Week 3 Tuesday

L-5

Abundances

Differential equations:

Continuity equation

$$\frac{dM_r(r)}{dr} = 4\pi r^2 \rho(r)$$

Hydrostatic equilibrium

$$\frac{dP(r)}{dr} = -\rho(r)\overbrace{\frac{GM_r(r)}{r^2}}^{g(r)}$$

Two equations, three unknowns (plus R)

Variables

$M_r(r)$
$\rho(r)$
$P(r)$

Boundary conditions

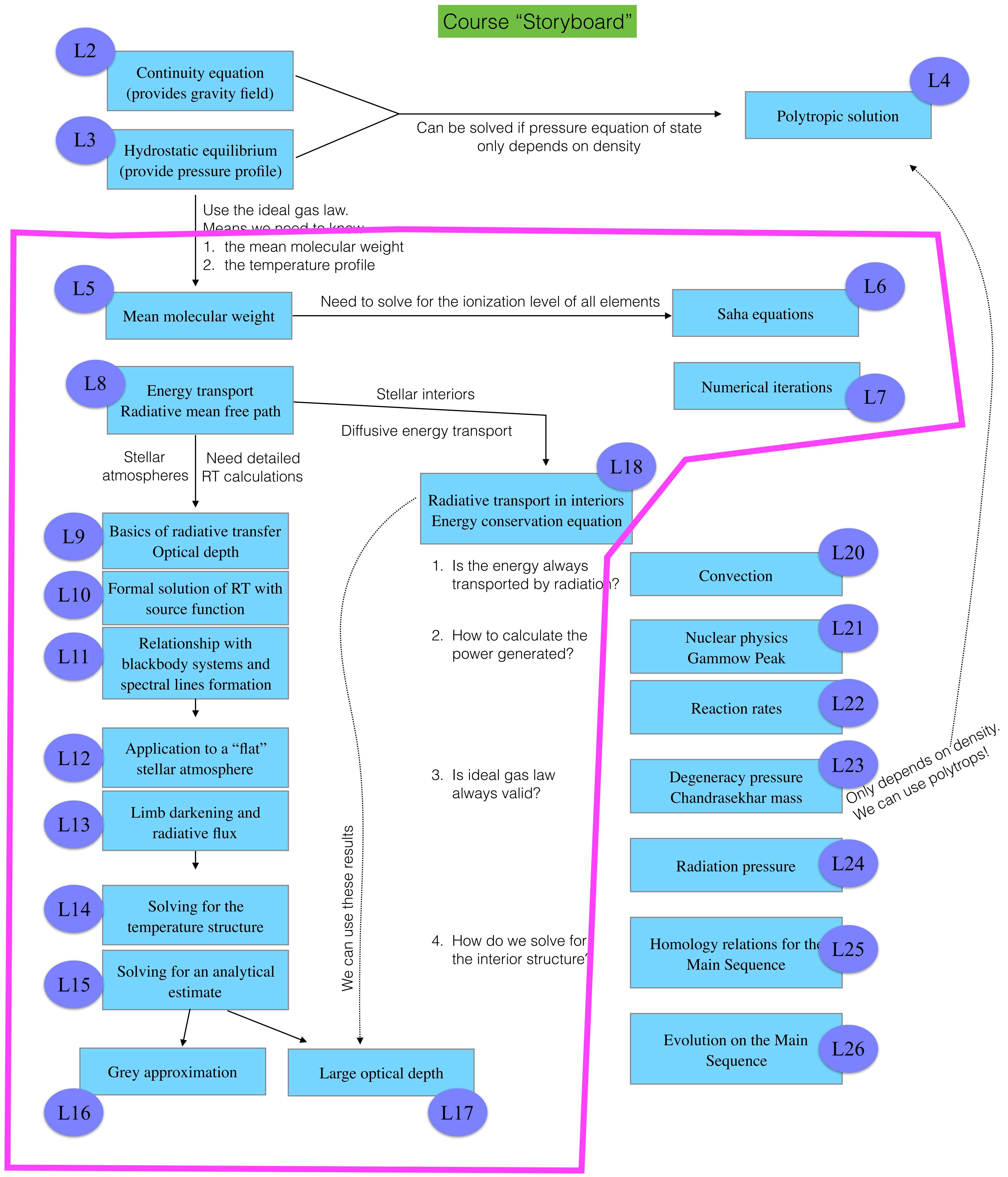
$M_r(r = 0) = 0$
$P(r = R) = 0$

We need a relationship between $P(r)$ and $\rho(r)$ (An "equation of state")

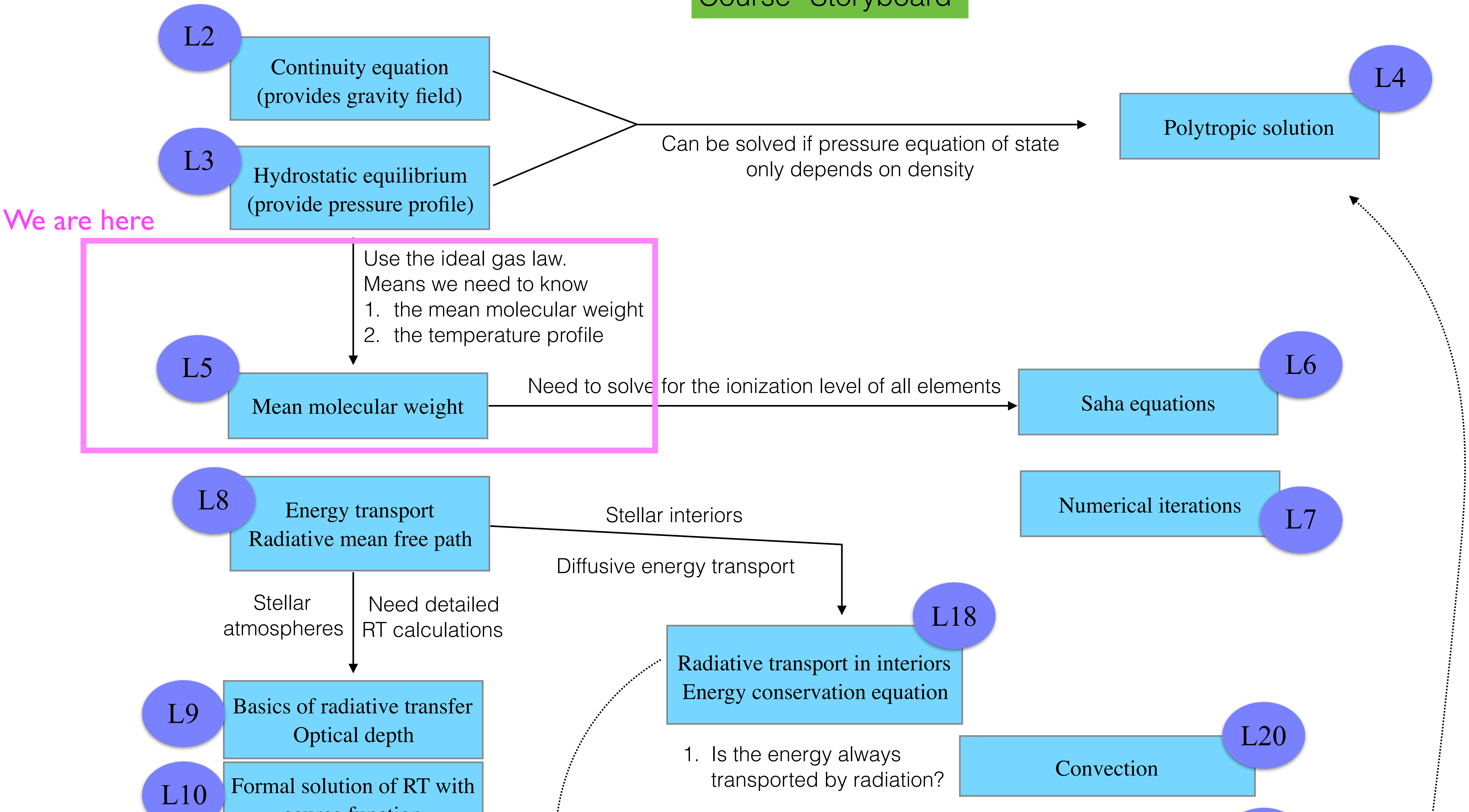
$$P = nkT \quad ?$$

If we consider ideal gas, OK… But we don't know T (yet).
And to relate n to ρ, we need to know the composition…

All of this is basically to get $T(r)$… yikes!



Course "Storyboard"



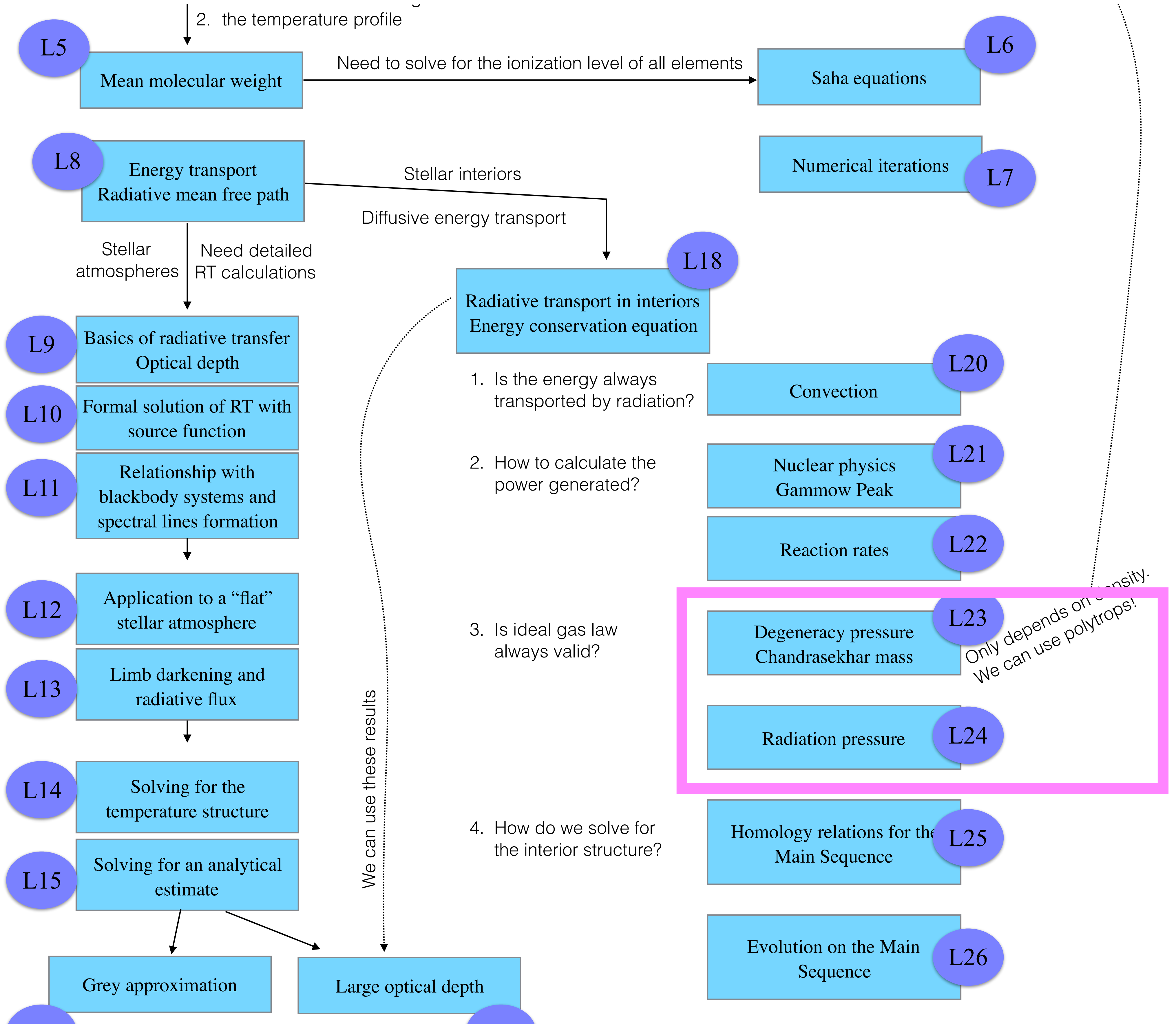
Equation of state:

Ideal gas law:

$$P_{\text{tot}}(r) \quad = \quad n(r) \;\; k \;\; T(r) \qquad + \; P_{\text{radiation}} \; + \; P_{\text{magnetic}} \; + \; P_{\text{degenerate}} + \;\; \dots$$

Note: gravity does not care where the pressure comes from!

What goes into $\dfrac{dP(r)}{dr} = -\rho(r)\dfrac{GM_r(r)}{r^2}$ is P_{tot}



Equation of state:

Ideal gas law:

$$P\ (r) = n(r)\ k\ T(r)$$

$n(r)$ is the concentration of **free** particles (particle / volume)

Our goal is to relate $n(r)$ with $\rho(r)$ (why?)

$n(r)$ is the concentration of **free** particles (particle / volume)

Our goal is to relate $n(r)$ with $\rho(r)$

<u>On the board</u>: introducing the mean molecular weight μ

$n(r)$ is the concentration of **free** particles (particle / volume)

Our goal is to relate $n(r)$ with $\rho(r)$

$$P(r) \;=\; n(r) \;\; kT(r)$$

$$n = \frac{\rho}{\mu m_H}$$

<u>On the board</u>: introducing the mean molecular weight μ

For a mixture of elements, we will make use of the **abundance**: mass fraction relative to total mass X_i

mass of 1 Si atom

# of Si atom / volume

total mass of Si / volume

e.g. $$X_{\text{Si}} = \frac{\text{Total mass of Si}}{\text{Total mass of gas}} = \frac{m_{\text{Si}} \;\; n_{\text{Si}}}{\rho}$$

total mass / volume

$n(r)$ is the concentration of **free** particles (particle / volume)

Our goal is to relate $n(r)$ with $\rho(r)$

$$P(r) = n(r) \; kT(r)$$

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<u>On the board</u>: introducing the mean molecular weight μ

For a mixture of elements, we will make use of the **abundance**: mass fraction relative to total mass X_i

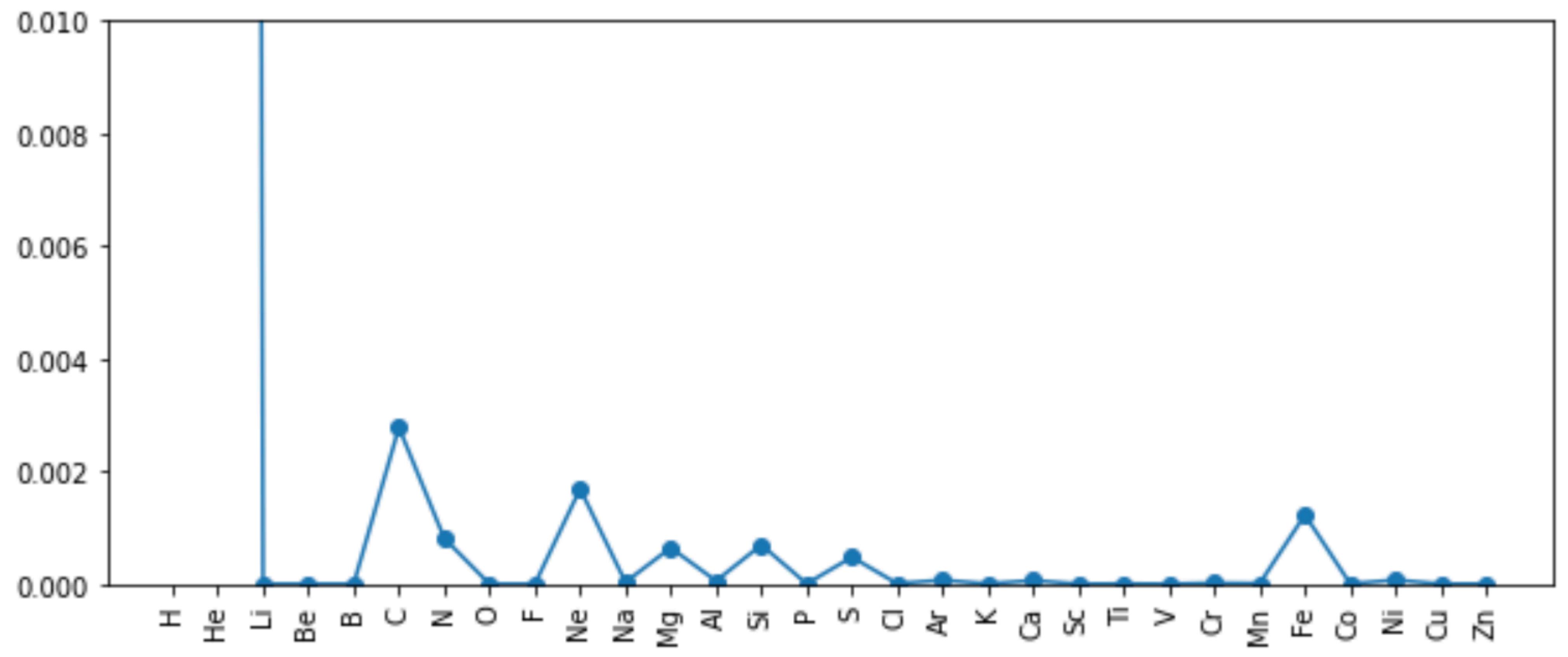
Atomic mass (e.g. 1 for H, 4 for He, etc)

$$X_i = \frac{(A_i m_H)\, n_i}{\rho}$$

We usually define:		for the Sun: (at surface)
X_H	$= X$	70%
X_{He}	$= Y$	28%
$X_{\text{everything else!}}$	$= Z$	2%

In notebook: Let's see what these abundances look like

* How to load in a text datafile
* How to add scatter points to a graph
* How to add text tick labels on an axis



In astro, you will see abundances expressed as:

$$n_i = \frac{X_i \rho}{A_i m_{\mathrm{H}}}$$

$$\log(\epsilon_i) = \log\left(\frac{n_i}{n_{\mathrm{H}}}\right) + 12$$

$$n_{\mathrm{H}} = \frac{X \rho}{m_{\mathrm{H}}}$$

Our goal is to relate $n(r)$ with $\rho(r)$

$$P(r) = n(r) \; kT(r)$$

$$n = \frac{\rho}{\mu m_H}$$

So now, if we know the abundances in the gas (all the X_i), what is the mean molecular weight μ?

$$X_i = \frac{(A_i m_H)\, n_i}{\rho}$$

$$n_i = \frac{\rho X_i}{(A_i m_H)}$$

$$\frac{1}{\mu m_H} = \frac{n_{\rm ion}}{\rho} + \frac{n_{\rm e}}{\rho}$$

$$\frac{1}{\mu_{\rm ion} m_H} \qquad \frac{1}{\mu_{\rm e} m_H}$$

(What is the conceptual meaning of $\mu_{\rm ion}$ and $\mu_{\rm e}$?)

$$\frac{1}{\mu_{\rm ion} m_H} = \frac{1}{\rho}(n_H + n_{He} + n_i \ldots)$$

$$\frac{1}{\mu_{\rm ion} m_H} = \frac{1}{\rho}\sum_i n_i$$

$$\frac{1}{\mu_{\rm ion} m_H} = \frac{1}{\rho}\sum_i \frac{\rho X_i}{(A_i m_H)}$$

$$\frac{1}{\mu_{\rm ion}} = \sum_i \frac{X_i}{A_i}$$

Let's imagine the gas is all ionized: each element contributes Z_i electrons

$$\frac{1}{\mu_{\rm e} m_H} = \frac{1}{\rho}(1\ n_H + 2\ n_{He} + 3\ n_{\rm Li} + \ldots Z_i\ n_i)$$

$$\frac{1}{\mu_{\rm e} m_H} = \frac{1}{\rho}\sum_i n_i (Z_i\ y_i)$$

$$\frac{1}{\mu_{\rm e} m_H} = \frac{1}{\rho}\sum_i \frac{\rho X_i}{(A_i m_H)} Z_i\ y_i$$

$$\frac{1}{\mu_{\rm e}} = \sum_i \frac{X_i}{A_i} Z_i\ y_i$$

Ionization fraction:
how many free electron per atom?
$y_i = 0$: completely neutral
$y_i = 1$: completely ionized

In summary:

$$\frac{1}{\mu} = \frac{1}{\mu_{\text{ion}}} + \frac{1}{\mu_{\text{e}}}$$

$$\frac{1}{\mu_{\text{ion}}} = \sum_i \frac{X_i}{A_i}$$

$$\frac{1}{\mu_{\text{e}}} = \sum_i \frac{X_i}{A_i} Z_i \ \ y_i$$

μ: average mass of a <u>free</u> particle (in m_H units)
μ_{ion}: average mass of the ions only (in m_H units)
μ_{e}: total mass divided by the number of free electron (in m_H units)

A few questions for you to think about:

1. If you know the total number of free particles (/volume), how do you get the density of the gas?
2. If you know the total number of free electrons (/volume), how do you get the density of the gas?
3. If the gas is entirely neutral, what happens to μ_{e} (it is **not** zero)? What are the consequences on the calculation of μ?
4. If the gas is entirely neutral, how do μ and μ_{ion} relate to each other?

<u>In notebook</u>: let's calculate the mean molecular weight at the surface of the Sun

In summary:

$$\frac{1}{\mu} = \frac{1}{\mu_{\rm ion}} + \frac{1}{\mu_{\rm e}}$$

$$\frac{1}{\mu_{\rm ion}} = \sum_i \frac{\boxed{X_i}}{A_i}$$

$$\frac{1}{\mu_{\rm e}} = \sum_i \frac{\boxed{X_i}}{A_i} Z_i \boxed{y_i}$$

μ: average mass of a <u>free</u> particle (in m_H units)
$\mu_{\rm ion}$: average mass of the ions only (in m_H units)
$\mu_{\rm e}$: total mass divided by the number of free electron (in m_H units)

Our goal is to relate $n(r)$ with $\rho(r)$

$$P(r) = n(r) \; kT(r)$$

$$n(r) = \frac{\rho(r)}{\mu(r) m_H}$$

We usually assume an (initial) abundance
(so known X_i)

During stellar evolution: $X_i \rightarrow X_i(r)$
(and also a function of time.. but remember the timescales)

Ionization depends on temperature: $y_i = y_i(r)$
(Topic of next lecture!)

In the textbooks:

We can calculate μ by using values for all of the X_i like we did for the notebook
(albeit for the completely ionized and completely neutral cases)

We can also get a good estimate by using an <u>approximation</u> that expresses μ with $X = X_{\mathrm{H}}$,
$Y = X_{\mathrm{He}}$, and $Z = X_{\text{metals=everything else}}$.

$$\mu_{\mathrm{neutral}} = \frac{1}{X + \frac{Y}{4}}$$

$$\mu_{\mathrm{ionized}} = \frac{2}{3X + \frac{Y}{2} + 1}$$