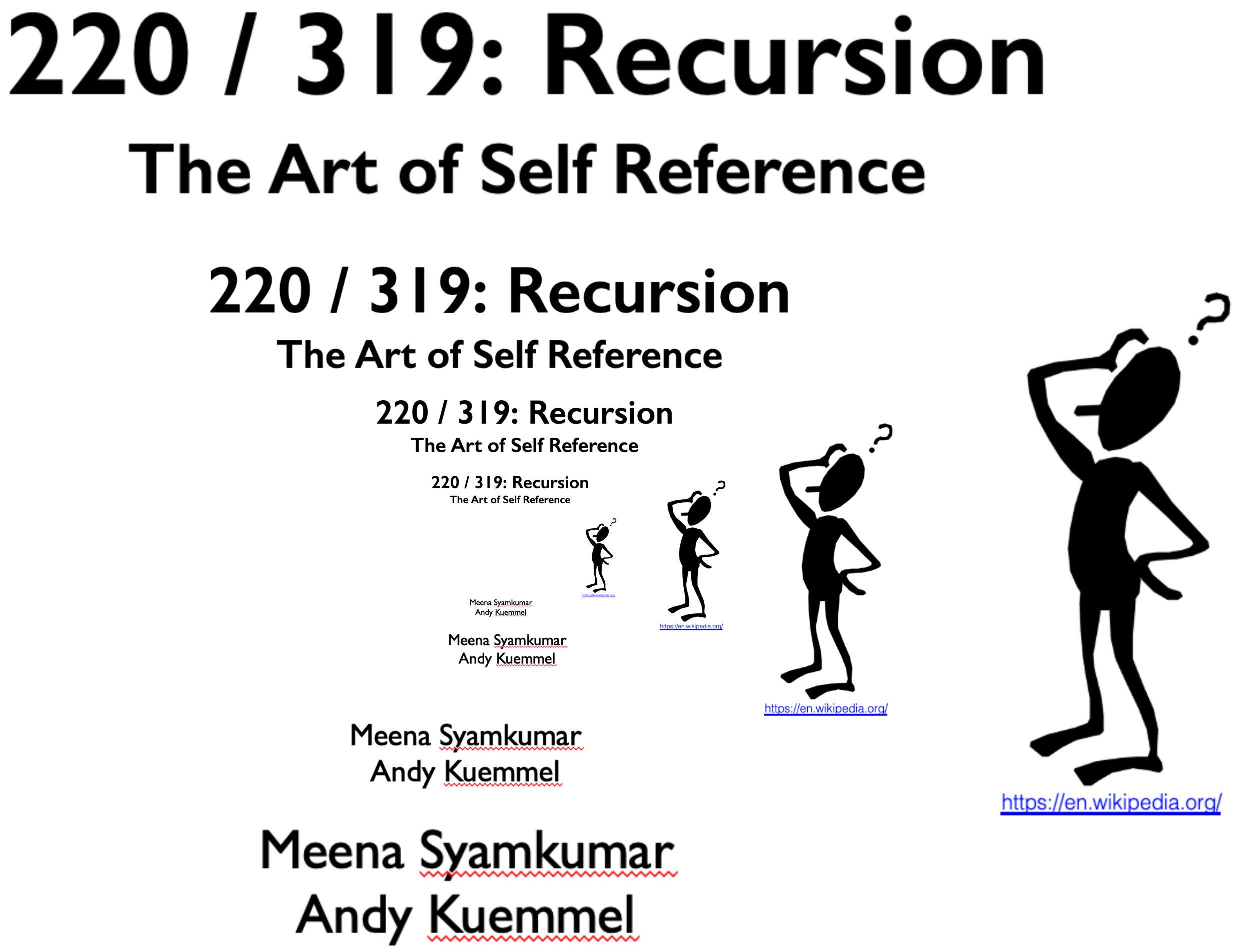
220 / 319: Recursion

The Art of Self Reference



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The Art of Self Reference

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https://en.wikipedia.org/

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Andy Kuemmel

Part 2 of CS220 - Data Structures

- Lists and Dictionaries
- CSV and JSON
- Objects and References
- Fancy Functions
    - Recursion
    - Generators
    - Functions are Objects
- Files
- Errors

Goal: use self-reference is a meaningful way

***Hofstadter's Law**: "It always takes longer than you expect, even when you take into account **Hofstadter's Law**."*

(From Gödel, Escher, Bach)

good advice for CS assignments!

"Dialectical Materialism is materialism that involves dialectic."

"The Marxist theory (adopted as the official philosophy of the Soviet communists) that political and historical events result from the conflict of social forces and are interpretable as a series of contradictions and their solutions. The conflict is believed to be caused by material needs."

https://en.wikipedia.org/wiki/Circular_definition

Goal: use self-reference is a meaningful way

***Hofstadter's Law**: "It always takes longer than you expect, even when you take into account **Hofstadter's Law**."*

(From Gödel, Escher, Bach)

***mountain**: "a landmass that projects conspicuously above its surroundings and is higher than a **hill**"*

***hill**: "a usually rounded natural elevation of land lower than a **mountain**"*

(Example of **unhelpful** self reference from Merriam-Webster dictionary)

https://en.wikipedia.org/wiki/Circular_definition

Overview: Learning Objectives

Recursive definitions and recursive information

- What is a **recursive definition/structure**?
- Arbitrarily vs. infinitely

Recursive code

- What is **recursive code**?
- Why write recursive code?
- Where do computers keep local variables for recursive calls?
- What happens to programs with **infinite recursion**?

Read *Think Python*

- ✦ Ch 5: “Recursion” through “Infinite Recursion”
- ✦ Ch 6: “More Recursion” through end

What is Recursion?

Recursive definitions

- Contain the term in the body
- Dictionaries, mathematical definitions, etc

A number **x** is a positive even number if:

- **x** is 2

  OR

- **x** equals another positive even number plus two

What is Recursion?

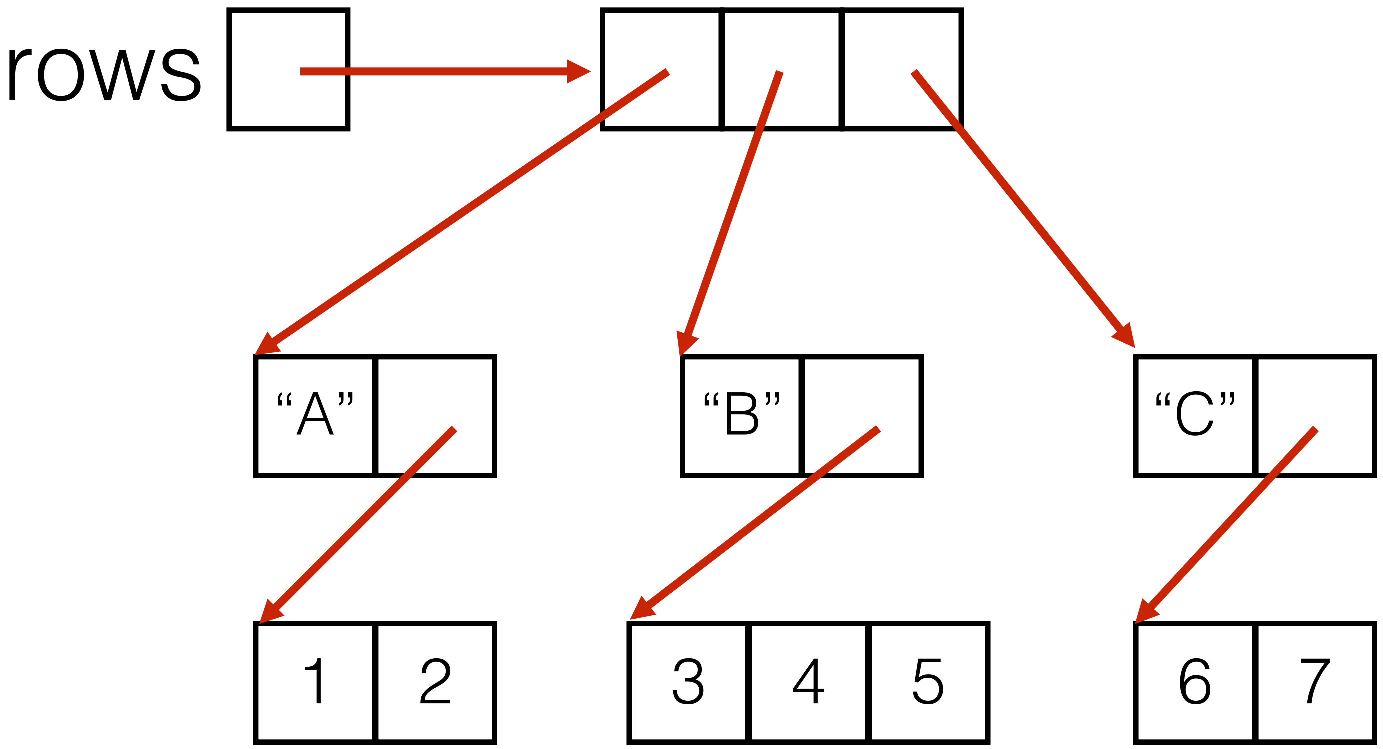
Recursive definitions

- Contain the term in the body
- Dictionaries, mathematical definitions, etc

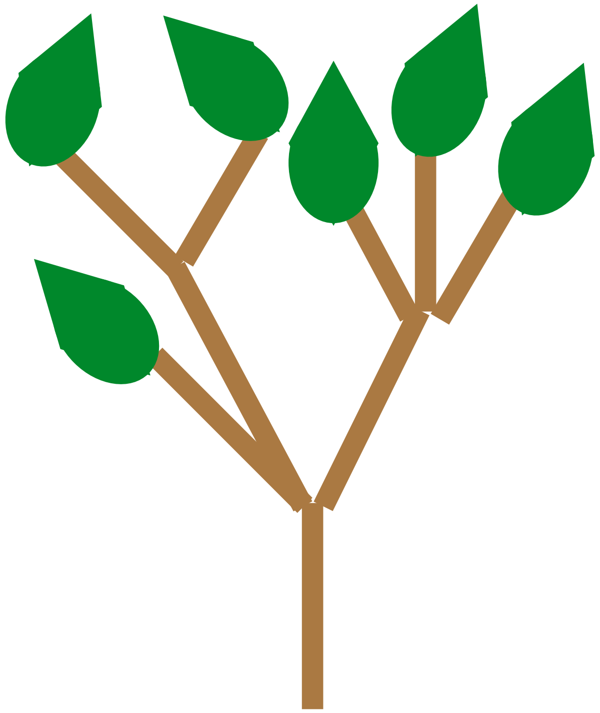
Recursive structures may refer to structures of the same type

- data structures or real-world structures

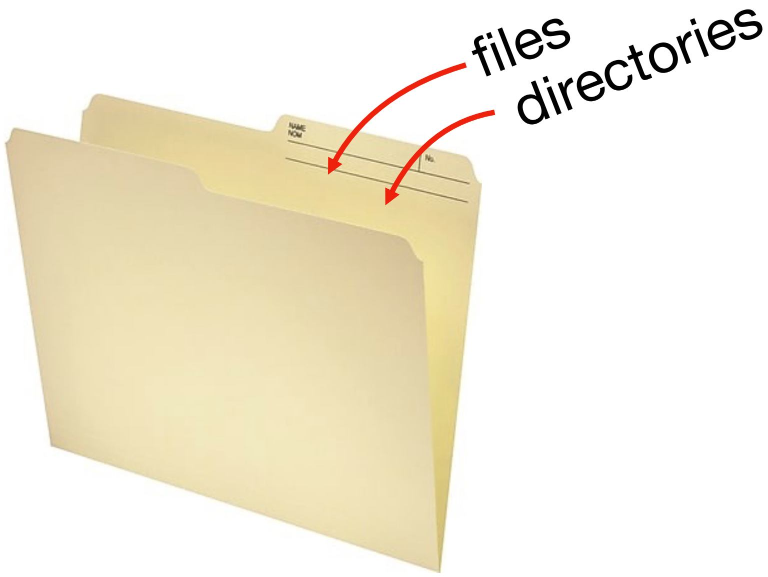
```
rows = [
   [“A”,[1,2]],
   [“B”,[3,4,5]],
   [“C”,[6,7]]
]
```



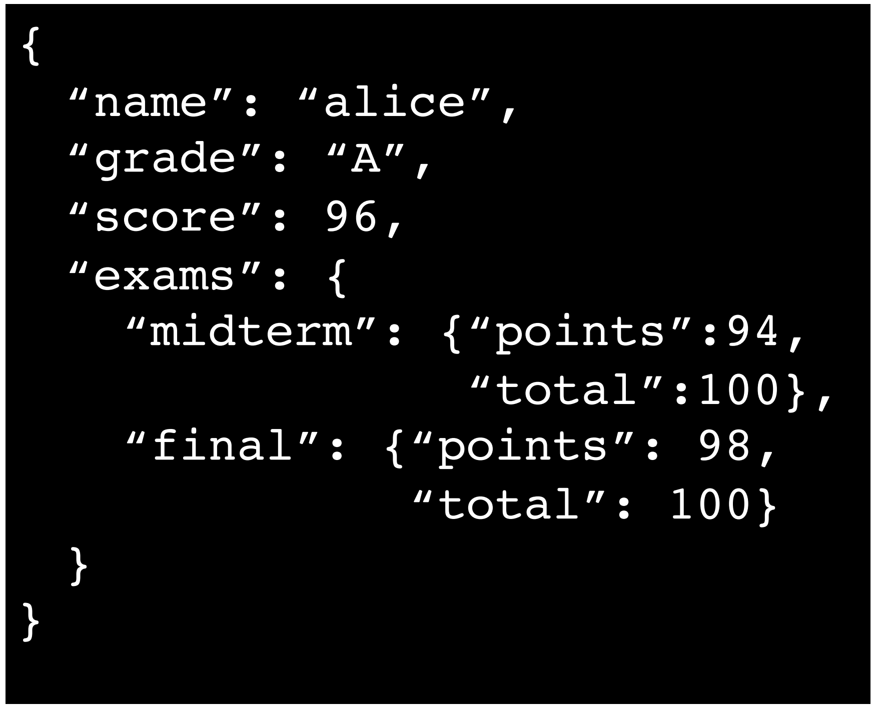
Recursive structures are EVERYWHERE!



nature



files



```
{
  “name”: “alice”,
  “grade”: “A”,
  “score”: 96,
  “exams”: {
    “midterm”: {“points”:94,
                “total”:100},
    “final”: {“points”: 98,
              “total”: 100}
  }
}
```

formats

Example: Trees (Finite Recursion)

Term: branch

Def: wooden stick, with an end splitting into other branches, OR terminating with a leaf

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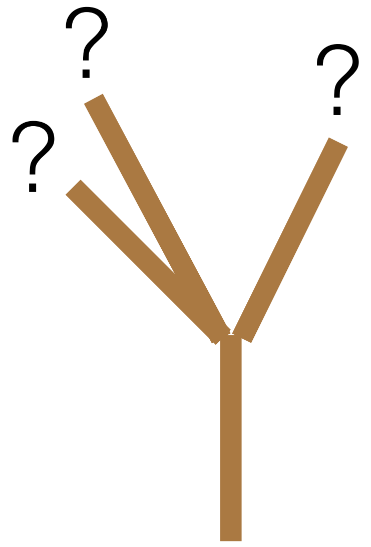


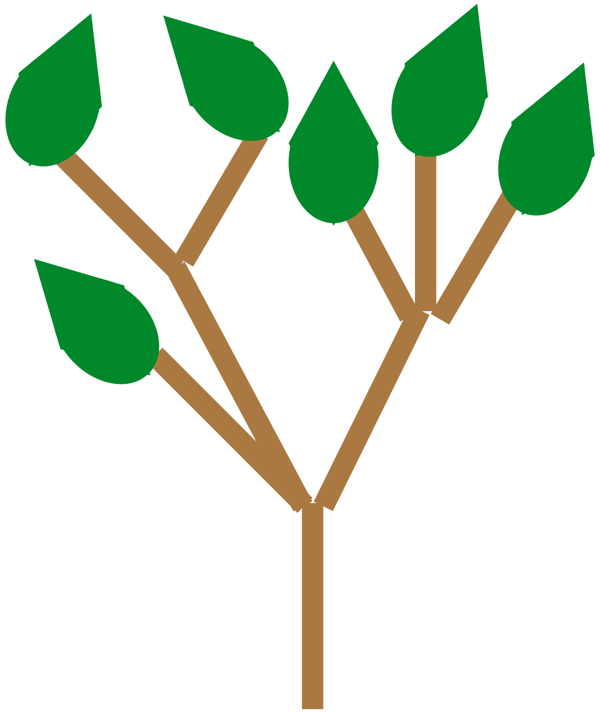
Example: Trees (Finite Recursion)

Term: branch

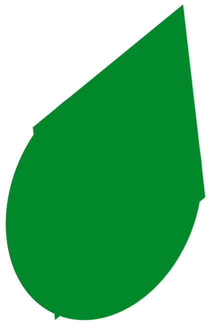
Def: wooden stick, with an end splitting into other branches, OR terminating with a leaf

trees are finite:
eventual **base case**
allows completion

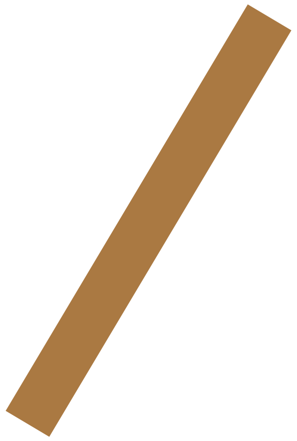
trees are arbitrarily large:
recursive case allows
indefinite growth



arbitrarily != infinitely



base case (leaf)



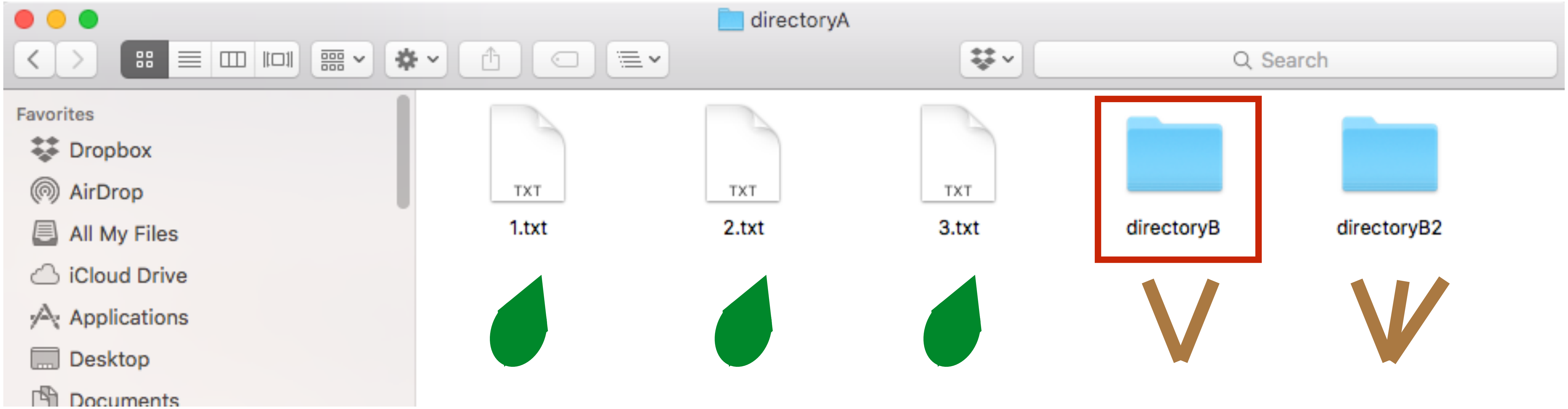
recursive case (branch)

Example: Directories (aka folders)

Term: directory

recursive because def contains term

Def: a collection of files and directories



file system tree

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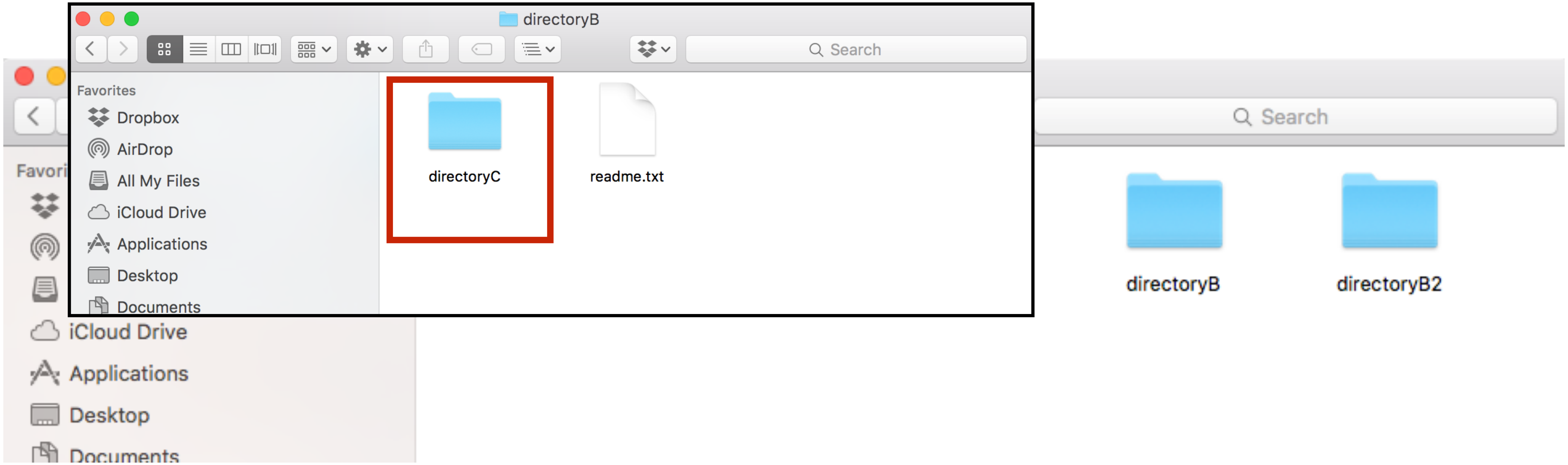

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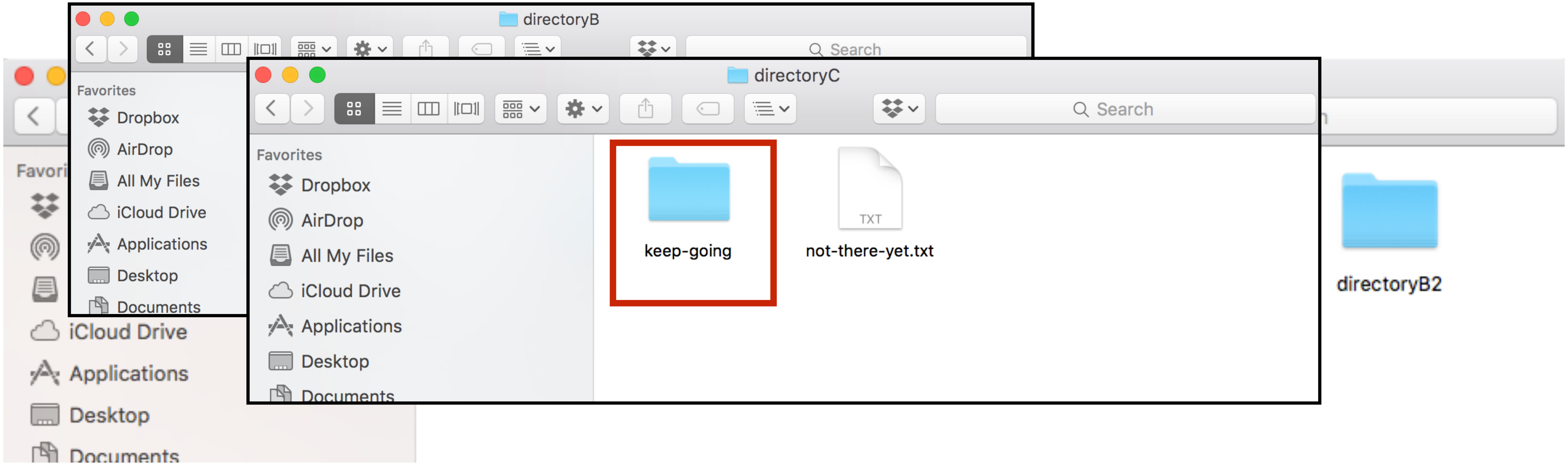

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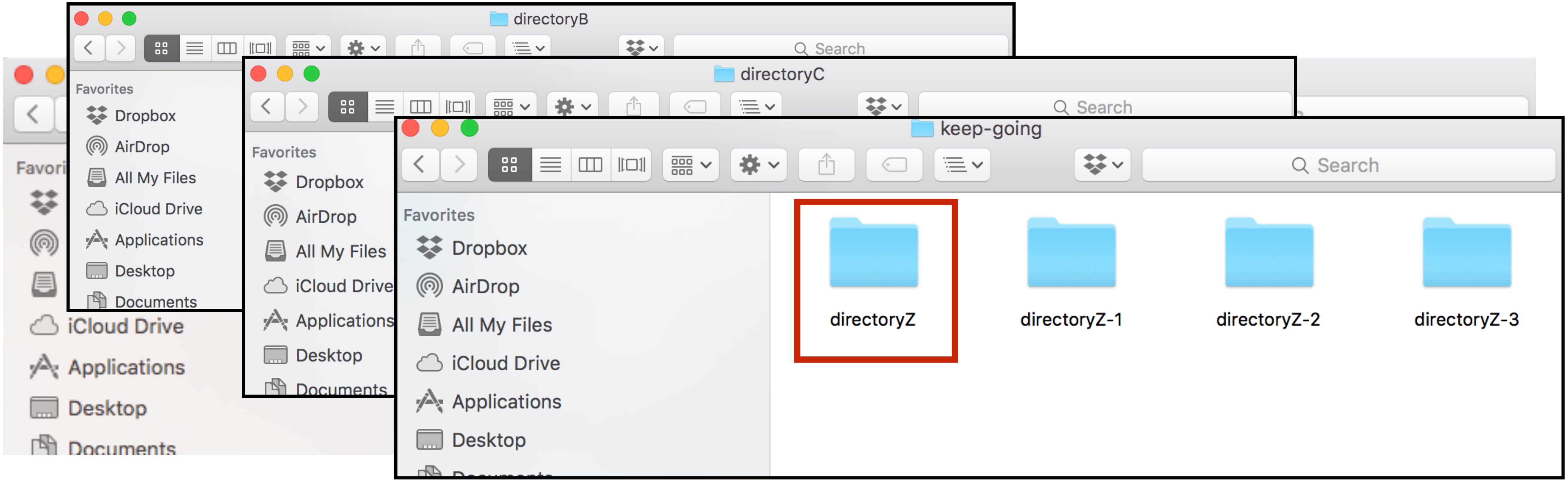
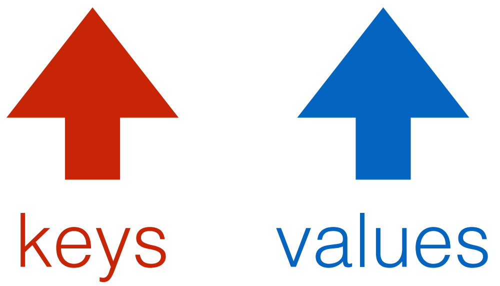


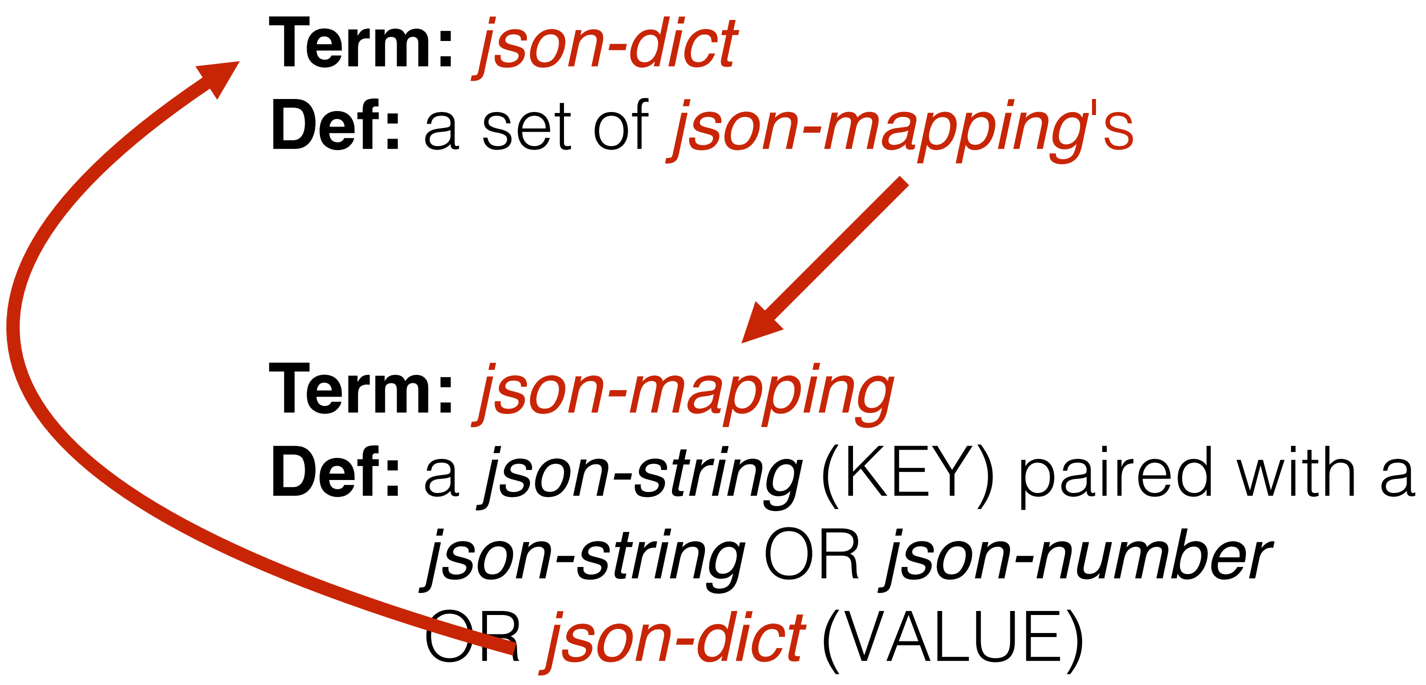
file system tree

Example: (simplified) JSON Format

Example JSON Dictionary:

```
{
 “name”: “alice”,
 “grade”: “A”,
 “score”: 96
}
```





Term: *json-dict*
Def: a set of *json-mapping*'s

Term: *json-mapping*
Def: a *json-string* (KEY) paired with a *json-string* OR *json-number* OR *json-dict* (VALUE)

recursive self reference isn’t always direct!

Example: (simplified) JSON Format

Example JSON Dictionary:

```
{
  “name”: “alice”,
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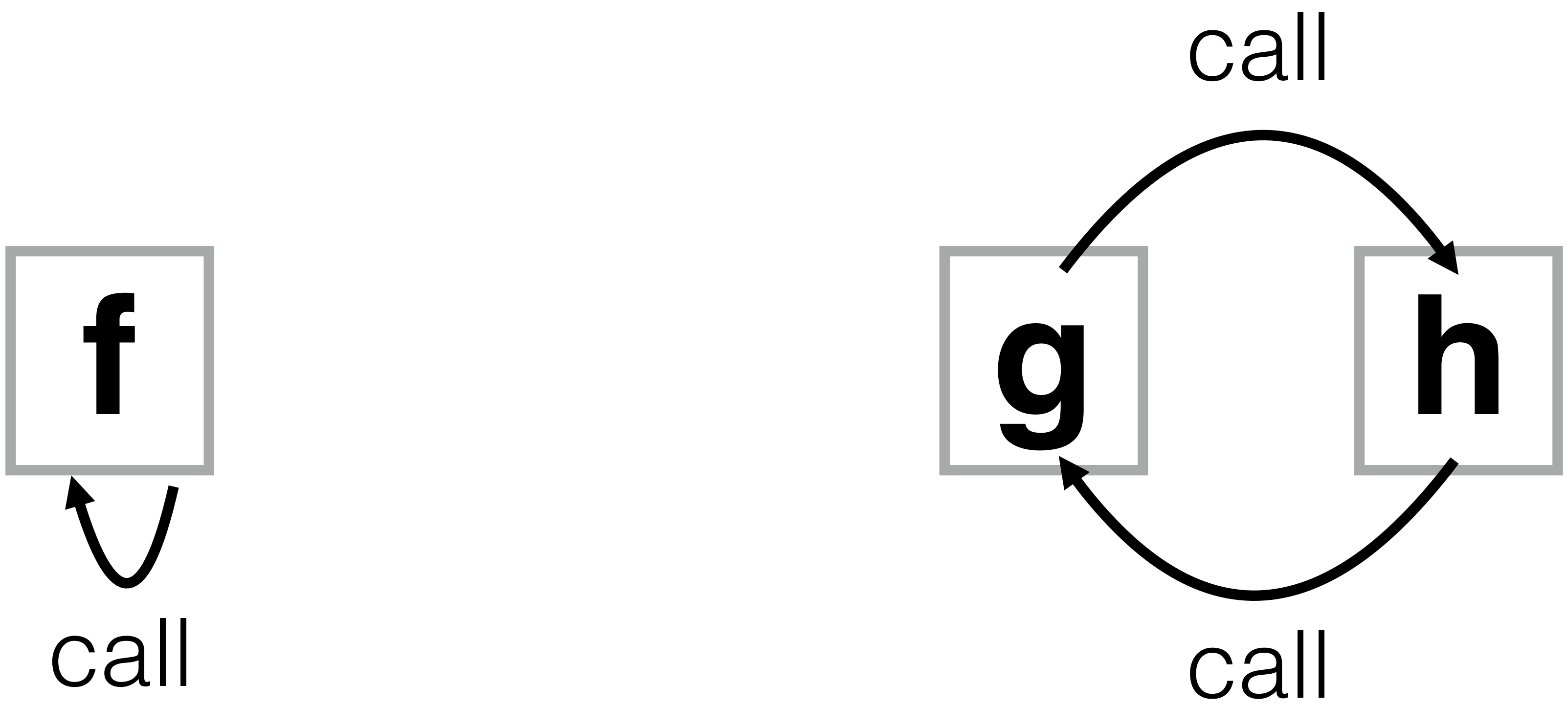
Recursive Code

What is it?

- A function that calls itself (possible indirectly)

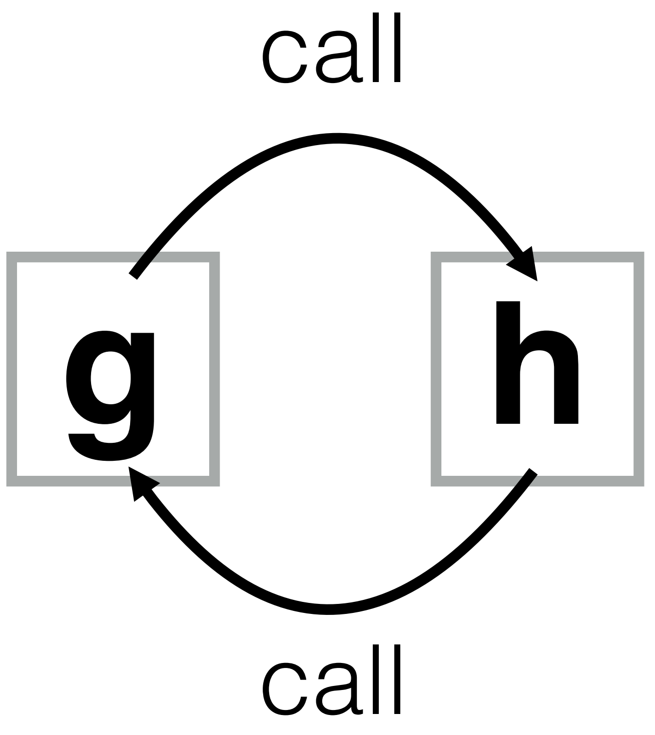


Recursive Code

What is it?

- A function that calls itself (possible indirectly)

```
def f():
    # other code
    f()
    # other code
```



Recursive Code

What is it?

- A function that calls itself (possible indirectly)

```
def f():
    # other code
    f()
    # other code
```

```
def g():
    # other code
    h()
    # other code

def h():
    # other code
    g()
    # other code
```

Recursive Code

What is it?

- A function that calls itself (possible indirectly)

Motivation: don't know how big the data is before execution

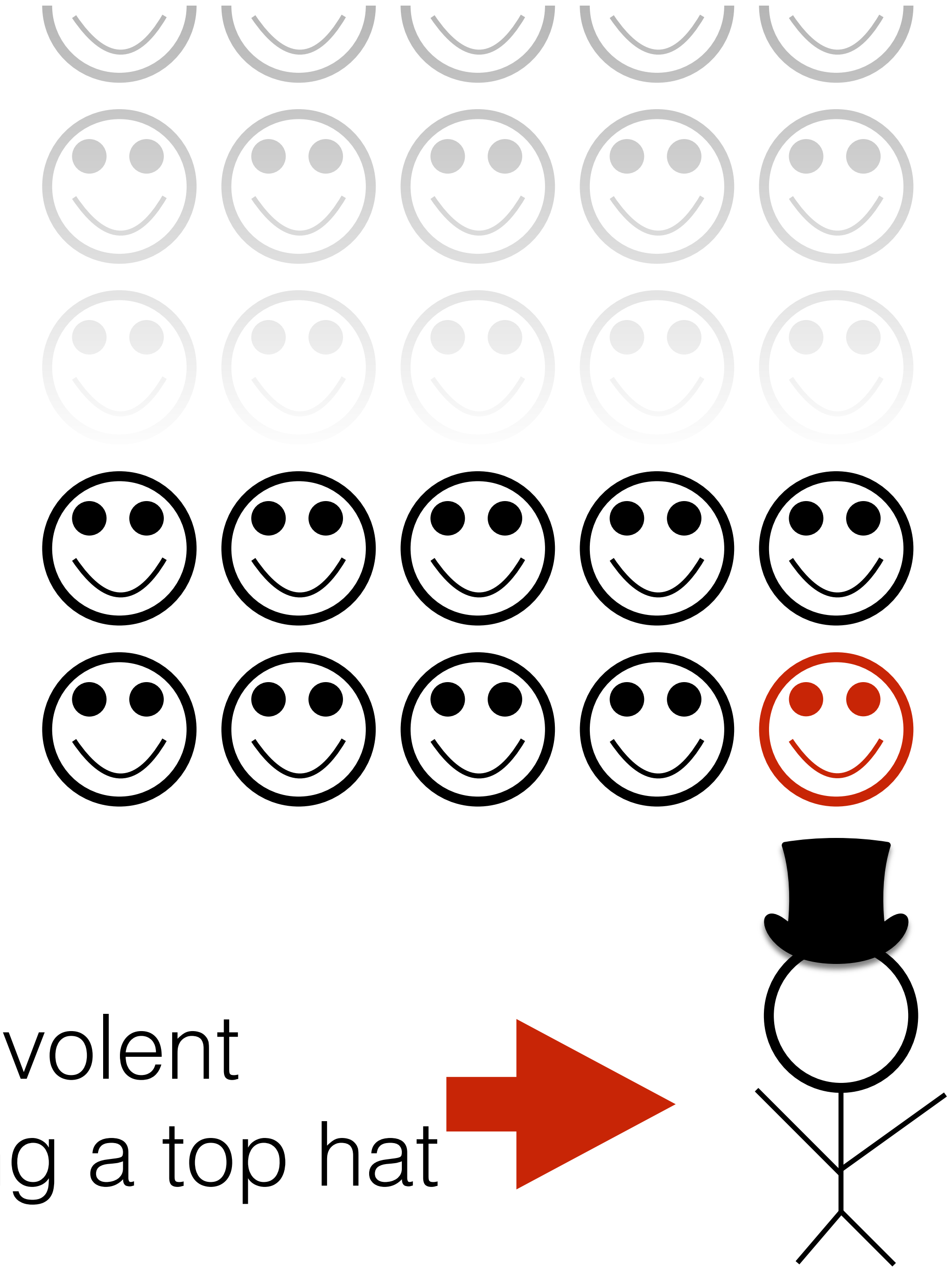
- Need either **iteration** or **recursion**
- In theory, these techniques are equally powerful

Why recurse? (instead of always iterating)

- in practice, often easier
- recursive code corresponds to recursive data
- reduce a big problem into a smaller problem

https://texastreesurgeons.com/services/tree-removal/

Recursive Student Counting



CS 220 students
in the front row

wise and benevolent
teacher wearing a top hat

Example from https://courses.cs.washington.edu/courses/cse143/17au/

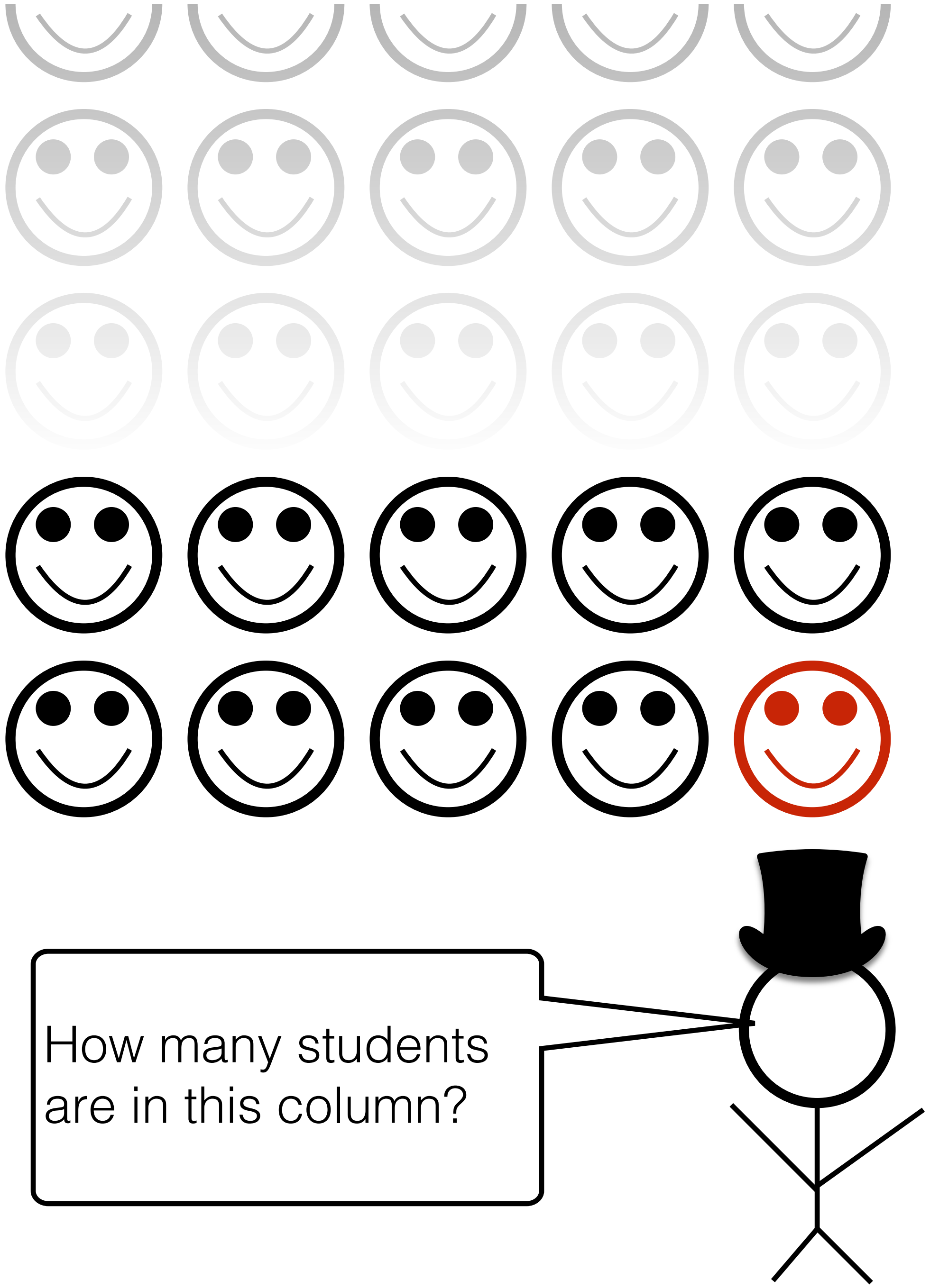
Recursive Student Counting

Imagine:

A teacher wants to know how many students are in a column. **What should each student ask the person behind them?**

Constraints:

- You can only talk to the student behind / in front of you



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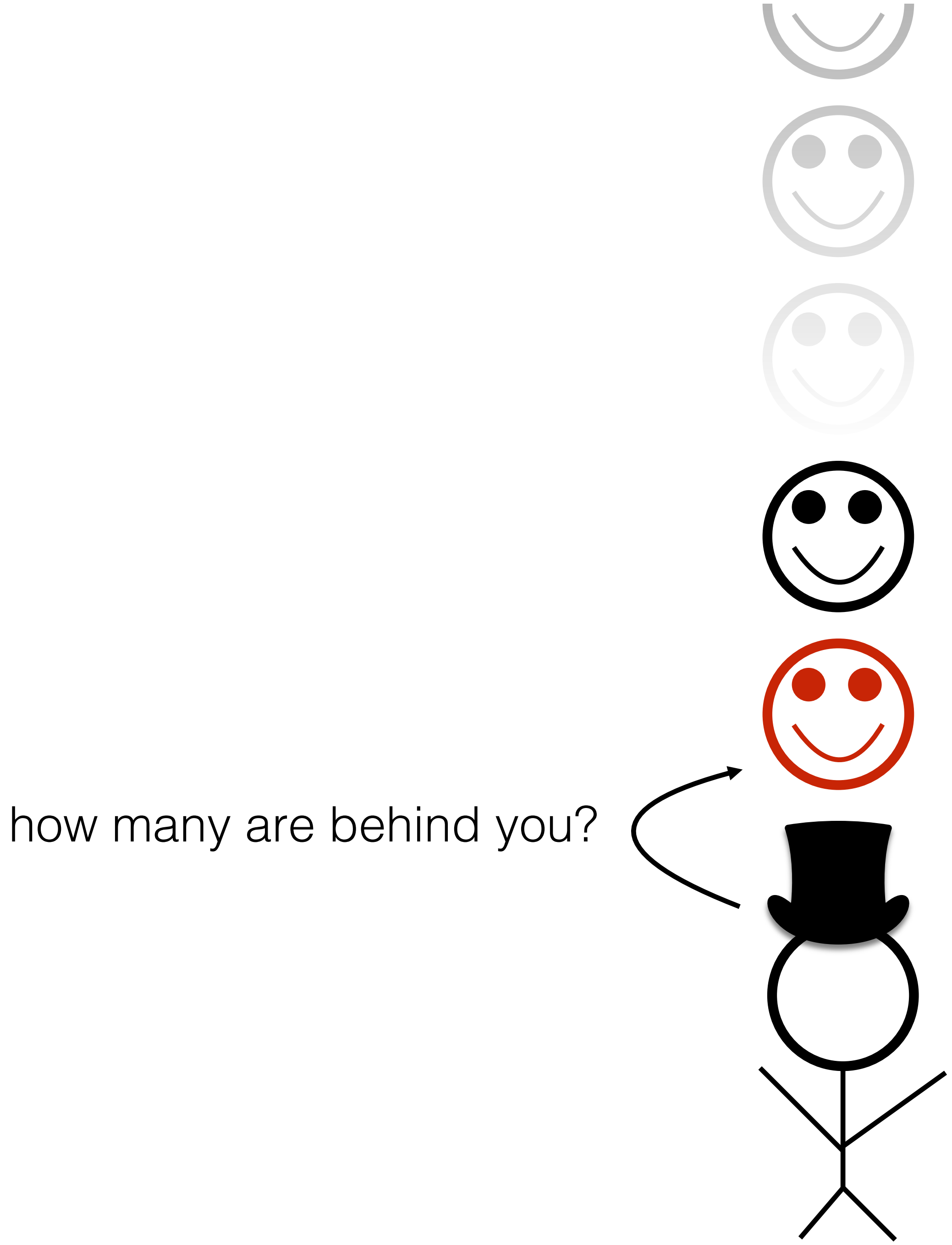
Recursive Student Counting

Strategy: reframe question as *"how many students are behind you?"*

Reframing is the hardest part



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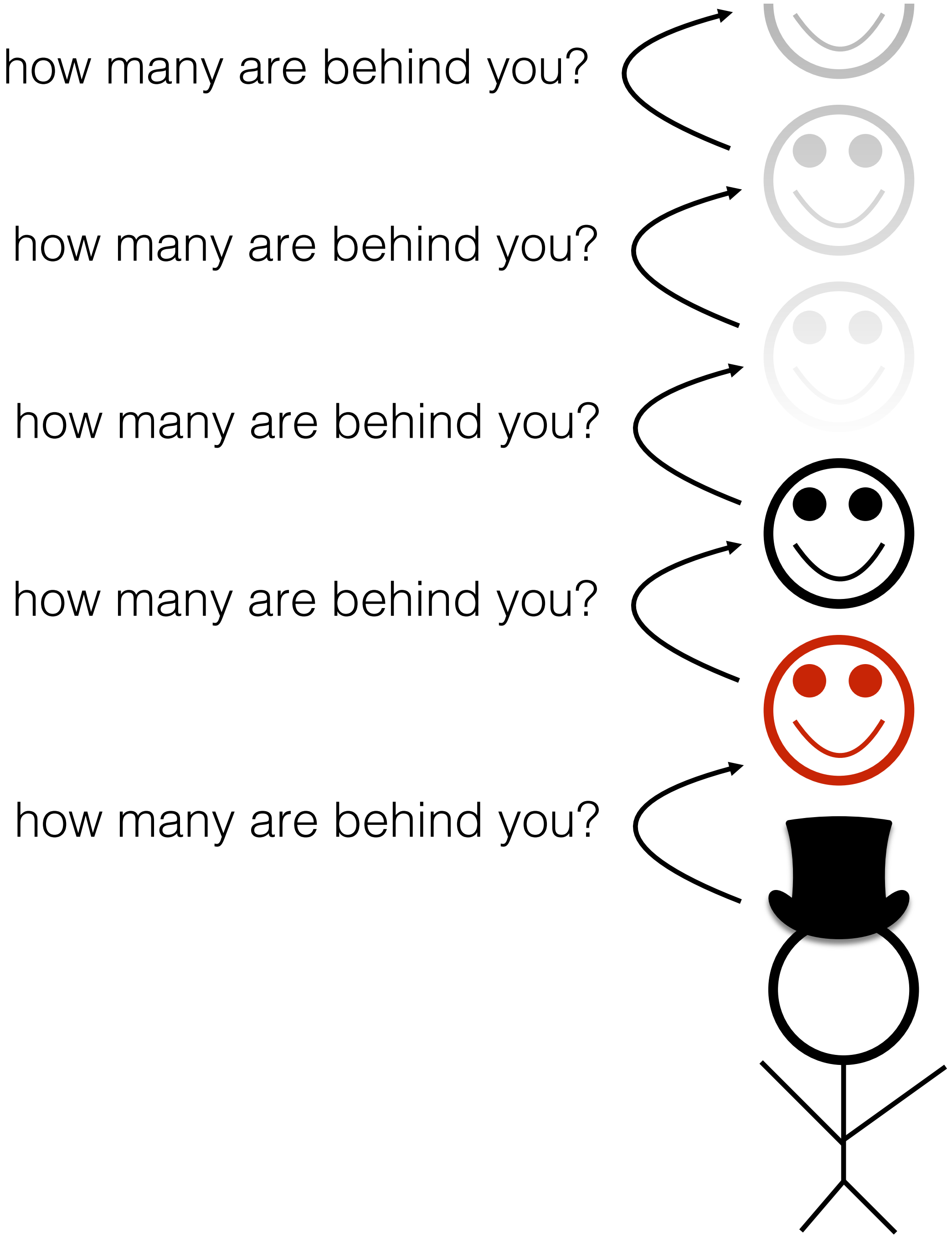
Recursive Student Counting

Strategy: reframe question as *"how many students are behind you?"*

Process:
if nobody is behind you: **say** 0
else: ask them, **say** their answer+1



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Recursive Student Counting

Strategy: reframe question as

"how many students are behind you?"

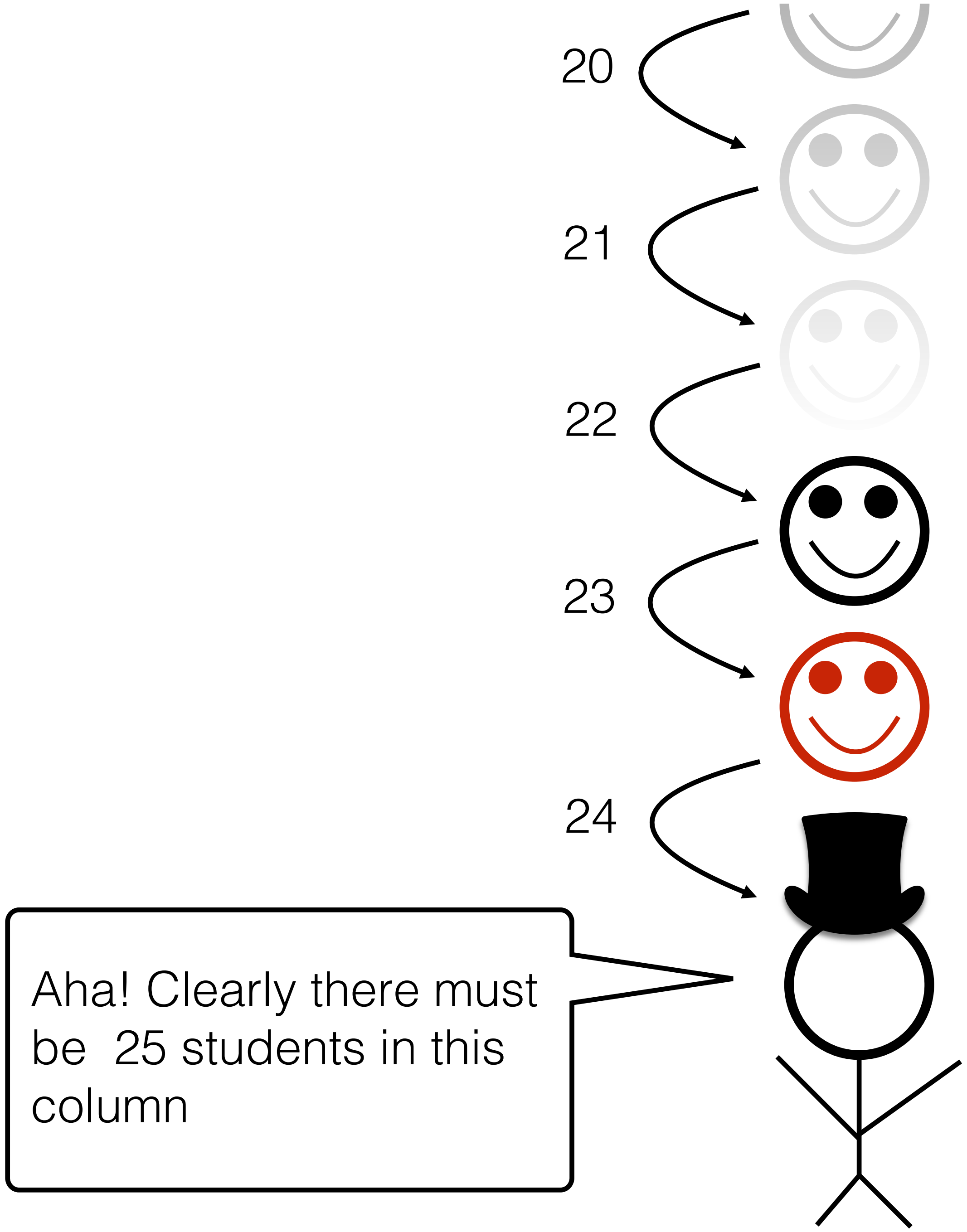
Process:

if nobody is behind you: **say** 0

else: ask them, **say** their answer+1

Observations:

- Each student runs the same "code"
- Each student has their own "state"



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Practice: Reframing Factorials

$$N! = 1 \times 2 \times 3 \times \ldots \times (N-2) \times (N-1) \times N$$

Example: Factorials

1. Examples:

`1! = 1` *simplest example*

`2! = 1*2 = 2`

`3! = 1*2*3 = 6`

`4! = 1*2*3*4 = 24`

`5! = 1*2*3*4*5 = 120`

2. Self Reference:

3. Recursive Definition:

4. Python Code:

```python
def fact(n):
    pass # TODO
```

Goal: work from examples to get to recursive code

Example: Factorials

1. Examples:

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1! = 1
2! = 1*2 = 2
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```

2. Self Reference:

look for patterns that allow rewrites with self reference

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Example: Factorials

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```

2. Self Reference:

```
1! = 1
2! = 1! * 2
3! = 2! * 3
4! = 3! * 4
5! = 4! * 5
```

don't need a pattern at the start

3. Recursive Definition:

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Example: Factorials

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```
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3. Recursive Definition:

*convert self-referring examples
to a recursive definition*

4. Python Code:

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Example: Factorials

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`1!` is `1`

`N!` is `(N-1)! * N` for N>1

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4. Python Code:

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    if n == 1:
        return 1
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def fact(n):
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```

Rule 1: Base case should always be defined and be terminal

Rule 2: Recursive case should make progress towards base case

Example: Factorials

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Let’s “run” it!

Tracing Factorial

```
def fact(n):
    if n == 1:
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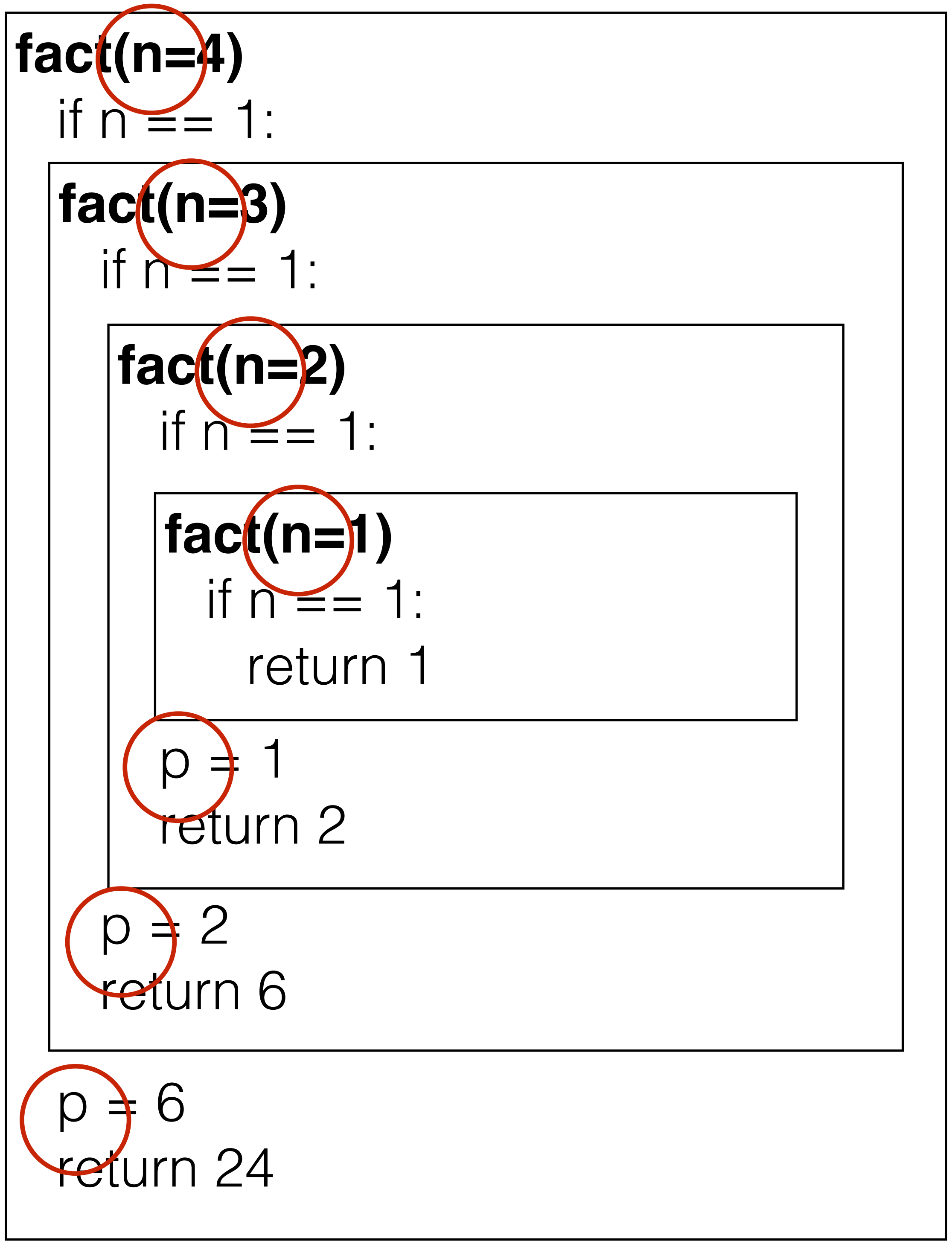
    return n * p
```

How does Python keep
all the variables separate?

frames to the rescue!



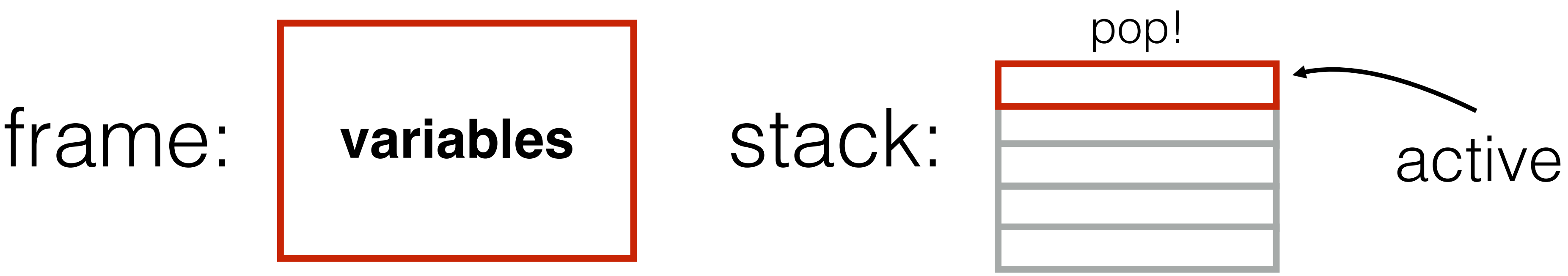
```
fact(n=4)
  if n == 1:
  fact(n=3)
    if n == 1:
    fact(n=2)
      if n == 1:
      fact(n=1)
        if n == 1:
          return 1
      p = 1
      return 2
    p = 2
    return 6
  p = 6
  return 24
```

Deep Dive: Invocation State

In recursion, each function invocation has its **own state**, but multiple invocations **share code**.

Variables for an invocation exist in a ***frame***

- the frames are stored in the **stack**
- one invocation is active at a time: its frame is on the top of stack



frame: **variables**

stack:

pop!

active

Deep Dive: Invocation State

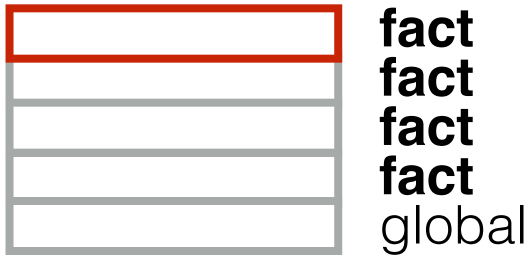
In recursion, each function invocation has its **own state**, but multiple invocations **share code**.

Variables for an invocation exist in a ***frame***

- the frames are stored in the **stack**
- one invocation is active at a time: its frame is on the top of stack
- if a function calls itself, there will be multiple frames at the same time for the multiple invocations of the same function

frame:

stack:



Deep Dive: Runtime Stack

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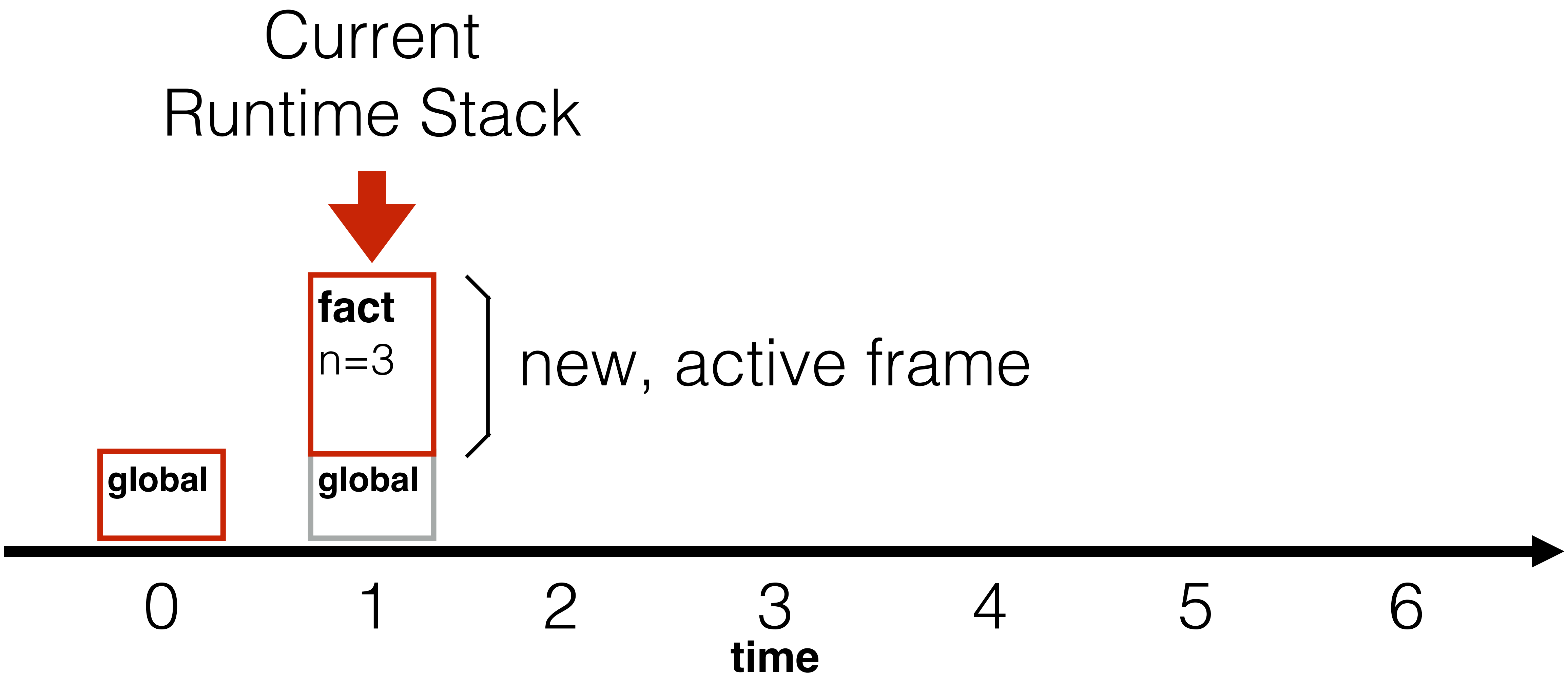

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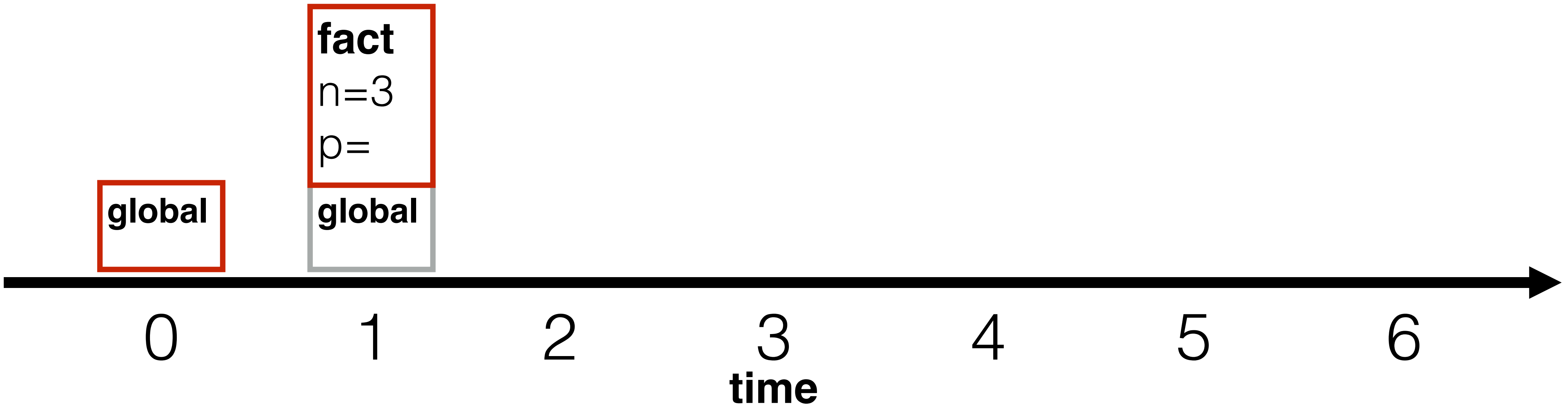

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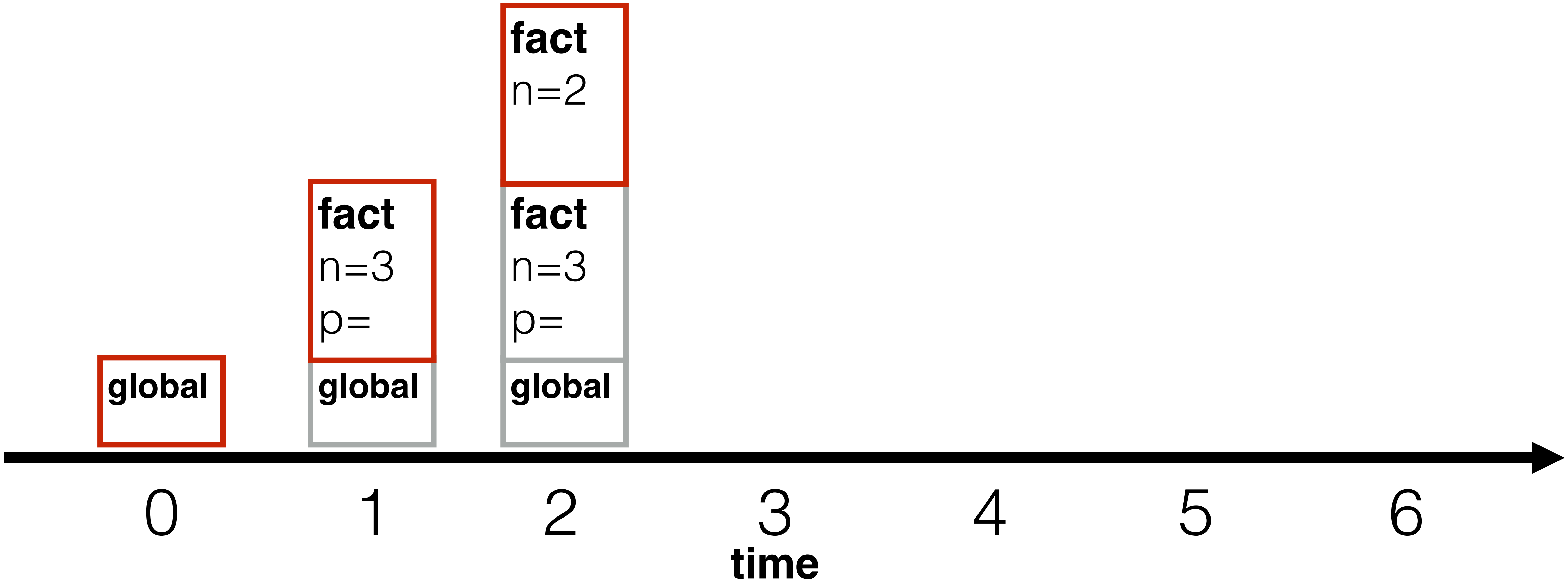

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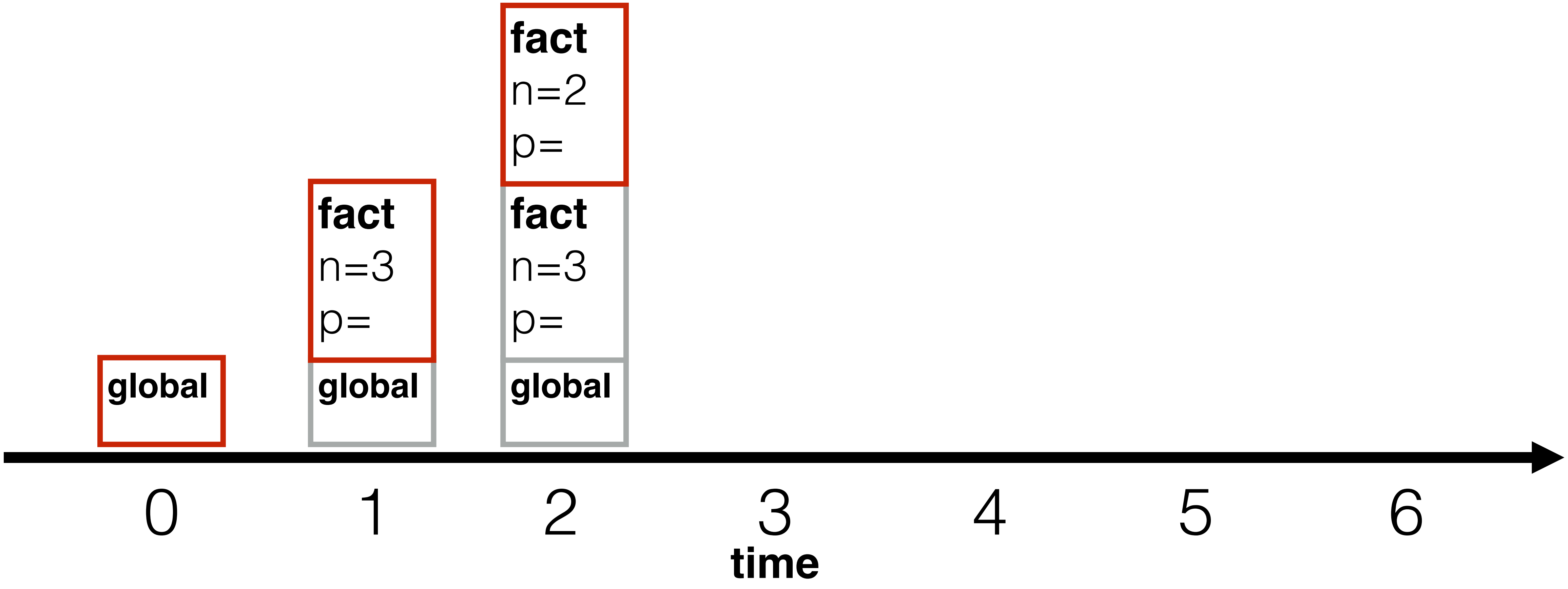

global

fact
n=3
p=

global

fact
n=2
p=

fact
n=3
p=

global

0 1 2 3 4 5 6

time

Deep Dive: Runtime Stack

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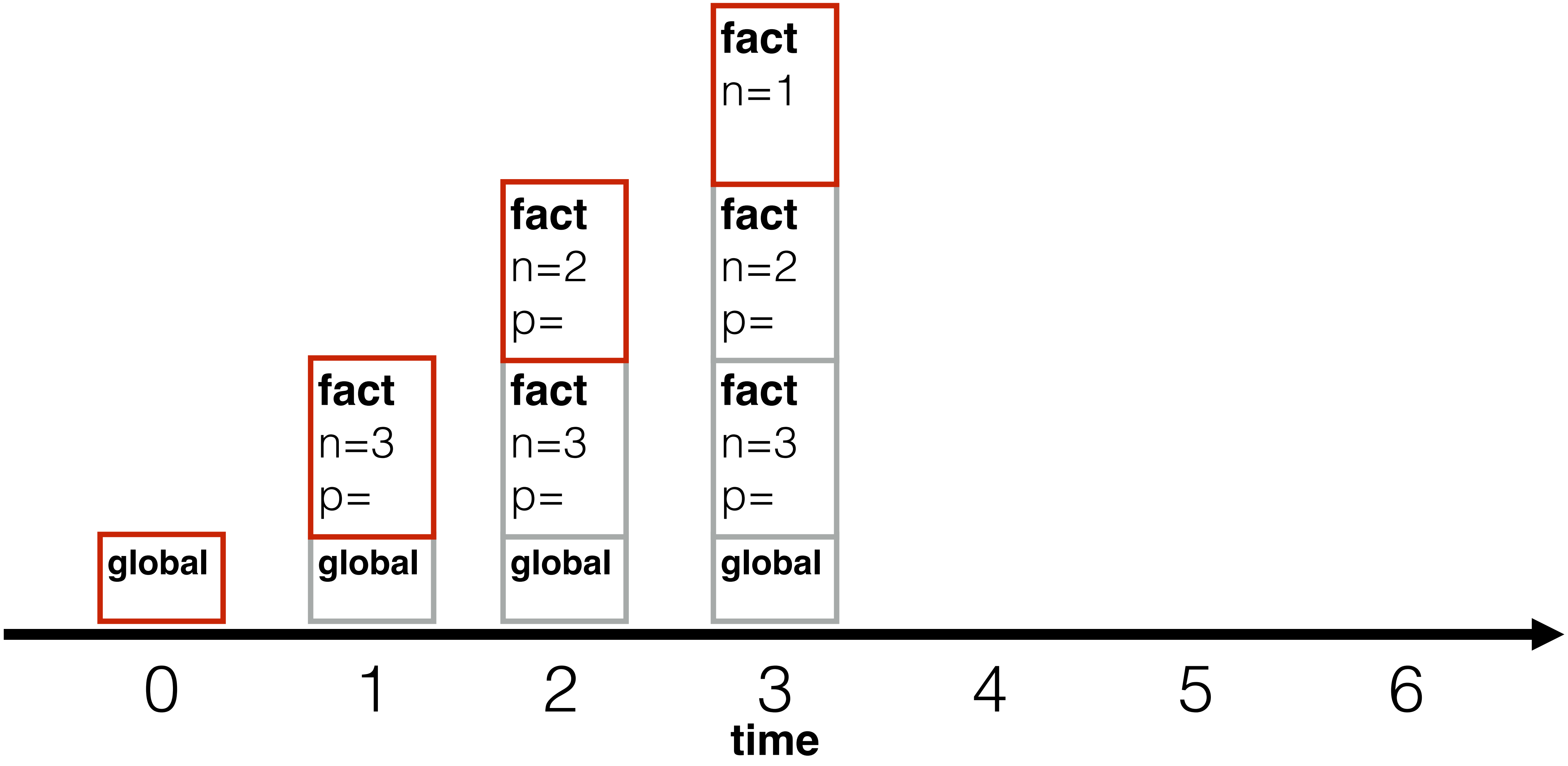

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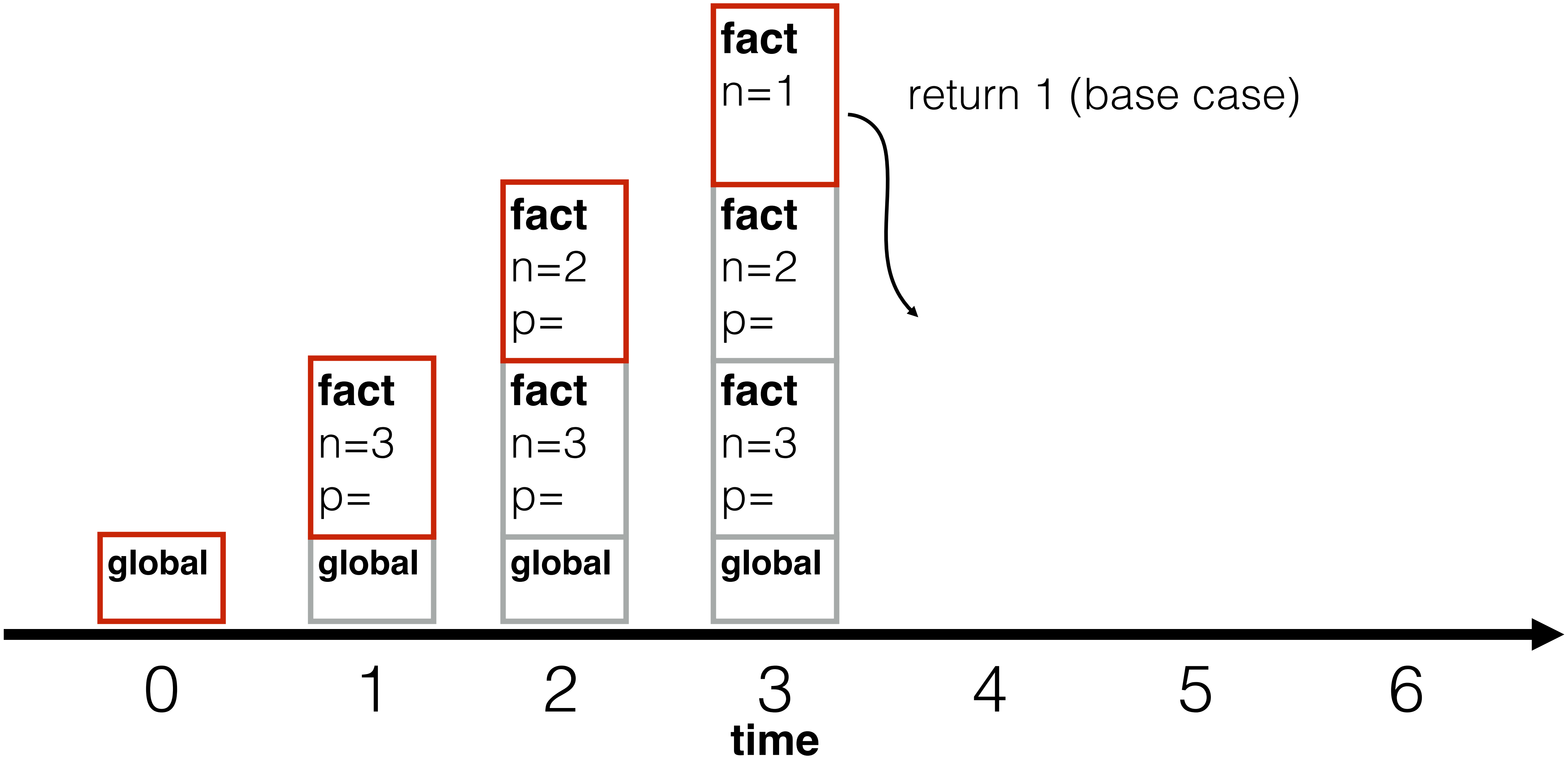

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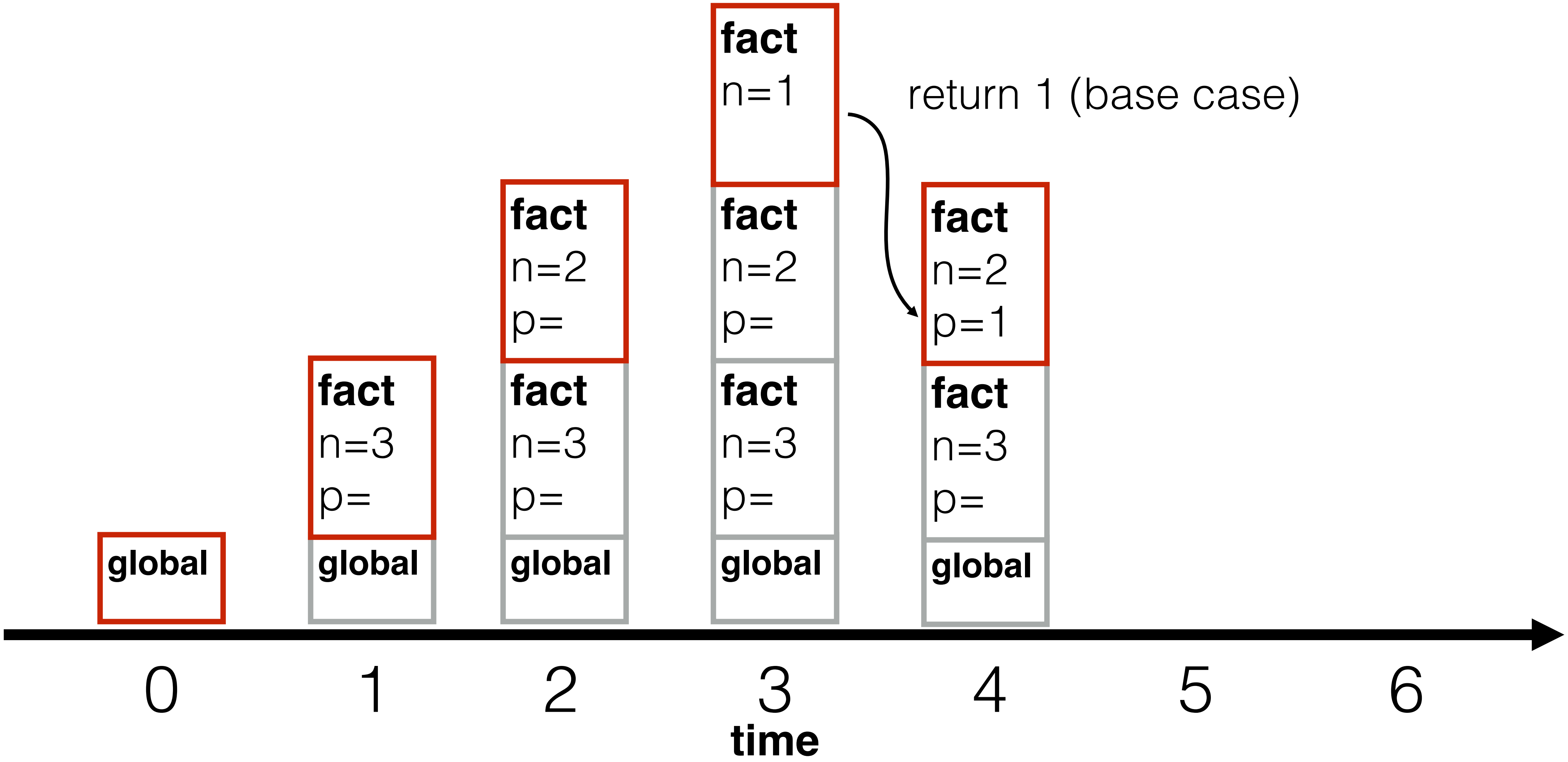
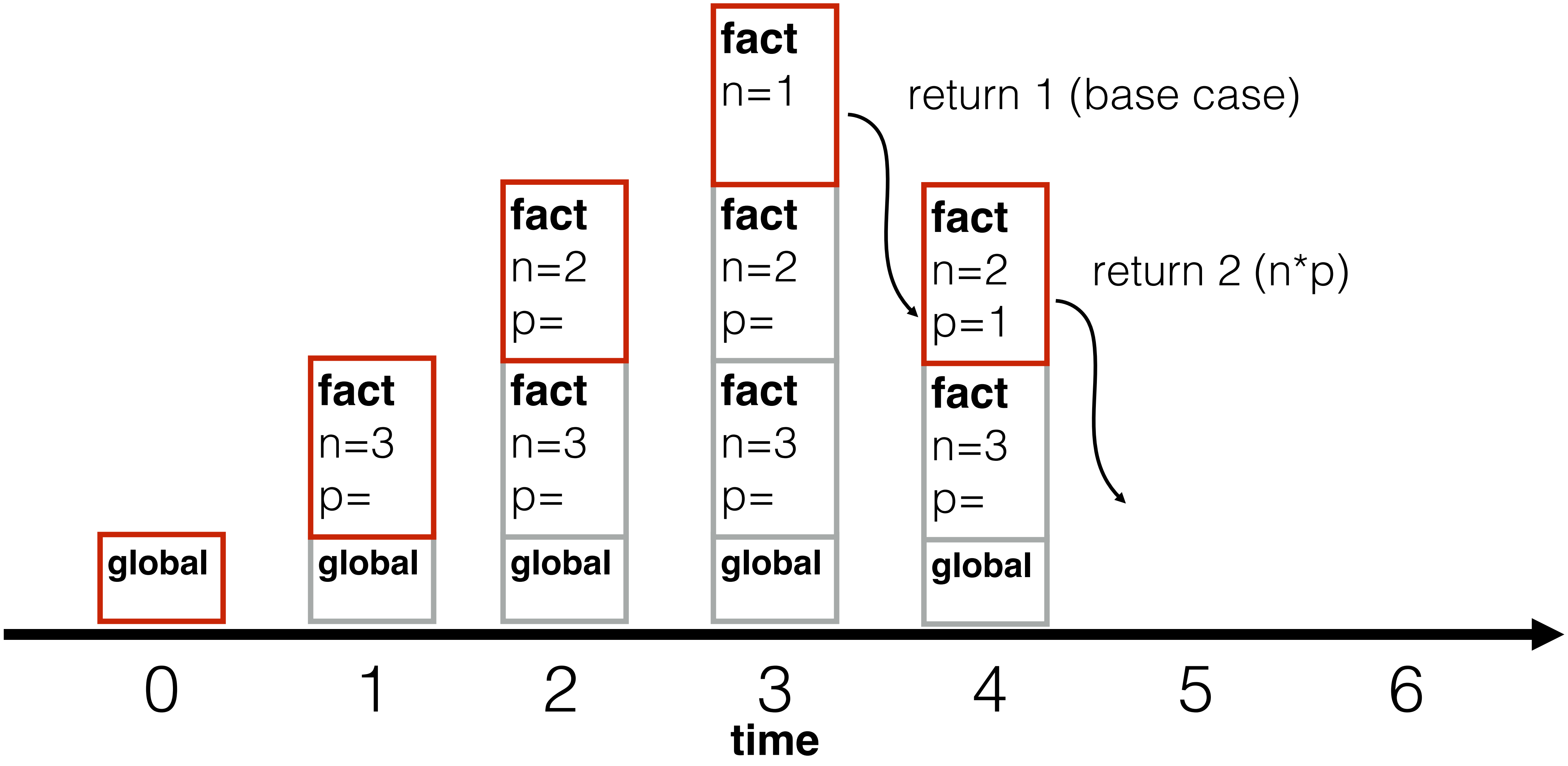
    return n * p
```



Deep Dive: Runtime Stack

```
def fact(n):
    if n == 1:
        return 1
    p = fact(n-1)
    return n * p
```



global

fact n=3 p=

global

fact n=2 p=

fact n=3 p=

global

fact n=1

fact n=2 p=

fact n=3 p=

global

return 1 (base case)

fact n=2 p=1

fact n=3 p=

global

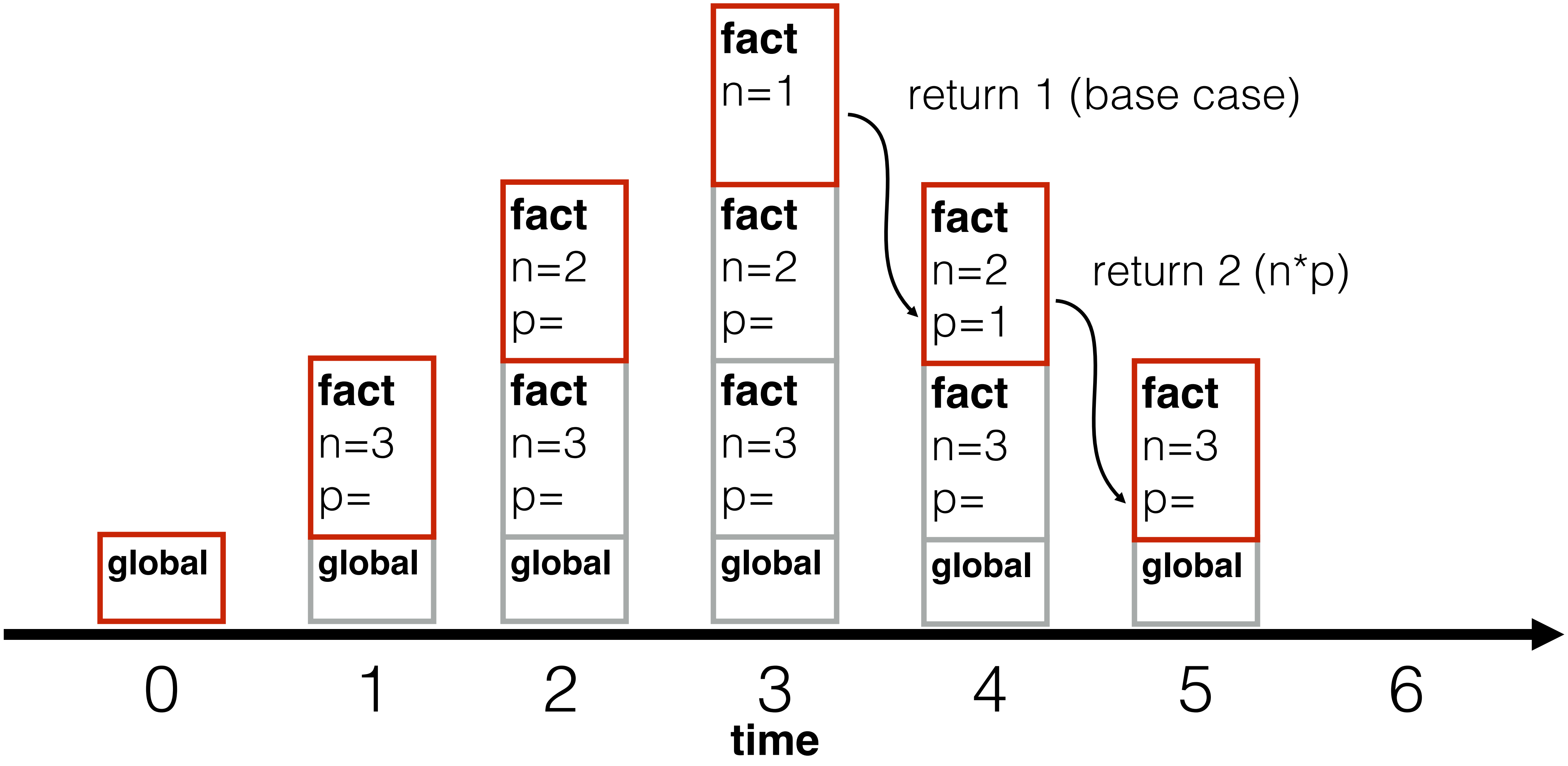
return 2 (n*p)

0 1 2 3 4 5 6

time

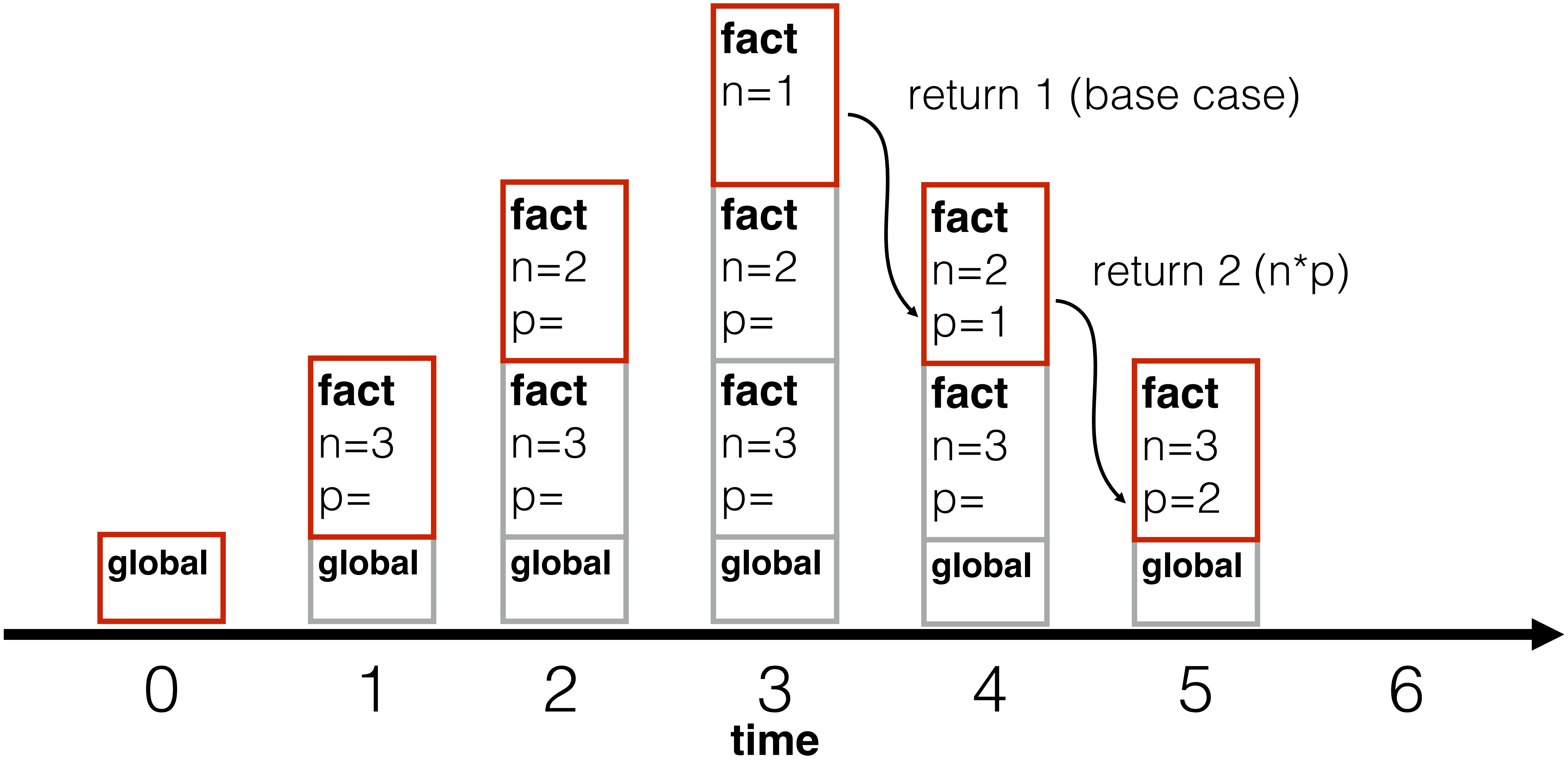
Deep Dive: Runtime Stack

```
def fact(n):
    if n == 1:
        return 1
    p = fact(n-1)
    return n * p
```



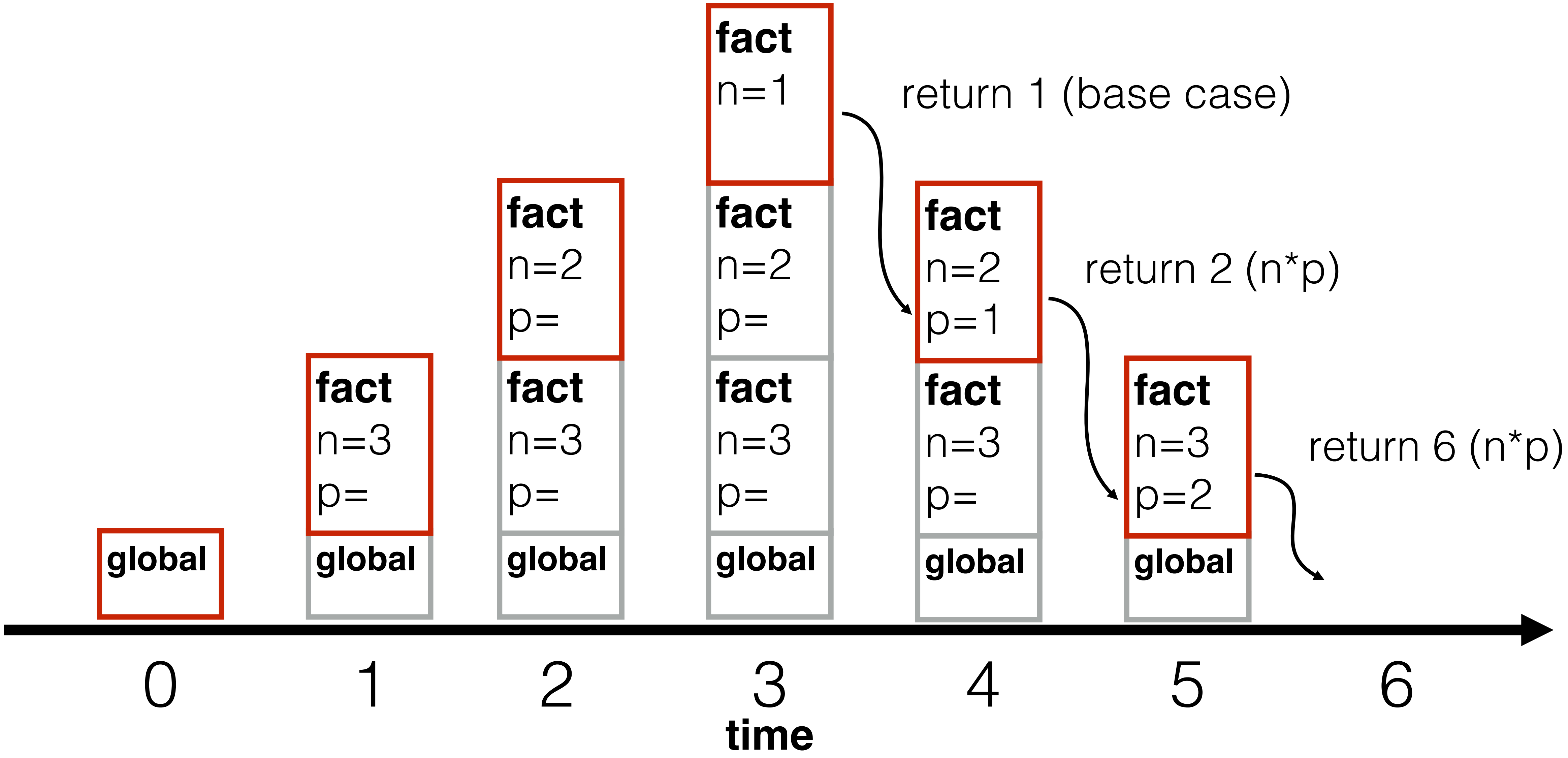
Deep Dive: Runtime Stack

```
def fact(n):
    if n == 1:
        return 1
    p = fact(n-1)
    return n * p
```



Deep Dive: Runtime Stack

```
def fact(n):
    if n == 1:
        return 1
    p = fact(n-1)
    return n * p
```



global

fact
n=3
p=
global

fact
n=2
p=
fact
n=3
p=
global

fact
n=1
fact
n=2
p=
fact
n=3
p=
global

return 1 (base case)

fact
n=2
p=1
fact
n=3
p=
global

return 2 (n*p)

fact
n=3
p=2
global

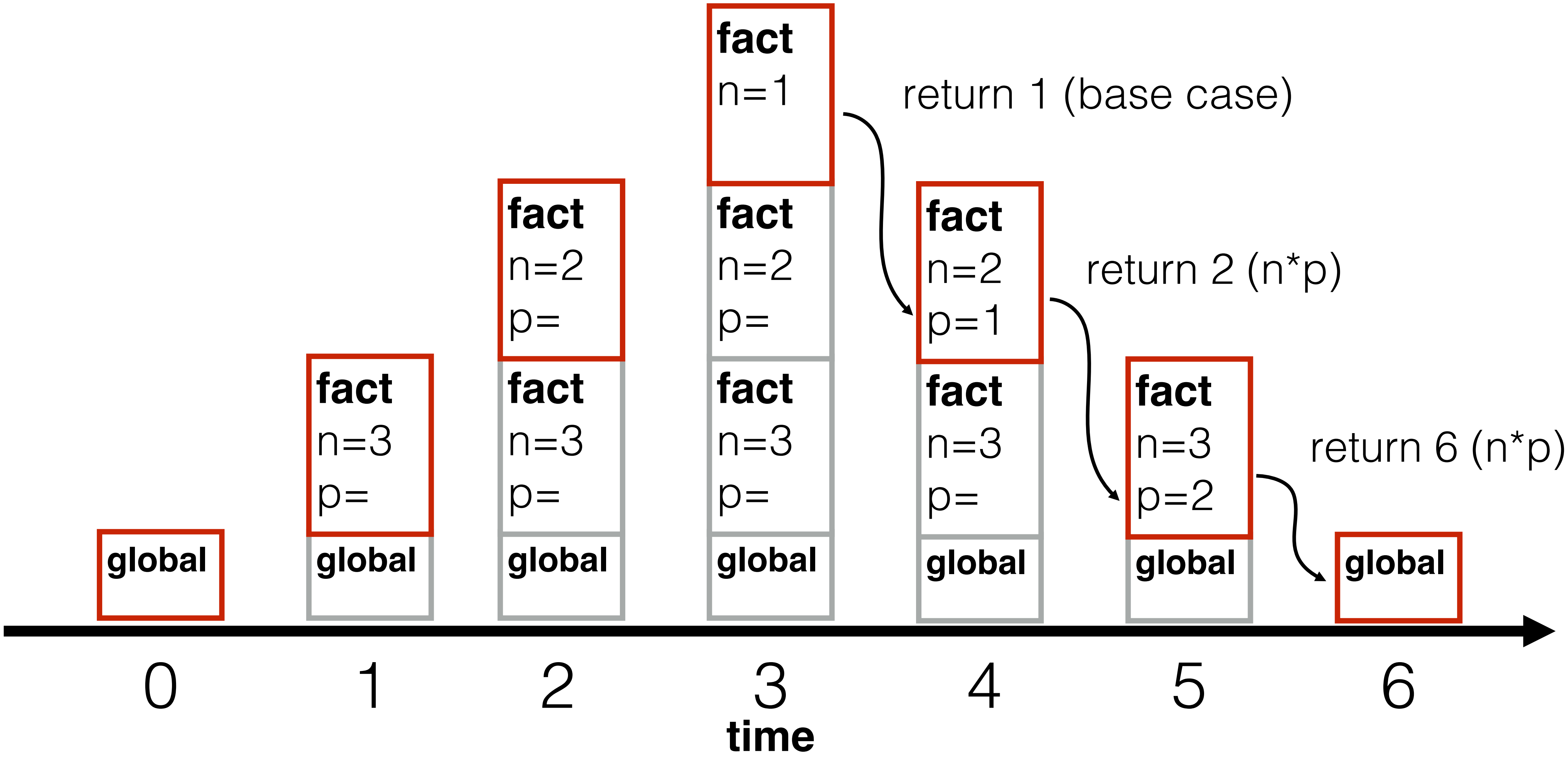
return 6 (n*p)

0 1 2 3 4 5 6

time

Deep Dive: Runtime Stack

```
def fact(n):
    if n == 1:
        return 1
    p = fact(n-1)
    return n * p
```



“Infinite” Recursion Bugs

What happens if:

- factorial is called with a negative number?
-

-1

```
def fact(n):
    if n == 1:
        return 1
    p = fact(n-1)
    return n * p
```

never terminates

“Infinite” Recursion Bugs

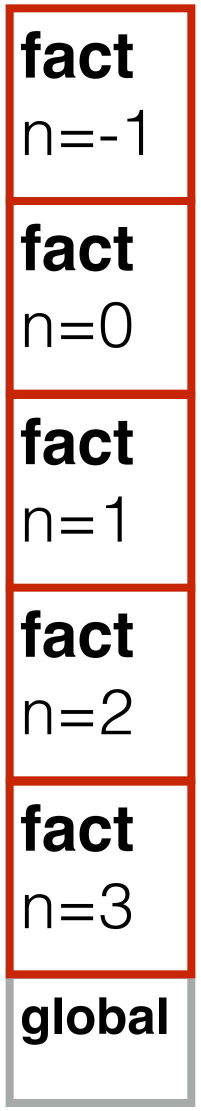
What happens if:

- factorial is called with a negative number?
- we forgot the “n == 1” check?

3

```
def fact(n):
    ~~if n == 1:~~
        ~~return 1~~
    p = fact(n-1)
    return n * p
```

never terminates



fact
n=-1

fact
n=0

fact
n=1

fact
n=2

fact
n=3

global

Coding Demos

Demo 1: Pretty Print

Goal: format nested lists of bullet points

Input:

- The recursive lists

Output:

- Appropriately-tabbed items

Example:

```
>>> pretty_print([“A”, [“1”, “2”, “3”,],
                  “B”, [“4”, [“i”, “ii”]]])
*A
  *1
  *2
  *3
*B
  *4
    *i
    *ii
```

Demo 2: Recursive List Search

Goal: does a given number exist in a recursive structure?

Input:

- A number
- A list of numbers and lists (which contain other numbers and lists)

Output:

- True if there’s a list containing the number, else False

Example:

```
>>> contains(3, [1,2,[4,[[3],[8,9]],5,6]])
True
>>> contains(12, [1,2,[4,[[3],[8,9]],5,6]])
False
```

Conclusion: Review Learning Objectives

Learning Objectives: Recursive Information

What is a recursive definition/structure?

- Definition contains term
- Structure refers to others of same type
- Example: a dictionary contains dictionaries (which may contain...)

/ recursive case

🍃 base case

Learning Objectives: Recursive Code

What is recursive code?

- Function that sometimes itself (maybe indirectly)

Why write recursive code?

- Real-world data/structures are recursive; intuitive for code to reflect data

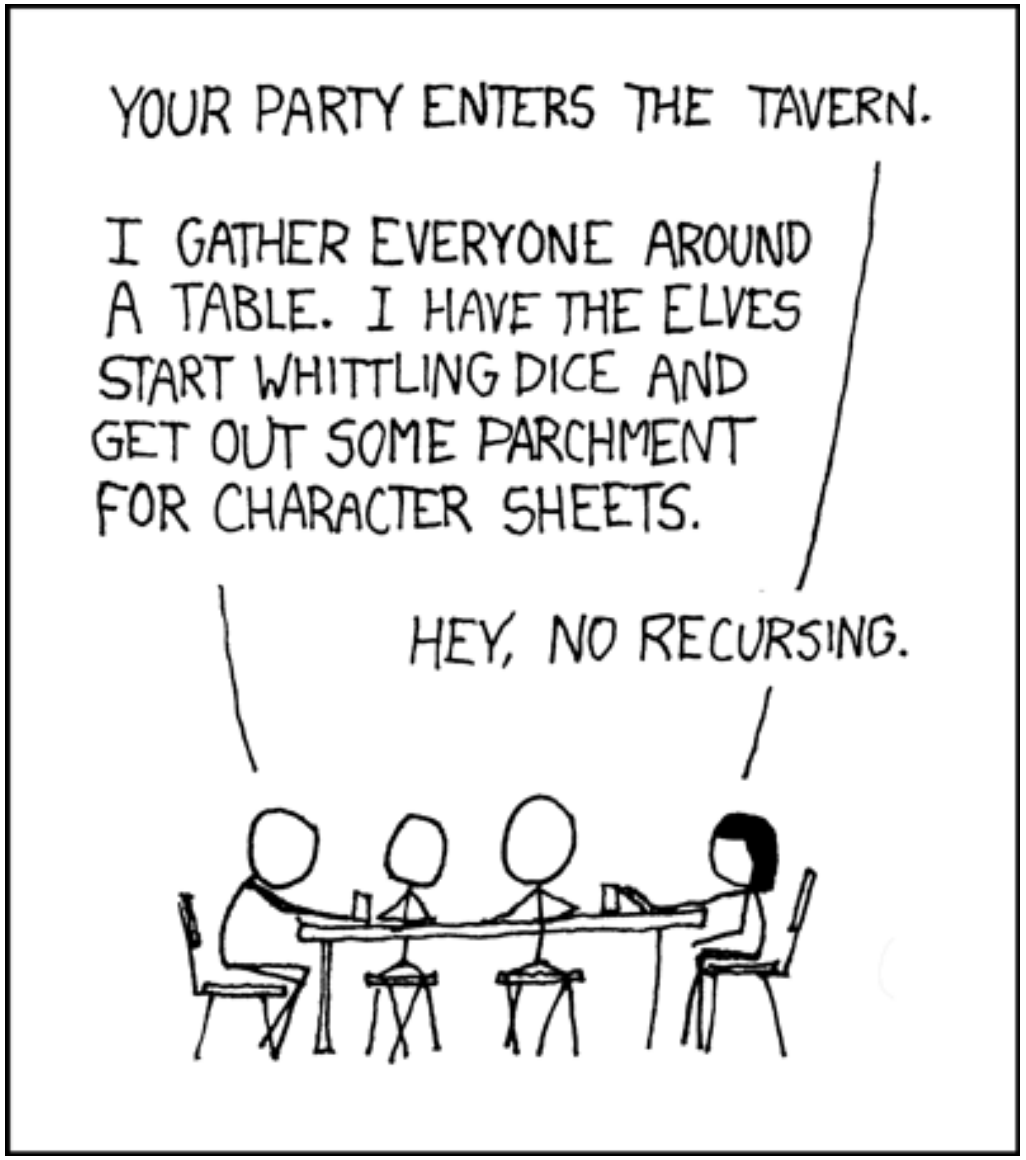
Where do computers keep local variables for recursive calls?

- In a section of memory called a “frame”
- Only one function is **active** at a time, so keep frames in a stack

What happens to programs with infinite recursion?

- Calls keep pushing more frames
- Exhaust memory, throw **StackOverflowError**

Questions?


YOUR PARTY ENTERS THE TAVERN.
I GATHER EVERYONE AROUND A TABLE. I HAVE THE ELVES START WHITTLING DICE AND GET OUT SOME PARCHMENT FOR CHARACTER SHEETS.
HEY, NO RECURSING.

https://xkcd.com/244/