Math Foundations of ML, Fall 2019

Homework #1

Due Thursday August 29, at the beginning of class

Reading: "Linear Algebra: A Concise Review" and Notes 1–3.

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

1. Sign up for Piazza if you have not already done it, https://piazza.com/gatech/fall2019/ece8843

2. Using you class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

3. (a) Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_K$ be a set of orthonormal vectors in $\mathbb{R}^N$:

$$\boldsymbol{u}_j^{\mathrm{T}} \boldsymbol{u}_i = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Show that given $\boldsymbol{x} \in \operatorname{Span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N\}$ we have

$$\boldsymbol{x} = \sum_{k=1}^{K} \alpha_k \boldsymbol{u}_k, \quad \text{where the } \{\alpha_k\} \text{ are comuted using } \alpha_k = \boldsymbol{u}_k^{\mathrm{T}} \boldsymbol{x}.$$

(Hint: what happens if I take the dot product of both sides of the equations above with $\boldsymbol{u}_1$?)

(b) Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_K$ be a set of orthonormal vectors in a subspace $\mathcal{T}$ of $\mathbb{R}^N$. Prove that if the only vector in $\mathcal{T}$ that is orthogonal to all of the $\boldsymbol{u}_k$ is $\mathbf{0}$, then $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_K\}$ is a basis for $\mathcal{T}$.

4. Suppose that $f(t)$ is a second-order spline that defined by the overlap of 5 B-splines:

$$f(t) = \sum_{k=0}^{4} \alpha_k b_2(t - k),$$

where $b_2(t)$ is defined as on page 11 of the notes,

$$b_2(t) = \begin{cases} (t + 3/2)^2/2 & -3/2 \le t \le -1/2 \\ -t^2 + 3/4 & -1/2 \le t \le 1/2 \\ (t - 3/2)^2/2 & 1/2 \le t \le 3/2 \\ 0 & |t| \ge 3/2 \end{cases}$$

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(a) Write a MATLAB[1] function

```
ft = piecepoly2(t, alpha);
```

which takes $\boldsymbol{\alpha} = \{\alpha_0, \ldots, \alpha_4\}$ and returns samples of $f(t)$ at the locations specified in the vector `t`. Turn in a plot of $f(t)$ for $\boldsymbol{\alpha} = \{3, 2, -1, 4, -1\}$. Sample t densely enough so that your plot looks like a smooth function.

(b) Suppose I tell you that

$$f(0) = 1, \quad f(1) = 2, \quad f(2) = -4, \quad f(3) = -5, \quad f(4) = -2.$$

What are the corresponding α_k? (Hint: you will have to construct a system of equations then solve it.)

(c) To generalize this, suppose that $f(t)$ is now a superposition of N B-splines:

$$f(t) = \sum_{n=0}^{N-1} \alpha_n b_2(t-n).$$

Describe how to construct the $N \times N$ matrix that maps the coefficients $\boldsymbol{\alpha}$ to the N samples $f(0), \ldots, f(N-1)$. That is, find $\boldsymbol{A}$ such that

$$\begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix} = \begin{bmatrix} & & \\ & \boldsymbol{A} & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{bmatrix}$$

(d) Argue that the matrix $\boldsymbol{A}$ from part (c) is invertible for all values of N. (Hint: $\boldsymbol{A}$ is invertible if an only if $\boldsymbol{Ax} = \mathbf{0}$ only when $\boldsymbol{x} = \mathbf{0}$, which means $\|\boldsymbol{Ax}\|_2 = 0$ only when $\boldsymbol{x} = \mathbf{0}$. Write $\boldsymbol{A}$ as $a\mathbf{I} + \boldsymbol{G}$ and so $\|\boldsymbol{Ax}\|_2 = \|a\boldsymbol{x} + \boldsymbol{Gx}\|_2 \geq a\|\boldsymbol{x}\|_2 - \|\boldsymbol{Gx}\|_2 \cdots$)

(e) To take this even further, suppose that

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n b_2(t-n),$$

so $f(t)$ is described by the (possibly infinite) sequence of numbers $\{\alpha_n\}_{n\in\mathbb{Z}}$. Show that there is a convolution operator that maps the sequence $\{\alpha_n\}$ to the sequence $\{f(n)\}$. That is, find a sequence of numbers $\{h_n\}_{n\in\mathbb{Z}}$ such that

$$f(n) = \sum_{\ell=-\infty}^{\infty} h_\ell \, \alpha_{n-\ell}.$$

(f) *Optional bonus problem:* Argue that the $\{\alpha_n\}$ can be computed from the $\{f(n)\}$ is a similar manner by showing how to compute a $\{g_n\}_{n\in\mathbb{Z}}$ such that

$$\alpha_n = \sum_{\ell=-\infty}^{\infty} g_\ell \, f(n-\ell).$$

[1]The code syntax in the homeworks will be for MATLAB, but please feel free to use Python instead if you are more comfortable.

5. In the file `hw01p5_nonuniform_samples`[2] there are a set of samples locations t_m and sample values y_m, $m = 1, \ldots, 9$.

   (a) Find a ninth-order polynomial that passes through these points. Plot your answer with the samples overlaid.

   (b) Find $f(t)$ such that

   $$f(t) = \sum_{k=0}^{9} \alpha_k \, b_2(t-k)$$

   and $f(t_m) = y_m$ for $m = 0, \ldots, 9$. Plot your answer with the samples overlaid.

6. (a) Let $f(t) = |t|^p$ for $p \geq 1$. Use the mean value theorem to prove that

   $$f\left(\frac{a+b}{2}\right) \;\leq\; \frac{f(a)+f(b)}{2}.$$

   (Hint: apply the MVT on the intervals $[a,c]$ and $[c,b]$ with $c = (a+b)/2$, and recognize that the derivative of f is monotonic.)

   (b) Let $\ell_p(\mathbb{N})$ be the set of all (infinite length) sequences

   $$\ell_p(\mathbb{N}) = \left\{ \{x_n\}_{n=1}^{\infty} \;:\; \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

   Show that $\ell_p(\mathbb{N})$ is indeed a linear vector space for $p \geq 1$. (You will probably find the assertion in part (a) handy.)

[2]You can read a .mat file into Python using `scipy.io.loadmat`.