Week 5 Thursday

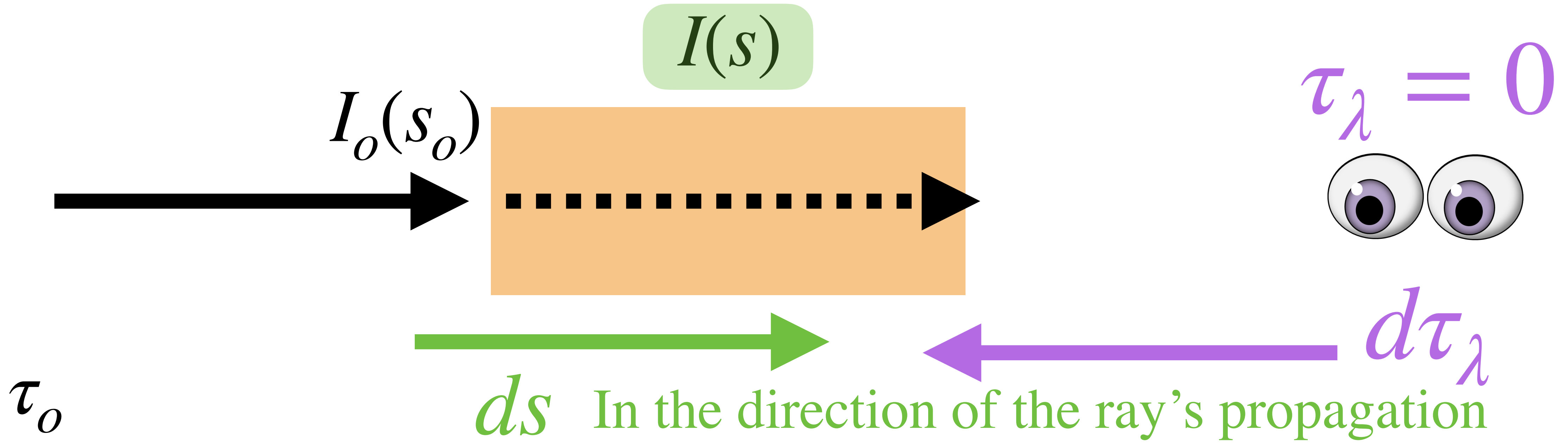
L-10

Source function

L8
Energy transport
Radiative mean free path
Stellar interiors
Diffusive energy transport
Numerical iterations
L7
Stellar atmospheres
Need detailed RT calculations
L18
Radiative transport in interiors
Energy conservation equation
L9
Basics of radiative transfer
Optical depth
We are here
L10
Formal solution of RT with source function
L11
Relationship with blackbody systems and spectral lines formation
L12
Application to a "flat" stellar atmosphere
L13
Limb darkening and radiative flux
L14
Solving for the temperature structure
L15
Solving for an analytical
We can use these results
1. Is the energy always transported by radiation?
L20
Convection
2. How to calculate the power generated?
L21
Nuclear physics
Gammow Peak
L22
Reaction rates
3. Is ideal gas law always valid?
L23
Degeneracy pressure
Chandrasekhar mass
Only depends on density.
We can use polytrops!
L24
Radiation pressure
4. How do we solve for the interior structure?
L25
Homology relations for the Main Sequence

	General	**Spherical, with azimuthal symmetry (u = cos θ)**
Mean intensity	$J_\lambda = \frac{1}{4\pi}\int I_\lambda d\Omega$	$J_\lambda = \frac{1}{2}\int_{-1}^{+1} I_\lambda(u)\ du$
Flux	$F_\lambda = \int I_\lambda \cos\theta d\Omega$	$\frac{F_\lambda}{4\pi} = \frac{1}{2}\int_{-1}^{+1} I_\lambda(u)\ u\ du$

Change in intensity



Optical depth

$$d\tau_\lambda = -\kappa_\lambda \rho ds$$

$$\tau_\lambda(s) - 0 = -\int_{s_{\text{obs}}}^{s} \kappa_\lambda(s)\ \rho(s)\ ds$$

Absorption

$$dI_\lambda = -\kappa_\lambda(s)\rho(s)dsI_\lambda$$

General Solution

$$\tau(s) = \int_{s'=s}^{s'=\infty} \kappa(s')\rho(s')ds'$$

$$I(s) = I_o e^{-\int_{s'=s_o}^{s'=s} \kappa(s')\rho(s')ds}$$

$$I(s) = I_o e^{\tau(s) - \tau(s=s_o)}$$

Constant properties in material

$$\tau(s) = \kappa\rho((s_o + d) - s)$$

$$I(s) = I_o e^{-\kappa\rho(s-s_o)}$$

Emission: how can we gain intensity?

1. Pure emission: create a photon

2. Scattering: deflect a photon

(Q: what are the units of κ_λ?)

Absorption: $$dI_\lambda = - \rho \; ds \; \kappa_\lambda \; I_\lambda$$

(Q: what is the meaning of $\rho \; ds \; \kappa_\lambda$?)

Emission:

Emission: how can we gain intensity?

1. Pure emission: create a photon

2. Scattering: deflect a photon

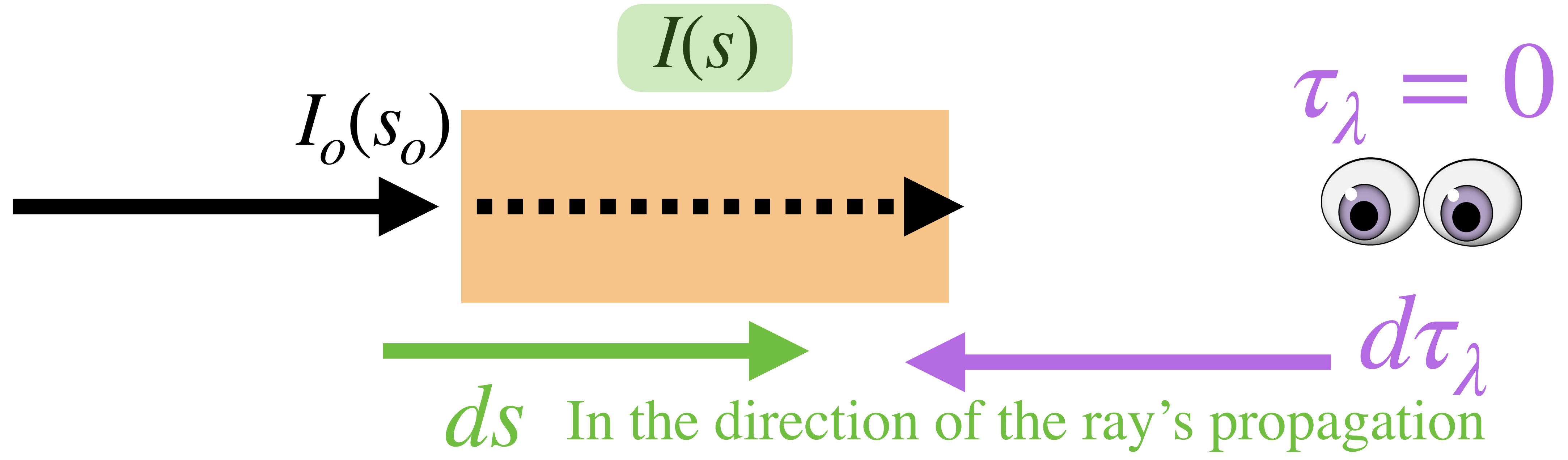
Absorption: $$dI_\lambda = - \rho \, ds \, \kappa_\lambda \, I_\lambda$$

(Q: why no I_λ?)

Emission: $$dI_\lambda = + \rho \, ds \, j_\lambda$$

(Q: what are the units of j_λ?)

Let's solve this equation, if we know $I(s_o) = I_o$



$$dI_\lambda = +j_\lambda(s)\ \rho(s)\ ds$$

Together on the board…

$$I(s) = I_o + \int_{s_o}^{s} j(s)\rho(s)ds$$

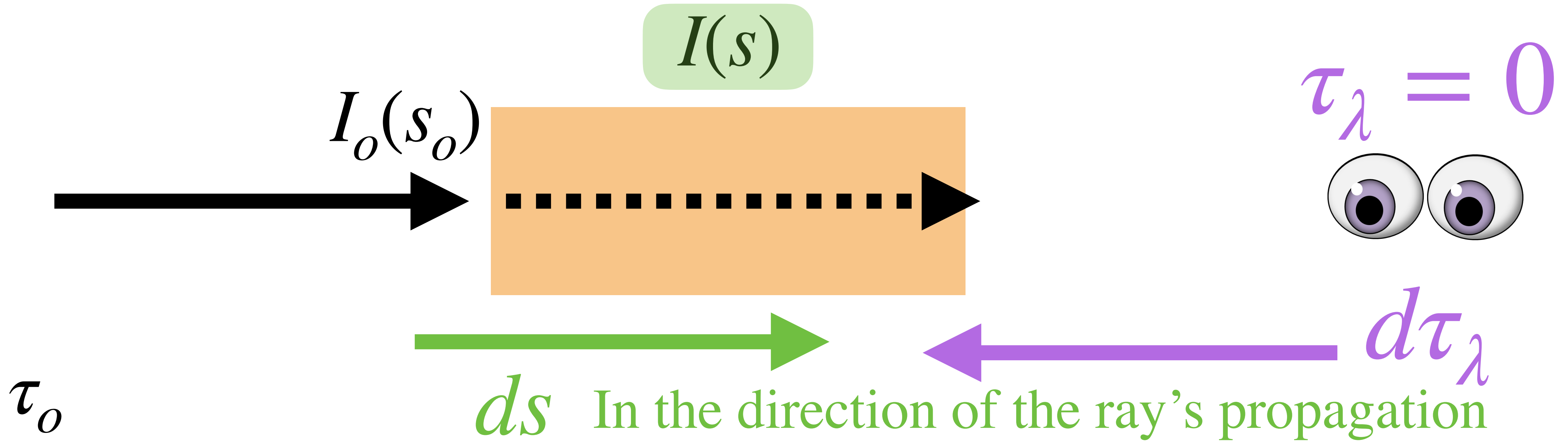
If the material has constant properties:

$$I(s) = I_o + j\rho[s - s_o]$$

In notebook:

- graph $I(s)$ for constant values
- graph $I(s)$ for linearly decreasing density

Change in intensity



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$$dI_\lambda = +\, j_\lambda(s)\rho(s)ds$$

General Solution

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$$I(s) = I_o e^{\tau(s) - \tau(s=s_o)}$$

$$I(s) = I_o + \int_{s'=s_o}^{s'=s} j(s')\rho(s')ds'$$

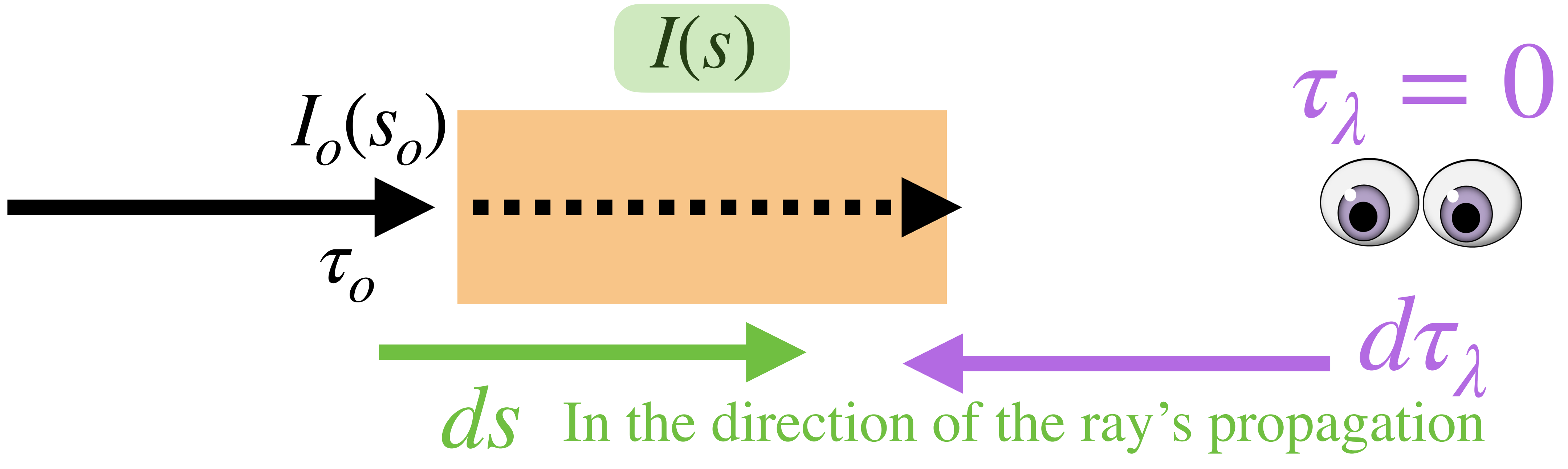
Constant properties in material

$$\tau(s) = \kappa\rho((s_o + d) - s)$$

$$I(s) = I_o e^{-\kappa\rho(s - s_o)}$$

$$I(s) = I_o + j\rho(s - s_o)$$

Now, what if we have both absorption and emission:
The “formal solution” of RT



(Q: why can’t we integrate this directly like earlier?)

$$\frac{dI_\lambda}{-\kappa_\lambda(s)\,\rho(s)\,ds} = \frac{-\kappa_\lambda(s)\,\rho(s)\,ds\,I_\lambda(s)}{-\kappa_\lambda(s)\,\rho(s)\,ds} + \frac{j_\lambda(s)\,\rho(s)\,ds}{-\kappa_\lambda(s)\,\rho(s)\,ds}$$

$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda(s) - \boxed{\frac{j_\lambda(s)}{\kappa_\lambda(s)}}$$

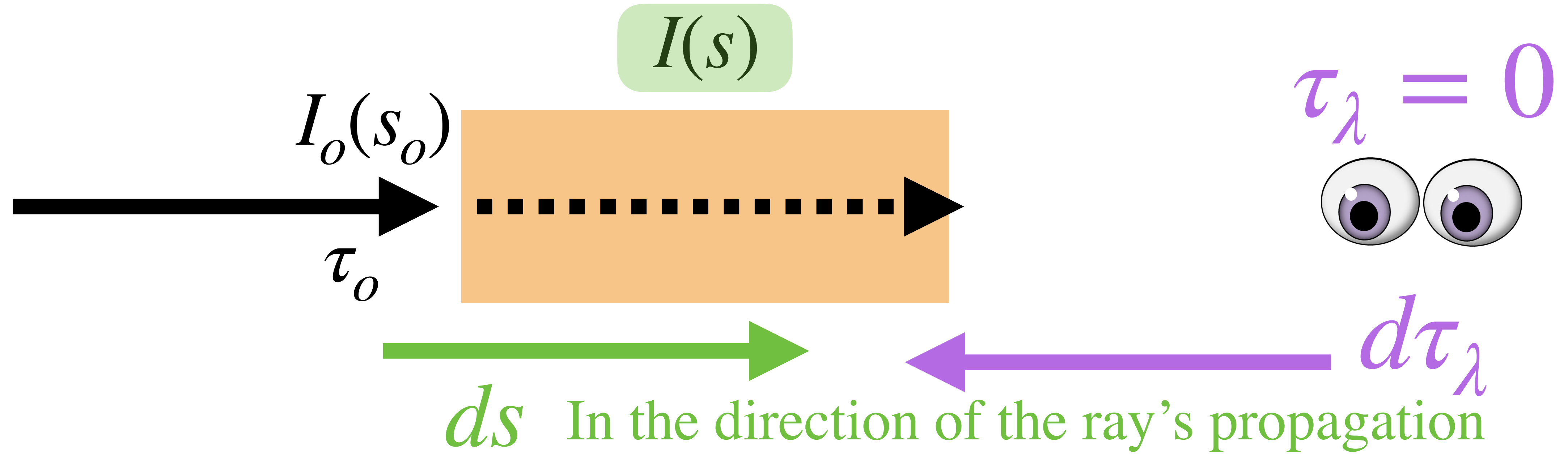
The “source function” $S_\lambda(s)$

(Q: what is the meaning of $S(\lambda)$?)

Here, I can use the optical depth as my “coordinate system” instead of the physical distance

$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda(\tau_\lambda) - S_\lambda(\tau_\lambda)$$

Now, what if we have both absorption and emission:
The “formal solution” of RT



$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda(\tau_\lambda) - S_\lambda(\tau_\lambda)$$

(Just dropping the λs for more clarity)

$$\frac{dI}{d\tau}\,e^{-\tau} - I(\tau)e^{-\tau} = -\,S(\tau)\,e^{-\tau}$$

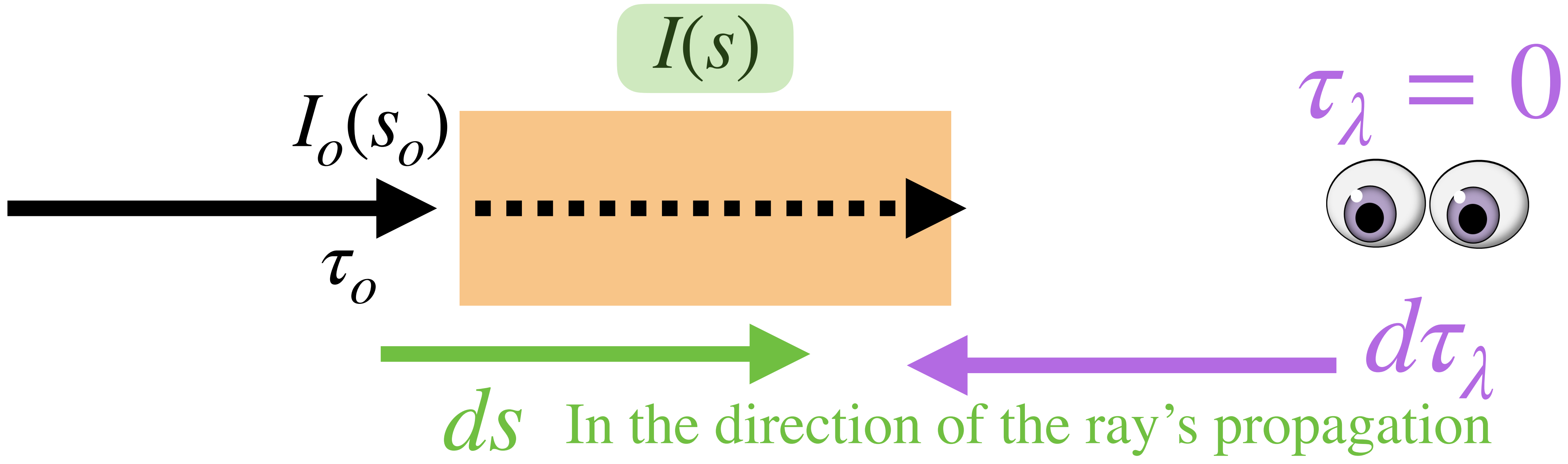
$$\frac{d}{d\tau}\left[I(\tau)\;e^{-\tau}\right] = -\,S(\tau)\,e^{-\tau}$$

$$\int_{\text{known position }(I_o,\tau_o)}^{\text{unknown position }(I,\tau)} d\left[I(\tau)\;e^{-\tau}\right] = -\int_{\tau_o}^{\tau} S(\tau)\,e^{-\tau}\,d\tau$$

$$I(\tau)e^{-\tau} - I_o e^{-\tau_o} = -\int_{\tau'=\tau_o}^{\tau'=\tau} S(\tau')\,e^{-\tau'}d\tau'$$

Here, I added explicit $'$ to the integrants, to distinguish it from the τ in the upper bound.
(This is the same as our integrals with a bound of r when we integrated the continuity and hydrostatic equations, but here it is a bit harder to visualize)

Now, what if we have both absorption and emission:
The “formal solution” of RT



$$I(\tau)e^{-\tau} - I_o e^{-\tau_o} = -\int_{\tau'=\tau_o}^{\tau'=\tau} S(\tau')\, e^{-\tau'} d\tau'$$

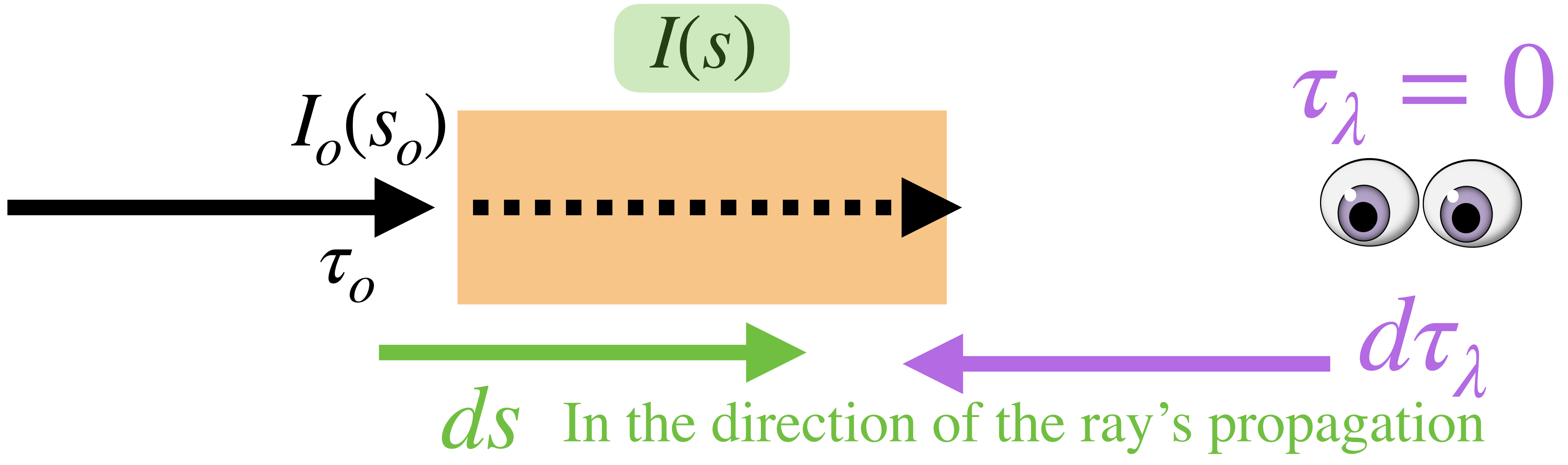
$$\frac{I(\tau)e^{-\tau}}{e^{-\tau}} = \frac{I_o e^{-\tau_o}}{e^{-\tau}} - \int_{\tau'=\tau_o}^{\tau'=\tau} S(\tau')\, \frac{e^{-\tau'}}{e^{-\tau}} d\tau'$$

Flip the bounds

$$I(\tau) = I_o e^{\tau-\tau_o} + \int_{\tau'=\tau}^{\tau'=\tau_o} S(\tau')\, e^{\tau-\tau'} d\tau'$$

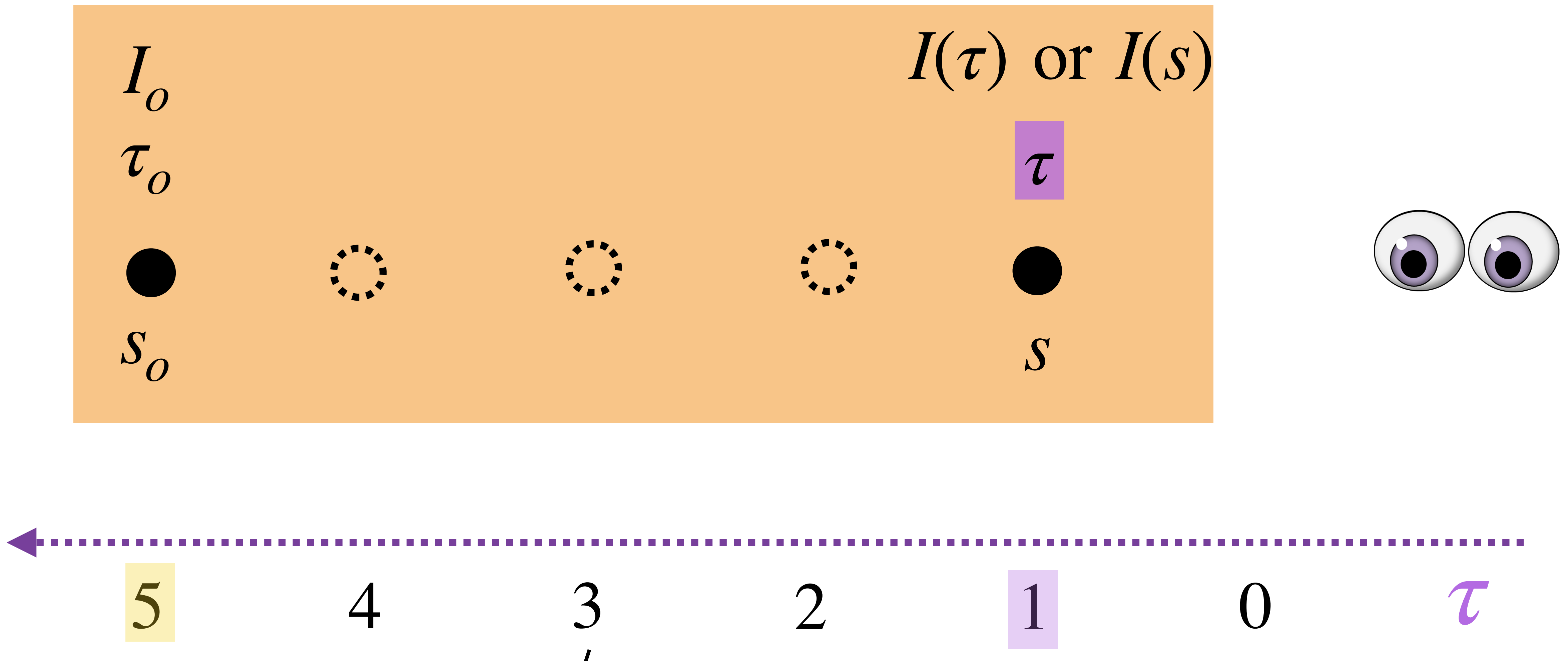
That is the “formal solution”

Now, what if we have both absorption and emission:
The "formal solution" of RT



$$I(\tau) = I_o e^{\tau - \tau_o} + \int_{\tau'=\tau}^{\tau'=\tau_o} S(\tau')\, e^{\tau - \tau'} d\tau'$$

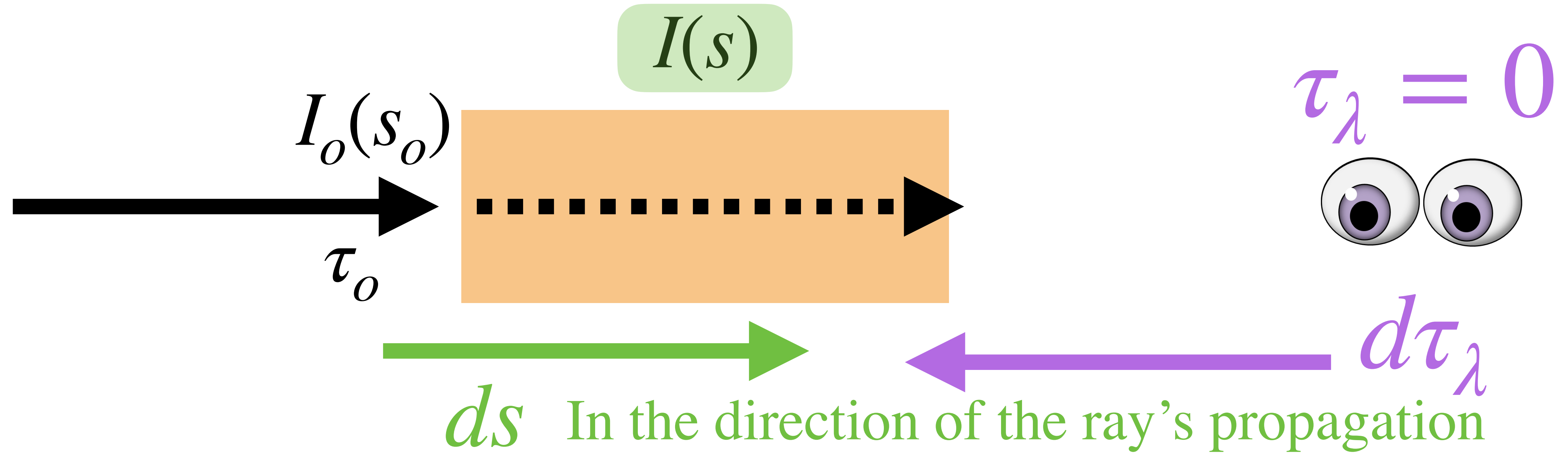
$$I(\tau = 1) = I_o e^{1-5} + \int_{\tau'=1}^{\tau'=5} S(\tau')\, e^{1 - \tau'} d\tau'$$



Replacing the integral by a sum, for illustration:

$$I(\tau = 1) = I_o e^{1-5} + S(\tau' = 2)\, e^{1-2} \Delta\tau' + S(\tau' = 3)\, e^{1-3} \Delta\tau' + \dots$$

1. What if the source function is constant $S(\tau) = S_o$?



$$I(\tau) \quad = \quad I_o e^{\tau-\tau_o} + \int_{\tau'=\tau}^{\tau'=\tau_o} S(\tau')\, e^{\tau-\tau'} d\tau'$$

1. Get the source function out of the integral
2. Note that $e^{\tau-\tau'} = e^{\tau}e^{-\tau'}$ and e^{τ} can also be taken out of the integral (not a function of τ')
3. We can integrate

$$I(\tau) \quad = \quad I_o e^{\tau-\tau_o} + S_o \left[1 - e^{\tau-\tau_o}\right]$$

2. What if the other material properties are also constant (κ_o, ρ_o)?

And we also would like to be able to plot $I(s)$ in our notebook, so we need to go back to that coordinate system!

$$I(s) \quad = \quad I_o e^{\tau(s)-\tau_o} + S_o \left[1 - e^{\tau(s)-\tau_o}\right]$$

$$\tau(s) - 0 = -\int_{s_{\rm obs}}^{s} \kappa(s)\, \rho(s)\, ds = -\kappa_o\rho_o\, [s - s_{\rm obs}]$$

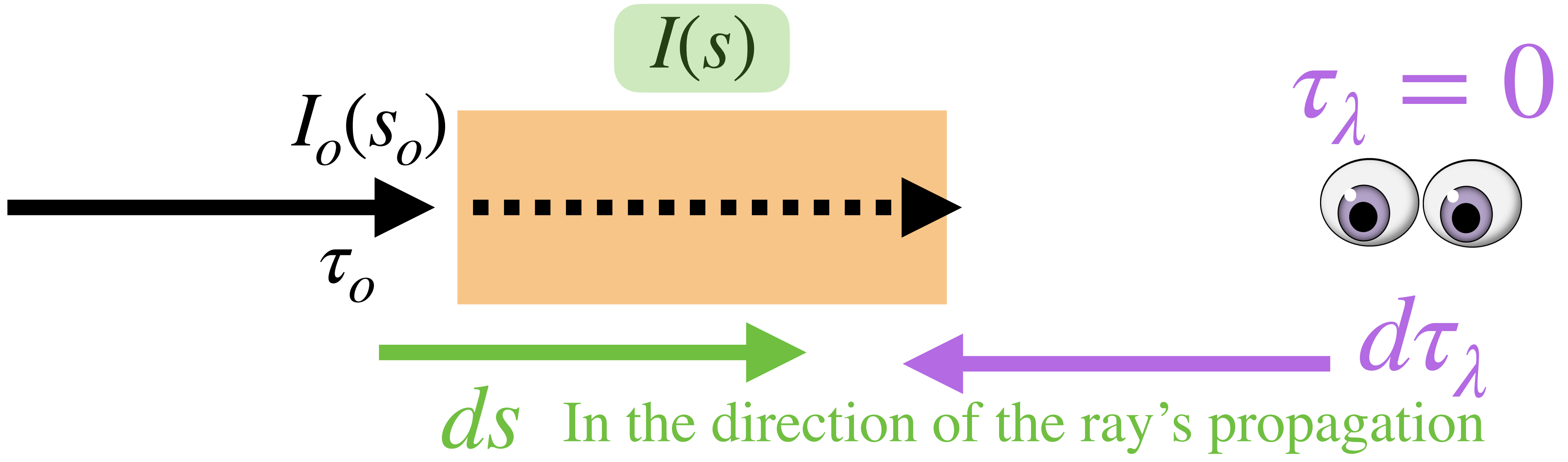
$$\tau(s_o) = -\kappa_o\rho_o\, [s_o - s_{\rm obs}]$$

$$\tau(s) - \tau(s_o) = -\kappa_o\rho_o\, [s - s_o]$$

$$I(s) \quad = \quad I_o e^{-\kappa_o\rho_o[s-s_o]} + S_o \left[1 - e^{-\kappa_o\rho_o[s-s_o]}\right]$$

$$\tau(s) - \tau_o = -\int_{s_o}^{s} \kappa(s)\, \rho(s)\, ds = -\kappa_o\rho_o[s - s_o]$$

$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda(\tau_\lambda) - S_\lambda(\tau_\lambda)$$

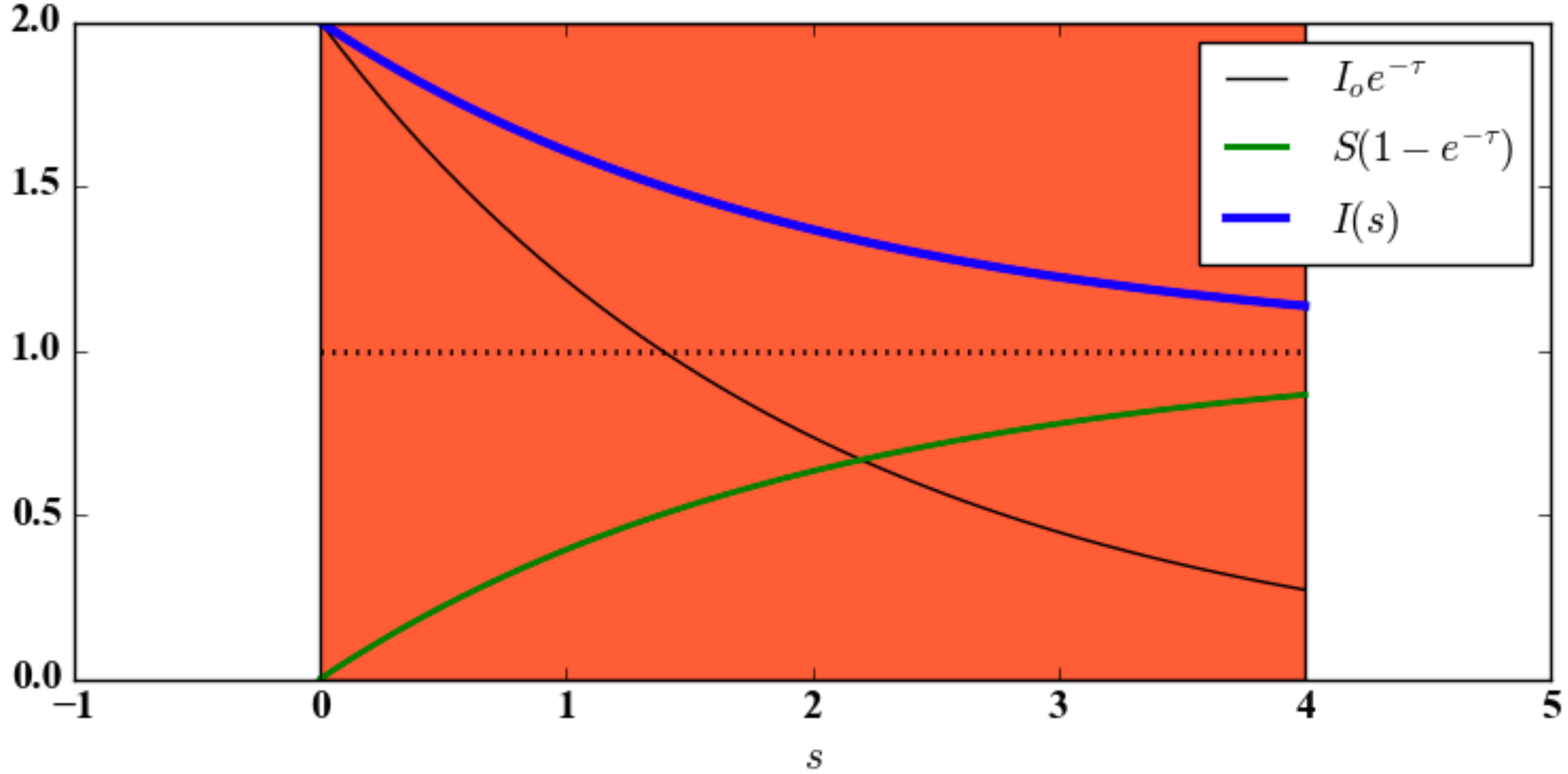


1. What if the source function is constant $S(\tau) = S_o$?

$$I(s) = I_o e^{\tau(s)-\tau_o} + S_o\left[1 - e^{\tau(s)-\tau_o}\right]$$

2. What if the other material properties are also constant (κ_o, ρ_o)?

$$I(s) = I_o e^{-\kappa_o\rho_o[s-s_o]} + S_o\left[1 - e^{-\kappa_o\rho_o[s-s_o]}\right]$$

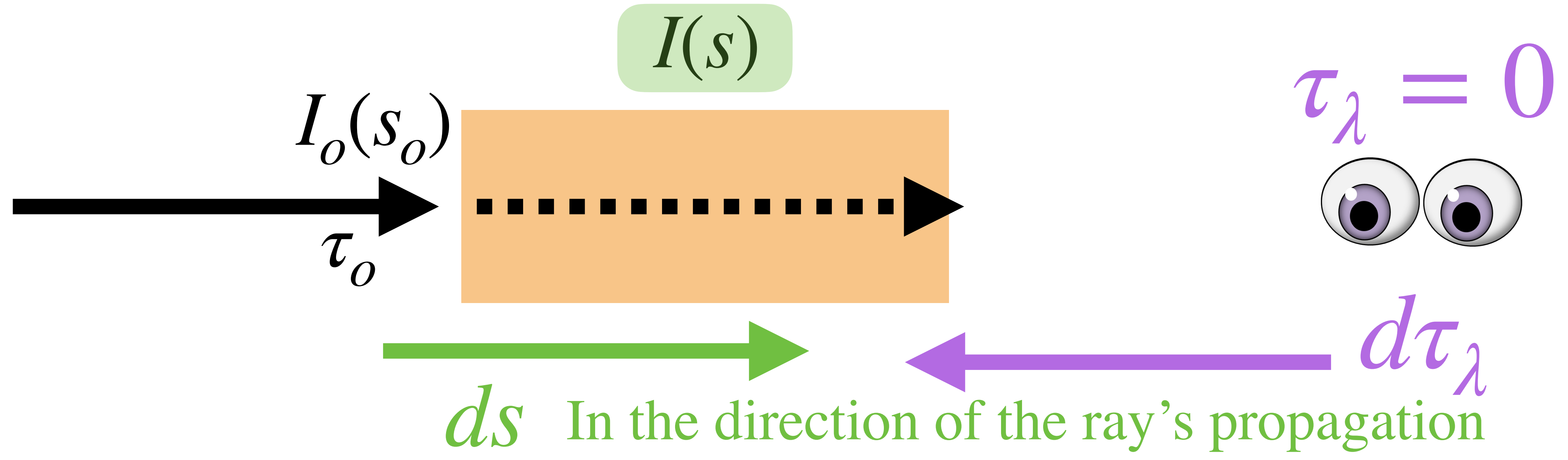


Q: What if $S_o = I_o$?

Q: What would happen to $I(s)$ if the slab was very long?

-> The source function is what the intensity will become, if given enough optical depth to reach it!

Change in intensity



Absorption + emission

$$\frac{dI_\lambda(\tau_\lambda)}{d\tau_\lambda} = I_\lambda(\tau_\lambda) - S_\lambda(\tau_\lambda)$$

General Solution

$$I(\tau(s)) = I_o\,\mathrm{e}^{\tau(s)-\tau_o} + \int_{\tau'=\tau(s)}^{\tau'=\tau_o} S(\tau')\,\mathrm{e}^{\tau(s)-\tau'}\,d\tau'$$

Constant source function

$$I(\tau(s)) = I_o\,\mathrm{e}^{\tau(s)-\tau_o} + S\left[1 - \mathrm{e}^{\tau(s)-\tau_o}\right]$$