1. Lorentz invariance of the Klein-Gordon eq. for scalars.

It is helpful to express the K-G eq. in 4-vector notation, i.e.

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = \left(-\hbar^2 c^2 \nabla^2 + m_0^2 c^4\right)\psi$$

or, $\left[\Box + \left(\frac{m_0 c}{\hbar}\right)^2\right]\psi(x_\mu) = 0$

where $\Box = \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu} \equiv \partial^\mu \partial_\mu$

In the transformed ~~system~~, the K-G eq. reads

$$\left[\frac{\partial}{\partial x'_\mu}\frac{\partial}{\partial x'^\mu} + K^2\right]\psi'(x'_\mu) = 0 \qquad K = \frac{m_0 c}{\hbar}$$

Since $\hat{p}_\mu = i\hbar \frac{\partial}{\partial x^\mu}$ and $\hat{p}_\mu \hat{p}^\mu \sim \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}$

and $p_\mu p^\mu = \text{const.}$ (as we have seen)

$\Rightarrow \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}$ is also Lorentz invariant.

For a scalar function $\psi'(x'_\mu) = \psi(x_\mu)$ also.

You can also consider a plane wave solution to confirm this: $\psi(x_\mu) = e^{i k_\mu x^\mu}$.

2. $\mathcal{L} = \bar{\Psi}(ic\hbar\gamma^\mu \partial_\mu - m_0 c^2)\psi$ — ①

$\bar{\Psi} = \psi^\dagger \gamma^0$

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial_\nu \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi})} \partial_\nu \bar{\Psi} - \delta^\mu{}_\nu \mathcal{L}. \quad - ②$$

Using ① in ②

$$T^\mu{}_\nu = \bar{\psi} ic\hbar \gamma^\mu \partial_\nu \psi - \delta^\mu{}_\nu \left[\bar{\psi} ic\hbar \gamma^\lambda \partial_\lambda \psi - \bar{\psi} m_0 c^2 \psi\right]$$

$$T^0{}_0 = \cancel{\bar{\Psi} ic\hbar\gamma^0 \partial_0 \psi} - \cancel{\bar{\psi} ic\hbar \gamma^0 \partial_0 \psi} - \bar{\psi} ic\hbar \gamma^i \partial_i \psi + \bar{\psi} m_0 c^2 \psi$$

Let me re-write in terms of α, β matrices like we had in the notes. $\beta = \gamma^0$, $\beta\alpha_i = \gamma^i$, $\beta^2 = 1$.

Therefore,
$$T^0{}_0 = -ic\hbar\, \psi^\dagger \beta\beta\alpha_i \partial_i \psi + \psi^\dagger \beta m_0 c^2 \psi$$
$$= \frac{\hbar c}{i} \psi^\dagger \left[\alpha_i \partial_i \psi + \beta m_0 c^2\right] \psi = \psi^\dagger H \psi$$

recall Eq. (7) in Dirac eq' notes.

As a consequence,
$$\int T^0{}_0\, d^3x = \langle \psi | H | \psi \rangle$$
→ energy! Expectation value of H in state ψ.

$$T^0{}_i = \bar{\psi} ic\hbar \gamma^0 \partial_i \psi = \psi^\dagger (ic\hbar) \gamma^0 \gamma^0 \partial_i \psi \qquad ((\gamma^0)^2 = 1)$$
$$= ic\hbar\, \psi^\dagger \partial_i \psi$$
$$= \psi^\dagger \hat{p} c \psi$$

$$\frac{1}{c}\int T^0{}_i \, d^3x = \langle \psi | \hat{P}_i | \psi \rangle \equiv \text{expectation value of momentum.}$$

$$T^\mu{}_\mu = \bar{\psi}\, i c\hbar \gamma^\mu \partial_\mu \psi - \left[\bar{\psi}\, i c\hbar \gamma^\lambda \partial_\lambda \psi - \bar{\psi} m_0 c^2 \psi\right]$$

$$= \bar{\psi} m_0 c^2 \psi = m_0 c^2 \bar{\psi}\psi .$$

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3. Recall that we found $\rho = \psi^\dagger \psi$ and $j^k = c\psi^\dagger \alpha^k \psi$ so that

$$j^\mu \equiv (c\rho, \vec{j}) \equiv \left\{ c\psi^\dagger\psi, c\psi^\dagger \vec{\alpha} \psi \right\}$$

$$= \left\{ c\psi^\dagger \gamma^0 \gamma^\mu \psi \right\}$$

Under Lorentz transformation, the current density transforms as,

$$j'^\mu(x') = c\psi'^\dagger(x')\gamma^0\gamma^\mu\psi'(x')$$

$$= c\psi^\dagger(x) S^\dagger \gamma^0 \gamma^\mu S\psi(x)$$

$$= c\psi^\dagger(x)\gamma^0 S^{-1}\gamma^\mu S\psi(x)$$

$$= c\Lambda^\mu{}_\nu \psi^\dagger \gamma^0 \gamma^\nu \psi(x)$$

$$= \Lambda^\mu{}_\nu j^\nu(x)$$

Since $S^{-1} = \gamma_0 S^\dagger \gamma_0$

and $S\gamma^\nu S^{-1} = \Lambda_\mu{}^\nu \gamma^\mu$

or

$\cancel{S\gamma^\mu S^{-1} = \Lambda^\nu{}_\mu \gamma^\nu}$

$S\gamma^\mu S^{-1}\Lambda^\nu{}_\mu = \gamma^\nu$

$\Rightarrow j^\mu$ is a 4-vector.

The continuity eq. ~~can~~ now be written as

$\partial_\mu j^\mu(x) = 0$ in the Lorentz invariant form.

Maxwell's eqs. in the 3-D notation.

4 (a)

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho_e \quad -① \qquad \vec{\nabla}\cdot\vec{B} = 0 \quad -②$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \quad -③ \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}_e + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \quad -④$$

$\rho_e(\vec{r})$ is a scalar and $\vec{J}_e(\vec{r})$ is a vector.

By requiring that ① is invariant under rotations and parity (e.m. interactions conserve parity) we have the electric field $\vec{E}(\vec{r})$ is a vector.

If $\vec{E}$ is a vector, since $\vec{\nabla}$ is a vector, $\vec{\nabla}\times\vec{E}$ will not change sign under parity and parity invariance of (3) requires that $\vec{B}$ is an axial vector.

(b)

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho_e \quad -1$$

$$\vec{\nabla}\cdot\vec{B} = 4\pi\rho_m \quad -2$$

$$\vec{\nabla}\times\vec{E} = \frac{4\pi}{c}\vec{J}_M - \frac{1}{c}\frac{\partial\vec{B}}{\partial t} \quad -3$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}_e + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \quad -4$$

Since $\vec{B}$ is an axial vector $\Rightarrow \rho_m \equiv$ pseudoscalar

3 is invariant under parity $\Rightarrow \vec{J}_m$ is an axial vector.

5.
$$\bar{\psi}'(x')\gamma^5\gamma^\mu\psi'(x') = [\psi'(x')]^\dagger\gamma^0\gamma^5\gamma^\mu\psi'(x')$$

$$= (\psi(x))^\dagger S^\dagger\gamma^0\gamma^5\gamma^\mu S\psi(x)$$

$$= \bar{\psi}\underbrace{\gamma^0 S^\dagger\gamma^0}_{S^{-1}}\gamma^5\gamma^\mu S\psi$$

$$= \bar{\psi}S^{-1}\cancel{\gamma^0}\gamma^5\gamma^\mu S\psi = \bar{\psi}S^{-1}\gamma^5 S S^{-1}\gamma^\mu S\psi$$

$$= \bar{\psi}\gamma^5\Lambda^\mu{}_\nu\gamma^\nu\psi$$

$$= \Lambda^\mu{}_\nu\,\bar{\psi}\gamma^5\gamma^\nu\psi$$ — transforms as a vector under Lorentz tr.

Under parity $\bar{\psi}'\gamma^5\gamma^\mu\psi'$

$$= \psi^\dagger\gamma^0\gamma^0\gamma^5\gamma^\mu\gamma^0\psi$$

$$= \bar{\psi}\gamma^0\gamma^5\gamma^\mu\gamma^0\psi$$

$$= (-1)\bar{\psi}(x)\gamma^5\gamma^0\gamma^\mu\gamma^0\psi(x)$$

$$= (-1)\bar{\psi}(x)\gamma^5(\gamma^\mu)^\dagger\psi(x)$$

Now $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^k)^\dagger = -\gamma^k$

Therefore the zeroth component gives $(-1)\bar{\psi}\gamma^5\gamma^0\psi$
and three other components give $+\bar{\psi}\gamma^5\gamma^k\psi$

$\Rightarrow$ axial vector

$\bar{\Psi}\gamma^5\gamma^\mu\psi\bar{\Psi}\gamma_\mu\psi$ → dot product of polar vector and axial vector
→ Lorentz invariant, negative parity

$\bar{\Psi}\gamma^5\psi\bar{\Psi}\gamma^5\psi$ → product of two pseudoscalars
→ Lorentz invariant, even under parity

$\bar{\Psi}\psi\bar{\Psi}\gamma^5\psi$ → Lorentz invariant, odd parity

$\bar{\Psi}\gamma^5\gamma^\mu\psi\bar{\Psi}\gamma^5\gamma_\mu\psi$ → Dot product of two axial vectors
— Lorentz invariant, even parity

$\bar{\Psi}\gamma^\mu\psi\bar{\Psi}\gamma^\mu\psi$ → dot product of 2 vectors
→ Lorentz invariant, even under parity.

6. (1) $\rho \to \pi^0\pi^0$ → identical bosons in the final state
→ wave function must be symmetric under interchange of particles.
(2) Parity must be conserved. Parity of neutral pions (intrinsic) is $(-1)\times(-1) = +1$. Since the ρ meson is (parity) 1^-, the orbital angular mom. must be with $(-1)^l$ with l odd. Pions are spin zero so from angular mom. conservation $l = 1$.
⇒ wavefunc. antisymmetric! Since $P\,Y_{lm} = (-1)^l Y_{lm}$

$\rho \to \pi^+ \pi^-$ can happen since the 2 pions are not identical particles.

7. The proton spin is $\frac{1}{2}$, pion spin is $0 \Rightarrow$ total spin $\frac{1}{2}$. Δ has spin $\frac{3}{2} \Rightarrow l = 1$.

8. $$I_{tot}^2 = (\vec{I}_1 + \vec{I}_2)^2 = \underbrace{I_1^2}_{\frac{1}{2}(\frac{1}{2}+1) = \frac{3}{4}} + I_2^2 + 2\,\vec{I}_1 \cdot \vec{I}_2$$

with $I_1 = I_2 = \frac{1}{2}$

$I_{tot}^2 = 0 \Rightarrow$ singlet

So, $0 = \frac{3}{4} + \frac{3}{4} + 2\,\vec{I}_1 \cdot \vec{I}_2 \Rightarrow \vec{I}_1 \cdot \vec{I}_2 = -\frac{3}{4}$

$I_{tot}^2 = 1(1+1) = 2$ in triplet case.

So, $2 = \frac{3}{4} + \frac{3}{4} + 2\,\vec{I}_1 \cdot \vec{I}_2 \Rightarrow \vec{I}_1 \cdot \vec{I}_2 = \frac{1}{4}$

9. When the total centre of mass energy is 1232 MeV, it is clear that the resonance Δ at that central ~~on~~ mass will dominate, i.e., $I = \frac{3}{2}$ channel dominates.

$\pi^- + p : |1\ -1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

$\pi^+ + p : |1\ 1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = |\tfrac{3}{2}\ \tfrac{3}{2}\rangle$

$K^0 + \Sigma^0 : |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle|1\ 0\rangle = \sqrt{\tfrac{2}{3}}|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

$K^+ + \Sigma^- : |\tfrac{1}{2}\ \tfrac{1}{2}\rangle|1\ -1\rangle = \sqrt{\tfrac{1}{3}}|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

$K^+ + \Sigma^+ : |\tfrac{1}{2}\ \tfrac{1}{2}\rangle|1\ 1\rangle = |\tfrac{3}{2}\ \tfrac{3}{2}\rangle$

~~$M_c = M_3 \ldots M_a$~~ $\langle \psi_f | H_1 + H_3 | \psi_i \rangle$ say.

For (a) $\pi^- + p \to K^0 + \Sigma^0$

$$\frac{\sqrt{2}}{3} M_{\frac{3}{2}} = \left\langle \left(\sqrt{\tfrac{2}{3}}\right) \tfrac{3}{2}\ -\tfrac{1}{2} \right| H_3 \left| \left(\sqrt{\tfrac{1}{3}}\right) \tfrac{3}{2}\ -\tfrac{1}{2} \right\rangle$$

$$-\frac{\sqrt{2}}{3} M_{1/2} = \left\langle \left(\sqrt{\tfrac{1}{3}}\right) \tfrac{1}{2}\ -\tfrac{1}{2} \right| H_1 \left| \left(-\sqrt{\tfrac{2}{3}}\right) \tfrac{1}{2}\ -\tfrac{1}{2} \right\rangle$$

$$\sigma_a = K\left|\frac{\sqrt{2}}{3} M_{3/2} - \frac{\sqrt{2}}{3} M_{1/2}\right|^2$$

(b) $\pi^- + p \to K^+ + \Sigma^- \Rightarrow \sigma_b = K\left|\frac{1}{3} M_{3/2} + \frac{2}{3} M_{1/2}\right|^2$

(c) $\pi^+ + p \to K^+ + \Sigma^+ \Rightarrow \sigma_c = K\left|M_{3/2}\right|^2$

When ~~I=3/2~~ $I = 3/2$ dominates $M_{3/2} \gg M_{1/2}$

$\sigma_a : \sigma_b : \sigma_c$ in this is $\frac{2}{9} : \frac{1}{9} : 1$ or $2 : 1 : 9$