Decomposition of Complete Tripartite Graphs Into 5-Cycles

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Abstract

A result of D. Sotteau on the necessary and sufficient conditions for decomposing the complete bipartite graphs into even cycles has been shown in many occasions, that it is a very important tool in the theory of graph decomposition into *even* cycles. In order to have similar tools in the case of odd cycle decomposition, obviously bipartite graphs are not suitable to be considered. Searching for such tools, we have considered decomposition of complete tripartite graphs, $K_{r,s,t}$, into 5-cycles. There are some necessary conditions that we have shown their sufficiency in the case of $r = t$, and some other cases. Our conjecture is that these conditions are always sufficient.

1 Introduction

A theorem of D. Sotteau [2] states that,

> Necessary and sufficient conditions in order that $K_{m,n}$ admits a decomposition into $2k$-cycles are:
>
> **(i)** $m \geq k, \quad n \geq k$;
>
> **(ii)** m and n both are even;
>
> **(iii)** $2k \mid mn$.

This theorem is a very useful tool in recursive construction in cycle decomposition problems. But it may be used only when cycles are of even length. So if we want to

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work with odd cycles we should look at other alternatives, say complete tripartite graphs, $K_{r,s,t}$.

Decomposition of $K_{r,s,t}$ into triangles is easy:

Proposition 1. *Necessary and sufficient conditions for $K_{r,s,t}$ to be decomposed into 3-cycles is that $r = s = t$.*

Proof. Necessity of the conditions is trivial and to show the sufficiency one may use a latin square of size r. ∎

Next case will be 5-*cycle* decomposition.

Example 1. The complete tripartite graph $K_{4,2,2}$ may be decomposed into 5-cycles. Suppose the sets $A = \{a_1, a_2, a_3, a_4\}$, $B = \{b_1, b_2\}$, and $C = \{c_1, c_2\}$ are three parts of this graph. Then a decomposition may be done as follows:

$$(b_1, a_1, c_2, a_3, c_1), (b_2, a_1, c_1, a_4, c_2), (c_2, a_2, b_2, a_3, b_1), \text{ and } (c_1, a_2, b_1, a_4, b_2).$$

2 Necessary Conditions

To decompose $K_{r,s,t}$ into 5-cycles some immediate necessary conditions follow.

Theorem 1. *Let $r \leq s \leq t$. If there exists a decomposition of $K_{r,s,t}$ into 5-cycles, then the following conditions are satisfied:*

(i) *r, s, and t all are even or all are odd;*

(ii) $5 \mid rs + rt + st$;

(iii) $t \leq 4rs/(r + s)$.

Proof. Each cycle uses two edges from each vertex, thus the degree of each vertex must be even. In other words $r + s$, $r + t$, and $s + t$ all are even. Thus (i) follows. To see (ii), note that $rs + rt + st$ is the total number of edges in $K_{r,s,t}$.

For (iii), we note that each 5-cycle uses *at least one* edge and *at most three* edges from between any two parts of $K_{r,s,t}$. This results in six inequalities:

$$\begin{aligned} 2st/3(s+t) &\leq r \leq 4st/(s+t), \\ 2rt/3(r+t) &\leq s \leq 4rt/(r+t), \\ 2rs/3(r+s) &\leq t \leq 4rs/(r+s). \end{aligned}$$

Since $r \leq s \leq t$, the inequality $t \leq 4rs/(r+s)$ implies other ones. ■

Conjecture. Three conditions given in Theorem 1 are also sufficient.

We have proved this conjecture in the case when two parts have equal number of vertices and satisfy the necessary conditions with one exception of $K_{5x,5x,z}$ (z not multiple of 5). Also we have some results for the remaining cases. For some small cases it has been shown to be true.

3 An Application

Before going further we show that how the decomposition of $K_{r,s,t}$ can be a *tool* for 5-cycle decompositions. In 1966 Alex Rosa [1] proved that,

> A complete graph on n vertices ,K_n, admits a 5-cycle decomposition if and only if $n \equiv 1$ or $5 \pmod{10}$.

We may prove the sufficiency of conditions in Rosa's theorem by applying the decomposition of $K_{r,s,t}$ into 5-cycles. (Necessity of the conditions is trivial as usual).

For n we have two cases:

(i) $n = 10l + 1$,

(ii) $n = 10l + 5$.

(i) In this case we let $r = 10l' + 5$ and $s = 10l'' + 1$ where:

$$l' = \lfloor l/3 \rfloor \quad \text{and} \quad l'' = l - 2\lfloor l/3 \rfloor - 1.$$

Then $n = r + r + s$. Now we proceed by induction, by decomposing K_r, K_r, K_s, and $K_{r,r,s}$ into 5-cycles and "patching" them together to obtain a decomposition for K_n.

(ii) In this case we let $r = 10l' + 5$ and $s = 10l'' + 5$, where l' and l'' are as above. And by a similar method we may decompose K_n.

4 Sufficiency of Conditions

In this section we prove some useful theorems and some results on the sufficiency of conditions. First an extension theorem:

Theorem 2. *If $K_{r,s,t}$ admits a 5-cycle decomposition, then so does $K_{ar,as,at}$ for each positive integer a.*

Proof. We apply Proposition 1 to the decomposition of $K_{a,a,a}$ into triangles. Indeed, each part in $K_{ar,as,at}$ may be partitioned into a classes, each with equal number of vertices. Then each class of vertices in each part may be considered as a vertex. This will give a $K_{a,a,a}$ which, in turn, can be decomposed into triangles. Each of these triangles represents a $K_{r,s,t}$. ■

Corollary 1. *$K_{r,r,r}$ admits a 5-cycle decomposition if and only if $5 \mid r$.*

Proof. Necessary conditions of Section 2, in this case, imply $5 \mid r$. To show sufficiency, according to Theorem 2, we just need a decomposition of $K_{5,5,5}$. Suppose that the sets $A = \{a_1, a_2, a_3, a_4, a_5\}$, $B = \{b_1, b_2, b_3, b_4, b_5\}$, $C = \{c_1, c_2, c_3, c_4, c_5\}$ are three parts of this graph. Then a decomposition may be done with the base cycles as follows:

$$(a_2, b_5, c_1, b_1, c_5), (b_2, c_5, a_1, c_1, a_5), \text{ and } (c_2, a_5, b_1, a_1, b_5).$$

In other words, we may generate other cycles by adding the indices of the above base cycles (mod 5). ■

Corollary 2. *For every positive integer n, $K_{2n,2n,4n}$ admits a 5-cycle decomposition.*

Proof. In the Example 1 of Section 1, we gave a decomposition for $K_{2,2,4}$. ■

Corollary 3. *For every positive integer n, $K_{m,3m,3m}$ admits a 5-cycle decomposition.*

Proof. $K_{1,3,3}$ may be decomposed easily as follows. Suppose that the sets $A = \{a_1\}$, $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2, c_3\}$ are three parts of this graph. Then a decomposition may be done with the base cycles as follows:

$$(a_1, b_i, c_i, b_{i+1}, c_{i+2}) \qquad (\text{mod } 3),$$

where, $i = 1, 2, 3$. ■

Next we use the idea of decomposition of $K_{2n,2n,4n}$ and $K_{m,3m,3m}$, which were done by *base cycles*, in the following theorem which is our main theorem.

Theorem 3. *Suppose that at least two parts in a complete tripartite graph have the same number of vertices, say $K_{r,r,s}$. And suppose that the triplet (r, r, s) satisfies all three necessary conditions given in Theorem 1. Then $K_{r,r,s}$ has a 5-cycle decomposition except possibly when r is a multiple of 5 but s is not.*

Proof. By simple arithmetics, from the necessary conditions we may deduce that: there exists m and n such that $r = 3m + 2n$ and $s = m + 4n$.

Now we give the following base cycles, assuming that three parts are $A = \{a_1, \ldots, a_s\}$, $B = \{b_1, \ldots, b_r\}$, and $C = \{c_1, \ldots, c_r\}$.

- r is even, which implies that s is also even:

$$\begin{gathered}(b_{i+2n-j}, a_{m+j}, c_{i+1}, a_{m+2n+2j-i+2\lfloor i/2\rfloor}, c_i)\\(c_{i+3m+n+j}, a_{m+n+j}, b_{i+1}, a_{m+2n+2j-i+2\lfloor i/2\rfloor}, b_i)\\(a_j, b_i, c_{i+3j}, b_{i+2}, c_{i+3j+1})\\i = 1, 2, \ldots, r;\;\; j = 1, 2, \ldots, m.\end{gathered}$$

  The indices of b's and c's are all to be computed modulo r.

- r and s are odd:

$$\begin{gathered}(b_{i+2n-j}, a_{m+j}, c_{i+1}, a_{m+2n+2j-i+2\lfloor i/2\rfloor}, c_i)\\(b_{r+2n-j}, a_{m+j}, c_{r+1}, a_{m+n+j}, c_r)\\i = 1, 2, \ldots, r-1;\;\; j = 1, 2, \ldots, n;\end{gathered}$$

$$\begin{gathered}(c_{i+1}, a_{m+n+j}, b_{i+n+2-j}, a_{m+2n+2j-i+2\lfloor i/2\rfloor}, b_{i+n+1-j})\\(c_r, a_{m+2n+2j-1}, b_{r+n+1-j}, a_{m+2n+2j}, b_{r+n-j})\\(c_1, a_{m+2n+2j}, b_{r+n+2-j}, a_{m+j+n}, b_{r+n+1-j})\\i = 1, 2, \ldots, r-2;\;\; j = 1, 2, \ldots, n;\end{gathered}$$

$$\begin{gathered}(a_j, b_i, c_{i+3j}, b_{i+2}, c_{i+3j+1})\\i = 1, 2, \ldots, r;\;\; j = 1, 2, \ldots, m.\end{gathered}$$

  Again the indices of b's and c's are all to be computed modulo r. ■

5 Searching for a decomposition in other cases

One may get discouraged and conjecture that the condition $r = t$ is also necessary! But it is not true. We show this in the following:

Lemma 1. *If $K_{a,b,b}$, $K_{a,c,c}$, and $K_{b,c,c}$ each admits a 5-cycle decomposition, then so does $K_{a+c,b+c,b+c}$.*

Proof. Consider the following 2×2 latin square:

	a	c
b	b	c
c	c	b

Using this latin square we may decompose $K_{a+c,b+c,b+c}$ into $K_{b,a,b}$, $K_{b,c,c}$, $K_{c,a,c}$, and $K_{c,c,b}$ each of which, by assumption, admits a 5-cycle decomposition. ∎

Also proofs of the following lemmas are immediate.

Lemma 2. *If $K_{a,b,b}$ and $K_{c,b,b}$ each admits a 5-cycle decomposition, then so does $K_{a+c,2b,2b}$.*

Lemma 3. *If $K_{a,b,b}$ and $K_{b,a,a}$ each admits a 5-cycle decomposition, then so does $K_{a+b,2a,2b}$.*

Example 2. $K_{30,20,40}$ admits a 5-cycle decomposition.

Proof. By Corollary 2, $K_{20,10,10}$ can be decomposed.

In the following we give two more decompositions:

- $K_{3,5,5}$, where its parts are $A = \{a, b, c\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{5, 6, 7, 8, 9\}$:

  $a0615$ $a1726$ $a2837$ $a3948$ $a4509$ $b5291$ $c5364$ $c6b70$ $b9c80$ $c3b47$ $b2c18$.

- $K_{4,10,10}$, where its parts are $A = \{a, b, c, d\}$, $B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and $C = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}\}$:

  $a1\bar{1}2\bar{2}$ $a2\bar{3}1\bar{4}$ $a3\bar{1}4\bar{3}$ $a4\bar{2}1\bar{5}$ $a5\bar{2}6\bar{1}$ $a6\bar{3}3\bar{6}$ $a7\bar{7}6\bar{0}$ $a8\bar{6}0\bar{8}$ $b1\bar{6}2\bar{4}$ $b2\bar{5}3\bar{7}$ $b3\bar{4}0\bar{5}$ $b4\bar{6}5\bar{3}$ $b5\bar{5}6\bar{6}$ $b6\bar{4}7\bar{8}$ $b7\bar{2}9\bar{1}$ $b8\bar{1}0\bar{2}$ $c1\bar{7}0\bar{9}$ $c\bar{0}1\bar{8}6$ $c3\bar{9}9\bar{3}$ $c\bar{1}5\bar{7}9$ $c\bar{6}9\bar{5}4$ $c2\bar{7}8\bar{8}$ $c5\bar{8}3\bar{2}$ $d1\bar{9}4\bar{8}$ $d3\bar{0}9\bar{4}$ $d\bar{0}5\bar{9}6$ $d7\bar{0}4\bar{7}$ $d2\bar{0}8\bar{3}$ $d\bar{6}7\bar{5}8$ $d\bar{1}7\bar{3}0$ $d\bar{9}2\bar{8}9$ $d5\bar{4}8\bar{2}$ $\bar{0}b9a0$ $\bar{9}b0c8$ $7c\bar{7}a\bar{9}$ $4d\bar{5}c\bar{4}$.

Thus, $K_{6,10,10}$ can be decomposed by Corollary 2. And $K_{10,20,20}$ may be decomposed by Lemma 2 and using $K_{6,10,10}$ and $K_{4,10,10}$. Therefore, $K_{30,20,40}$ admits a decomposition by Lemma 3 and using $K_{10,20,20}$ and $K_{20,10,10}$. ∎

References

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