

Week 5 Thursday
L-10

General

Mean intensity

$$J_{\lambda} = \frac{1}{4\pi} \int I_{\lambda} d\Omega$$

Flux

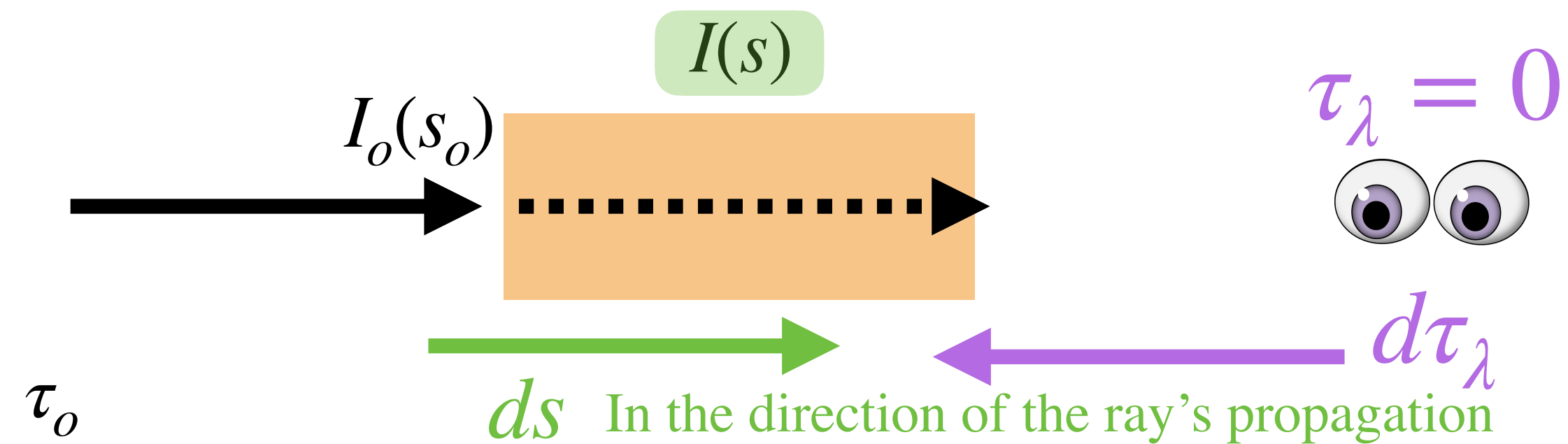
$$F_{\lambda} = \int I_{\lambda} \cos \theta d\Omega$$

Spherical, with azimuthal symmetry ($u = \cos \theta$)

$$J_{\lambda} = \frac{1}{2} \int_{-1}^{+1} I_{\lambda}(u) \, du$$

$$\frac{F_{\lambda}}{4\pi} = \frac{1}{2} \int_{-1}^{+1} I_{\lambda}(u) \, u \, du$$

Change in intensity



Optical depth

$$d\tau_\lambda = -\kappa_\lambda \rho ds$$

$$\tau_\lambda(s) - 0 = - \int_{s_{\text{obs}}}^s \kappa_\lambda(s) \rho(s) ds$$

Absorption

$$dI_\lambda = -\kappa_\lambda(s) \rho(s) ds I_\lambda$$

General Solution

$$\tau(s) = \int_{s'=s}^{s'=\infty} \kappa(s') \rho(s') ds'$$

$$I(s) = I_o e^{- \int_{s'=s_o}^{s'=s} \kappa(s') \rho(s') ds}$$

$$I(s) = I_o e^{\tau(s) - \tau(s=s_o)}$$

Constant properties in material

$$\tau(s) = \kappa \rho ((s_o + d) - s)$$

$$I(s) = I_o e^{-\kappa \rho (s - s_o)}$$

Emission: how can we gain intensity?

1. Pure emission: create a photon



2. Scattering: deflect a photon



(Q: what are the units of κ_λ ?)

Absorption: $dI_\lambda = - \rho \, ds \, \kappa_\lambda \, I_\lambda$

(Q: what is the meaning of $\rho \, ds \, \kappa_\lambda$?)

Emission:

Emission: how can we gain intensity?

1. Pure emission: create a photon



2. Scattering: deflect a photon



Absorption: $dI_\lambda = - \rho \, ds \, \kappa_\lambda \, I_\lambda$

Emission: $dI_\lambda = + \rho \, ds \, j_\lambda$

(Q: why no I_λ ?)

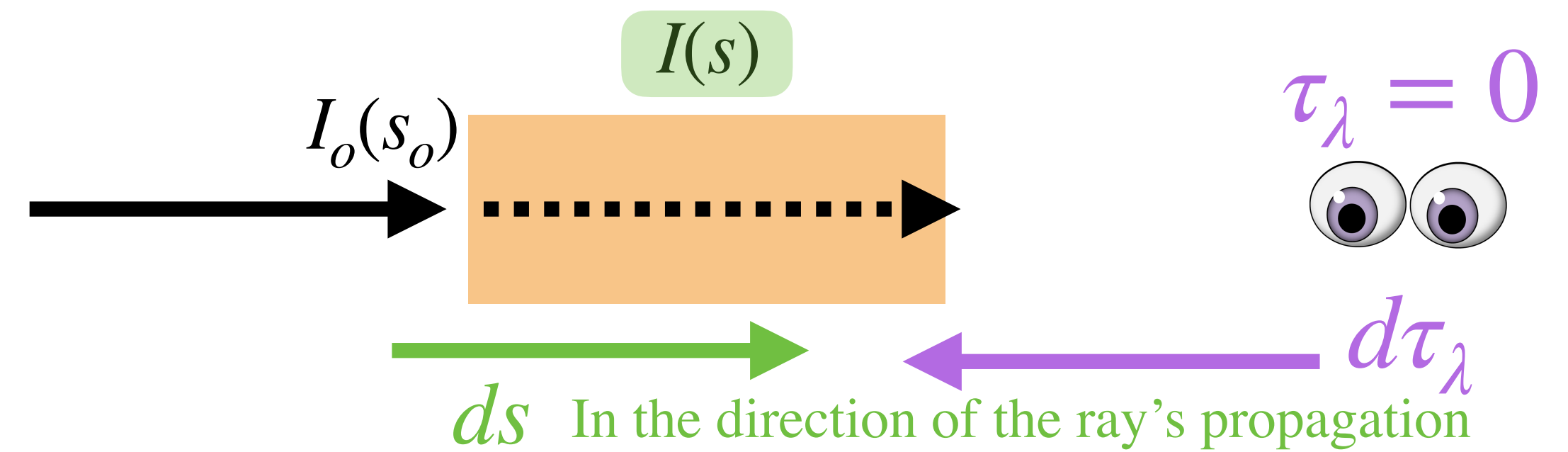
(Q: what are the units of j_λ ?)

Let's solve this equation, if we know $I(s_o) = I_o$

$$dI_\lambda = +j_\lambda(s) \rho(s) ds$$

Together on the board...

$$I(s) = I_o + \int_{s_o}^s j(s)\rho(s)ds$$



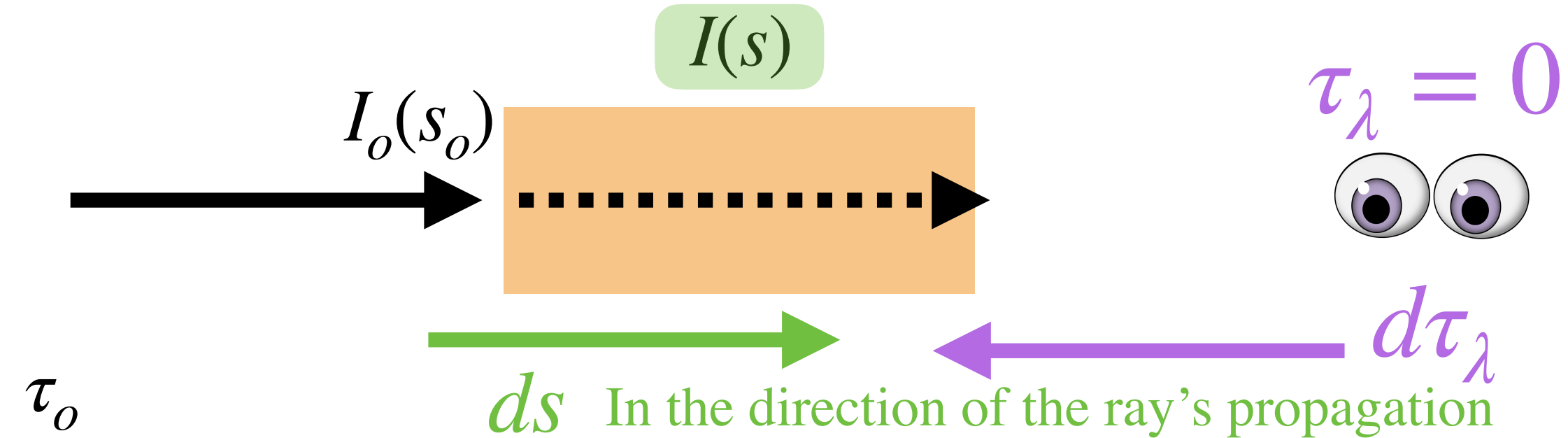
If the material has constant properties:

$$I(s) = I_o + j\rho[s - s_o]$$

In notebook:

- graph $I(s)$ for constant values
- graph $I(s)$ for linearly decreasing density

Change in intensity



Optical depth

$$d\tau_\lambda = -\kappa_\lambda \rho ds$$
$$\tau_\lambda(s) - 0 = - \int_{s_{\text{obs}}}^s \kappa_\lambda(s) \rho(s) ds$$

Absorption

$$dI_\lambda = -\kappa_\lambda(s) \rho(s) ds I_\lambda$$

Emission

$$dI_\lambda = + j_\lambda(s) \rho(s) ds$$

General Solution

$$\tau(s) = \int_{s'=s}^{s'=\infty} \kappa(s') \rho(s') ds'$$

$$I(s) = I_o e^{- \int_{s'=s_o}^{s'=s} \kappa(s') \rho(s') ds}$$

$$I(s) = I_o e^{\tau(s) - \tau(s=s_o)}$$

$$I(s) = I_o + \int_{s'=s_o}^{s'=s} j(s') \rho(s') ds'$$

Constant properties in material

$$\tau(s) = \kappa \rho ((s_o + d) - s)$$

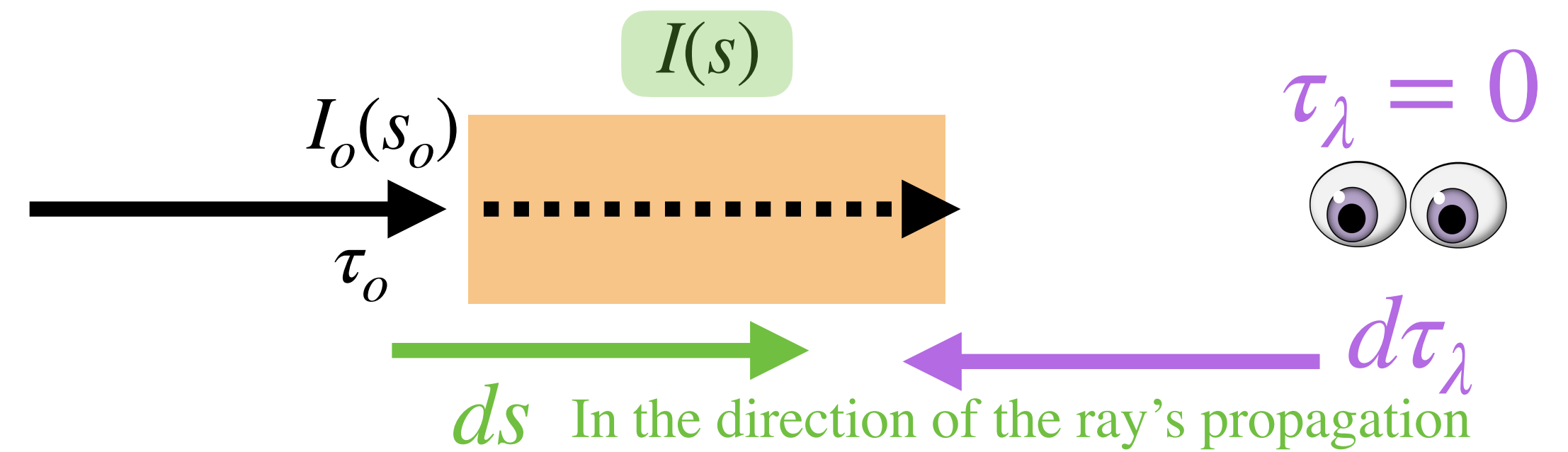
$$I(s) = I_o e^{-\kappa \rho (s - s_o)}$$

$$I(s) = I_o + j \rho (s - s_o)$$

Now, what if we have both absorption and emission:
The “formal solution” of RT

(Q: why can't we integrate this directly like earlier?)

$$\frac{dI_\lambda}{-\kappa_\lambda(s) \rho(s) ds} = \frac{-\kappa_\lambda(s) \rho(s) ds I_\lambda(s)}{-\kappa_\lambda(s) \rho(s) ds} + \frac{j_\lambda(s) \rho(s) ds}{-\kappa_\lambda(s) \rho(s) ds}$$



$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda(s) - \boxed{\frac{j_\lambda(s)}{\kappa_\lambda(s)}} \quad \text{The “source function” } S_\lambda(s) \quad (\text{Q: what is the meaning of } S(\lambda)?)$$

$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda(\tau_\lambda) - S_\lambda(\tau_\lambda)$$

Here, I can use the optical depth as my “coordinate system” instead of the physical distance

Now, what if we have both absorption and emission:
The “formal solution” of RT

$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda(\tau_\lambda) - S_\lambda(\tau_\lambda)$$

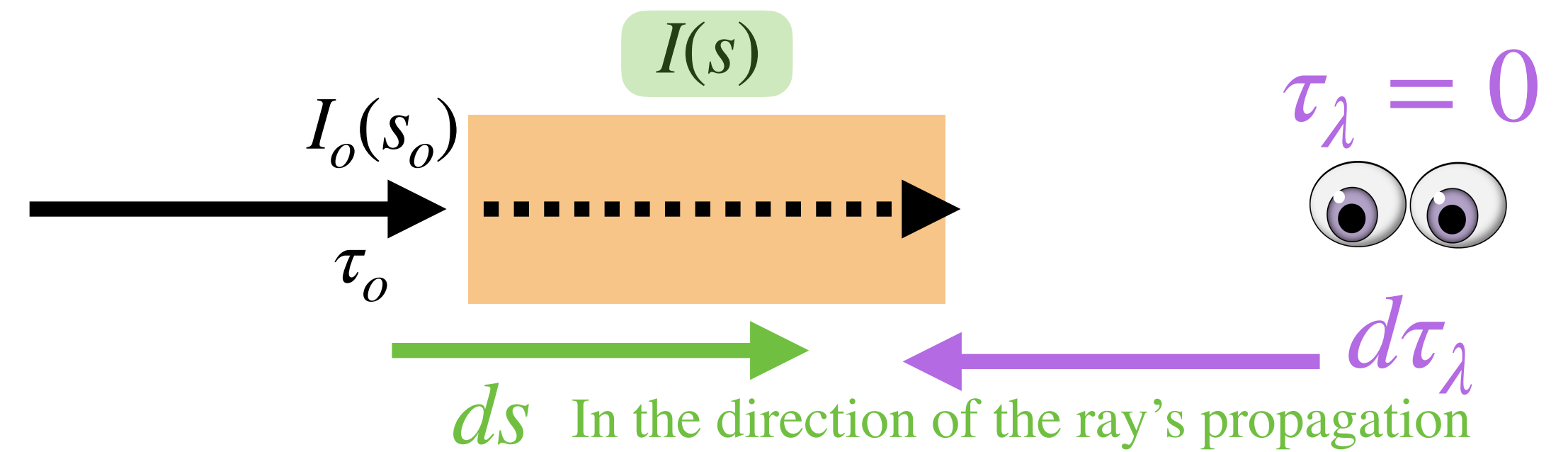
(Just dropping the λ s for more clarity)

$$\frac{dI}{d\tau} e^{-\tau} - I(\tau) e^{-\tau} = -S(\tau) e^{-\tau}$$

$$\frac{d}{d\tau} [I(\tau) e^{-\tau}] = -S(\tau) e^{-\tau}$$

$$\int_{\text{known position } (I_o, \tau_o)}^{\text{unknown position } (I, \tau)} d [I(\tau) e^{-\tau}] = - \int_{\tau_o}^{\tau} S(\tau) e^{-\tau} d\tau$$

$$I(\tau) e^{-\tau} - I_o e^{-\tau_o} = \int_{\tau'=\tau_o}^{\tau'=\tau} S(\tau') e^{-\tau'} d\tau'$$



Here, I added explicit ' to the integrands, to distinguish it from the τ in the upper bound.

(This is the same as our integrals with a bound of r when we integrated the continuity and hydrostatic equations, but here it is a bit harder to visualize)

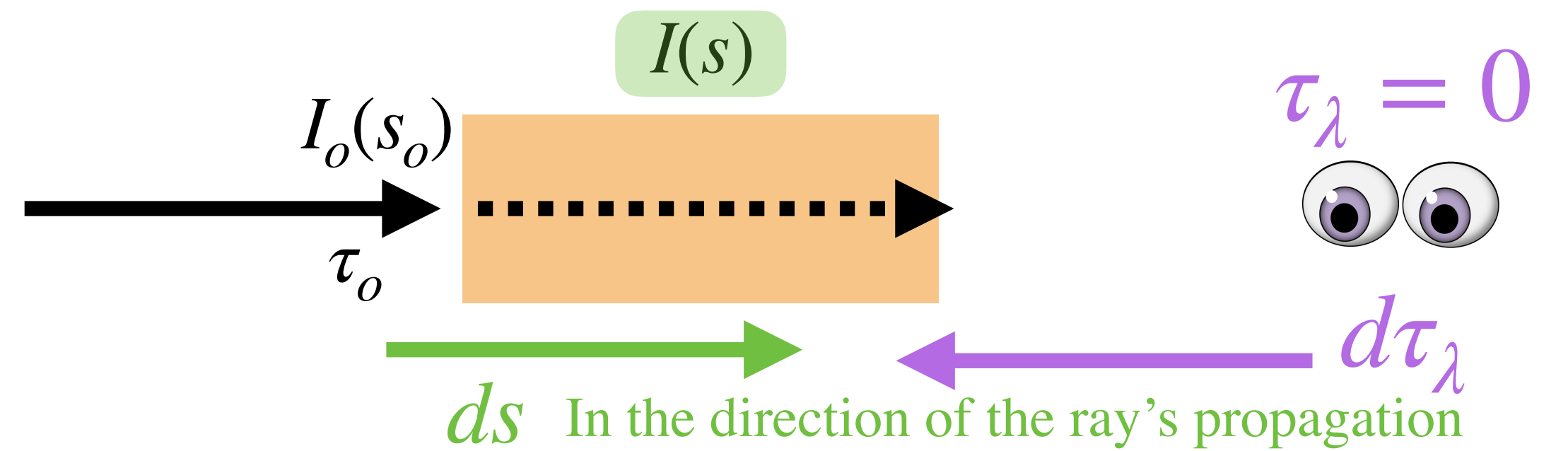
Now, what if we have both absorption and emission:
The “formal solution” of RT

$$I(\tau)e^{-\tau} - I_o e^{-\tau_o} = \int_{\tau'=\tau_o}^{\tau'=\tau} S(\tau') e^{-\tau'} d\tau'$$

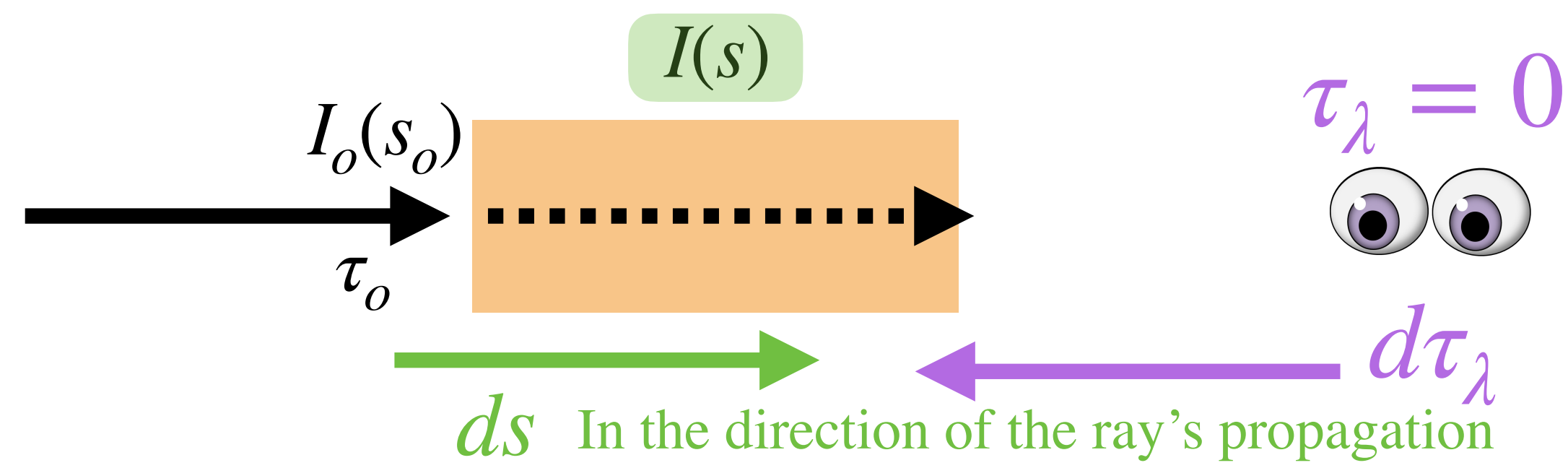
$$\frac{I(\tau)e^{-\tau}}{e^{-\tau}} = \frac{I_o e^{-\tau_o}}{e^{-\tau}} + \int_{\tau'=\tau_o}^{\tau'=\tau} \frac{S(\tau') e^{-\tau'}}{e^{-\tau}} d\tau'$$

$$I(\tau) = I_o e^{\tau-\tau_o} + \int_{\tau'=\tau_o}^{\tau'=\tau} S(\tau') e^{\tau-\tau'} d\tau'$$

That is the “formal solution”

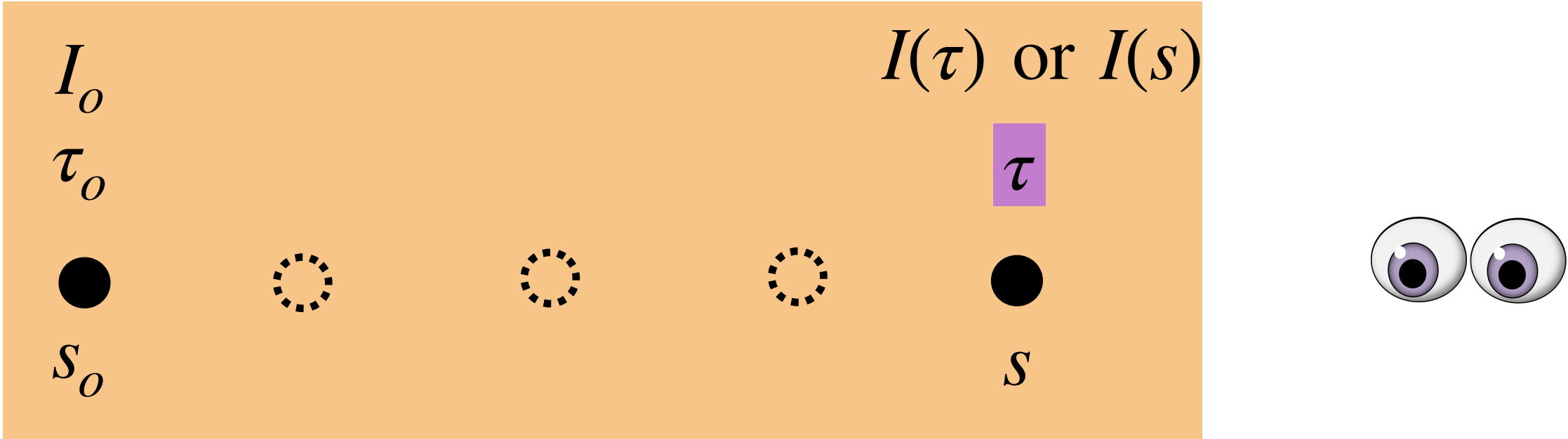


Now, what if we have both absorption and emission:
 The “formal solution” of RT



$$I(\tau) = I_o e^{\tau - \tau_o} + \int_{\tau' = \tau_o}^{\tau' = \tau} S(\tau') e^{\tau - \tau'} d\tau'$$

$$I(\tau = 1) = I_o e^{1 - 5} + \int_{\tau' = 5}^{\tau' = 1} S(\tau') e^{1 - \tau'} d\tau'$$



Replacing the integral by a sum, for illustration:

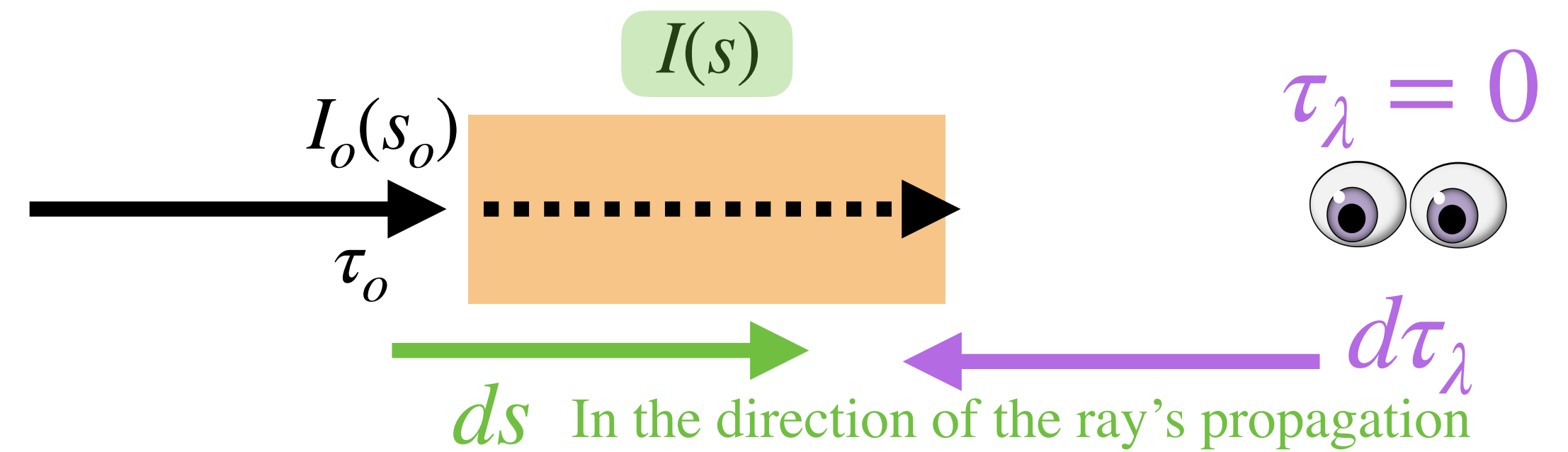
$$I(\tau = 1) = I_o e^{1 - 5} + S(\tau' = 4) e^{1 - 4} \Delta\tau' + S(\tau' = 3) e^{1 - 3} \Delta\tau' + \dots$$

1. What if the source function is constant $S(\tau) = S_o$?

$$I(\tau) = I_o e^{\tau - \tau_o} + \int_{\tau' = \tau_o}^{\tau' = \tau} S(\tau') e^{\tau - \tau'} d\tau'$$

1. Get the source function out of the integral
2. Note that $e^{\tau - \tau'} = e^{\tau} e^{-\tau'}$ and e^{τ} can also be taken out of the integral (not a function of τ')
3. We can integrate

$$I(\tau) = I_o e^{\tau - \tau_o} + S_o [1 - e^{\tau - \tau_o}]$$



2. What if the other material properties are also constant (κ_o, ρ_o)?

And we also would like to be able to plot $I(s)$ in our notebook, so we need to go back to that coordinate system!

$$I(s) = I_o e^{\tau(s) - \tau_o} + S_o [1 - e^{\tau(s) - \tau_o}]$$

$$\tau(s) - 0 = - \int_{s_{\text{obs}}}^s \kappa(s) \rho(s) ds = - \kappa_o \rho_o [s - s_{\text{obs}}]$$

$$\tau(s_o) = - \kappa_o \rho_o [s_o - s_{\text{obs}}]$$

$$\tau(s) - \tau(s_o) = - \kappa_o \rho_o [s - s_o]$$

$$I(s) = I_o e^{-\kappa_o \rho_o [s - s_o]} + S_o [1 - e^{-\kappa_o \rho_o [s - s_o]}]$$

$$\tau(s) - \tau_o = - \int_{s_o}^s \kappa(s) \rho(s) ds = - \kappa_o \rho_o [s - s_o]$$

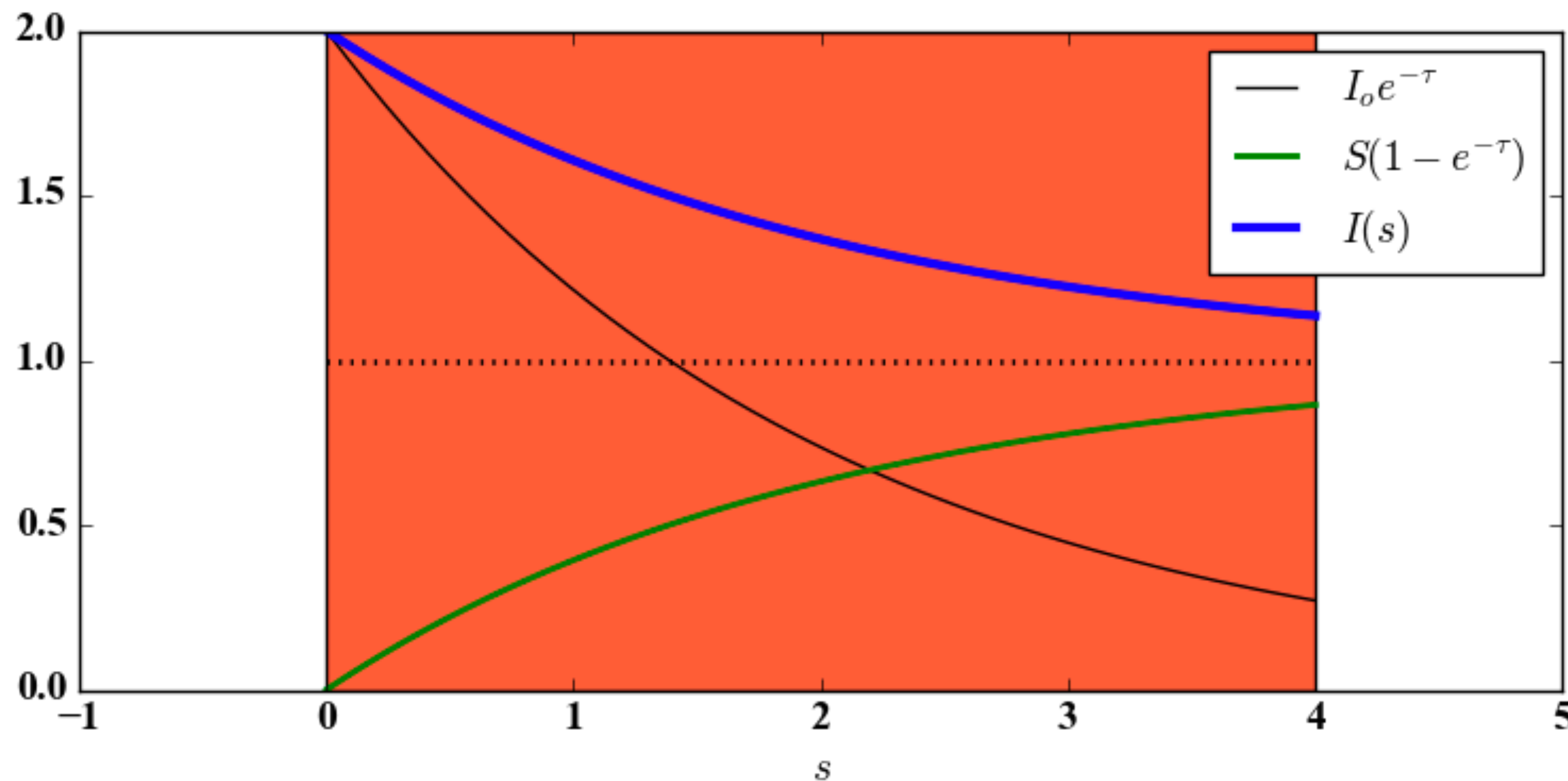
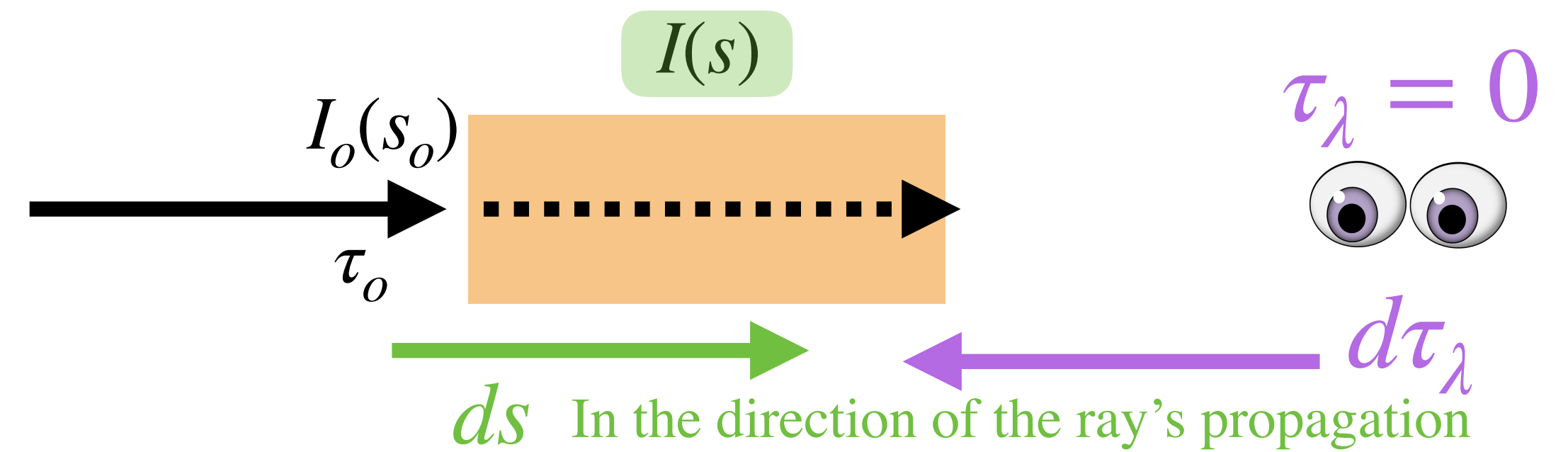
$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda(\tau_\lambda) - S_\lambda(\tau_\lambda)$$

1. What if the source function is constant $S(\tau) = S_o$?

$$I(s) = I_o e^{\tau(s)-\tau_o} + S_o [1 - e^{\tau(s)-\tau_o}]$$

2. What if the other material properties are also constant (κ_o, ρ_o)?

$$I(s) = I_o e^{-\kappa_o \rho_o [s-s_o]} + S_o [1 - e^{-\kappa_o \rho_o [s-s_o]}]$$

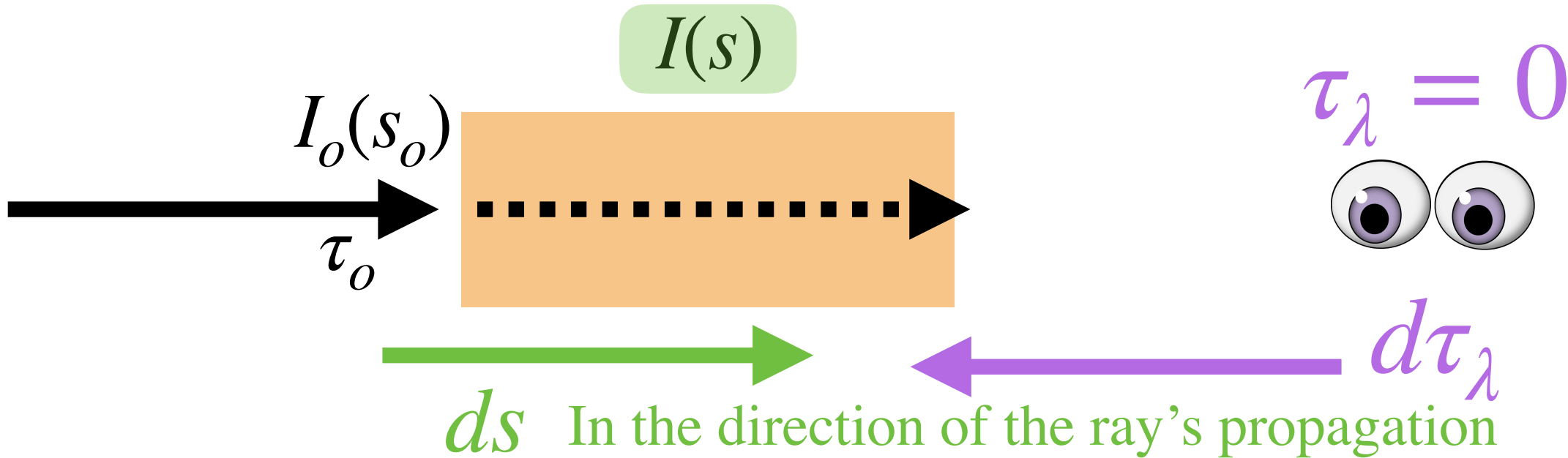


Q: What if $S_o = I_o$?

Q: What would happen to $I(s)$ if the slab was very long?

-> The source function is what the intensity will become, if given enough optical depth to reach it!

Change in intensity



Absorption + emission

$$\frac{dI_\lambda(\tau_\lambda)}{d\tau_\lambda} = I_\lambda(\tau_\lambda) - S_\lambda(\tau_\lambda)$$

General Solution

$$I(\tau(s)) = I_o e^{\tau(s)-\tau_o} + \int_{\tau'=\tau(s)}^{\tau'=\tau_o} S(\tau') e^{\tau(s)-\tau'} d\tau'$$

Constant source function

$$I(\tau(s)) = I_o e^{\tau(s)-\tau_o} + S \left[1 - e^{\tau(s)-\tau_o} \right]$$