Assignment 3: Detection in Communications

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1 Introduction

As already introduced in the lecture, telecommunications is a field where detection based on the Bayesian decision theory can directly be applied. Thanks to convincing theoretical statistical models, we are even able to analytically derive decision rules—at least if we restrict to the simple AWGN model.

In this assignment we will analyze the capabilities of the popular digital modulation schemes QAM (Quadrature Amplitude Modulation) and PSK (Phase-Shift Keying), starting with the simple AWGN baseband model. Furthermore, we will look into issues of wireless transmission and discuss some approaches to overcome these issues.

The main focus of the assignment is the detection problem, also known as *demapping* in the field of communications. Further theoretical discussions are presented in the lectures of the *Institute of Telecommunications (INUE)* headed by Prof. ten Brink.

2 Fundamentals of Digital Communications

Transmission paths can be described by the entities *transmitter (TX)*, *channel* and *receiver (RX)*. For our purposes, it is sufficient to describe the complete path in the *complex baseband* in discrete time. Side-effects caused by the conversion to analog, transmission in a radio-frequency passband and back-conversion to digital are neglected.

2.1 Complex baseband

All symbols used during transmission can be described by a complex number $s^{(k)} \in \mathbb{C}$ (with k being the discrete time index). The allowed symbols are graphically represented by constellation diagrams. We focus on the QAM and the PSK, see Figure 1 for examples of both modulation schemes (red dots).

In the following, we interpret $\mathbb{C}$ as $\mathbb{R}^2$; this is sufficient and convenient for our detection problem:

$$s^{(k)} = s_{\mathrm{Re}}^{(k)} + \mathrm{j} s_{\mathrm{Im}}^{(k)} \quad \hat{=} \quad \boldsymbol{s}^{(k)} = \left[s_{\mathrm{Re}}^{(k)}, s_{\mathrm{Im}}^{(k)} \right]^T .$$

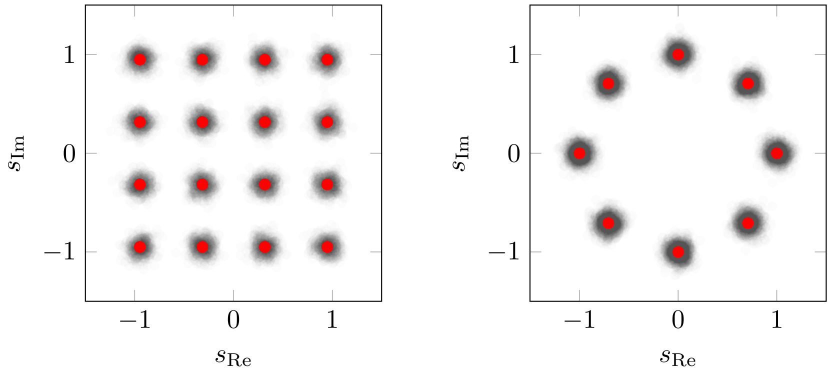

Figure 1: Illustration of 16-QAM and 8-PSK in the IQ-plane

2.2 AWGN channel

During transmission, noise cannot be avoided: Signal degregation due to attenuation of the channel reduces the ratio of transmission and noise power, the *signal to noise ratio (SNR)*.

In the AWGN (additive white Gaussian noise) channel model, we assume that all noise sources can be subsumed under one noise source which produces uncorrelated, normally distributed samples. The noise is added to the symbols we want to transmit. Using an SNR of 30 dB, this results in samples as depicted in Figure 1.

Throughout the assignment, we normalize the symbols to have an average power of one; the power of a transmitted symbol $\boldsymbol{s}^{(k)}$ is given by

$$\left|s^{(k)}\right|^2 = \left[s_{\mathrm{Re}}^{(k)}\right]^2 + \left[s_{\mathrm{Im}}^{(k)}\right]^2.$$

The received symbols

$$r^{(k)} = s^{(k)} + n^{(k)}$$

are a sum of the transmitted symbols $s^{(k)} \in \mathbb{C}$ and complex Gaussian distributed noise samples $n^{(k)} \sim \mathcal{CN}$ (this is equivalent to considering two independent noise variables $n_{\mathrm{Re}}^{(k)} \sim \mathcal{N}(0, \sigma^2)$ and $n_{\mathrm{Im}}^{(k)} \sim \mathcal{N}(0, \sigma^2)$).

Using the formulation with vectors $\in \mathbb{R}^2$ instead of complex symbols, we get

$$\boldsymbol{r}^{(k)} = \boldsymbol{s}^{(k)} + \boldsymbol{n}^{(k)}$$

with $\boldsymbol{n}^{(k)} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$.

In this setup, the signal to noise ratio is given by[1]

$$\mathrm{SNR} = \frac{\mathbb{E}[s^2]}{\mathbb{E}[n^2]} = \frac{1}{2\sigma^2}.$$

2.3 Rayleigh fading channel

In wireless communications, noise due to the receiver's electronics is not the most severe problem. Multipath propagation in none-line-of-sight scenarios (e.g. in

[1]The 2 in the denominator is due to the *complex* Gaussian distribution; we do not go further into the reason for this.

cities) causes the received signal power to vary widely. In the worst case, all propagation paths interfere such that the overall received useful signal vanishes (destructive interference).

One possibility to deal with this issue is *MIMO (mulitple input, multiple output)*. In this assignment we restrict to the case of one transmitter and two receivers. For simplicity, we assume both receivers to be far away from each other such that their signal attenuation is uncorrelated.

The received symbols are

$$\begin{aligned} \boldsymbol{r}_1 &= h_1 \boldsymbol{s} + \boldsymbol{n}_1, \\ \boldsymbol{r}_2 &= h_2 \boldsymbol{s} + \boldsymbol{n}_2. \end{aligned}$$

The Rayleigh distributed coefficients h_1 and h_2 are estimated before passing $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ to our detector.

3 Demapping

To detect (or demap) the received symbols, we use a maximum likelihood (ML) detector. The decision boundaries are simple geometric forms; therefore our detector takes on a simple closed form expression.

3.1 Intersymbol interference

Let's assume that the received symbol does not only depend on the currently transmitted symbol but also on the past. This is called *intersymbol interference* and can be caused by a variety of undesired effects, e.g. imperfect sampling by the receiver, a bad designed pulse shaper, multipath propagation in wireless communications, ...

For simplicity, we only regard a dependence of $\boldsymbol{r}^{(k)}$ on $\boldsymbol{s}^{(k)}$ and the most recent past sample $\boldsymbol{s}^{(k-1)}$. This can be modelled by

$$\boldsymbol{r}^{(k)} = \boldsymbol{s}^{(k)} + \boldsymbol{n}^{(k)} + \alpha \left(\boldsymbol{s}^{(k-1)} + \boldsymbol{n}^{(k-1)} \right) \tag{1}$$

with some known (or estimated) $\alpha \in \mathbb{R}$.

The receiver now shows constellation diagrams as illustrated in Figure 2. Needless to say, detection under this condition is more challenging, especially if α is large.

4 Tasks

4.1 Theoretical discussion

1. Why do we typically assume a uniform prior distribution over all constellation symbols? What does this imply regarding the relationship of MAP and ML detection?

2. Derive the coordinates of the normalized constellation symbols (the constellations are of course constructed such that all symbols are equally spaced in the IQ-plane) of 4-QAM, 16-QAM, 8-PSK and 16-PSK.

   Hint: The average power of a symbol is given by $\mathbb{E}[\|\boldsymbol{s}_i\|^2]$.

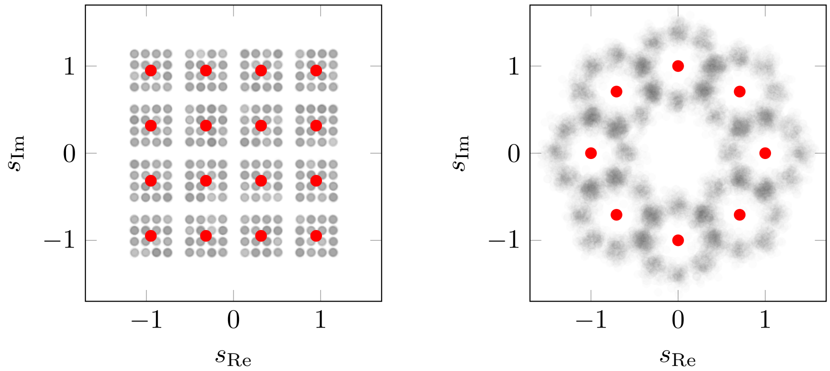

Figure 2: Received symbols if intersymbol interference is present

3. Determine the likelihood $p(\boldsymbol{r} \mid \boldsymbol{s}_i)$ for each symbol $\boldsymbol{s}_i$.

   Hint: How is the distribution of the noise vector transformed if we add a deterministic vector $\boldsymbol{s}_i$?

4. Derive the decision boundaries of 4-QAM and 8-PSK by equalizing appropriate discriminant functions for the constellation symbols. Generalize your results for M-QAM and M-PSK.

   Hint: We only need to consider neighbouring symbols for the decision boundaries as the likelihood of symbols far away from the regarded region is far below the likelihoods of the neigbouring symbols.

5. Show that the (symbol) error rate under your previously derived ML detector for 4-QAM can be approximated by

$$P_{\mathrm{s}} \approx 2Q\left(\sqrt{\mathrm{SNR}}\right).$$

   Hint: Use the approximation $x^2 + x \approx x$ if $x \ll 1$.

6. *Optional:* Intersymbol interference can be overcome if applying an appropriate digital filter before decision making. Give the transfer function $H(z)$ of the filter.

7. Alternatively, we can derive the new likelihoods $p(\boldsymbol{r} \mid \boldsymbol{s}_i)$. Show that $\boldsymbol{r}$ is distributed according to a Gaussian mixture model.

   Hint: Take a careful look at Equation (1). The likelihood $p(\boldsymbol{r}^{(k)} \mid \boldsymbol{s}_i^{(k)})$ now depends on $\boldsymbol{s}^{(k-1)}$ which is *not* deterministic.

8. *Optional:* Propose a detection algorithm that makes use of two independent receivers as discussed in Section 2.3.

4.2 Programming/Simulation tasks

9. Write a method that outputs an array of the possible symbols of an arbitrary M-QAM and M-PSK constellation scheme. The symbols should be normalized to have an average transmission power of one. Visualize the constellation points in the IQ-plane.

10. Estimate the symbol error rate of QAM and PSK modulation schemes for a given SNR.

    This can be done by a *Monte Carlo simulation*: Randomly generate a large number N of transmission symbols, corrupt them by additive white Gaussian noise of the desired variance (determined by the SNR) and estimate the transmission symbol. Then count the number K of falsely classified symbols. An estimate of the symbol error rate is given by $P_\mathrm{s} \approx K/N$. The larger N, the better the estimate.

    Compare your results to Task 5.

    Which modulation scheme would you recommend in communication systems with a high SNR, which in one with poor SNR?

    Compare the performance of 16-PSK to 16-QAM. Explain the difference in performance.

    Hint: To compare modulation schemes, it is common to plot P_s in dependence of the SNR. Such a plot can be obtained by calculating estimates for P_s for a list of equally spaced SNR, e.g.

    $$\mathrm{SNR} \in \{0\,\mathrm{dB}, 5\,\mathrm{dB}, 10\,\mathrm{dB}, \ldots, 25\,\mathrm{dB}, 30\,\mathrm{dB}\}.$$

11. Complete the method which outputs the likelihoods of a received symbol. The method should be capable to handle intersymbol interference modelled by (1).

    Is your maximum likelihood detector able to correctly classify most symbols, even if α is large? Compare your symbol error rates to the ones obtained if intersymbol interference is not present.

12. *Optional:* Visualize the decision regions for each symbol using `matplotlib`.

5 Solutions

1. It does not make sense to send some symbols more often than others. There is a source encoder in the pipeline before transmission which outputs almost equally probable bits. Those are mapped to symbols which are therefore uniformly distributed.

   MAP and ML are equivalent under this condition.

2. *4-QAM:* The four unnormalized symbols have coordinates $\boldsymbol{s}_1' = [1,1]^T$, $\boldsymbol{s}_2' = [-1,1]^T$, $\boldsymbol{s}_3' = [1,-1]^T$ and $\boldsymbol{s}_4' = [-1,-1]^T$. They all share the transmission power $\|\boldsymbol{s}_i'\|^2 = 2$. The normalized symbols $\boldsymbol{s}_i = a\boldsymbol{s}_i'$ are defined by $\mathbb{E}[\|\boldsymbol{s}_i\|^2] = 1$. Thus $a = 1/\sqrt{2}$ and

$$\begin{aligned}
\boldsymbol{s}_1 &= \left[1/\sqrt{2}, 1/\sqrt{2}\right]^T \approx [0.7, 0.7]^T, \\
\boldsymbol{s}_2 &= \left[-1/\sqrt{2}, 1/\sqrt{2}\right]^T \approx [-0.7, 0.7]^T, \\
\boldsymbol{s}_3 &= \left[1/\sqrt{2}, -1/\sqrt{2}\right]^T \approx [0.7, -0.7]^T, \\
\boldsymbol{s}_4 &= \left[-1/\sqrt{2}, -1/\sqrt{2}\right]^T \approx [-0.7, -0.7]^T.
\end{aligned}$$

   16-QAM: Analogously, we obtain the normalized symbols for higher QAM constellations as 16-QAM. The unnormalized symbols are all possible combinations of ± 1 and ± 3 for both coordinates. Therefore, there are four symbols of power 2, four symbols of power 18 and eight symbols of power 10. The average power is 10, the normalization constant $a = 1/\sqrt{10}$.

   8-PSK: PSK constellations consist of equally spaced symbols on the unit circle, so the average power is one by definition. The symbols are defined by their angle ϕ with respect to the real axis: $\phi_i = i \cdot 2\pi/8 = i \cdot \pi/4$. The real and the imaginary part can be obtained using $s_{i,\mathrm{Re}} = \cos\phi_i$ and $s_{i,\mathrm{Im}} = \sin\phi_i$. Thus 8-PSK consists of the symbols

$$\begin{aligned}
\boldsymbol{s}_1 &= [1,0]^T, & \boldsymbol{s}_2 &= \left[1/\sqrt{2}, 1/\sqrt{2}\right]^T \approx [0.7, 0.7]^T, \\
\boldsymbol{s}_3 &= [0,1]^T, & \boldsymbol{s}_4 &= \left[-1/\sqrt{2}, 1/\sqrt{2}\right]^T \approx [-0.7, 0.7]^T, \\
\boldsymbol{s}_5 &= [-1,0]^T, & \boldsymbol{s}_6 &= \left[-1/\sqrt{2}, -1/\sqrt{2}\right]^T \approx [-0.7, -0.7]^T, \\
\boldsymbol{s}_7 &= [0,-1]^T, & \boldsymbol{s}_8 &= \left[1/\sqrt{2}, -1/\sqrt{2}\right]^T \approx [0.7, -0.7]^T.
\end{aligned}$$

   16-PSK: Similar derivation, no simple exact algorithmic expression for the coordinates.

3. All received symbols are normally distributed: $\boldsymbol{r} \mid \boldsymbol{s}_i \sim \mathcal{N}(\boldsymbol{s}_i, \sigma^2 \boldsymbol{I})$. The likelihood is given by

$$p(\boldsymbol{r} \mid \boldsymbol{s}_i) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2}\|\boldsymbol{r} - \boldsymbol{s}_i\|^2\right).$$

4. Equalizing the likelihoods, we obtain straight decision boundaries passing through the middle point between neighbouring constellation symbols.

   4-QAM: The decision boundaries are $s_{\mathrm{Re}} = 0$ and $s_{\mathrm{Im}} = 0$.

   8-PSK: The decision boundaries are straight lines passing through

$$\begin{aligned} \boldsymbol{m}_i &= \frac{1}{2}\big(\boldsymbol{s}_i + \boldsymbol{s}_{i+1}\big) \\ &= \frac{1}{2}\left[\cos\left(\frac{\pi}{4}i\right) + \cos\left(\frac{\pi}{4}i + \frac{\pi}{4}\right), \sin\left(\frac{\pi}{4}i\right) + \sin\left(\frac{\pi}{4}i + \frac{\pi}{4}\right)\right]^T \end{aligned}$$

   and orthogonal to

$$\begin{aligned} \boldsymbol{n}_i &= \boldsymbol{s}_i - \boldsymbol{s}_{i+1} \\ &= \left[\cos\left(\frac{\pi}{4}i\right) - \cos\left(\frac{\pi}{4}i + \frac{\pi}{4}\right), \sin\left(\frac{\pi}{4}i\right) - \sin\left(\frac{\pi}{4}i + \frac{\pi}{4}\right)\right]^T. \end{aligned}$$

   They are given by

$$\boldsymbol{n}_i^T[\boldsymbol{r} - \boldsymbol{m}_i] = 0.$$

   Similar results for higher constellations.

5. Given that $\boldsymbol{s}_1$ was transmitted, the probability of a correct decision is

$$\begin{aligned} P(\hat{\boldsymbol{s}} = \boldsymbol{s}_1 \,|\, \boldsymbol{s} = \boldsymbol{s}_1) &= P(r_{\mathrm{Re}} > 0, r_{\mathrm{Im}} > 0 \,|\, \boldsymbol{s} = \boldsymbol{s}_1) \\ &= \left(\int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(r - 1/\sqrt{2}\right)^2\right) \mathrm{d}r\right)^2 \\ &= \left[\int_{-1/(\sqrt{2}\sigma)}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\tilde{r}^2\right) \mathrm{d}\tilde{r}\right]^2 \\ &= \left[Q\left(-\frac{1}{\sqrt{2\sigma^2}}\right)\right]^2 \\ &= \left[1 - Q\left(\frac{1}{\sqrt{2\sigma^2}}\right)\right]^2 \\ &= 1 - 2Q\left(\frac{1}{\sqrt{2\sigma^2}}\right) + \left[Q\left(\frac{1}{\sqrt{2\sigma^2}}\right)\right]^2 \\ &\approx 1 - 2Q\left(\frac{1}{\sqrt{2\sigma^2}}\right). \end{aligned}$$

   Exploiting the symmetry of our problem, we finally get the symbol error rate

$$\begin{aligned} P_{\mathrm{s}} &= 1 - \sum_{i=1}^{4} P(\hat{\boldsymbol{s}} = \boldsymbol{s}_i \,|\, \boldsymbol{s} = \boldsymbol{s}_i) P(\boldsymbol{s} = \boldsymbol{s}_i) \\ &= 1 - P(\hat{\boldsymbol{s}} = \boldsymbol{s}_1 \,|\, \boldsymbol{s} = \boldsymbol{s}_1) \\ &\approx 2Q\left(\frac{1}{\sqrt{2\sigma^2}}\right) = 2Q\left(\sqrt{\mathrm{SNR}}\right). \end{aligned}$$

6. Intersymbol interference can be modelled by an FIR filter, in our case
$$G(z) = 1 + \alpha z^{-1}.$$
We can invert this by applying an IIR filter
$$H(z) = \frac{1}{G(z)} = \frac{1}{1 + \alpha z^{-1}}$$
(which can also be approximated by an FIR filter).

7. We model the received symbols by
$$\begin{aligned} \boldsymbol{r}^{(k)} &= \boldsymbol{s}^{(k)} + \boldsymbol{n}^{(k)} + \alpha \left(\boldsymbol{s}^{(k-1)} + \boldsymbol{n}^{(k-1)} \right) \\ &= \boldsymbol{s}^{(k)} + \alpha \boldsymbol{s}^{(k-1)} + \underbrace{\boldsymbol{n}^{(k)} + \alpha \boldsymbol{n}^{(k-1)}}_{\tilde{\boldsymbol{n}}}. \end{aligned}$$
$\tilde{\boldsymbol{n}}$, a linear combination of Gaussian distributed variables, is normally distributed as well. Subsequent symbols $\boldsymbol{s}^{(k)}$ and $\boldsymbol{s}^{(k-1)}$ are independent of each other. Conditioned on the current sample $\boldsymbol{s}^{(k)}$, $\boldsymbol{s}^{(k)} + \alpha \boldsymbol{s}^{(k-1)}$ is uniformly distributed on
$$\left\{ \boldsymbol{s}^{(k)} + \alpha \boldsymbol{s}_i \,\middle|\, \boldsymbol{s}_i \in \mathcal{S} \right\}.$$
Therefore, $\boldsymbol{r}^{(k)}$ is distributed according to a mixture of Gaussians.

8. The higher h_i, the better the SNR on the received path $i \in \{1, 2\}$. A simple method is to detect the symbols only based on the path with the higher SNR, i.e. the higher h_i.

   Detection can be improved if we use *maximal ratio combination* [**?**, section 3.2]. To get some insight, it is sufficient to only analyze the convenient case of one-dimensional symbols (i.e. PAM). The received symbols are
$$\begin{aligned} r_1 &= h_1 s + n_1, \\ r_2 &= h_2 s + n_2. \end{aligned}$$
The joint density of the received symbols given the transmitted symbol s_i can be written as
$$p(r_1, r_2 \,|\, s_i) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2} \left((r_1 - h_1 s_i)^2 + (r_2 - h_2 s_i)^2 \right) \right).$$
The decision rule between neighbouring symbols $s_i < s_j$ can be simplified,
$$\begin{aligned} p(r_1, r_2 \,|\, s_i) &\underset{s_j}{\overset{s_i}{\gtrless}} p(r_1, r_2 \,|\, s_j) \\ (r_1 - h_1 s_i)^2 + (r_2 - h_2 s_i)^2 &\underset{s_i}{\overset{s_j}{\gtrless}} (r_1 - h_1 s_j)^2 + (r_2 - h_2 s_j)^2 \\ -2(r_1 h_1 + r_2 h_2) s_i + (h_1^2 + h_2^2) s_i^2 &\underset{s_j}{\overset{s_j}{\gtrless}} -2(r_1 h_1 + r_2 h_2) s_i + (h_1^2 + h_2^2) s_j^2 \\ 2(r_1 h_1 + r_2 h_2) &\underset{s_i}{\overset{s_j}{\gtrless}} (s_i + s_j)(h_1^2 + h_2^2) \\ \frac{r_1 h_1 + r_2 h_2}{h_1^2 + h_2^2} &\underset{s_i}{\overset{s_j}{\gtrless}} \frac{s_i + s_j}{2}. \end{aligned}$$

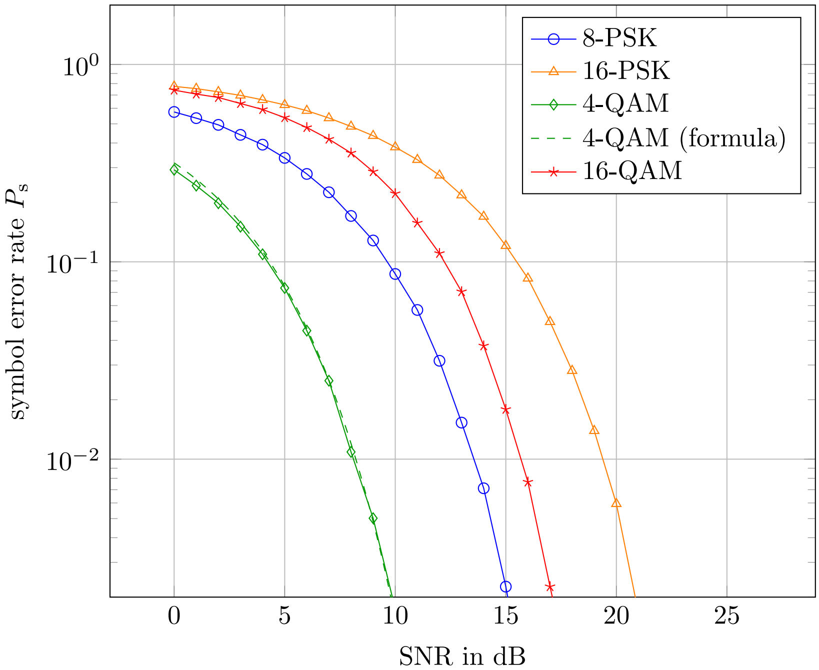

Figure 3: Symbol error rates of some selected constellations

Instead of looking which constellation point our single received symbol is closest to, we now combine both received information according to

$$r_{\text{combined}} = \frac{r_1 h_1 + r_2 h_2}{h_1^2 + h_2^2}.$$

9. See code.

10. See code and Figure 3 for simulation results.

    The theoretical result on 4-QAM is validated by our simulation.

    If SNR is high enough, high modulation schemes are favorable because of the higher data rates; if SNR is poor, reliable communications only possible with low modulation scheme.

    16-PSK performs worse than 16-QAM for a fixed SNR because the constellation points are not that thight.

11. See code. Our detector can handle even a high amount of intersymbol interference. For increasing α it's more likely that received symbols of different sent symbols are exactly overlapping, even on high SNR. This, of course, increases the symbol error rate and leads to a saturation of P_s for high SNRs.

12. See code.