PRÁCTICA 3

1) $\underbrace{2K}_{} = 2K+1$ $K \in \mathbb{Z}$

$\omega = \omega + 1$ Absurdo

2) $a|1 \rightarrow a = 1 \vee a = -1$ VERDADERO

a) $a|1$ si $\exists c \in \mathbb{Z} : 1 = a.c$

si $a = 1 \rightarrow c = 1$ $(\in \mathbb{Z}) \Rightarrow \frac{1}{c} = a = \frac{1}{1} = 1$

si $a = -1 \rightarrow c = -1$ $(\in \mathbb{Z}) \Rightarrow \frac{1}{c} = a = \frac{1}{-1} = -1$

b) $a|b$ y $b|c$ entonces $a|c$ VERDADERO

$$\left.\begin{array}{l} b|c \rightarrow \exists d \in \mathbb{Z} : c = bd \\ a|b \rightarrow \exists e \in \mathbb{Z} : b = ae \end{array}\right\} \rightarrow c = (\overbrace{ae}^{b}).d$$

↓ PROPIEDAD ASOCIATIVA

$$c = a(\underbrace{e.d}_{\in \mathbb{Z}})$$

c) $a(a-1)$ es par VERDADERO

2 opciones → $a = 2K \longrightarrow a(a-1) = 2K(2K-1) = 2\overbrace{[K(2K-1)]}^{w \in \mathbb{Z}}$

↳ $a = 2K+1 \longrightarrow a(a-1) = 2K+1 . 2K = 2\underbrace{[K(2K+1)]}_{z \in \mathbb{Z}}$

d) $x|y$ y $y|z \rightarrow x|yz$

$y|z \rightarrow \exists c \in \mathbb{Z} : z = y.c$ $x|yz \rightarrow \exists h \in \mathbb{Z} : yz = xh$

$x|y \rightarrow \exists d \in \mathbb{Z} : y = x.d$

$yz = y.z = (xd).(y.c)$

↓ Propiedad Asociativa

$$yz = x(\underbrace{d.y.c}_{h \in \mathbb{Z}})$$

3) $N = 5q_1 + 3$

$N = 7q_2 + 4$

$N = 35q + r$

$7N = 35q_1 + 21 \rightarrow 2(7N) = 35(q_1 . 2) + 21 . 2 = 35(2q_1) + 42$

$5N = 35q_2 + 20 \rightarrow 3(5N) = 35(q_2 3) + 20 . 3 = 35(3q_2) + 60$

$15N \rightarrow 35(3q_1) + 60$

$- 14N \rightarrow - 35(2q_2) + 42$

N

$35q_3 + 18 = N$

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4) $a = 11q_1 + 4$

$b = 11q_2 + 7$

$(a + b^2) = 11q + r$

$[(11q_1 + 4) + (11q_2 + 7)^2] = 11q + r$

$[11q_1 + 4 + 121q_2^2 + 2(11q_2).7 + 49] = 11q + r$

$11q_1 + 121q_2^2 + 11q_2 . 14 + 53 = 11q + r$

$11(\underbrace{q_1 + 11q_2^2 + 14q_2}_{K \in \mathbb{Z}}) + 53 = 11q + r$

$11.(\underbrace{K + 4}_{q}) + 9 = 11q + r$

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5) a)

98 |8

18 12 |8

4 1 |8

4 0

1 = 144

b)

44 |8

5 |8

4 0

5 = 54

c)

20 |8

2 |8

4 0

2 = 24

6) i) (16, 24) Método "Subir por los restos"

$24 = 16.1 + 8$

$16 = 8.2 + 0$ → Como el resto es 0, tenemos que 8 es el MCD de (16, 24)

$(24,16) = (16,8) = (8,0)$

ii) (70, 50)

$70 = 50.1 + 20$

$(70,50) = (50,20) = (20,10) = (10,0)$

$50 = 20.2 + 10$

$20 = 10.2 + 0$ MCD = 20

iii) (121, 88)

$121 = 88.1 + 33$

$88 = 33.2 + 22$

$(121,88) = (88,33) = (33,22) = (22,11) = (11,0)$

$33 = 22.1 + 11$

$22 = 11.2 + 0$ MCD = 11

iv) (-90, 90) $(-90,90) = (90,0)$

$-90 = 90.(-1) + 0$ MCD = 90

v) (980, 224)

$980 = 224.4 + 84$

$224 = 84.2 + 56$ $(980,224) = (224,84) = (84,56) = (56,28) =$

$84 = 56.1 + 28$ $= (28,0)$

$56 = 28.2 + 0$ MCD = 28

7) a) $a+b$ es coprimo con a

$d=(a+b,\ a) \quad d\in\mathbb{Z}$

$$\left.\begin{matrix} d \mid a+b \\ d \mid a \end{matrix}\right\} \rightarrow d \mid b$$

$d \mid a+b \Rightarrow a+b=d.c$

$d \mid a \Rightarrow a=d.e$

$(a+b)-a=dc-de=d(c-e)$

$b=d\underbrace{(c-e)}_{\in\mathbb{Z}} \rightarrow d \mid b$

por definición:

$d \mid mcd(a,b)$

pero si a y b son coprimos ($MCD(a,b)=1$)

$\rightarrow d \mid 1 \therefore d=1$

b) si a es no nulo, $(a,0)=|a|$

$a \mid a$ y $a \mid 0 \rightarrow mcd(a,0)=|a|$

(siendo $a\neq 0$)

c) $(a,b)=1$ entonces $ma+nb=K$ con m, n y $K \in\mathbb{Z}$

por hipótesis: $(a,b)=1$

Por el teorema de Bézout, existen m y $n \in\mathbb{Z}$ tales que:

$$1=ma+nb$$

$$K.1=K(ma+nb)$$

$$K=\underbrace{(Km)}_{\in\mathbb{Z}}a+\underbrace{(Kn)}_{\in\mathbb{Z}}b$$

8) mcd$(5K+3, 3K+2)$, para cualquier $K \in \mathbb{Z}$

$5K+3 = (3K+2) + 2K+1$

$3K+2 = (2K+1) + K+1$

$2K+1 = (K+1) + K$

$K+1 = (K) + 1$ mcd$(K, 1) = 1$ son coprimos

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9) $a, b \in \mathbb{Z}$ y p primo. si $p|ab \rightarrow p|a \vee p|b$

Lema de Euclides: Si p es primo y $p|ab \rightarrow p|a$ o $p|b$

Caso 1: $p|a$, por ende se cumple la condición $\Rightarrow p|a \Rightarrow a = pe$

$a.b = (pe).b =$
$= p(\underline{eb})$
$\in \mathbb{Z}$

Caso 2: $p \nmid a \rightarrow a$ y p son coprimos [mcd = 1]

si a y p son coprimos, por el teorema de Bézout si mcd$(a,p) = 1 \ni$ dos $\mathbb{Z}$ tales que: $ax + py = 1$ ↳(x e Y)

multiplicamos por b

$abx + pby = b$

[como $p|ab \rightarrow p|axb$ $p|pby$] $\rightarrow p|b$

Aclaración: Si $p \nmid a$ entonces $p|b$ si o si.

Si p no es primo:

$p|ab \rightarrow p \nmid a$ y $p \nmid b$ quiere decir que $p = xy$ $(x, y \in \mathbb{Z})$ compuesto.

Ej: $p = 6$, $a = 2$ y $b = 3$

$6|(2 \times 3) = 6 \rightarrow 6|ab$

pero $6 \nmid 2$ y $6 \nmid 3$

Entonces si p no es primo no se puede asegurar que: $p|ab \rightarrow p|a \vee p|b$

10) $q: 7290q = w^3$ $\quad q, w \in \mathbb{Z}$

w^3 debe ser divisible por $7290 = 2 \times 3^6 \times 5$

↪ $2 \times 3^2 \times 5 \mid w$ $\quad 2 \times 3^2 \times 5 = 90$ $\quad w = 90.z$ $\quad z \in \mathbb{Z}$

Por lo tanto, podemos decir que si w es múltiplo de 90, entonces q existe y es $\mathbb{Z}$

por ej: si $z = 1$ entonces

↪ $w^3 = 729.000$ y $q = 100$

$7290 \times 100 = 729.000$

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11) a y $b \in \mathbb{Q} / a < b \rightarrow \exists\, x \in \mathbb{Q} / a < x < b$

$a < b$ $\qquad a < b$

$a + a < b + a$ $\qquad a + b < b + b$

$2a < a + b$ $\qquad a + b < 2b$

$2a < a + b < 2b$

$a < \boxed{\frac{a+b}{2}} < b$ → $x \in \mathbb{Q}$

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12) $\nexists\, x \in \mathbb{Q} / x^3 = 2$

suponemos $\sqrt[3]{2} \in \mathbb{Q}$

$\frac{a}{b} = x \rightarrow \left(\frac{a}{b}\right)^3 = x^3 = 2 = \sqrt[3]{2} \rightarrow \sqrt[3]{2} = \frac{c}{d} \Rightarrow 2 = \frac{c^3}{d^3}$

↪ $2d^3 = c^3$ ← c debe ser por

↪ $2d^3 = (2k)^3 =$

$\Rightarrow 2d^3 = 8k^3$

↪ $d^3 = 4k^3$ ← b también debe ser por

contradice la suposición inicial

↪ (si $x \in \mathbb{Q}$ entonces a y b deben ser primos entre sí)

$\boxed{x \in \mathbb{I}}$

13)

a) $\sqrt{-49}$ $\quad z = a + ib \quad a = Re(z) \quad b = Im(z)$

$i = \sqrt{-1} \quad \sqrt{-49} = \sqrt{-1 \times 49} = \sqrt{-1} \times \sqrt{49} = i7$

$a = 0 \quad b = 7 \qquad z = 0 + i7$

b) $\sqrt{-20}$ $\quad \sqrt{-20} = \sqrt{-1 \times 20} = i \times \sqrt{4 \times 5} = i \times 2\sqrt{5}$

$z = 0 + i.2\sqrt{5} \quad a = 0 \quad b = 2\sqrt{5}$

c) $\sqrt{\frac{-9}{16}}$ $\quad \sqrt{\frac{-9}{16}} = \sqrt{-1 \cdot \frac{9}{16}} = \sqrt{-1} \cdot \frac{3}{4} = i\frac{3}{4}$

$a = 0 \quad b = \frac{3}{4} \qquad z = 0 + i\frac{3}{4}$

d) $z = -8 \quad a = -8 \quad b = 0 \quad z = -8 + i.0$

e) $z = 7i \quad a = 0 \quad b = 7 \quad z = 0 + i7$

f) $z = (3+i) + (5-4i) \quad z = 8 - 3i \quad a = 8 \quad b = -3$

g) $z = 3i - (5-2i) \quad z = -5 + 5i \quad a = -5 \quad b = 5$

h) $\frac{1+3i}{3-i}$ → cociente entre complejos $\frac{z_1}{z_2} = z_1 . z_2^{-1}$

$\hookrightarrow \frac{(1+3i).(3-(-i))}{(3-i).(3-(-i))} = \frac{(1.3 + 3.(-1)) + i(-1.(-1) + 3.3)}{3^2 + (-1)^2} =$

$= \frac{i10}{10} = i \qquad a = 0 \quad b = 1 \quad z = 0 + 1i$

i) $\frac{1-i}{(1+i)^2}$ → binomio ⇒ $\frac{1-i}{1 + 2.1.i + \underset{= -1}{i^2}} = \frac{1-i}{2i}$

$= \frac{(1+(-i)).(0-2i)}{(0+i2).(0-i2)} = \frac{(0+(-1).2) + i(-1.2 + 0.(-1))}{0^2 + 2^2} =$

$= \frac{-2 + i(-2)}{4} = -\frac{1}{2} - \frac{1}{2}i \qquad a = -\frac{1}{2} \quad b = -\frac{1}{2}$

14) $z + \overline{z} = -8$

módulo de $z = |z|$

↳ $\sqrt{a^2 + b^2}$

$|z| + |\overline{z}| = 10$

$(a + ib) + (a - ib) = -8$

$2a = -8$

$a = -4$

$\sqrt{a^2 + b^2} + \sqrt{a^2 + b^2} = 2\sqrt{a^2 + b^2} = 10$

$= \sqrt{a^2 + b^2} = 5$

$= a^2 + b^2 = 5^2$

$(-4)^2 + b^2 = 25$

$b^2 = 25 - 16$

$b^2 = 9$

$z = -4 + 3i$

$b = 3$

15) a) $x - 15i = 9 + 5yi$

parte real $x = 9$

$x = 9 \quad y = -3$

parte imaginaria $-15 = 5y \Rightarrow -15/5 = y = -3$

b) $2x + 3yi = 6 + yi$

$x = 3 \quad y = 0$

parte real $2x = 6 \Rightarrow x = 3$

parte imaginaria $3y = y \Rightarrow 3y - y = 0 \Rightarrow 2y = 0 = y$

16) $x \in \mathbb{R} \,/\, Re(z) = Im(z) \wedge z = \frac{x + 2i}{4 - 3i}$

$\boxed{z . z^{-1}}$

$$\frac{(x + 2i).(4 - (-3).i)}{(4 + (-3)i).(4 - (-3)i)} = \frac{(x4 + 2(-3)) + i\,(-x.(-3) + 4.2)}{4^2 + (-3)^2} = \frac{(4x - 6) + i(3x + 8)}{16 + 9} =$$

$$= \frac{4x - 6 + 3xi + 8i}{25}$$

$Re(z) = \frac{4x - 6}{25}$

$Im(z) = \frac{3x + 8}{25}$

Como $Re(z) = Im(z) \Longrightarrow \frac{4x - 6}{25} = \frac{3x + 8}{25}$

$4x - 3x - 6 = 8$

$x = 8 + 6 \longrightarrow x = 14$

PRACTICA 3 ⊛

17) $K \in \mathbb{R} \,/\, \frac{2-(1+K)i}{1-Ki} \in \mathbb{R}$

$$\frac{(2-(1+K)i).(1-(-K)i)}{(1+(-K)i).(1-(-K)i)} = \frac{(2.1-(1+K).(-K)) + i(-2(-K) + 1.(1+K))}{1^2+(-K)^2} =$$

$$= \frac{2+K+K^2 + i(2K+1-K)}{1+K^2} = \frac{2+K(1+K)+i(2K-(1+K))}{1+K^2}$$

$Re(z) = \frac{2+K(1+K)}{1+K^2} = \frac{2+K+K^2}{1+K^2}$

$Im(z) = \frac{2K-(1+K)}{1+K^2} = \frac{2K-K-1}{1+K^2} = \frac{K-1}{1+K^2}$

para que $z \in \mathbb{R}$, su parte imaginaria debe ser 0.

$\rightarrow \frac{K-1}{1+K^2} = 0 \rightarrow K-1=0$

$\boxed{K=1}$

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18) a) i^{489} 489 mod 4 => resto = 1

$\hookrightarrow i^1 = i$

b) $-i^{1026}$ 1026 mod 4 => resto = 2

$-i^2 = -(-1) = 1$

c) $(3i)^{168} = 3^{168}.\, i^{168}$ 168 mod 4 = 0

$= 3^{168}.1 = 3^{168}$

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19)

a) $z_1 + z_7 = (3+3i)+(-4-4i) = -1-1i$

b) $z_5 - z_3 = 5i + 5 + 4i = 5+9i$

c) $z_9 . z_6 = (2-2i).(-7) = 2.(-7)+(-2i).(-7) = -14+14i$

d) $z_8 / z_{10} = (-8i)/(3-4i) = \frac{(-8i).(3-(-4i))}{(3+(-4i)).(3-(-4i))} = \frac{((-8).(-4))+i(3.(-8))}{3^2+(-4)^2} =$

$$= \frac{32+(-24)i}{9+16} = \frac{32-24i}{25}$$

e) $z_3 + z_6 = (5+4i)+(-7) = -2+4i$

f) $z_2 - z_6 = (-1+i)-(-7) = 6+i$

g) $z_3 \cdot z_{10} = (5+4i).(3-4i) = 5.3 + 5.(-4i) + 4i.3 + 4i.(-4i) =$

$= 15 + (-20)i + 12i - 16i^2 = 15 + 16 + i(-20+12) = 31 - 8i$

h) $z_1^3 = (3+3i)^3 = (3+3i).(3+3i).(3+3i) = (9+9i+9i+9i^2).(3+3i)$

$(9 + 2(9i) + 9.(-1)).(3+3i) = 18i.(3+3i) = 54i + 54i^2 = -54 + 54i$

i) $z_9^9 = (2-2i)^9 \Rightarrow z^n = (|z|e^{i\alpha})^n = |z|^n.e^{in\alpha} = (2\sqrt{2})^9.\ e^{i9\cdot\frac{\pi}{4}}$

$r = |z| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

$\alpha = \arg(z) = \tan^{-1}(\frac{-2}{2}) = \tan^{-1}(-1) = -\frac{\pi}{4}$

$n = 9$

j) $z_8^{15} = (5i)^{15} = 5^{15}.\ i^{15} = 5^{15}.\ i^3 = 5^{15}.\ (-i) = -5^{15}i$

k) $z_{10}^3 = (3-4i)^3 = (3-4i).(3-4i).(3-4i) = (9 - 4i3 - 4i3 + 16i^2).(3-4i)$

$= (9 - 2.4i3 + 16i^2).(3-4i) = (9 - 24i + 16.(-1)).(3-4i) =$

$= (-7-24i).(3-4i) = -21 + 28i - 72i + 96i^2 = -21 - 44i + 96(-1) =$

$= -117 - 44i$

l) $\sqrt[4]{z_2} = \sqrt[4]{-1+i} =$

$r = |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ $\qquad z_2 = \sqrt{2}\, e^{i\frac{3}{4}\pi}$

$\alpha = \arg(z) = \tan^{-1}(\frac{1}{-1}) = \frac{3}{4}\pi$

$n = 4$

$|\omega| = \sqrt[4]{|z|} = \sqrt[4]{\sqrt{2}} = \sqrt[8]{2}$

$\varphi_K = \frac{\frac{3}{4}\pi + 2K\pi}{4} \qquad 0 \leq K < 4$

$\varphi_0 = \frac{\frac{3}{4}\pi + 2.0\pi}{4} = \frac{\frac{3\pi}{4}}{4} = \frac{3}{16}\pi$

$\varphi_1 = \frac{\frac{3}{4}\pi + 2.1\pi}{4} = \frac{\frac{3}{4}\pi + 2\pi}{4} =$

$= \frac{11}{16}\pi$

$\varphi_3 = \frac{\frac{3}{4}\pi + 2.3\pi}{4} = \frac{\frac{3}{4}\pi + 6\pi}{4} =$

$= \frac{27}{16}\pi$

$\varphi_2 = \frac{\frac{3}{4}\pi + 2.2\pi}{4} = \frac{\frac{3}{4}\pi + 4\pi}{4} =$

$= \frac{19}{16}\pi$

$\omega_0 = \sqrt[8]{2}\, e^{i\frac{3}{16}\pi} \quad \omega_1 = \sqrt[8]{2}\, e^{i\frac{11}{16}\pi} \quad \omega_2 = \sqrt[8]{2}\, e^{i\frac{19}{16}\pi} \quad \omega_3 = \sqrt[8]{2}\, e^{i\frac{27}{16}\pi}$

m) $\sqrt[3]{z_4} = \sqrt[3]{9} =$

$r = |z| = \sqrt{9^2 + 0^2} = \sqrt{81} = 9$

$\alpha = \tan^{-1}\left(\frac{0}{9}\right) = 0$ $\qquad z_4 = 9e^{i0}$

$|w| = \sqrt[3]{|z|} = \sqrt[3]{9}$ $\qquad N = 3$

$\varphi_K = \dfrac{0 + 2K\pi}{3}$ $\qquad 0 \leq K < 3$

$\varphi_0 = \dfrac{0 + 2.0\pi}{3} = 0$ $\qquad \varphi_1 = \dfrac{0 + 2.1\pi}{3} = \dfrac{2}{3}\pi$

$\varphi_2 = \dfrac{0 + 2.2\pi}{3} = \dfrac{4}{3}\pi$

$w_0 = 9^{\frac{1}{3}} . e^{i0}$

$w_1 = 9^{\frac{1}{3}} . e^{i\frac{2}{3}\pi}$

$w_2 = 9^{\frac{1}{3}} . e^{i\frac{4}{3}\pi}$

n) $\sqrt[7]{i}$

$r = |z| = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$

$\alpha = \dfrac{\pi}{2}$ $\qquad z = 1 . e^{i\frac{\pi}{2}}$

$|w| = \sqrt[7]{1} = 1$ $\qquad N = 7$

$\varphi_K = \dfrac{\frac{\pi}{2} + 2K\pi}{7}$ $\qquad 0 \leq K < 7$

$\varphi_0 = \dfrac{\frac{\pi}{2} + 2.0\pi}{7} = \dfrac{\frac{\pi}{2}}{7} = \dfrac{1}{14}\pi$ $\qquad \varphi_1 = \dfrac{\frac{\pi}{2} + 2.1\pi}{7} = \dfrac{\frac{5}{2}\pi}{7} = \dfrac{5}{14}\pi$

$\varphi_2 = \dfrac{\frac{\pi}{2} + 2.2\pi}{7} = \dfrac{\frac{9}{2}\pi}{7} = \dfrac{9}{14}\pi$ $\qquad \varphi_3 = \dfrac{\frac{\pi}{2} + 2.3\pi}{7} = \dfrac{13}{14}\pi$

$\varphi_4 = \dfrac{\frac{\pi}{2} + 2.4\pi}{7} = \dfrac{17}{14}\pi$ $\qquad \varphi_5 = \dfrac{\frac{\pi}{2} + 2.5\pi}{7} = \dfrac{21}{14}\pi = \dfrac{3}{2}\pi$

$\varphi_6 = \dfrac{\frac{\pi}{2} + 2.6\pi}{7} = \dfrac{25}{14}\pi$

$w_0 = e^{i\frac{\pi}{14}}$ $\qquad w_1 = e^{i\frac{5}{14}\pi}$

$w_2 = e^{i\frac{9}{14}\pi}$ $\qquad w_3 = e^{i\frac{13}{14}\pi}$

$w_4 = e^{i\frac{17}{14}\pi}$ $\qquad w_5 = e^{i\frac{3}{2}\pi}$

$w_6 = e^{i\frac{25}{14}\pi}$