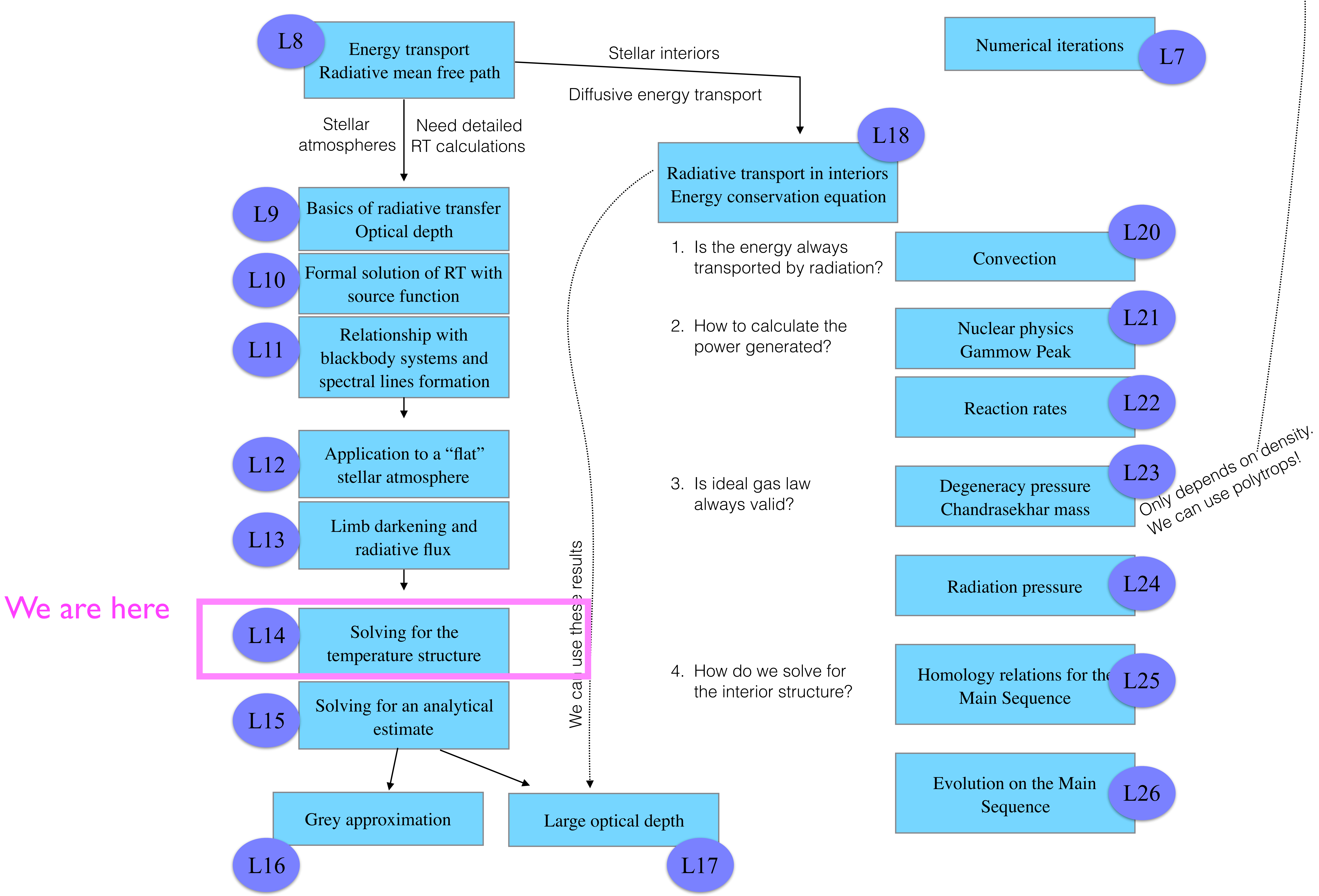
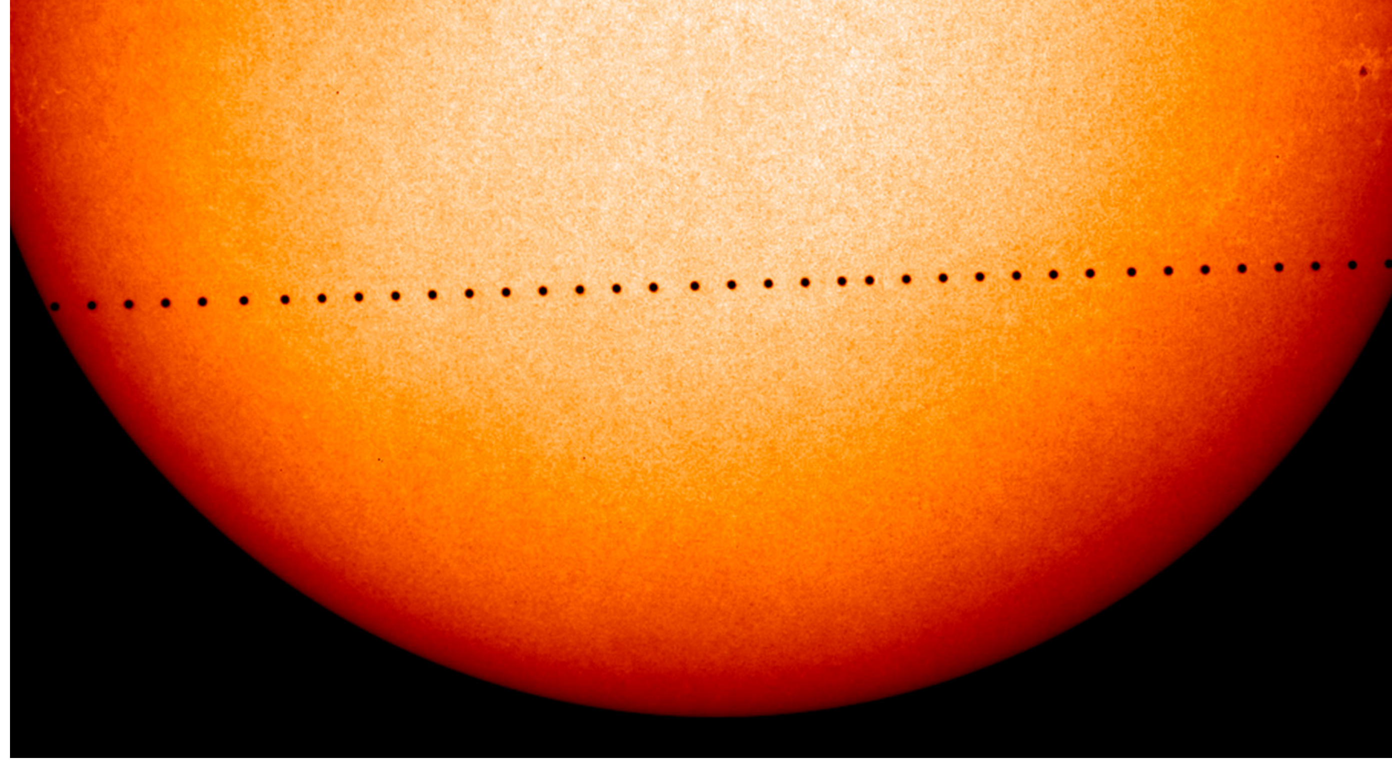
Week 8 Thursday
L-14
Flux in flat atmospheres
(Solving for T, yah!)



L8
Energy transport
Radiative mean free path
Stellar interiors
Diffusive energy transport
Numerical iterations
L7
Stellar atmospheres
Need detailed RT calculations
L18
Radiative transport in interiors
Energy conservation equation
L9
Basics of radiative transfer
Optical depth
L10
Formal solution of RT with source function
L11
Relationship with blackbody systems and spectral lines formation
1. Is the energy always transported by radiation?
L20
Convection
2. How to calculate the power generated?
L21
Nuclear physics
Gammow Peak
L22
Reaction rates
L12
Application to a “flat” stellar atmosphere
L13
Limb darkening and radiative flux
3. Is ideal gas law always valid?
L23
Degeneracy pressure
Chandrasekhar mass
Only depends on density.
We can use polytrops!
L24
Radiation pressure
We are here
L14
Solving for the temperature structure
4. How do we solve for the interior structure?
Homology relations for the Main Sequence
L25
L15
Solving for an analytical estimate
Evolution on the Main Sequence
L26
Grey approximation
L16
Large optical depth
L17

Intensity for a flat, semi-infinite atmosphere



Reminder

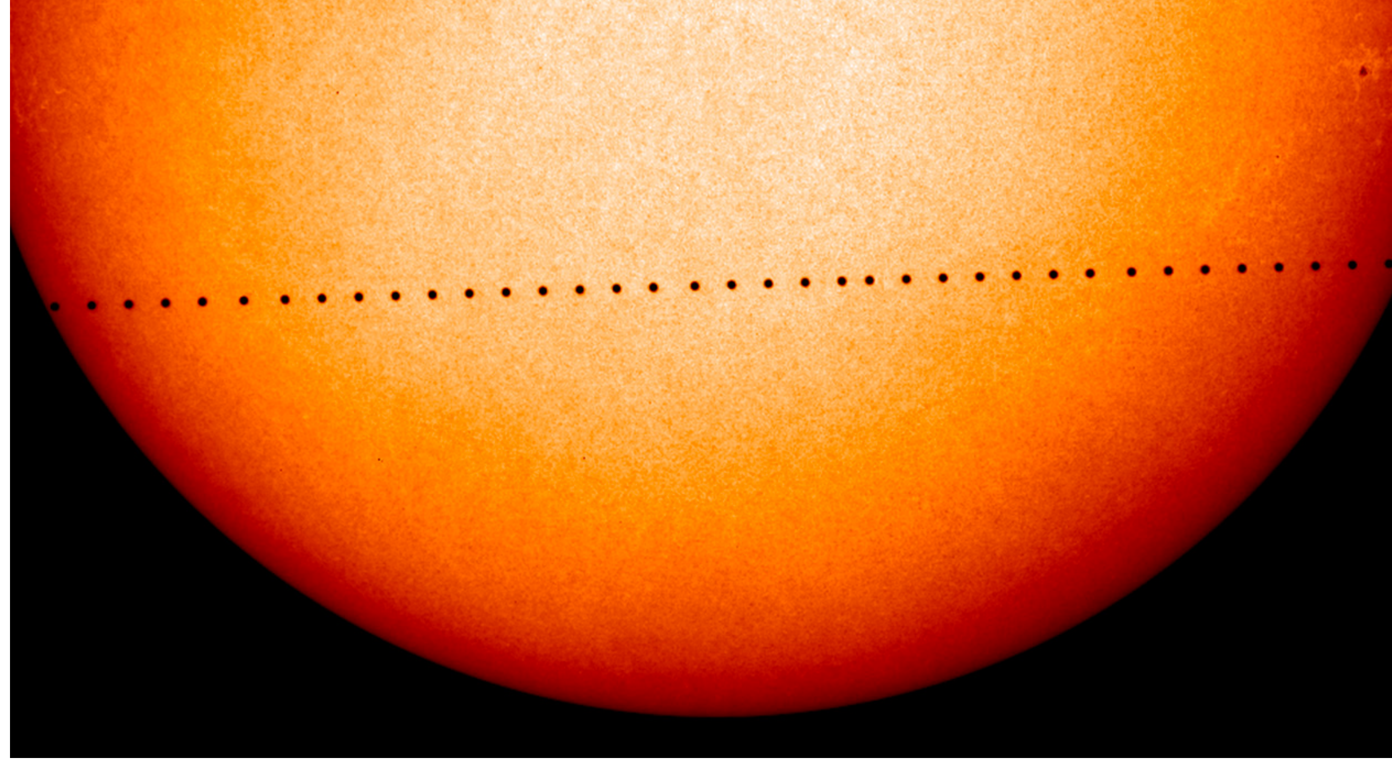
out

$$I(\tau_z, u > 0) = \int_{\tau_z'=\tau_z}^{\tau_z'=\infty} S(\tau_z')\ \mathrm{e}^{\frac{\tau_z - \tau_z'}{u}}\ d\tau_z$$

in

$$I(\tau_z, u < 0) = \int_{\tau_z'=\tau_z}^{\tau_z'=0} S(\tau_z')\ \mathrm{e}^{\frac{\tau_z - \tau_z'}{u}}\ d\tau_z$$

Intensity for a flat, semi-infinite atmosphere



Reminder

out

$$I(\tau_z = 0, u > 0) = \int_{\tau_z'=\tau_z}^{\tau_z'=\infty} \boxed{S(\tau_z')}\, e^{\frac{\tau_z - \tau_z'}{u}}\, d\tau_z$$

Means we can find $S(\tau)$ (and $T(z)$)

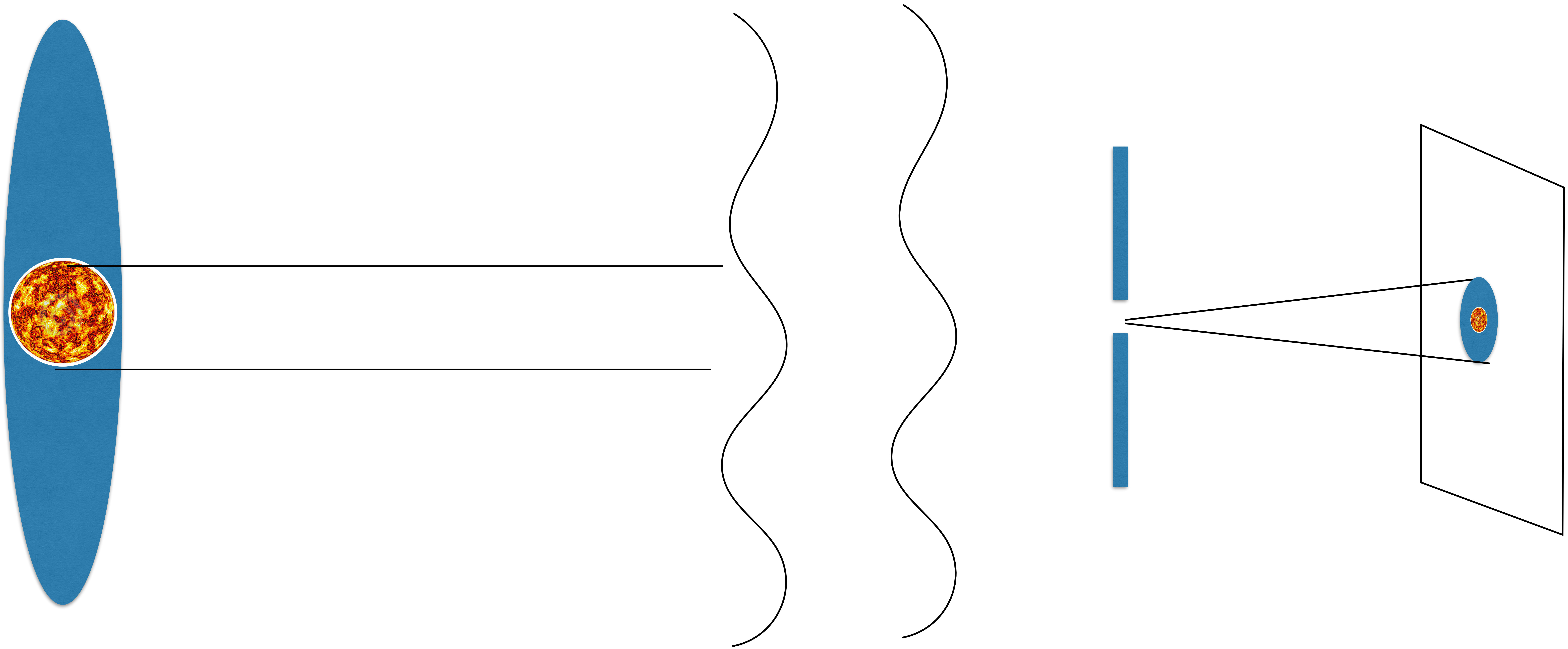
in

$$I(\tau_z = 0, u < 0) = \int_{\tau_z'=\tau_z}^{\tau_z'=0} \boxed{S(\tau_z')}\, e^{\frac{\tau_z - \tau_z'}{u}}\, d\tau_z$$

For the Sun, we can measure!

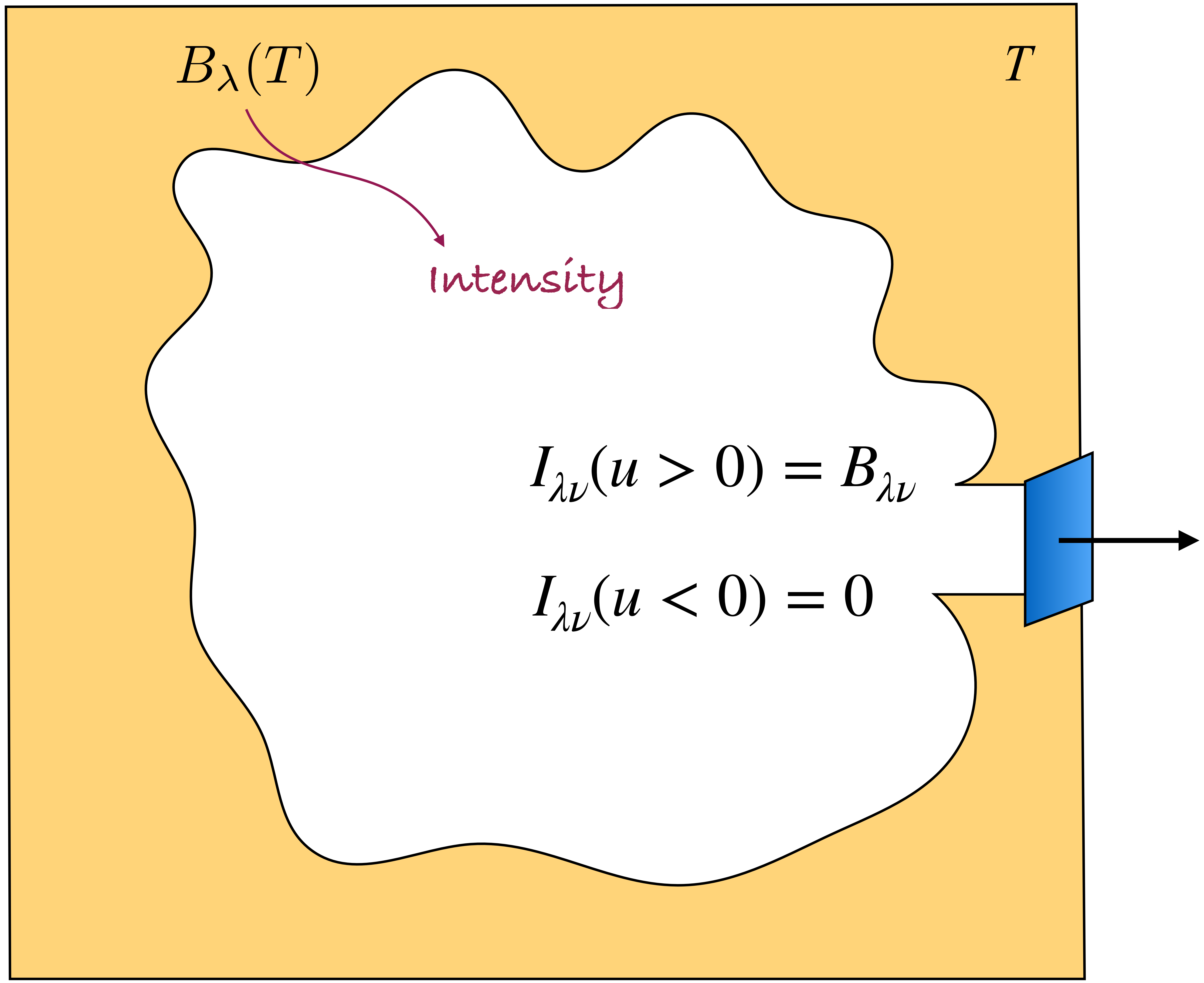
For other stars, our telescope cannot resolve their stellar disk
= we measure the surface flux.
So we cannot use limb-darkening to find T(z)

Reminder



Flux through a surface in a blackbody radiation field

Reminder



$$F_{\lambda\nu} = 2\pi \int_{\cancel{-1}\,0}^{+1} B_{\lambda\nu}\, u du$$

$$= 2\pi B_{\lambda\nu} \int_{0}^{+1} u du$$

$$= \pi B_{\lambda\nu}$$

$$F_{\text{tot}} = \pi \int_0^\infty B_{\lambda\nu} d\nu$$

$$= \pi \int_0^\infty \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu$$

$$= \frac{2\pi^5 k^4}{15 h^3 c^2} T^4$$

$$= \sigma T^4$$

The ‘effective’ temperature $T_{\rm eff}$

Reminder

The temperature of a BB that has
the same surface (wavelength-integrated) flux
as the star

$$F_{\lambda\nu} = 2\pi \int_{\cancel{-1}\;0}^{+1} B_{\lambda\nu}\, u du$$

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$$= \pi B_{\lambda\nu}$$

$$F_{\rm tot} = \pi \int_0^\infty B_{\lambda\nu} d\nu$$

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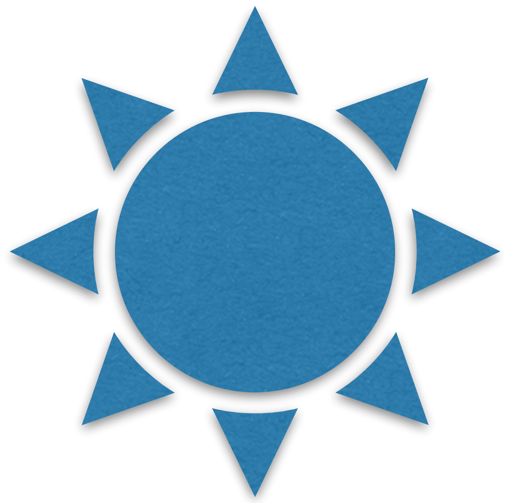
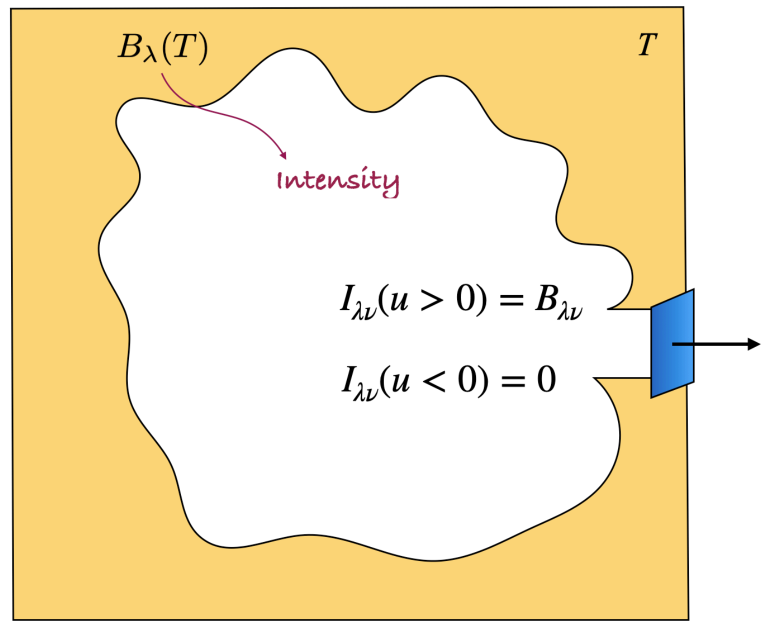
$$= \frac{2\pi^5 k^4}{15 h^3 c^2} T^4$$

$$= \sigma T^4$$

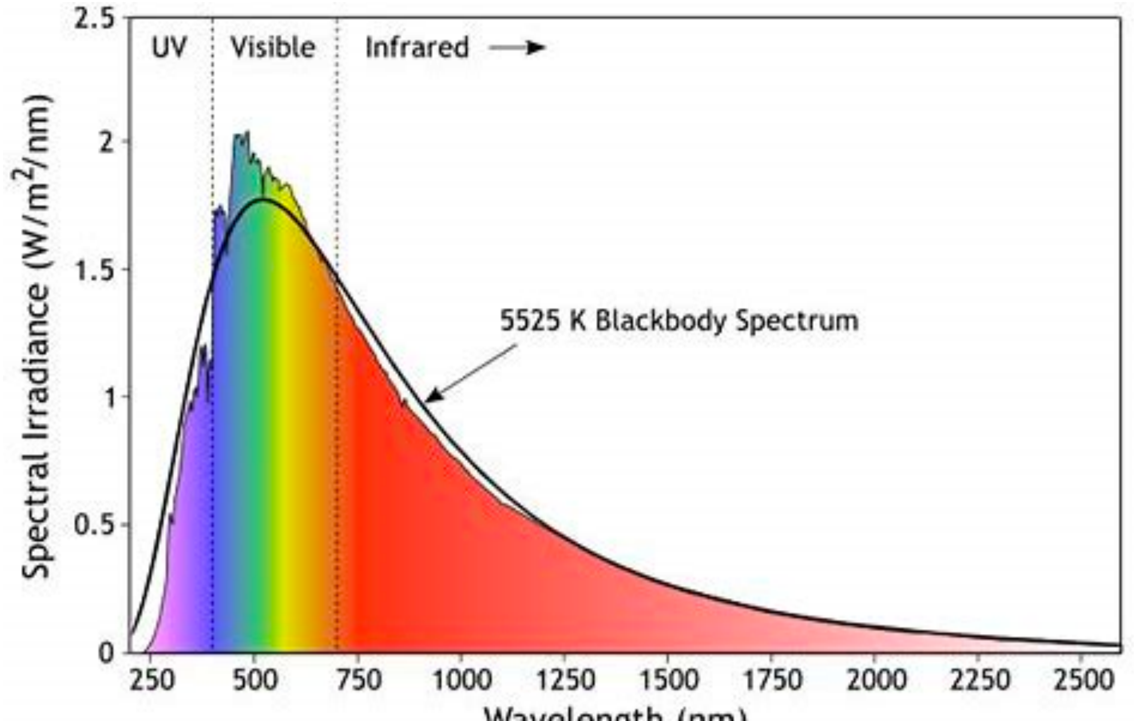
For the Sun: 63 MegaWatt / m² $\longrightarrow$ **For a BB to have 63 MegaWatt / m²**





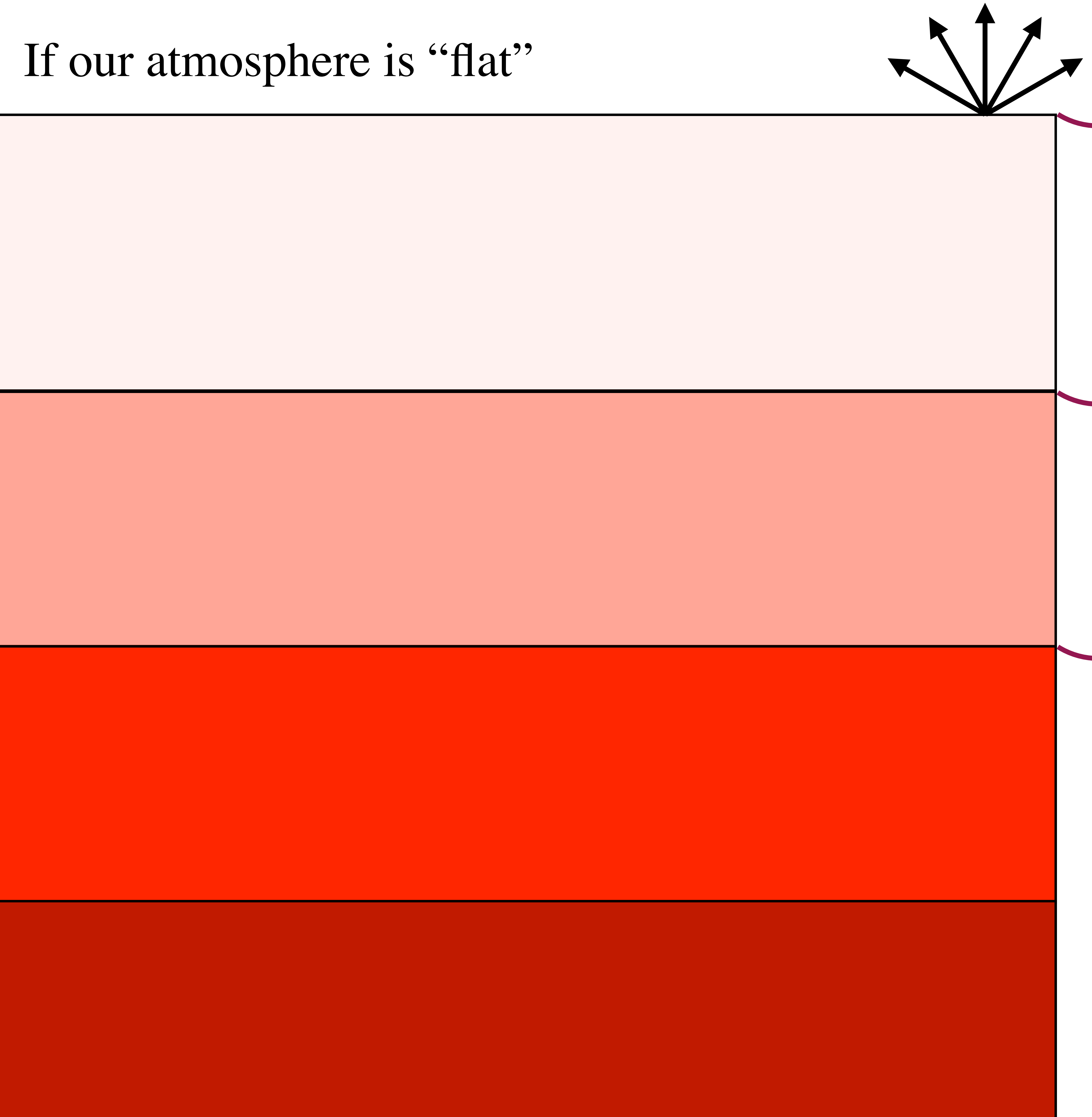
Thus $T_{\rm eff}$ for the sun is ~ 5700 K. $\longleftarrow$ **T needs to be ~ 5700 K**

Note: this does NOT mean that
$F_{\lambda,\odot} = B_\lambda(T = 5700K)$



The (wavelength-integrated) flux is <u>Power per area</u>

If our atmosphere is "flat"



Flux through that surface $= \sigma T_{eff}^4$

For the Sun: 63 MegaWatt / m² <u>upward</u>

Flux through that surface also

Flux through that surface also

$$\frac{dF}{dz} = 0$$

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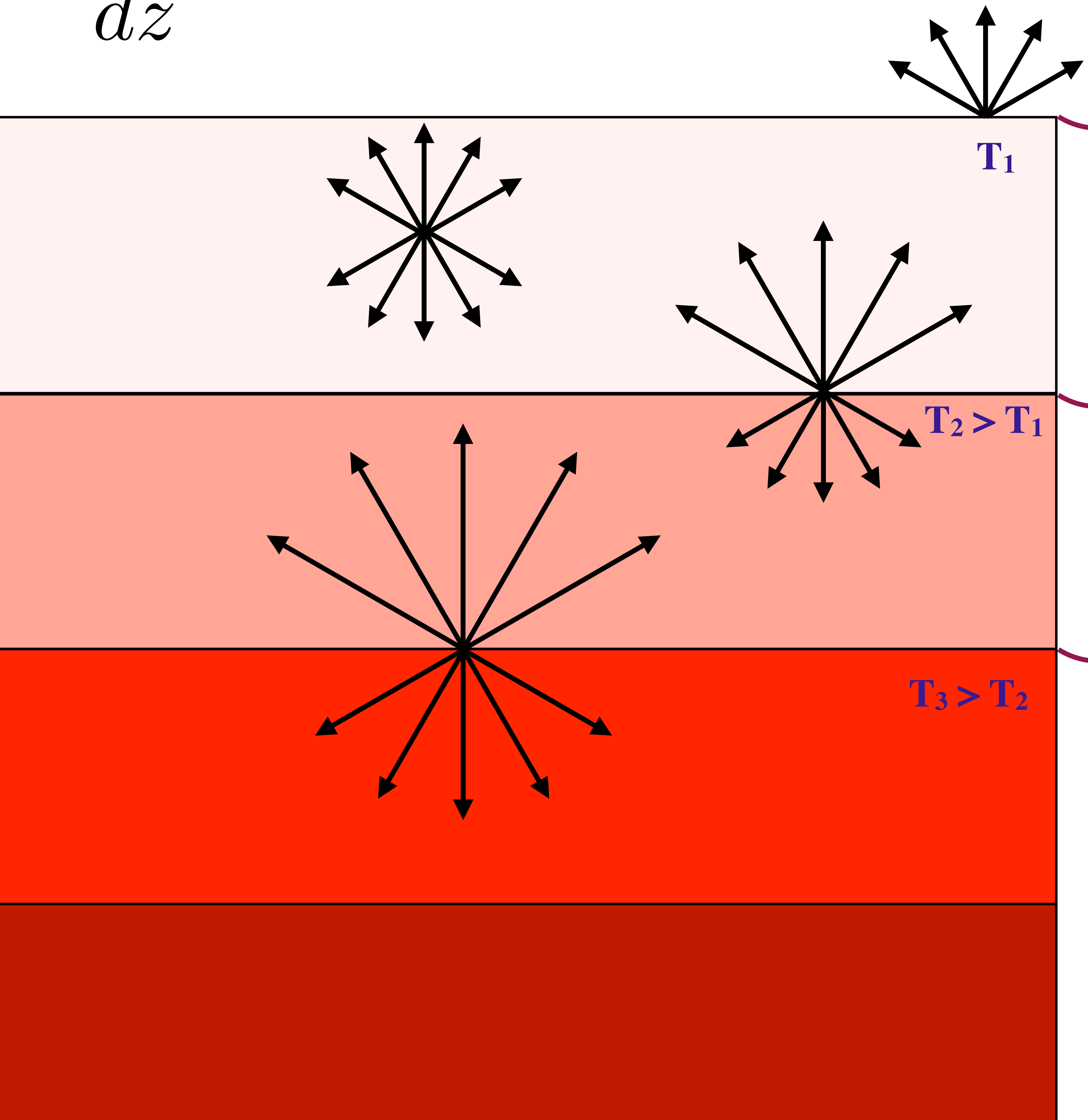


Flux through that surface = σT_{eff}^4

For the Sun: 63 MegaWatt / m² upward

Flux through that surface also

Flux through that surface also

So we need to relate the flux to source function.

Spherical, with azimuthal symmetry ($u = \cos\theta$)

General

Flux

$$F_\lambda = \int I_\lambda \cos\theta \, d\Omega$$

$$\frac{F_\lambda}{4\pi} = \frac{1}{2}\int_{-1}^{+1} I_\lambda(u)\ u\ du$$

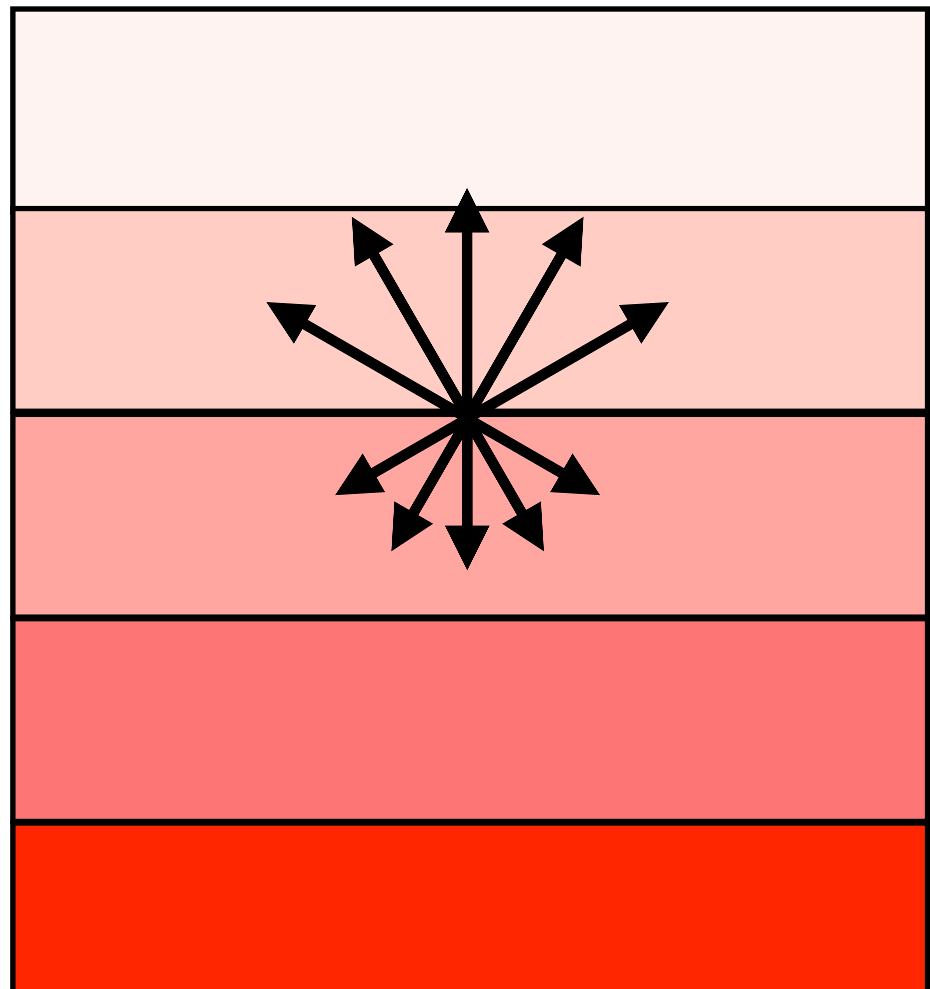
Solution for a flat, semi-infinite atmosphere:

out

$$I(\tau_z, u > 0) = \int_{\tau_z'=\tau_z}^{\tau_z'=\infty} S(\tau_z')\ e^{\frac{\tau_z - \tau_z'}{u}}\ d\tau_z$$

in

$$I(\tau_z, u < 0) = \int_{\tau_z'=\tau_z}^{\tau_z'=0} S(\tau_z')\ e^{\frac{\tau_z - \tau_z'}{u}}\ d\tau_z$$



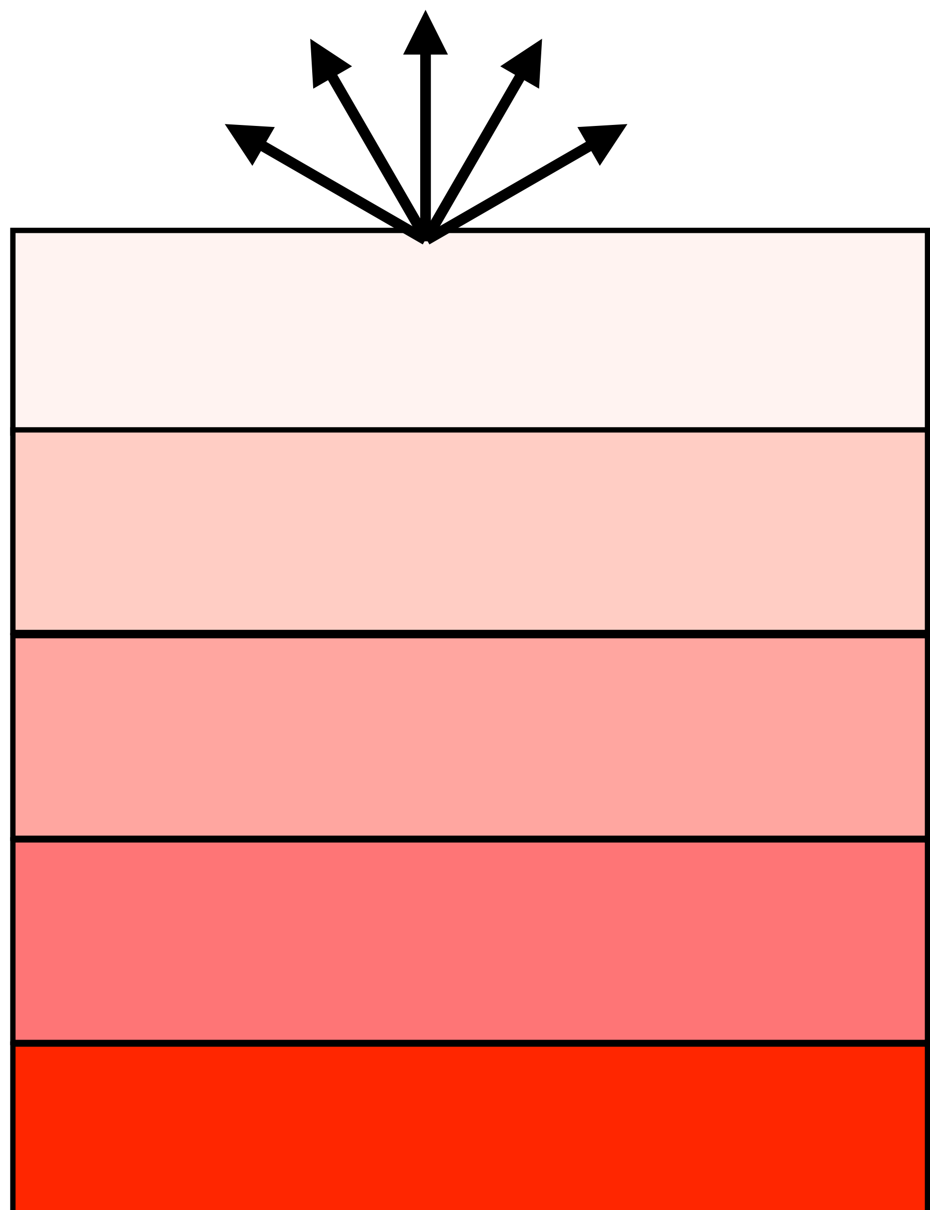
First a parenthesis: For the 'surface', the Eddington-Barbier relation

	General	**Spherical, with azimuthal symmetry ($u = \cos\theta$)**
Flux	$F_\lambda = \int I_\lambda \cos\theta d\Omega$	$\frac{F_\lambda}{4\pi} = \frac{1}{2}\int_{-1}^{+1} I_\lambda(u)\ u\ du$

Solution for a flat, semi-infinite atmosphere:

out

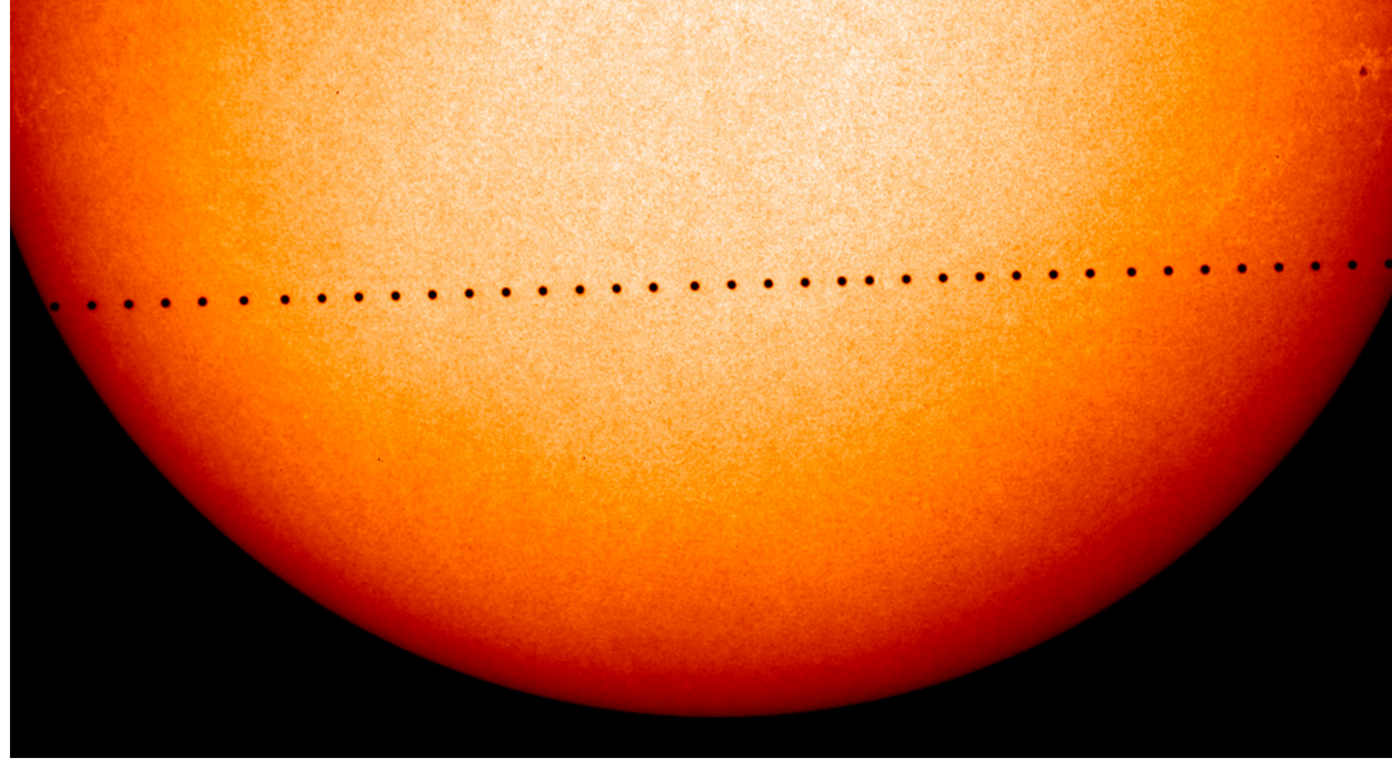
$$I_{\text{Obs}}(u > 0) \quad = \quad I(\tau_z = 0, u > 0) = \int_{\tau_z'=\tau_z}^{\tau_z'=\infty} S(\tau_z')\ \mathrm{e}^{\frac{\tau_z - \tau_z'}{u}}\ d\tau_z$$



Let's make the approximation that
S increases linearly with τ_z

Intensity for a flat, semi-infinite atmosphere

Reminder



$$S(\tau_z') = S_0 + S_1\ \tau_z'$$

5. At home: Formal solution with source functio increases linearly with optical depth

Let's assume that the density in the slab is constant, such that $\kappa\rho = 2.0$ per un

The source function is a function of τ such that:

$$S(\tau) = S_0 + S_1\tau$$

where $S_0 = 0.5$ intensity unit, and $S_1 = 1.3$ intensity units per optical depth un

There is no intial intensity entering the slab so $I_o = 0$.

Prepare your code such that you can vary the values of the paramters.

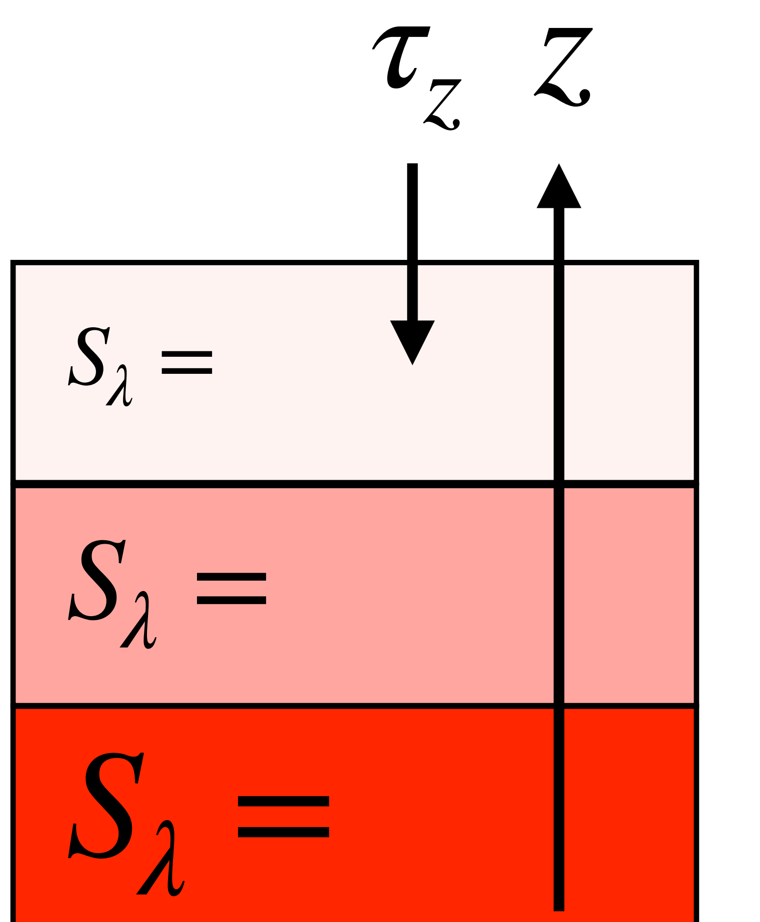
$$I_{\rm Obs}(u > 0) \quad = \quad I(\tau_z = 0, u > 0) = \int_{\tau_z'=\tau_z}^{\tau_z'=\infty} S(\tau_z')\ e^{\frac{\tau_z - \tau_z'}{u}}\ d\tau_z$$

Let's make the approximation that S increases linearly with τ_z

… math here :)

$$I(\tau_z = 0,\ u > 0) = S_0 + S_1\ u$$

$$= S(\tau_z = u)$$

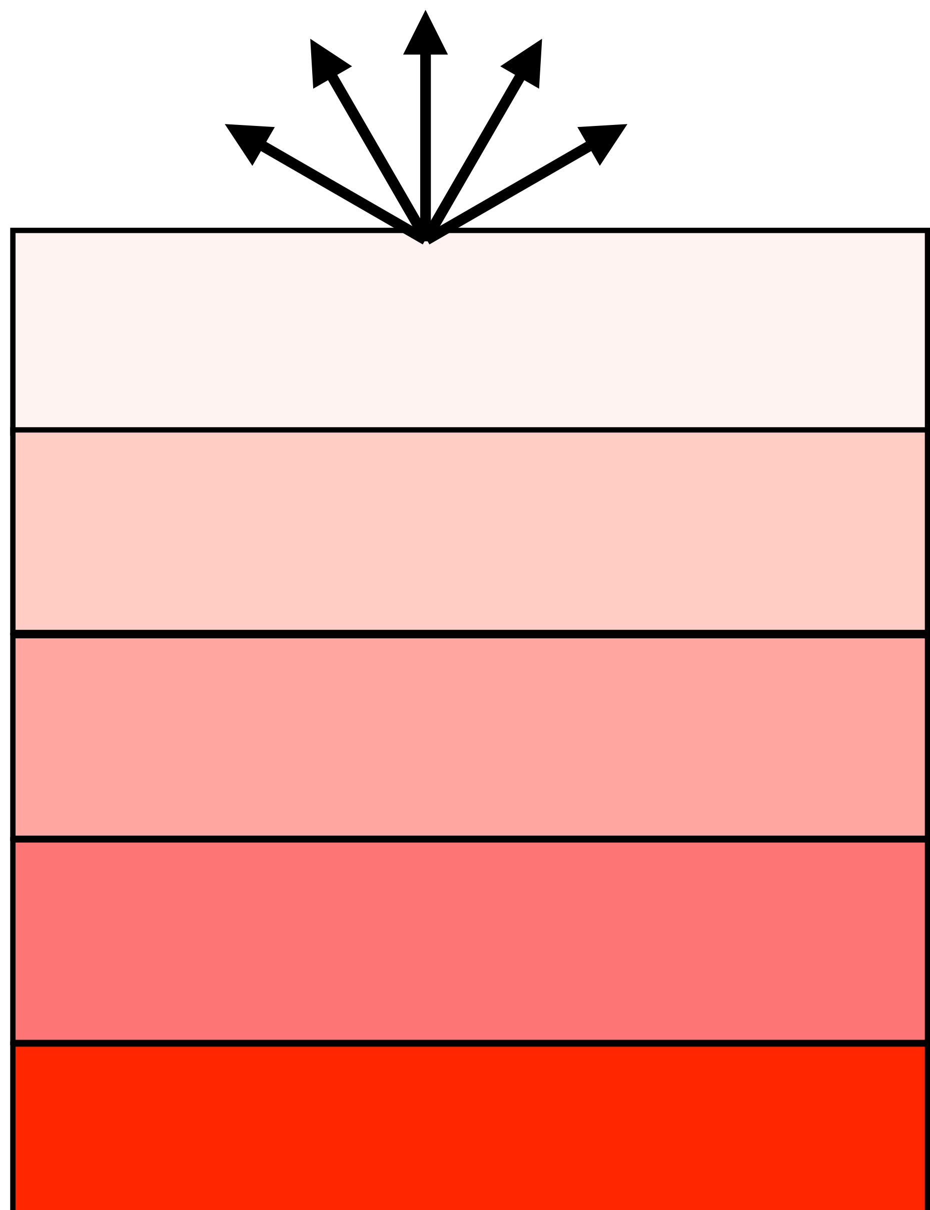


So the intensity for a given u ray is equal to the value of the source function S at the layer where the vertical optional depth τ_z is equal to u

First a parenthesis: For the ‘surface’, the Eddington-Barbier relation

On the board

The surface flux is a measure of the value of the source function at an optical depth of 2/3.



So we need to relate the flux to source function.

Spherical, with azimuthal symmetry (u = cos θ)

General

Flux

$$F_\lambda = \int I_\lambda \cos\theta \, d\Omega$$

$$\frac{F_\lambda}{4\pi} = \frac{1}{2}\int_{-1}^{+1} I_\lambda(u)\ u\ du$$

Put the solution for $I(\tau_z, u)$ in the flux equation. On the board

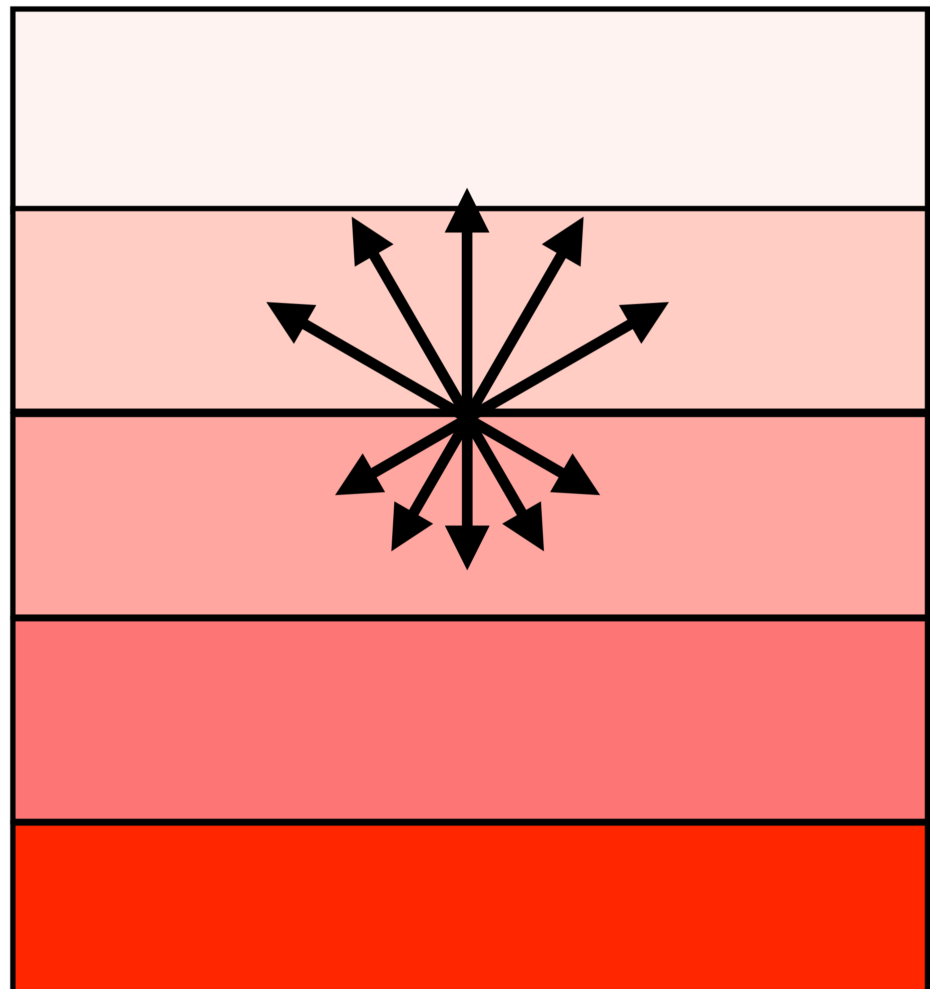
Solution for a flat, semi-infinite atmosphere:

out

$$I(\tau_z, u > 0) = \int_{\tau_z'=\tau_z}^{\tau_z'=\infty} S(\tau_z')\ \mathrm{e}^{\frac{\tau_z - \tau_z'}{u}}\ d\tau_z$$

in

$$I(\tau_z, u < 0) = \int_{\tau_z'=\tau_z}^{\tau_z'=0} S(\tau_z')\ \mathrm{e}^{\frac{\tau_z - \tau_z'}{u}}\ d\tau_z$$



So we need to relate the flux to source function.

On the board:

Step1: Separate the integral in the flux equation into two pieces

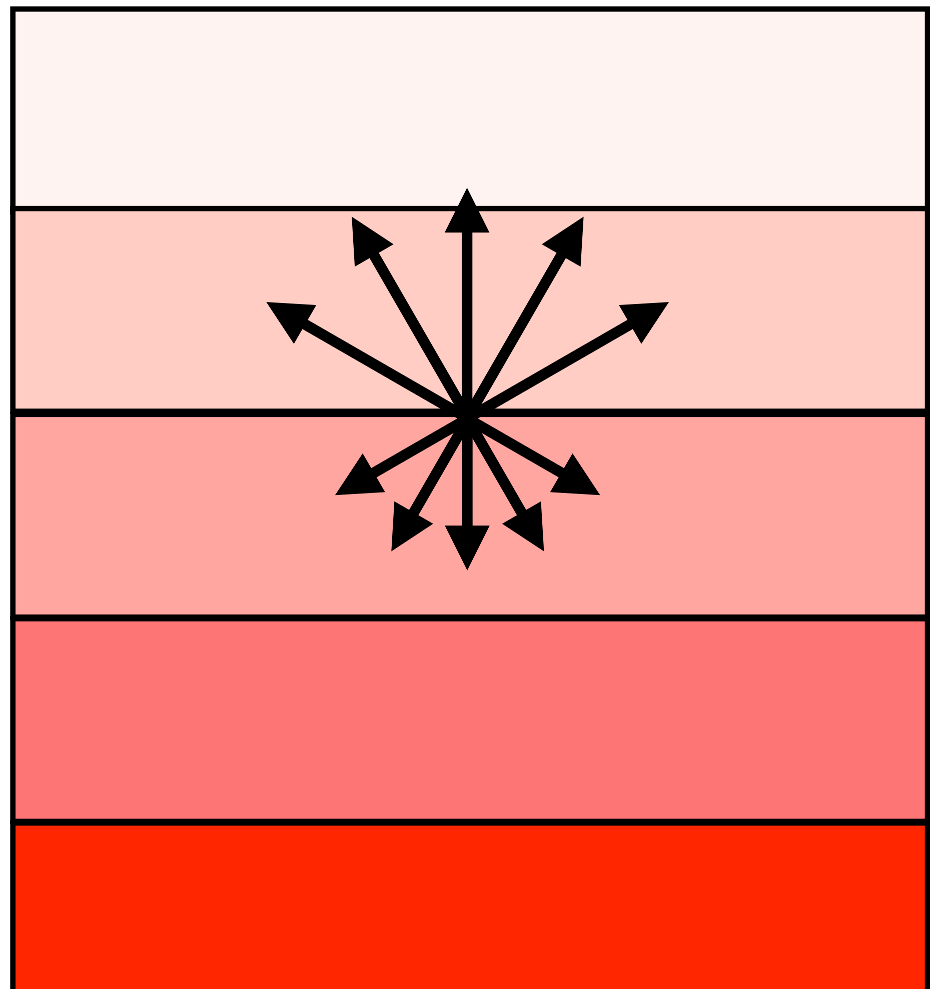
Step2: Switch the u and the τ_z integrals

Step3: Assume that the source function is isotropic (not a function of u)

Step4: Make a substitution in the 1st term for $a = (\tau_z' - \tau_z)$, and $x = 1/u$
Step5: Make a substitution in the 2nd term for $a = (\tau_z - \tau_z')$, and $x = -1/u$

Step6: Replace the x integral for "exponential functions"

$$\frac{F_\lambda(\tau)}{4\pi} = -\frac{1}{2}\int_{\tau'=0}^{\tau} S_\lambda(\tau')E_2(\tau-\tau')d\tau' + \frac{1}{2}\int_{\tau'=\tau}^{\infty} S_\lambda(\tau')E_2(\tau'-\tau)d\tau'$$



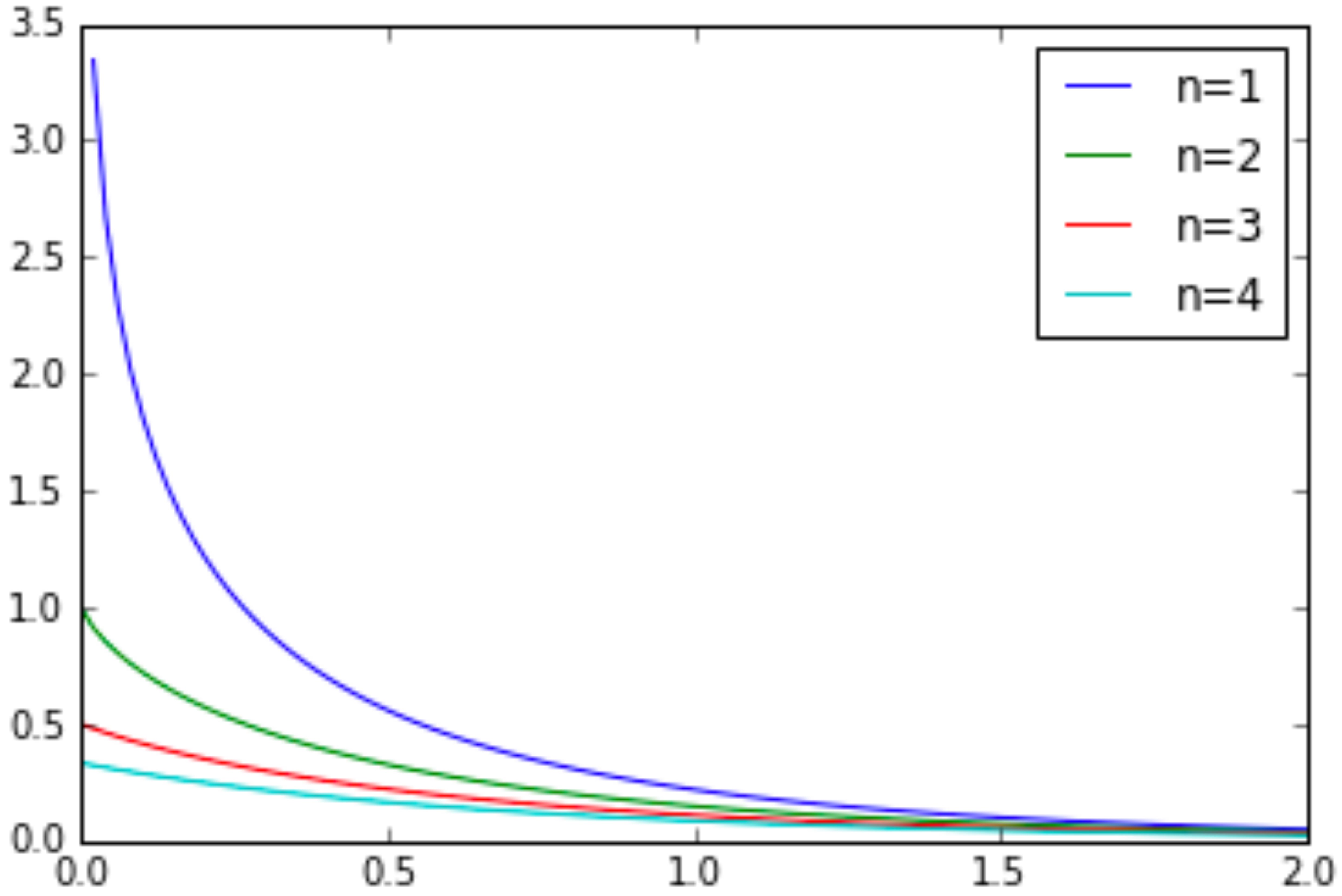
Exponential functions [python: scipy.special.expn(n, a)]

$$E_n(a) = \int_1^\infty \frac{\mathrm{e}^{-ax}}{x^n} dx$$

$$E_n(a=0) = \frac{1}{n-1}$$

$$\frac{dE_n(a)}{da} = -E_{n-1}(a)$$

$$nE_{n+1}(a) = \mathrm{e}^{-a} - aE_n(a)$$



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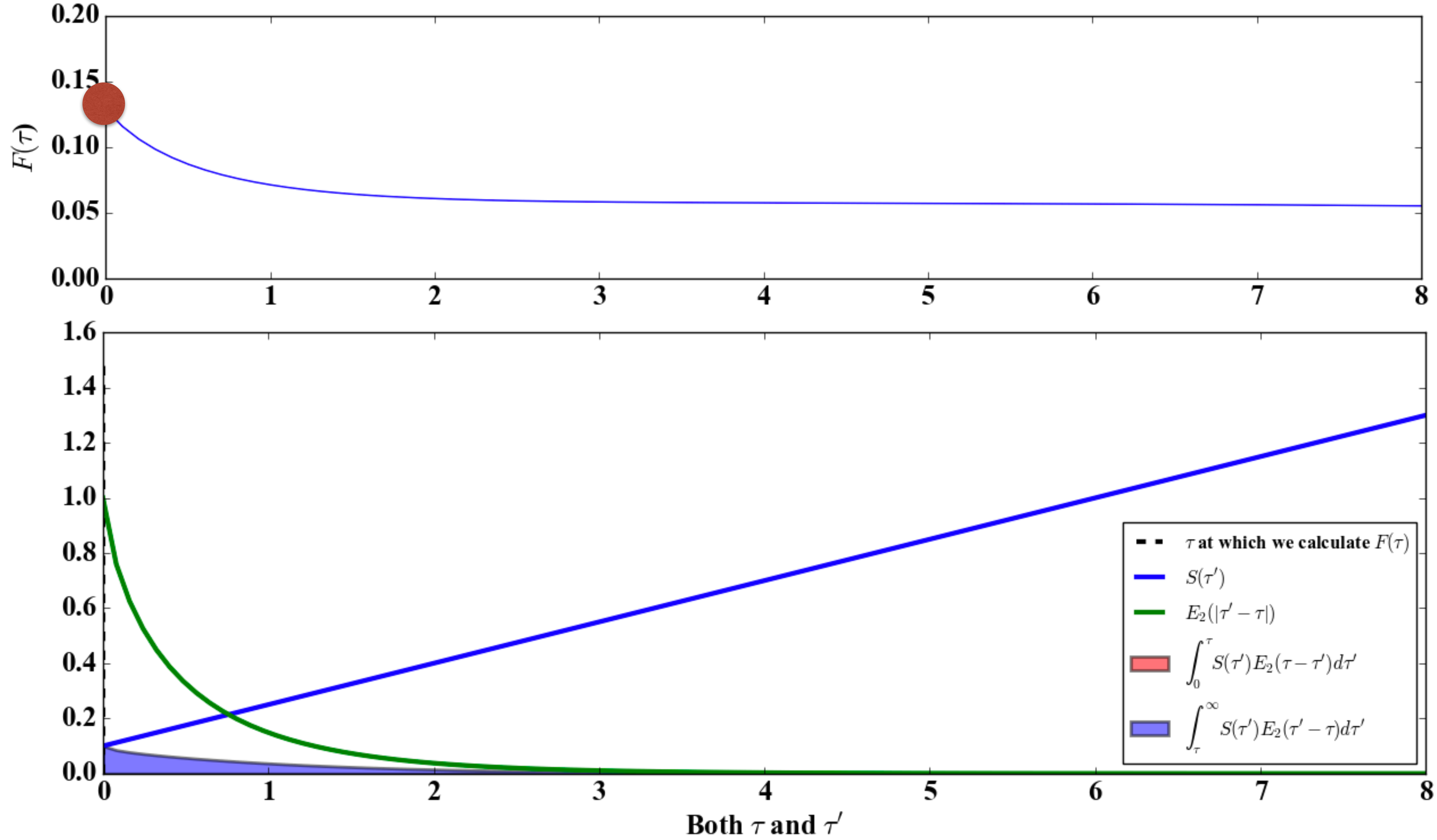

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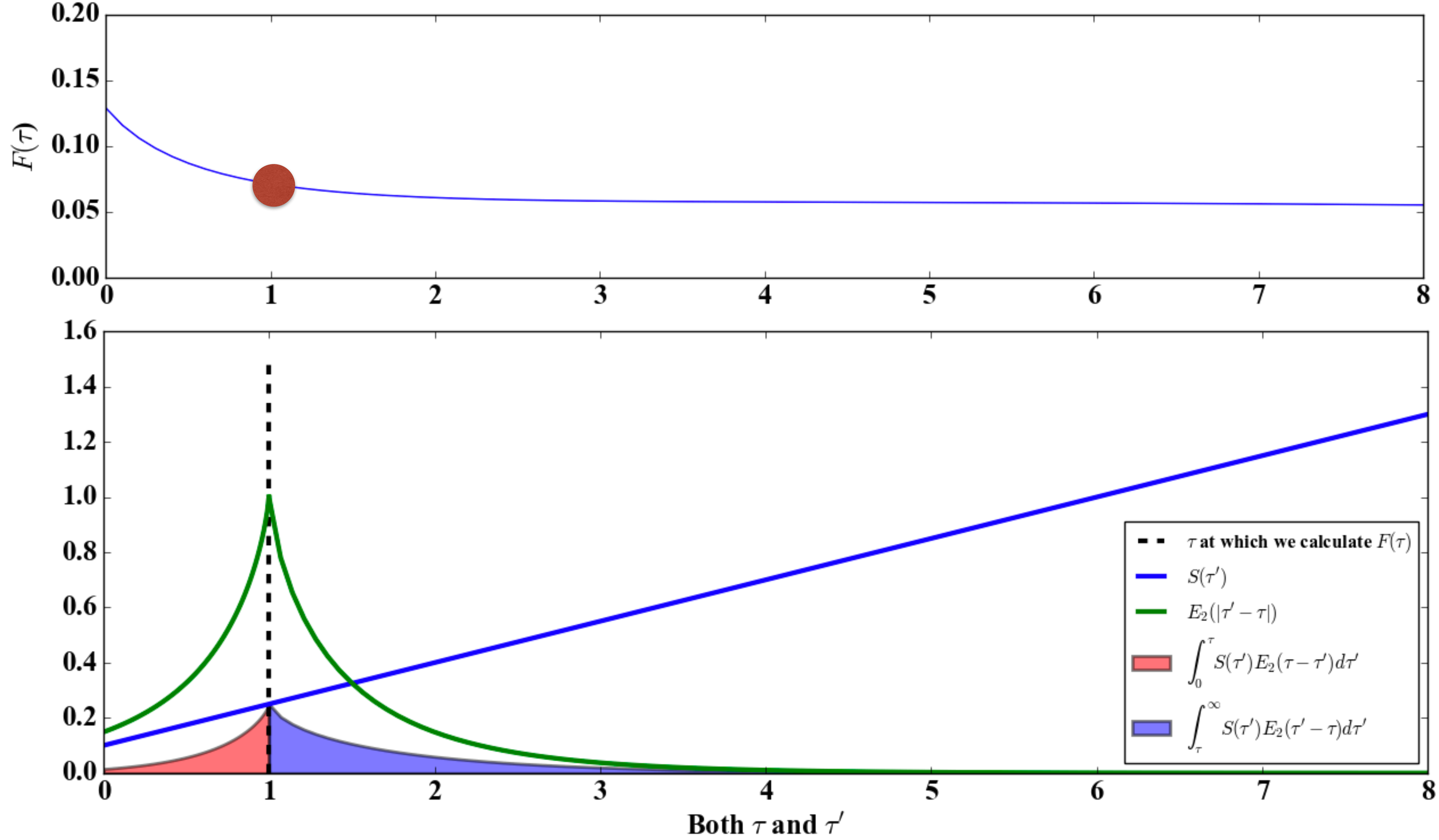

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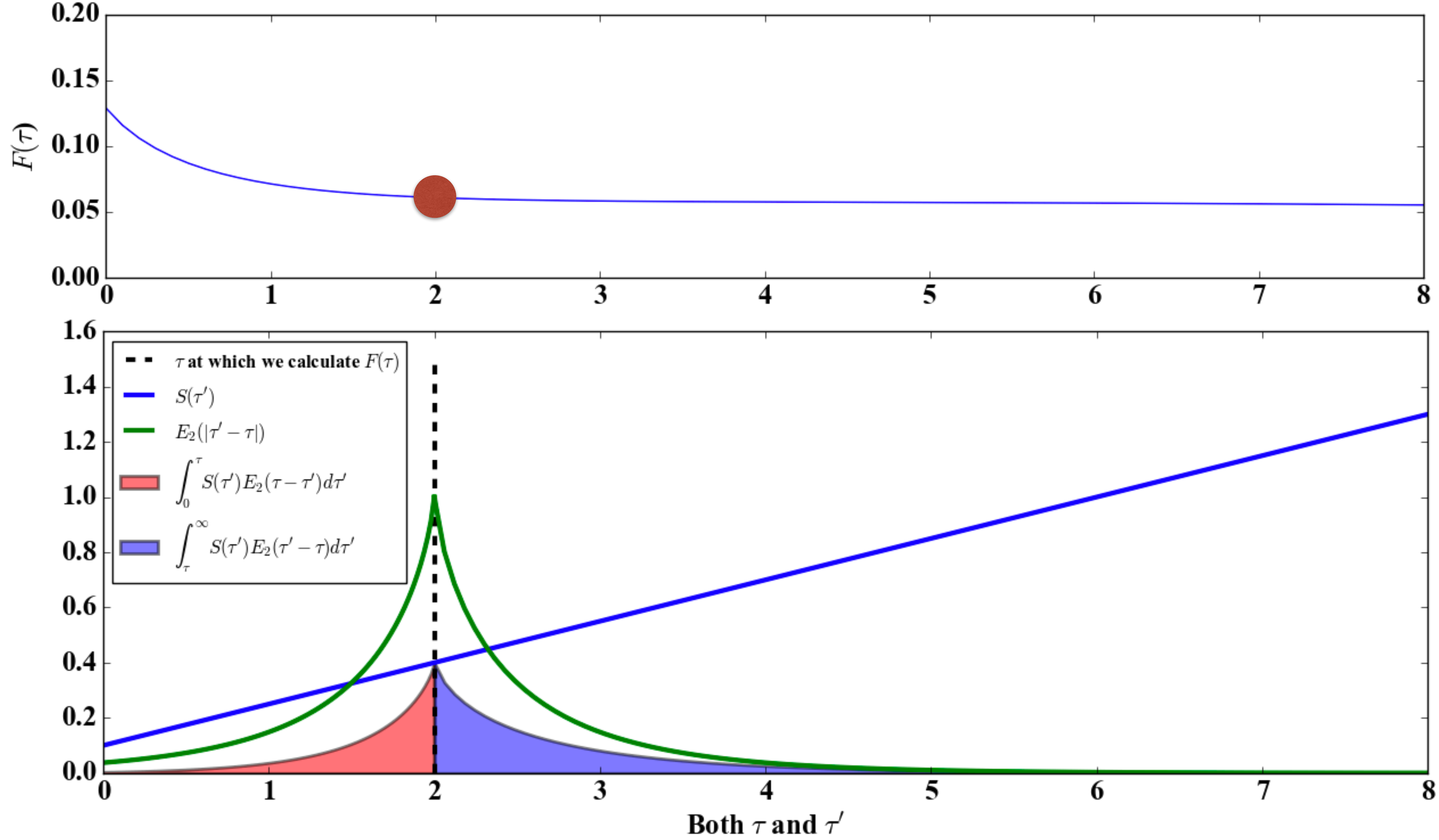

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2 $$F(\lambda,\tau_\lambda) = 2\pi \int_{\tau'=\tau}^{\tau'=\infty} S(\lambda,\tau'_\lambda)E_2(\tau'_\lambda-\tau_\lambda)d\tau'_\lambda - 2\pi \int_{\tau'=0}^{\tau'=\tau} S(\lambda,\tau'_\lambda)E_2(\tau_\lambda-\tau'_\lambda)d\tau'_\lambda$$ $S(\lambda,\tau_\lambda)$

$z(\tau_\lambda)$ $T(z)$

3 $$S(\lambda,\tau_\lambda) = B(\ \lambda, T(z(\tau_\lambda))\ )$$

$\kappa_\lambda(z)$ $\rho(z)$

4 $$d\tau_\lambda(z) = -\kappa_\lambda(z)\rho(z)dz$$

5 $$\kappa_\lambda(z) = f(\text{composition}, \text{T(z)})$$ composition

$P(z)$ $\mu(z)$

6 $$P(z) = \frac{\rho(z)kT(z)}{\mu(z)m_H}$$

7 $$\mu(z) = f(\text{composition}, \text{T(z)}, P(z))$$

8 $$\frac{dP(z)}{dz} = -g(z)\rho(z)$$ $g(z)$

9 $$g(z) \simeq g_\star$$ $g_\star$

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