\frac{dP(M_r)}{dM_r} = -\frac{GM_r}{4\pi\ r^4(M_r)}$$

$$\frac{dL_r(M_r)}{dM_r} = \epsilon(M_r)$$

$$\frac{dT(M_r)}{dM_r} = -\frac{3\kappa(M_r)\ L_r(M_r)}{64\pi^2 ac\ r^4(M_r)\ T^3(M_r)}$$

Homology relations

$$q = \frac{M_r}{M_\star} \qquad dM_r = M_\star dq$$

$$r = M_\star^{a_r}\ \tilde{r}(q)$$

$$P = M_\star^{a_P}\ \tilde{P}(q)$$

$$L_r = M_\star^{a_L}\ \tilde{L}_r(q)$$

$$T = M_\star^{a_T}\ \tilde{T}(q)$$

Enclosed mass coordinates

$$\frac{dP(M_r)}{dM_r} = -\frac{GM_r}{4\pi\; r^4(M_r)}$$

$$\frac{M_\star^{a_P}}{M_\star}\frac{d\tilde{P}}{dq} = -\frac{G\;(\;qM_\star)}{4\pi\left(M_\star^{4a_r}\;\tilde{r}^4\right)}$$

$$M_\star^{(\;4a_r + a_p - 2\;)} = -\frac{G\;q}{4\pi\;\tilde{r}^4\;(d\tilde{P}/dq)}$$

$$(\;4a_r\; + \;a_p\; - \;2\;) = 0$$

Homology relations

$$q = \frac{M_r}{M_\star} \qquad dM_r = M_\star dq$$

$$r = M_\star^{a_r}\;\;\tilde{r}(q)$$

$$P = M_\star^{a_P}\;\;\tilde{P}(q)$$

$$L_r = M_\star^{a_L}\;\;\tilde{L}_r(q)$$

$$T = M_\star^{a_T}\;\;\tilde{T}(q)$$

Enclosed mass coordinates

$$\frac{dr(M_r)}{dM_r} = \frac{1}{4\pi\, r^2(M_r)\, \rho(M_r)}$$

$$\frac{M_\star^{a_r}}{M_\star}\frac{d\tilde{r}}{dq} = \frac{1}{4\pi\, M_\star^{2a_r}\, \tilde{r}^2(q)\, \boxed{\rho}}$$

$$\boxed{\rho} = \frac{1}{4\pi\tilde{r}^2(d\tilde{r}/dq)}\, M_\star^{1-3a_r}$$

$$\propto \; M_\star^{1-3a_r}$$

Homology relations

$$q = \frac{M_r}{M_\star} \qquad dM_r = M_\star dq$$

$$r = M_\star^{a_r}\;\; \tilde{r}(q)$$

$$P = M_\star^{a_P}\;\; \tilde{P}(q)$$

$$L_r = M_\star^{a_L}\;\; \tilde{L}_r(q)$$

$$T = M_\star^{a_T}\;\; \tilde{T}(q)$$

Enclosed mass coordinates

$$\frac{dL_r(M_r)}{dM_r} = \epsilon(M_r)$$

$$\frac{M_\star^{a_L}}{M_\star}\frac{d\tilde{L}}{dq} = \boxed{\epsilon}$$

$$\boxed{\epsilon} = M_\star^{(a_L - 1)} \frac{d\tilde{L}_r}{dq}$$

$$\boxed{\epsilon \propto M_\star^{(a_L - 1)}}$$

Homology relations

$$q = \frac{M_r}{M_\star} \qquad dM_r = M_\star dq$$

$$r = M_\star^{a_r} \tilde{r}(q)$$

$$P = M_\star^{a_P} \tilde{P}(q)$$

$$L_r = M_\star^{a_L} \tilde{L}_r(q)$$

$$T = M_\star^{a_T} \tilde{T}(q)$$

Enclosed mass coordinates

$$\boxed{\kappa \propto M_\star^{(\,4a_T + 4a_r - a_L - 1\,)}}$$

$$\frac{dT(M_r)}{dM_r} = -\frac{3\boxed{\kappa}(M_r)\,L_r(M_r)}{64\pi^2 ac\, r^4(M_r)\, T^3(M_r)}$$

Homology relations

$$q = \frac{M_r}{M_\star} \qquad dM_r = M_\star dq$$

$$r = M_\star^{a_r}\ \tilde{r}(q)$$

$$P = M_\star^{a_P}\ \tilde{P}(q)$$

$$L_r = M_\star^{a_L}\ \tilde{L}_r(q)$$

$$T = M_\star^{a_T}\ \tilde{T}(q)$$

Enclosed mass coordinates

$$\frac{dr(M_r)}{dM_r} = \frac{1}{4\pi\ r^2(M_r)\ \rho(M_r)}$$

$$\frac{dP(M_r)}{dM_r} = -\frac{GM_r}{4\pi\ r^4(M_r)}$$

$$\frac{dL_r(M_r)}{dM_r} = \epsilon(M_r)$$

$$\frac{dT(M_r)}{dM_r} = -\frac{3\kappa(M_r)\ L_r(M_r)}{64\pi^2 ac\ r^4(M_r)\ T^3(M_r)}$$

Homology relations

$$\rho \propto M_\star^{1-3a_r}$$

$$(4a_r + a_p - 2) = 0$$

$$\epsilon \propto M_\star^{(a_L - 1)}$$

$$\kappa \propto M_\star^{(4a_T + 4a_r - a_L - 1)}$$

But:

Ideal gas law says:

$$\rho \propto \frac{P}{T} \qquad P = M_\star^{a_P} \tilde{P}(q) \qquad T = M_\star^{a_T} \tilde{T}(q) \qquad \propto M_\star^{(a_P - a_T)}$$

Homology relations

$$\rho \propto M_\star^{1-3a_r}$$

$$(4a_r + a_p - 2) = 0$$

$$\epsilon \propto M_\star^{(a_L - 1)}$$

$$\kappa \propto M_\star^{(4a_T + 4a_r - a_L - 1)}$$

$$(a_P - a_T) = 1 - 3a_r$$

$$3a_r + 1a_P - 1a_T = 1$$

But:

Ideal gas law says:

$$\rho \propto \frac{P}{T} \qquad P = M_\star^{a_P} \; \tilde{P}(q) \qquad T = M_\star^{a_T} \; \tilde{T}(q) \qquad \propto M_\star^{(\,a_P - a_T\,)}$$

The p-p chain says:

$$\epsilon \underset{\sim}{\propto} \rho \, T^4 \qquad \propto M_\star^{(\,1 - 3a_r\,)} M_\star^{4a_T}$$

Dominant opacity processes for (ρ, T) in the Sun:

$$\kappa \propto \rho \, T^{-7/2}$$

Homology relations

$$\rho \propto M_\star^{(\,1 - 3a_r\,)} \qquad 3a_r + 1a_P - 1a_T = 1$$

$$(\,4a_r + a_p - 2\,) = 0 \qquad 4a_r + a_p = 2$$

$$\epsilon \propto M_\star^{(\,a_L - 1\,)} \qquad 3a_r + a_L - 4a_T = 2$$

$$\kappa \propto M_\star^{(\,4a_T + 4a_r - a_L - 1\,)} \qquad 14a_r - 2a_L + 15a_T = 4$$

4	a_r	1	a_p	0	a_L	0	a_T	=	2
3	a_r	1	a_p	0	a_L	-1	a_T	=	1
14	a_r	0	a_p	-2	a_L	15	a_T	=	4
3	a_r	0	a_p	1	a_L	-4	a_T	=	2

$$a_r = \frac{1}{13} \qquad a_P = \frac{22}{13} \qquad a_L = \frac{71}{13} \qquad a_T = \frac{12}{13}$$

$$3a_r + 1a_P - 1a_T = 1$$

$$4a_r + a_p = 2$$

$$3a_r + a_L - 4a_T = 2$$

$$14a_r - 2a_L + 15a_T = 4$$

$$q = \frac{M_r}{M_\star}$$

$$M_{\text{array}} = [1,\ 5,\ 10,\ 15, \ldots.]\ M_\odot$$

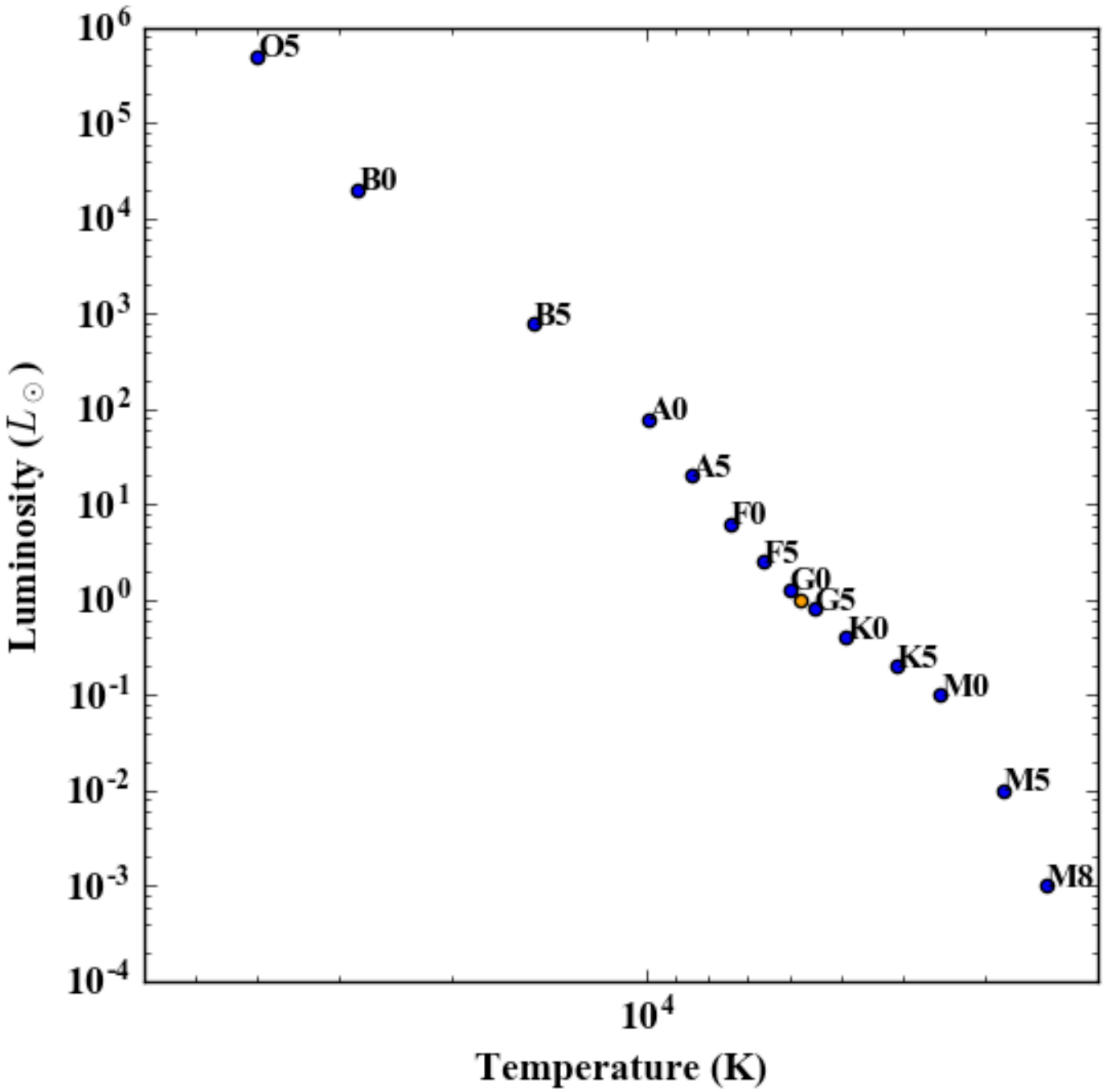
$$a_r = \frac{1}{13} \qquad r = M_\star^{a_r}\ \tilde{r}(q)$$

$$a_P = \frac{22}{13} \qquad P = M_\star^{a_P}\ \tilde{P}(q)$$

$$a_L = \frac{71}{13} \qquad L_r = M_\star^{a_L}\ \tilde{L}_r(q)$$

$$\frac{L_\star}{L_\odot} = \left(\frac{M_\star}{M_\odot}\right)^{71/13}$$

$$a_T = \frac{12}{13} \qquad T = M_\star^{a_T}\ \tilde{T}(q)$$



$$q = \frac{M_r}{M_\star}$$

$$M_{\text{array}} = [1,\ 5,\ 10,\ 15, \ldots.]\ M_\odot$$

$$a_r = \frac{1}{13} \qquad r = M_\star^{a_r}\ \tilde{r}(q)$$

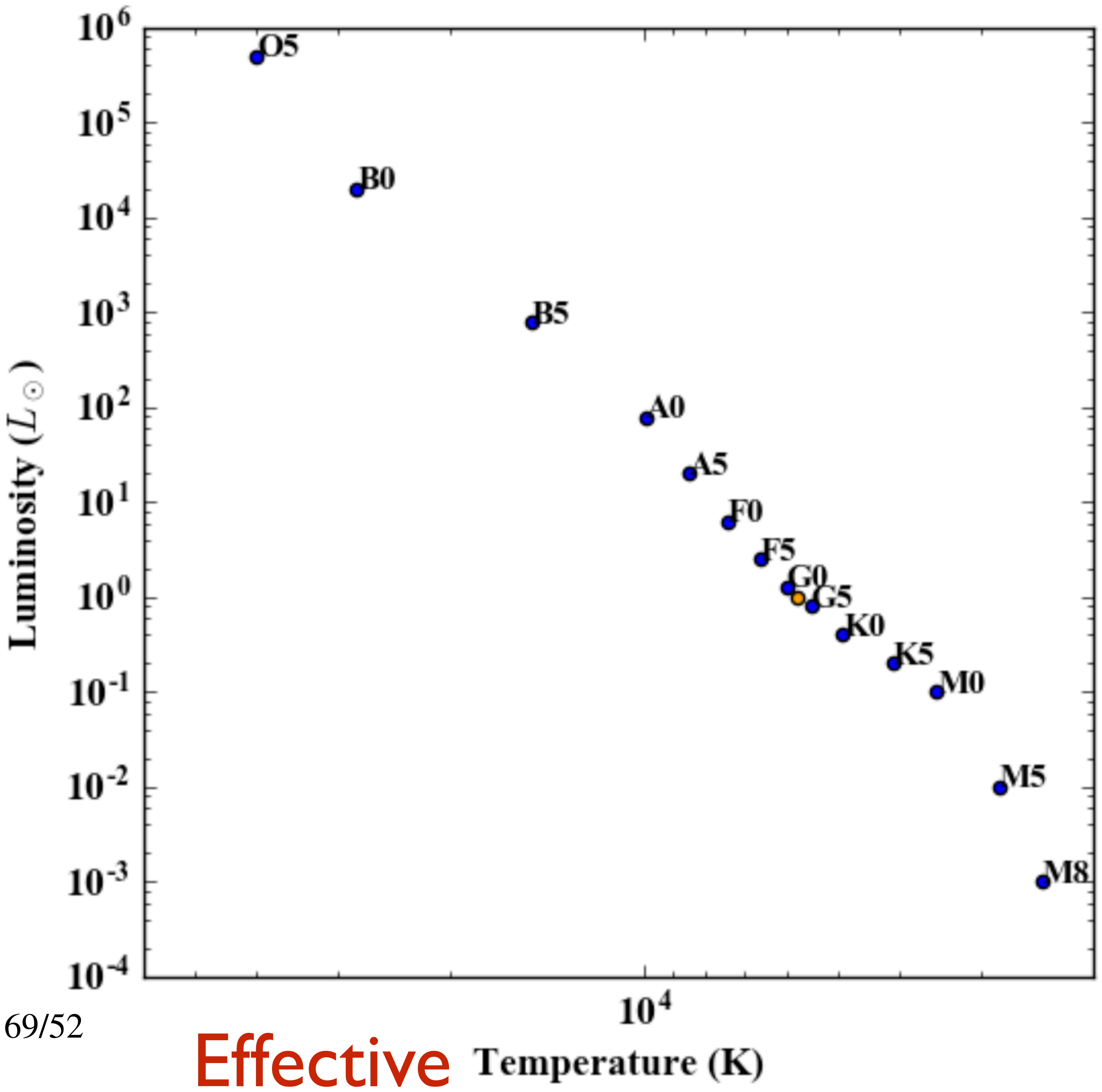
$$a_P = \frac{22}{13} \qquad P = M_\star^{a_P}\ \tilde{P}(q)$$

$$a_L = \frac{71}{13} \qquad L_r = M_\star^{a_L}\ \tilde{L}_r(q)$$

$$a_T = \frac{12}{13} \qquad T = M_\star^{a_T}\ \tilde{T}(q)$$

The effective temperature of a star is the ________ of a _________ with the same __________

$$\frac{T_{\text{eff},\star}}{T_{\text{eff},\odot}} = \left(\frac{M_\star}{M_\odot}\right)^{(a_L - 2a_r)/4} = \left(\frac{M_\star}{M_\odot}\right)^{69/52}$$

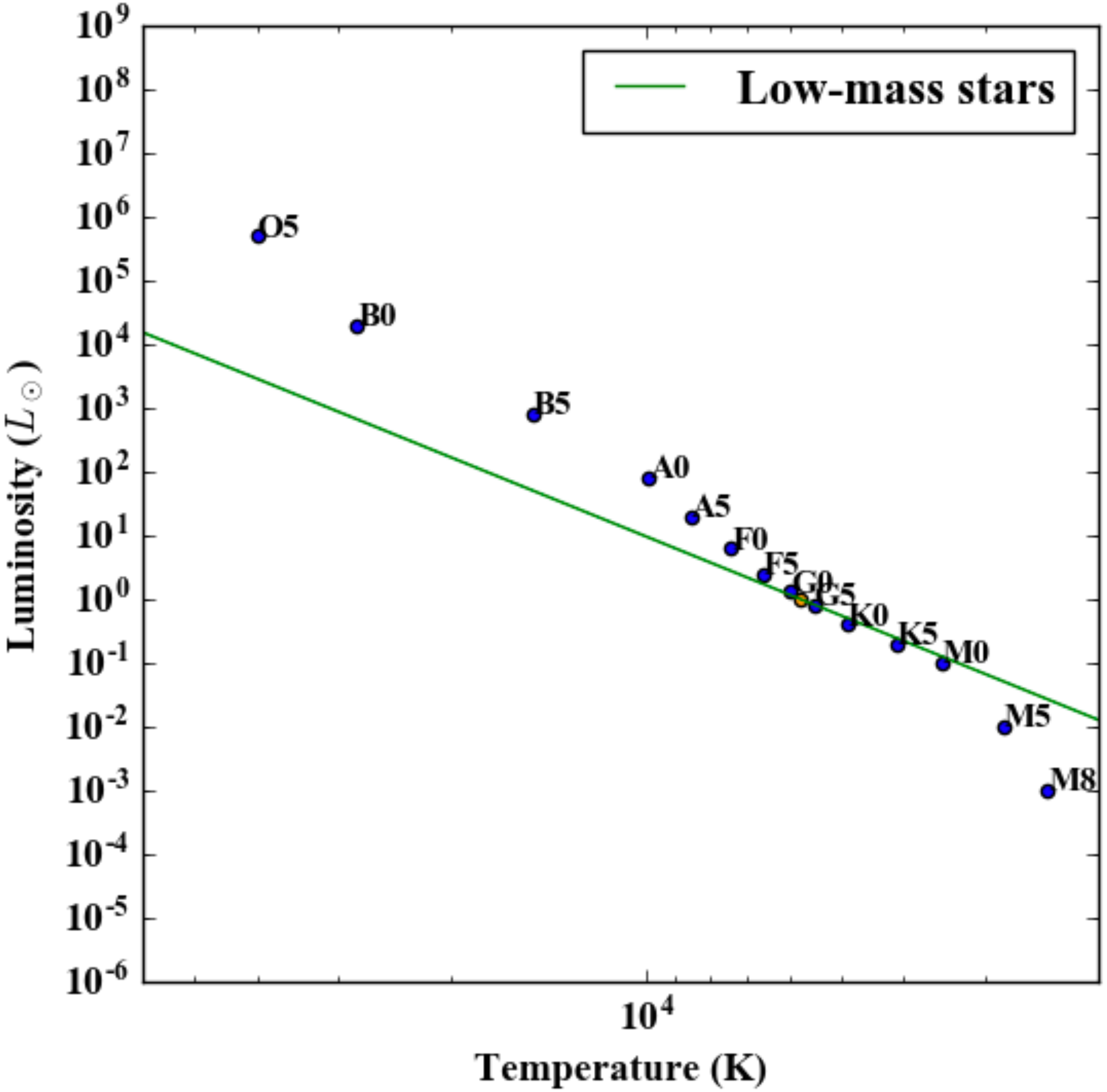


Effective

$M_{\text{array}} =$ `np.linspace(0.1, 40, 100)`

$$\frac{L_\star}{L_\odot} = \left(\frac{M_\star}{M_\odot}\right)^{71/13}$$

$$\frac{T_{\text{eff},\star}}{T_{\text{eff},\odot}} = \left(\frac{M_\star}{M_\odot}\right)^{(a_L - 2a_r)/4} = \left(\frac{M_\star}{M_\odot}\right)^{69/52}$$



But:

Ideal gas law says:

$$\rho \propto \frac{P}{T} \propto M_\star^{(a_P - a_T)}$$

What needs to change for more massive stars?

The p-p chain says:

$$\epsilon \lesssim \rho\, T^4 \propto M_\star^{(1-3a_r)} M_\star^{4a_T}$$

$$\epsilon \propto \rho\, T^{18} \propto M_\star^{(1-3a_r)} M_\star^{4a_T}$$

Dominant opacity processes for (ρ, T) in the Sun:

$$\kappa \propto \rho\, T^{-7/2}$$

$$\kappa \propto \text{constant}$$

Homology relations

$$\rho \propto M_\star^{(1-3a_r)}$$

$$3a_r + 1a_P - 1a_T = 1$$

$$(4a_r + a_p - 2) = 0$$

$$4a_r + a_p = 2$$

$$\epsilon \propto M_\star^{(a_L - 1)}$$

$$3a_r + a_L - 4a_T = 2$$

$$3a_r + a_L - 18a_T = 2$$

$$\kappa \propto M_\star^{(4a_T + 4a_r - a_L - 1)}$$

$$14a_r - 2a_L + 15a_T = 4$$

$$4a_r - 1a_L + 4a_T = 1$$

4	a_r	1	a_p	0	a_L	0	a_T	=	2
3	a_r	1	a_p	0	a_L	-1	a_T	=	1
4	a_r	0	a_p	-1	a_L	4	a_T	=	1
3	a_r	0	a_p	1	a_L	-18	a_T	=	2

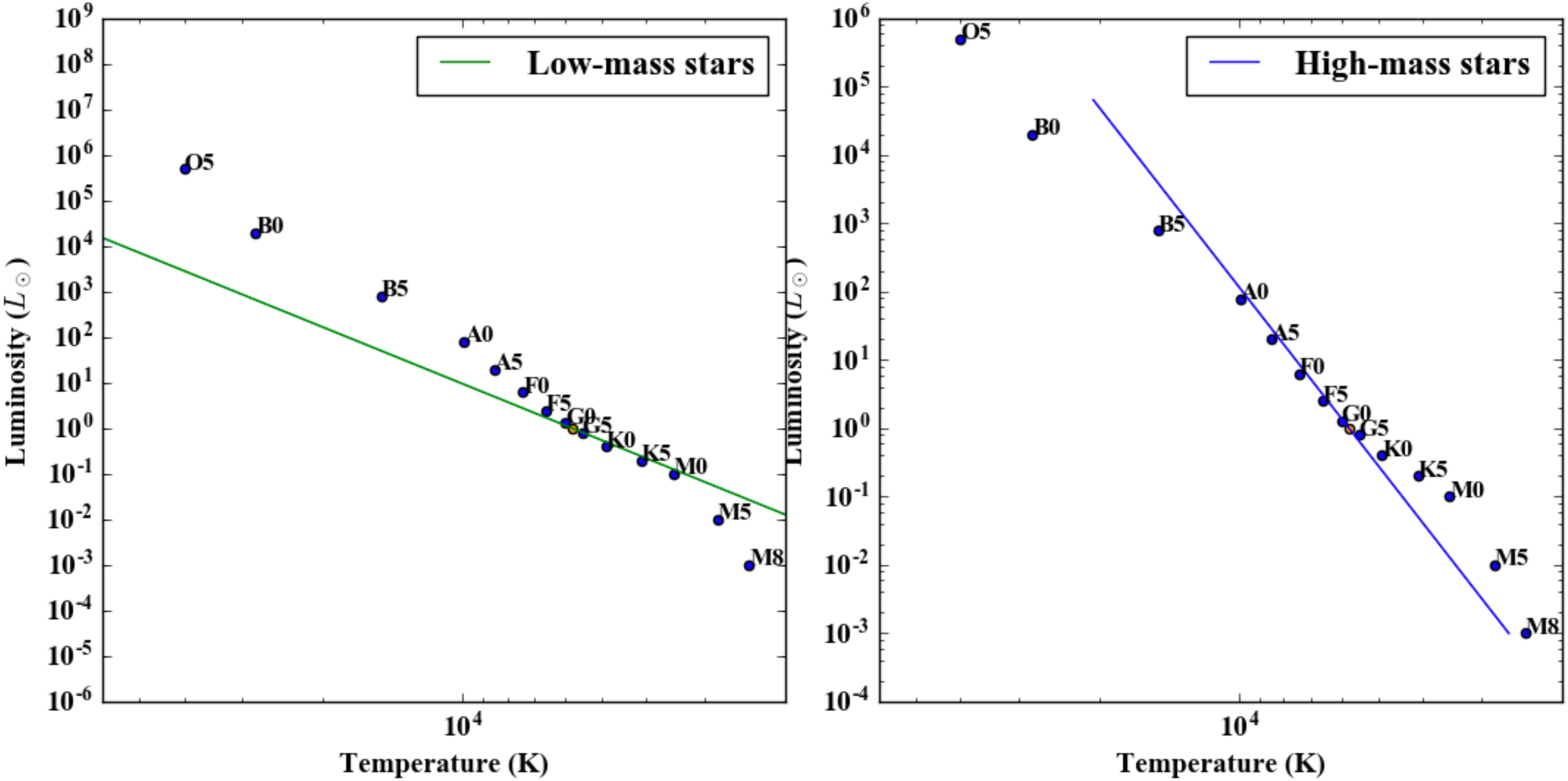
$a_r = \frac{17}{21}$ $\quad a_P = -\frac{26}{21}$ $\quad a_L = 3$ $\quad a_T = \frac{4}{21}$

$$3a_r + 1a_P - 1a_T = 1$$

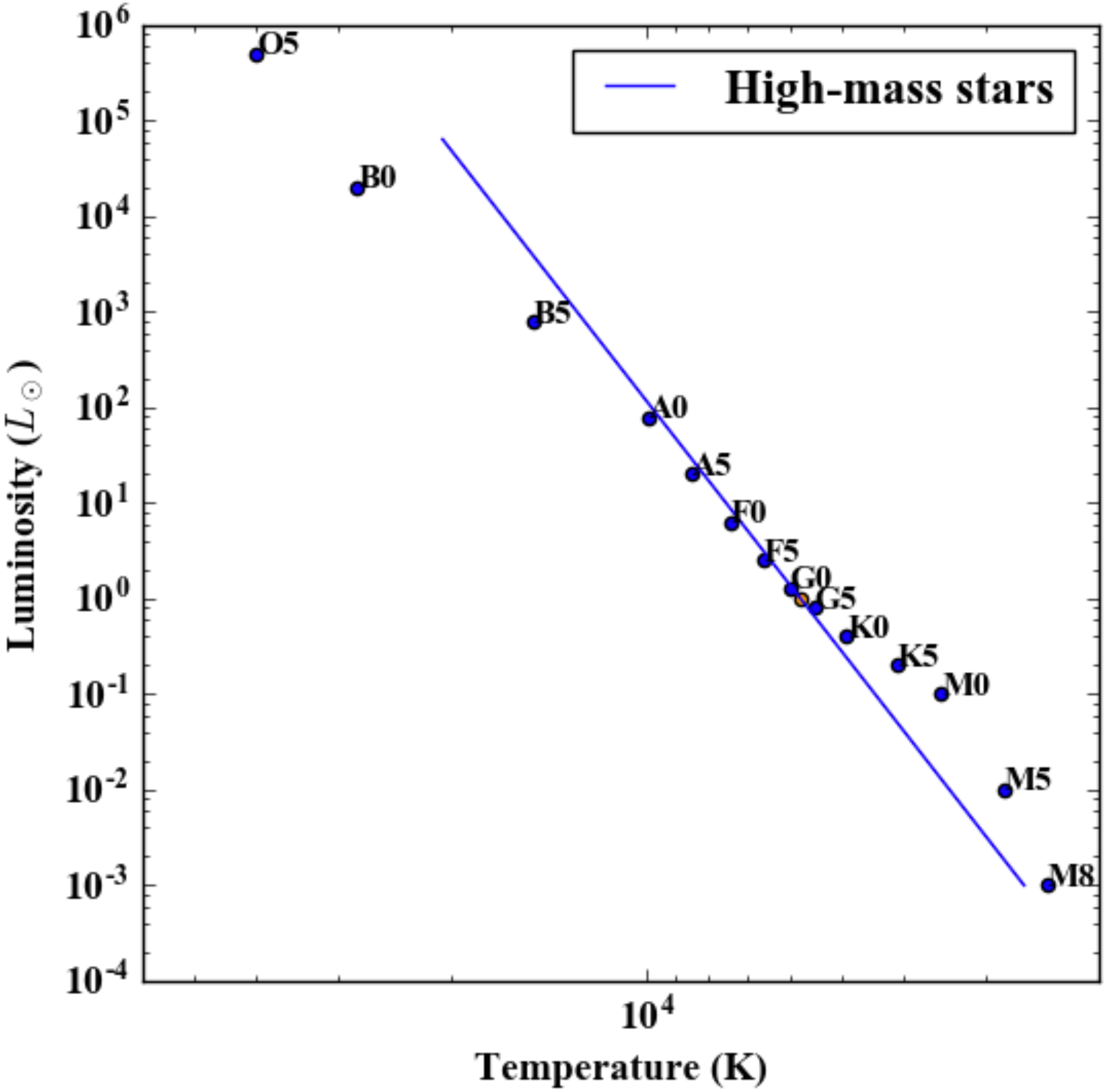
$$4a_r + a_p = 2$$

$$3a_r + a_L - 18a_T = 2$$

$$4a_r - 1a_L + 4a_T = 1$$



Low-mass stars
High-mass stars
Luminosity ($L_\odot$)
Temperature (K)
O5
B0
B5
A0
A5
F0
F5
G0
G5
K0
K5
M0
M5
M8



What would happen if a massive stars was powered by PP chain instead of CNO cycle?

Let's define a generalized case:

$$\varepsilon = \varepsilon_0 \rho T^{\eta}$$

$$\kappa = \kappa_0(\alpha)\left(\rho T^{-7/2}\right)^{\alpha}$$

$$\begin{bmatrix} 4 & 1 & 0 & 0 \\ 3 & 1 & 0 & -1 \\ (8+6\alpha) & 0 & -2 & (8+7\alpha) \\ 3 & 0 & 1 & -\eta \end{bmatrix} \begin{bmatrix} a_r \\ a_p \\ a_L \\ a_T \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2(\alpha+1) \\ 2 \end{bmatrix}$$

Let's define a generalized case:

$$\varepsilon = \varepsilon_0 \rho T^{\eta}$$

$$\kappa = \kappa_0(\alpha) \left(\rho T^{-7/2}\right)^{\alpha}$$

The solution is:

$$a_r = \frac{2\eta - 5\alpha - 2}{2\eta - \alpha + 6}$$

$$a_p = \frac{-4\eta + 18\alpha + 20}{2\eta - \alpha + 6}$$

$$a_L = \frac{(6+4\alpha)\eta + 13\alpha + 18}{2\eta - \alpha + 6}$$

$$a_T = \frac{4\alpha + 8}{2\eta - \alpha + 6}$$

Let's define a generalized case:

$$\frac{L_\star}{L_\odot} = \left(\frac{M_\star}{M_\odot}\right)^{a_L}$$

$$\varepsilon = \varepsilon_0 \rho T^{\eta}$$

$$\kappa = \kappa_0(\alpha)\left(\rho T^{-7/2}\right)^{\alpha}$$

Let's look at the luminosity relation:

$$a_L = \frac{(6+4\alpha)\eta + 13\alpha + 18}{2\eta - \alpha + 6} = 3 + 2\alpha + \frac{2\alpha(\alpha+2)}{2\eta + 6 - \alpha}$$

For low mass stars, $\alpha = 1$

$$a_L = 5 + \frac{6}{2\eta + 5}$$

For low mass stars, $\alpha = 0$

$$a_L = 3 \; !$$

Independent of the T scaling of the nuclear reactions !!

Same luminosity, but hotter!