A spectral curve approach to Lawson symmetric CMC surfaces of genus 2

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Abstract Minimal and CMC surfaces in S^3 can be treated via their associated family of flat $SL(2, \mathbb{C})$-connections. In this the paper we parametrize the moduli space of flat $SL(2, \mathbb{C})$-connections on the Lawson minimal surface of genus 2 which are equivariant with respect to certain symmetries of Lawson's geometric construction. The parametrization uses Hitchin's abelianization procedure to write such connections explicitly in terms of flat line bundles on a complex 1-dimensional torus. This description is used to develop a spectral curve theory for the Lawson surface. This theory applies as well to other CMC and minimal surfaces with the same holomorphic symmetries as the Lawson surface but different Riemann surface structure. Additionally, we study the space of isospectral deformations of compact minimal surface of genus $g \geq 2$ and prove that it is generated by simple factor dressing.

Mathematics Subject Classification (2000) 53A10 · 53C42 · 53C43 · 14H60

1 Introduction

The study of minimal surfaces in three dimensional space forms is among the oldest subjects in differential geometry. While minimal surfaces in Euclidean 3-space are never compact, there exist compact minimal surfaces in S^3. In fact, it has been shown by Lawson [21] that for every genus g there exists at least one embedded closed minimal surface in the 3-sphere. A slightly more general surface class is given by constant mean curvature (CMC) surfaces. Due to the Lawson correspondence the partial differential equations describing minimal and CMC surfaces in S^3 can be treated in a uniform way. Compact minimal and CMC surfaces of genus 0 and 1

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are well-understood by now: The only CMC 2-spheres in S^3 are the totally umbilic spheres as the Hopf differential vanishes. Furthermore, Brendle [5] has recently shown that the only embedded minimal torus in S^3 is the Clifford torus up to isometries. This was extended by Andrews and Li [1] who proved that the only embedded CMC tori in S^3 are the unduloidal rotational Delaunay tori. Nevertheless, there exist compact immersed minimal and CMC tori in S^3 which are not congruent to the Clifford torus respectively to the Delaunay tori. First examples have been constructed by Hitchin [15] via integrable systems methods. Moreover, all CMC tori in S^3 are constructed from algebro-geometric data defined on their associated spectral curve, see [4,15,24].

The study of minimal surfaces via integrable system methods is based on the associated $\mathbb{C}^*$-family of flat $SL(2, \mathbb{C})$-connections ∇^λ, $\lambda \in \mathbb{C}^*$. Flatness of ∇^λ for all λ in the spectral plane $\mathbb{C}^*$ is the gauge theoretic reformulation of the harmonic map equation. Knowing the family of flat connections is tantamount to knowing the minimal surface, as the minimal surface is given by the gauge between the trivial connections ∇^1 and ∇^{-1}. Slightly more generally, there also exists a family of flat connections associated to CMC surfaces in S^3. They are given as the gauge between ∇^{λ_1} and ∇^{λ_2} for $\lambda_1 \neq \lambda_2 \in S^1 \subset \mathbb{C}^*$ and have mean curvature $H = i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$. In the abelian case of CMC 2-tori ∇^λ splits for generic λ into a direct sum of flat connections on a line bundle and its dual. Therefore, the $\mathbb{C}^*$-family of flat $SL(2, \mathbb{C})$-connections associated to a CMC torus is characterized by a spectral curve parametrizing the corresponding family of flat complex line bundles. On surfaces of genus $g \geq 2$ flat $SL(2, \mathbb{C})$-connections are generically irreducible and therefore they have non-abelian monodromy. In fact, every (compact) branched CMC surface of genus $g \geq 2$ whose associated family of flat connections has abelian holonomy factors through a CMC torus or is a branched conformal covering of a round sphere [12]. Thus the abelian spectral curve theory for minimal and CMC tori are no longer applicable in the case of compact immersed minimal and CMC surfaces of genus $g \geq 2$.

The aim of this paper is to develop what might be called an integrable systems theory for compact higher genus minimal and CMC surfaces in S^3 based on its associated family of flat connections. The main benefit of this approach is that one can divide the construction and the study of minimal or CMC surfaces into three steps:

1. Write down (enough) flat $SL(2, \mathbb{C})$-connections on a given Riemann surface.
2. Construct a family $\tilde{\nabla}^\lambda$ of flat $SL(2, \mathbb{C})$-connections gauge equivalent (where the gauge is allowed to depend on the spectral parameter λ) to a family of flat connections associated to a CMC surface in S^3. To ensure this, $\tilde{\nabla}^\lambda$ needs to be unitarizable for $\lambda \in S^1$, and trivial for $\lambda_1 \neq \lambda_2 \in S^1 \subset \mathbb{C}^*$ and must have a special asymptotic behavior as $\lambda \to 0$.
3. Reconstruct an associated family of flat connections of a CMC surface from the gauge equivalent family.

In a certain sense these steps occur in the integrable system approach to CMC tori [15]. Here the gauge class of a generic flat $SL(2, \mathbb{C})$-connection is determined by the holonomy of one of the eigenlines. The spectral curve parametrizes these holonomies and the gauge to the associated family can be determined with the help of the eigenline bundle on the spectral curve.

Similarly, the loop group approach to CMC surfaces put forward in [10], sometimes called the DPW method, starts with a family of holomorphic (or meromorphic) $SL(2, \mathbb{C})$-connections on a Riemann surface. Typically, these connections are given by a λ-dependent holomorphic (or meromorphic) $\mathfrak{sl}(2, \mathbb{C})$-valued 1-form called the DPW potential. The DPW potential has a special asymptotic behavior for $\lambda \to 0$ which guarantees the construction of a minimal surface as follows: a (λ-dependent) parallel frame for the family of holomorphic (or meromorphic) flat connections can be split into its unitary and positive parts by the loop group Iwasawa decomposition. The unitary part is characterized by the property that it is unitary on the unit circle $S^1 \subset \mathbb{C}^*$ and the positive part extends to $\lambda = 0$ in a special way. Then, the positive part is the gauge one is looking for, or equivalently, the unitary part is a (λ-dependent) parallel frame for a family of flat connections associated to a minimal surface.

It is well-known that every flat (smooth) $SL(2, \mathbb{C})$-connection on a compact Riemann surface is gauge equivalent (via a gauge which might have singularities) to a flat meromorphic connection, i.e., to a connection whose connection 1-form with respect to an arbitrary holomorphic frame is meromorphic. Nevertheless, it is impossible to parametrize meromorphic connections in a way such that one obtains a unique representative for every gauge class of flat $SL(2, \mathbb{C})$-connections. Therefore, the DPW potential does not need to exist for all $\lambda \in \mathbb{C}^*$. Moreover, the meromorphic connections (given by the DPW potential) need to be unitarizable for $\lambda \in S^1$ (i.e., unitary with respect to an appropriate λ-dependent unitary metric). This reality condition leads to the problem of computing the monodromies of meromorphic connections, which cannot be done by now. The aim of this paper is to overcome these problems, at least partially.

The moduli space of flat $SL(2, \mathbb{C})$-connections on a compact Riemann surface of genus 2 has, at its smooth points, dimension $6g - 6$. There exist singular points, corresponding to reducible flat connections, which have to be dealt with carefully, see [14]. As we are studying holomorphic families of connections (in the sense that the connection 1-forms with respect to a fixed connection depend holomorphically on λ), the moduli space needs to be equipped with a compatible complex structure. Moreover, we need to determine the asymptotic behavior of the family of (gauge equivalence classes of) flat connections for $\lambda \to 0$. This seems to be difficult in the setup of character varieties, i.e., if we identify a gauge equivalence class of flat connections with the conjugacy class of the induced holonomy representation of the fundamental group of the compact Riemann surface. A more adequate picture of the moduli space of flat $SL(2, \mathbb{C})$-connections is given as an affine bundle over the "moduli space" of holomorphic structures of rank 2 with trivial determinant. The projection of this bundle is given by taking (the isomorphism class of) the complex anti-linear part of the connection. This complex anti-linear part is a holomorphic structure, and for a generic flat connections it is even stable. Elements in a fiber of this affine bundle, which can be represented by two flat connections with the same induced holomorphic structure, differ by a holomorphic 1-form with values in the trace free endomorphism bundle. These 1-forms are called Higgs fields and, as a consequence of Serre duality, they are in a natural way the cotangent vectors of the moduli space of holomorphic structures, at least at its smooth points. The bundle is an affine holomorphic bundle and not isomorphic to a holomorphic vector bundle because it does not admit a holomorphic

section. A famous theorem of Narasimhan and Seshadri [23] states that for every stable holomorphic structure on a complex vector bundle of degree 0 over a compact Riemann surface there exists a unique hermitian metric such that the corresponding Chern connection is flat. Thus, the affine bundle of gauge equivalence classes of flat connections over the moduli space of stable holomorphic structures has a smooth (in fact real analytic) section which is induced by the one-to-one correspondence between stable holomorphic structures and unitary flat connections.

In addition to the study of the moduli spaces, we want to construct families of flat connections explicitly. This can be achieved by using Hitchin's abelianization, see [16,17]. The eigenlines of Higgs fields (with respect to some holomorphic structure $\bar{\partial}$) whose determinant is given by the Hopf differential of the CMC surface are well-defined on a double covering of the Riemann surface. They determine points in an affine Prym variety and as line subbundles they intersect each other over the umbilics of the minimal surface. Moreover, a flat connection with holomorphic structure $\bar{\partial}$ determines a meromorphic connection on the direct sum of the two eigenlines of the Higgs field. The residue of this meromorphic connection can be computed explicitly, and the flat meromorphic connection is determined by algebraic geometric data on the double covering surface. Moreover, $\mathbb{C}^*$-families of flat connections can be written down in terms of a spectral curve which double covers the spectral plane $\mathbb{C}^*$. A double covering is needed as a holomorphic structure with a Higgs field corresponds to two different eigenlines and these eigenlines come together at discrete spectral values.

The spectral curve parametrizes the eigenlines of Higgs fields

$$\Psi_\lambda \in H^0(M, K\,End_0(V, \bar{\partial}^\lambda))$$

with $\det \Psi_\lambda = Q$, where the holomorphic structure $\bar{\partial}^\lambda$ is the complex anti-linear part of the connection ∇^λ.

In order to fix the (gauge equivalence classes of the) flat connections ∇^λ additional spectral data are needed. They are given by anti-holomorphic structures on the eigenlines, or, after fixing a special choice of a flat meromorphic connection on a line bundle in the affine Prym variety, by a lift into the affine bundle of gauge equivalence classes of flat line bundle connections. Then, analogous to the case of tori, the asymptotic behavior for $\lambda \to 0$ of the family of flat connections can be understood explicitly: the spectral curve branches over 0 and the family of flat line bundle connections has a first order pole over $\lambda = 0$, see Theorem 5. The spectral data must satisfy a certain reality condition imposed by the property that the connections ∇^λ are unitary for $\lambda \in S^1$. In contrast to the case of CMC tori this reality condition is hard to determine explicitly. Nevertheless, the reality condition is closely related to the geometry of the moduli spaces, see Theorem 2. Once one has constructed such families of (gauge equivalence classes of) flat connections, one can construct minimal and CMC surfaces in S^3 by loop group factorization methods analogous to the DPW method. It would be very interesting to see whether these loop group factorizations can be made as explicit as in the case of tori via the eigenline construction of Hitchin [15].

In this paper we only carry out the details of this program for the Lawson surface of genus 2. These methods easily generalize to the case of Lawson symmetric minimal and CMC surfaces of genus 2, i.e., those surfaces with the same holomorphic and space

orientation preserving symmetries as the Lawson surface but with possibly different conformal type (determined by the cross ratio of the branch images of the threefold covering $M \to M/\mathbb{Z}_3 \cong \mathbb{P}^1$). We briefly discuss this generalization in chapter 7. As explained there one could in principle always exchange *minimal* by *CMC* and *Lawson surface* by *Lawson symmetric surface* within the paper. Moreover, the definition of the spectral curve makes sense even in the case of a compact minimal or CMC surface of genus $g \geq 2$ as long as the Hopf differential has simple zeros. In that case the asymptotic of the spectral data is analogous to Theorem 5. The main difference to the general case is that we can describe the moduli space of flat connections as an affine bundle over the moduli space of holomorphic structures explicitly, see Theorem 1.

In Sect. 2 we study the moduli space of those holomorphic structures of rank 2 with trivial determinant that admit a flat connection whose gauge equivalence classes are invariant under the symmetries of the Lawson surface of genus 2. We show that this space is a projective line with a double point. In Sect. 3 we parametrize representatives of each isomorphism class in the above moduli space by using the eigenlines of special Higgs fields. This method is called Hitchin's abelianization. In our situation the space of all eigenlines is given by the 1-dimensional square torus. By Hitchin's abelianization this torus double covers the moduli space of holomorphic structures away from the double point. This covering map will be crucial for the construction of a spectral curve later on. We use this description in Sect. 4 in order to parametrize the moduli space of flat $SL(2, \mathbb{C})$-connections whose gauge equivalence classes are invariant under the symmetries of the Lawson surface. In Theorem 1 we prove an explicit 2:1 correspondence (away from a co-dimension 1 subset corresponding to the double point of the moduli space of holomorphic structures) between flat $\mathbb{C}^*$-connections on the above mentioned square torus and the moduli space of flat connections whose gauge equivalence classes are invariant under the symmetries of the Lawson surface of genus 2. This study will be completed in Sect. 5 where we consider flat connections whose underlying holomorphic structures do not admit Higgs fields whose determinant is equal to the Hopf differential of the Lawson surface.

In Sect. 6 we define the spectral curve associated to a minimal surface in S^3 which has the conformal type and the holomorphic symmetries of the Lawson surface of genus 2 (Proposition 6). The spectral curve is equipped with a meromorphic lift into the affine bundle of isomorphy classes of flat line bundle connections on the square torus. This lift determines the gauge equivalence classes of the flat connections. The spectral data satisfy two important properties, see Theorem 5. Firstly, they have a certain asymptotic at $\lambda = 0$. Moreover, the spectral data must satisfy a reality condition which is related to the geometry of the moduli space of stable bundles, see Theorem 2. We prove a general theorem (6) about the reconstruction of minimal surfaces out of those families of flat connections $\tilde{\nabla}^\lambda$ as described in step 2 above.

Similar to the case of tori, compact minimal and CMC surfaces of higher genus are in general not uniquely determined by the knowledge of the gauge equivalence classes of ∇^λ for all $\lambda \in \mathbb{C}^*$. We show in Theorem 7 that all different minimal immersions with the same map $\lambda \mapsto [\nabla^\lambda]$ into the moduli space of flat connections and with the same induced Riemann surface structure are generated by simple factor dressing (as defined for example in [8]). Finally, we prove in Theorem 8 that minimal surfaces with the symmetries of the Lawson genus 2 surface can be reconstructed uniquely from

spectral data satisfying the conditions of Theorem 5. Moreover, we give an energy formula for those minimal surfaces in terms of their spectral data.

In the appendix, we briefly recall the gauge theoretic reformulation of the minimal surface equations in S^3 due to Hitchin [15] which leads to the associated family of flat connections. We also describe the construction of the Lawson minimal surface of genus 2.

2 The moduli space of Lawson symmetric holomorphic structures

Before studying the associated family of flat $SL(2, \mathbb{C})$-connections

$$\lambda \mapsto \nabla^\lambda$$

for a given compact oriented minimal or CMC surface in S^3 (see Appendix A or chapter 7 in the case of CMC surfaces), we need to understand the moduli space of gauge equivalence classes of flat $SL(2, \mathbb{C})$-connections on the surface. We consider it as an affine bundle over the moduli space of isomorphism classes of holomorphic structures $(V, \bar{\partial})$ of rank 2 with trivial determinant over the Riemann surface. The complex structure is the one induced by the minimal immersion and the projection is given by taking the complex anti-linear part

$$\nabla'' := \frac{1}{2}(\nabla + i * \nabla)$$

of the flat connection ∇. The difference $\Psi = \nabla^2 - \nabla^1 \in \Gamma(M, K\,End_0(V))$ between two flat $SL(2, \mathbb{C})$-connections ∇^1 and ∇^2 with the same underlying holomorphic structure $\bar{\partial} = (\nabla^i)''$ satisfies

$$0 = F^{\nabla^2} = F^{\nabla^1} + d^\nabla \Psi = \bar{\partial}\Psi.$$

Therefore, the fiber of the affine bundle over a fixed isomorphism class of holomorphic structures (represented by the holomorphic structure $\bar{\partial}$) is given by the space of Higgs fields

$$H^0(M, K\,End_0(V, \bar{\partial})),$$

i.e., the space of holomorphic trace free endomorphism valued 1-forms on M. By Serre duality, this is naturally isomorphic to the cotangent space of the moduli space of holomorphic structures, at least at its smooth points.

In this paper we mainly focus on the Lawson minimal surface M of genus 2. Therefore we start by studying those holomorphic structures of rank 2 on M which can occur as the complex anti-linear parts of a connection ∇^λ in the associated family of M. As we will see, this simplifies the study of the moduli spaces and allows us to find explicit formulas for flat connections with a given underlying holomorphic structure.

The complex structure of the Lawson surface of genus 2 is given by (the compactification of) the complex curve

$$y^3 = \frac{z^2 - 1}{z^2 + 1}. \tag{1}$$

As a surface in S^3 it has a large group of extrinsic symmetries, see Appendix B. We will focus on the symmetries which are holomorphic on M and orientation preserving in S^3. The reason for this restriction relies on the fact that only those give rise to symmetries of the individual flat connections ∇^λ. As a group, they are generated by the following automorphisms, where the equations are written down with respect to the coordinates y and z of (1):

– the hyper-elliptic involution φ_2 of the surface of genus 2 which is given by

$$(y, z) \mapsto (y, -z);$$

– the automorphism φ_3 satisfying

$$\varphi_3(y, z) = (e^{\frac{2}{3}\pi i} y, z);$$

– the composition τ of the reflections at the spheres S_1 and S_2 is given by

$$(y, z) \mapsto \left(e^{\frac{1}{3}\pi i} \frac{1}{y}, \frac{i}{z} \right).$$

Every single connection ∇^λ is gauge equivalent to $\varphi_2^* \nabla^\lambda$, $\varphi_3^* \nabla^\lambda$ and $\tau^* \nabla^\lambda$. This can be deduced from the construction of the associated family of flat connections, see [18] for details.

Definition 1 A $SL(2, \mathbb{C})$-connection ∇ on M is called Lawson symmetric, if ∇ is gauge equivalent to $\varphi_2^* \nabla$, $\varphi_3^* \nabla$ and $\tau^* \nabla$. Similarly, a holomorphic structure $\bar{\partial}$ of rank 2 with trivial determinant on M is called Lawson symmetric if it is isomorphic to $\varphi_2^* \bar{\partial}$, $\varphi_3^* \bar{\partial}$ and $\tau^* \bar{\partial}$.

We first determine which holomorphic structures occur in the family

$$\lambda \mapsto \bar{\partial}^\lambda := (\nabla^\lambda)'' = \frac{1}{2}(\nabla^\lambda + i * \nabla^\lambda)$$

associated to the Lawson surface. As ∇^λ is generically irreducible (see [19]), and special unitary for $\lambda \in S^1$, $\bar{\partial}^\lambda$ is generically stable: A holomorphic bundle of rank 2 of degree 0 is (semi-)stable if every holomorphic line sub-bundle has negative (non-positive) degree, see [23] or [22]. On a compact Riemann surface of genus 2 the moduli space of stable holomorphic structures with trivial determinant on a vector bundle of rank $r = 2$ can be identified with an open dense subset of a projective 3-dimensional space, see [22]: the set of those holomorphic line bundles, which are dual to a holomorphic line subbundle of degree -1 in the holomorphic rank 2 bundle, is given by the support of a divisor which is linearly equivalent to twice the Θ-divisor in the Picard variety $Pic_1(M)$ of holomorphic line bundles of degree 1. This

divisor uniquely determines the rank 2 bundle up to isomorphism if the bundle is stable. Therefore the moduli space of stable holomorphic structures of rank 2 with trivial determinant can be considered as a subset of the projective space of the 4-dimensional space $H^0(Jac(M), L(2\Theta))$ of Θ functions of rank 2 on the Jacobian of M. The complement of this subset in the projective space is the Kummer surface associated to the Riemann surface of genus 2. The points on the Kummer surface can be identified with the S-equivalence classes of strictly semi-stable holomorphic structures. Recall that the S-equivalence class of a stable holomorphic structure is just its isomorphism class but that S-equivalence identifies the strictly semi-stable holomorphic direct sum bundles $V = L \oplus L^*$ (where $deg(L) = 0$) with nontrivial extensions $0 \to L \to V \to L^* \to 0$. An extension $0 \to L \to V \to L^* \to 0$ (where L is allowed to have arbitrary degree) is given by a holomorphic structure of the form

$$\bar{\partial} = \begin{pmatrix} \bar{\partial}^L & \gamma \\ 0 & \bar{\partial}^{L^*} \end{pmatrix},$$

where $\gamma \in \Gamma(M, \bar{K}Hom(L^*, L))$. It is called non-trivial if the holomorphic structure is not isomorphic to the holomorphic direct sum $L \oplus L^*$. This is measured by the extension class $[\gamma] \in H^1(M, Hom(L^*, L))$. Note that the isomorphism class of the holomorphic bundle V given by an extension $0 \to L \to V \to L^* \to 0$ with extension class $[\gamma]$ is already determined by L and $\mathbb{C}[\gamma] \in \mathbb{P}H^1(M, Hom(L^*, L))$.

Proposition 1 *Let $\mathcal{M} \subset \mathbb{P}^3 = \mathbb{P}H^0(Jac(M), L(2\Theta))$ be the space of S-equivalence classes of semi-stable Lawson symmetric holomorphic structures over the Lawson surface M. Then the connected component $\mathcal{S}$ of $\mathcal{M}$ containing the trivial holomorphic structure $(\mathbb{C}^2, d'')$ is given by a projective line in $\mathbb{P}^3$.*

Proof The fixed point set of any of these three symmetries is given by the union of projective subspaces of $\mathbb{P}^3$. Clearly, the common fixed-point set contains a projective subspace of dimension ≥ 1, as $\lambda \mapsto \bar{\partial}^\lambda$ is a non-constant holomorphic map into this space.

The space of S-equivalence classes of semi-stable non-stable bundles is the Kummer surface of M in $\mathbb{P}^3$. It has degree 4, and 16 double points. These double points are given by extensions of self-dual line bundles L by itself. In order to see that $\mathcal{S}$ is a projective line it is enough to show that the only strictly semi-stable bundles V, whose isomorphism classes are invariant under φ_2, φ_3 and τ, are the trivial rank two bundle $\mathbb{C}^2$ (which is a double point in the Kummer surface) and the direct sum bundles

$$L(P_1 - P_2) \oplus L(P_2 - P_1), \quad L(P_1 - P_4) \oplus L(P_4 - P_1),$$

where $P_1, .., P_4 \in M$ are the zeros of the Hopf differential of M. So let L be a holomorphic line sub-bundle of V of degree 0. Because M has genus 2 there exists two points $P, Q \in M$ such that L is given as the line bundle $L(P - Q)$ associated to the divisor $P - Q$. If $P = Q$ then V is in the S-equivalence class of $\mathbb{C}^2$. If $P \neq Q$ then $\varphi_2^* L(P - Q)$ is either isomorphic to $L(P - Q)$ or $L(Q - P)$, as $\varphi_2^* V$ and V are S-equivalent. Clearly, the same holds for τ, and as φ_3 is of order 3 we even get that

$\varphi_3^* L(P - Q) = L(P - Q)$. From these observations we deduce that the points P and Q are fixed points of φ_3, and as a consequence V is S-equivalent to one of the above mentioned direct sum bundles. □

The next proposition shows that we do not need to care about S-equivalence of holomorphic bundles.

Proposition 2 *Every Lawson symmetric strictly semi-stable holomorphic rank 2 bundle $V \to M$ is isomorphic to the direct sum of two holomorphic line bundles.*

Proof As we have seen in the proof of the previous theorem V is S-equivalent to one of the holomorphic rank 2 bundles $\underline{\mathbb{C}}^2$, $L(P_1 - P_2) \oplus L(P_2 - P_1)$ and $L(P_1 - P_4) \oplus L(P_4 - P_1)$. As

$$\varphi_2^* L(P_i - P_j) = L(P_j - P_i) \neq L(P_i - P_j)$$

for $i \neq j$ we see that V cannot be a non-trivial extension of $L(P_i - P_j)$ by its dual $L(P_j - P_i)$. It remains to consider the case where V is S-equivalent to $\underline{\mathbb{C}}^2$. Then the holomorphic structure of V is given by

$$\begin{pmatrix} \bar{\partial}^{\mathbb{C}} & \gamma \\ 0 & \bar{\partial}^{\mathbb{C}} \end{pmatrix}.$$

Here $\gamma \in \Gamma(M, \bar{K})$ and the projective line of its cohomology class in $H^1(M, \mathcal{O})$ is an invariant of the isomorphism class of V. This projective line is determined by its annihilator in $H^0(M, K) = H^1(M, \mathcal{O})^*$. The annihilator of $[\gamma]$ is $H^0(M, K)$ exactly in the case where V is (isomorphic to) the holomorphic direct sum $\mathbb{C}^2 \to M$, and otherwise it is a line in $H^0(M, K)$. Since V is isomorphic to $\varphi_2^* V, \varphi_3^* V$ and $\tau^* V$ this line would be invariant under φ_2, φ_3 and τ which leads to a contradiction. □

2.1 Non semi-stable holomorphic structures

It was shown in [19] that for a generic $\lambda \in \mathbb{C}^*$ the holomorphic structure $\bar{\partial}^\lambda$ is stable. Nevertheless there can exist special $\lambda \in \mathbb{C}^*$ such that $\bar{\partial}^\lambda$ is neither stable nor semi-stable. We now study which non semi-stable holomorphic structures admit Lawson symmetric flat connections.

Let ∇ be a flat, Lawson symmetric $SL(2, \mathbb{C})$-connection on a complex rank 2 bundle over M such that $\nabla'' = \bar{\partial}$ is not semi-stable. By assumption, there exists a holomorphic line subbundle L of $(V, \bar{\partial})$ of degree ≥ 1. The second fundamental form

$$\beta = \pi^{V/L} \circ \nabla_{|L} \in \Gamma(K\,Hom(L, V/L)) = \Gamma(KL^{-2})$$

of L with respect to ∇ is holomorphic by flatness of ∇. As $deg(L) \geq 1$, L cannot be a parallel subbundle because in that case it would inherit a flat connection. This implies $0 \neq \beta \in H^0(M, KL^{-2})$, and because $\deg(KL^{-2}) = 2 - 2\deg(L) \leq 0$ we obtain

that $L^2 = K$. This means that L is a spin bundle of M. The only spin bundle S of M which is isomorphic to $\varphi_2^* S$, $\varphi_3^* S$ and $\tau^* S$ is given by

$$S = L(Q_1 + Q_3 - Q_5),$$

see [18]. As there exists a flat connection with underlying holomorphic structure $\bar{\partial}$, the bundle $(V, \bar{\partial})$ cannot be isomorphic to the holomorphic direct sum $S \oplus S^* \to M$. Therefore it is given by a non-trivial extension $0 \to S \to V \to S^* \to 0$. As $H^1(M, S^2)$ is 1-dimensional, a non semi-stable holomorphic structure admitting a flat, Lawson symmetric $SL(2, \mathbb{C})$-connection is already unique up to isomorphism. A particular choice of such a flat connection ∇ is given by the uniformization connection, see [17]: Consider the holomorphic direct sum $V = S \oplus S^* \to M$, where S is the spin bundle mentioned above. On M there exists a unique Riemannian metric of constant curvature -4 in the conformal class of the Riemann surface M. This Riemannian metric induces spin connections and unitary metrics on S and S^*. Let $\Phi = 1 \in H^0(M, K Hom(S, S^*))$ and $\Phi^* = vol$ be its dual with respect to the metric. Then

$$\nabla^u = \nabla = \begin{pmatrix} \nabla^{spin} & vol \\ 1 & \nabla^{spin^*} \end{pmatrix}, \tag{2}$$

is flat. Moreover, it is also Lawson symmetric. This can easily be deduced from the uniqueness of the conformal Riemannian metric of constant curvature -4. The holomorphic structure ∇'' is clearly given by the non-trivial extension $0 \to S \to V \to S^* \to 0$.

Proposition 3 *Every flat, Lawson symmetric connection ∇ on M, whose underlying holomorphic structure ∇'' is not semi-stable, is gauge equivalent to*

$$\begin{pmatrix} \nabla^{spin} & CQ + vol \\ 1 & \nabla^{spin^*} \end{pmatrix},$$

where ∇^{spin} and vol are the spin connection and the volume form of the conformal metric of constant curvature -4 on M, $C \in \mathbb{C}$ and Q is the Hopf differential of the Lawson surface.

Proof Every other flat $SL(2, \mathbb{C})$-connection ∇, whose underlying holomorphic structure is $\bar{\partial}$, is given by $\nabla = \nabla^u + \Psi$ where

$$\Psi \in H^0(M, K End_0(V, \bar{\partial}))$$

is a Higgs field. An arbitrary section $\Psi \in \Gamma(M, K End_0(V, \bar{\partial}))$ is given by

$$\Psi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

with respect to the decomposition $V = S \oplus S^*$ and the matrix entries are thus sections $a \in \Gamma(M, K), b \in \Gamma(M, K^2)$ and $c \in \Gamma(M, \mathbb{C})$. Then

$$\bar{\partial} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} \bar{\partial}^K a + c vol & \bar{\partial}^{K^2} b + 2a vol \\ \bar{\partial}^{\mathbb{C}} c & -\bar{\partial}^K a - c vol \end{pmatrix}.$$

This shows that $c = 0$ if Ψ is holomorphic. Moreover, for a holomorphic 1-form $\alpha \in H^0(M, K)$ the gauge

$$g := \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

is holomorphic with respect to $\bar{\partial}$ and satisfies

$$g^{-1} \nabla^u g - \nabla^u = \begin{pmatrix} -\alpha & \partial^K \alpha \\ 0 & \alpha \end{pmatrix} \in H^0(M, K End_0(V, \bar{\partial})).$$

Therefore, we can restrict our attention to the case of Higgs fields which have the following form

$$\Psi := \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

where b is a holomorphic quadratic differential by holomorphicity of Ψ. The Hopf differential Q of the Lawson surface is (up to constant multiples) the only holomorphic quadratic differential on M which is invariant under φ_2, φ_3 and τ. From this it easily follows that if the gauge equivalence class of $\nabla^u + \Psi$ is invariant under φ_2, φ_3 and τ then b must be a constant multiple of Q. □

Remark 1 The orbits under the group of gauge transformations of the above mentioned non semi-stable holomorphic structure and $\bar{\partial}^0$ get arbitrarily close to each other: Consider the families of holomorphic structures

$$\bar{\partial}_t = \begin{pmatrix} \bar{\partial}^S & t\, vol \\ Q^* & \bar{\partial}^{S^*} \end{pmatrix} \quad \text{and} \quad \tilde{\bar{\partial}}_t = \begin{pmatrix} \bar{\partial}^S & vol \\ t\, Q^* & \bar{\partial}^{S^*} \end{pmatrix}$$

on $V = S \oplus S^*$. Clearly, $\bar{\partial}_t$ and $\tilde{\bar{\partial}}_t$ are gauge equivalent for $t \neq 0$. On the other hand $\bar{\partial}_0 = \bar{\partial}^0$ and $\tilde{\bar{\partial}}_0 = \bar{\partial}$ which are clearly not isomorphic. We will see later how to distinguish such families of isomorphism classes of holomorphic structures if they are equipped with corresponding families of gauge equivalence classes of flat connections.

3 Hitchin's abelianization

A very useful construction for the study of a moduli space of holomorphic (Higgs) bundles is given by Hitchin's integrable system [16,17]. We do not describe this

integrable system in detail but apply some of the methods in order to construct the moduli space $\mathcal{S}$, which was studied in the previous chapter, explicitly. The main idea is the following: A holomorphic structure of rank 2 equipped with a Higgs field is already determined by the eigenlines of the Higgs field (which are in general only well-defined on a double covering of the Riemann surface). In fact, the rank 2 bundle is the push forward of the dual of an eigenline bundle. In our situation, appropriate Higgs fields of a Lawson symmetric holomorphic structure are basically unique up to a multiplicative constant by Lemma 1 and its proof. In general the two eigenlines are given by points in a Prym variety which are dual to each other. This Prym variety turns out to be the Jacobian of a 1-dimensional square torus in the case of Lawson symmetric holomorphic structures with symmetric Higgs fields, see Lemmas 2 and 3. Moreover, this Jacobian double covers the moduli space $\mathcal{S}$ in a natural way (Proposition 4).

Lemma 1 *Let $\bar{\partial}$ be a Lawson symmetric, semi-stable holomorphic structure on a rank 2 bundle over M which is not isomorphic to $\bar{\partial}^0$. Then there exists a Higgs field $\Psi \in H^0(M, K\,End_0(V, \bar{\partial}))$ with*

$$\det \Psi = Q \in H^0(M, K^2)$$

which satisfies $\varphi^* \Psi = g^{-1} \Psi g$ for every Lawson symmetry φ, where g is the isomorphism between the holomorphic structures $\bar{\partial}$ and $\varphi^* \bar{\partial}$. This Higgs fields is unique up to sign.

Proof By Proposition 2 every Lawson symmetric, semi-stable and non-stable holomorphic structure is the holomorphic direct sum of two line bundles. For these bundles, it is easy to construct a Higgs field Ψ with $\det \Psi = Q$. Moreover, this Higgs field Ψ can be constructed such that its pull-back $\varphi^* \Psi$ for a Lawson symmetry φ is conjugated to Ψ.

All stable holomorphic structures give rise to smooth points in the moduli space of holomorphic structures. Let $[\mu] \in H^1(M, End_0(V))$ be a non-zero tangent vector of the isomorphism class of the stable holomorphic structure $\bar{\partial}$ in $\mathcal{S}$. By the non-abelian Hodge theory (see for example [3]) and the Narasimhan–Seshadri theorem, the class $[\mu]$ can be represented by an endomorphism-valued complex anti-linear 1-form $\mu \in \Gamma(M, \bar{K}\,End_0(V))$ which is parallel with respect to the (unique) unitary flat connection ∇ with $\nabla'' = \bar{\partial}$. Let φ be one of the symmetries φ_2, φ_3 or τ and g be a gauge, i.e., a smooth isomorphism, of V such that $\varphi^* \bar{\partial} = g^{-1} \bar{\partial} g$. As $\bar{\partial}$ is stable g is unique up to multiplication with a constant multiple of the identity. We claim that $\varphi^* \mu = g^{-1} \mu g$. To see this note that $g^{-1} \mu g$ represents (with respect to $g^{-1} \bar{\partial} g$) the same tangent vector in $T_{[\bar{\partial}]}\mathcal{S}$ as μ (with respect to $\bar{\partial}$) and as $\varphi^* \mu$ (with respect to $\varphi^* \bar{\partial} = g^{-1} \bar{\partial} g$). Therefore $[g^{-1} \mu g - \varphi^* \mu] = 0 \in T_{[\bar{\partial}]}\mathcal{S}$, and by non-abelian Hodge theory $g^{-1} \mu g - \varphi^* \mu$ is in the image of $g^{-1} \nabla g$. Moreover the unitary flat connections $g^{-1} \nabla g$ and $\varphi^* \nabla$ coincide by the uniqueness of the Narasimhan-Seshadri theorem. Hence the difference $g^{-1} \mu g - \varphi^* \mu$ is parallel. This is only possible if $g^{-1} \mu g - \varphi^* \mu = 0$ as claimed.

Consider the (non-zero) adjoint $\Psi = \mu^* \in H^0(M, K\,End_0(V))$ which clearly satisfies $\varphi^* \Psi = g^{-1} \Psi g$ for φ and g as above. Therefore the holomorphic quadratic

differential $\det \Psi \in H^0(M; K^2)$ is invariant under φ_2, φ_3 and τ. If $\det \Psi \neq 0$ this implies that it is a constant non-zero multiple of the Hopf differential of the Lawson surface. If $\det \Psi = 0$ consider the holomorphic line bundle $L = \ker \Psi \subset V$. As Ψ is trace-free it defines $0 \neq \tilde{\Psi} \in H^0(M, K\,Hom(V/L, L)) = H^0(M, KL^2)$. Because $\deg(L) \leq -1$ as $\bar{\partial}$ is stable, L must be dual to a spin bundle, and because $\varphi^*\Psi = g^{-1}\Psi g$ it is even the dual of the holomorphic spin bundle $S = L(Q_1 + Q_3 - Q_5)$. This easily implies that $\bar{\partial}$ is isomorphic to $\bar{\partial}^0$ in the case of $\det \Psi = 0$. $\square$

Definition 2 The Higgs fields of Lemma 1 are called symmetric Higgs fields.

3.1 The eigenlines of symmetric Higgs fields

The zeros of the Hopf differential Q of the Lawson surface M are simple. As a Higgs field is trace free by definition, the eigenlines of a symmetric Higgs field Ψ (for a Lawson symmetric holomorphic structure $\bar{\partial}$) with $\det \Psi = Q$ are not well-defined on the Riemann surface M. Following Hitchin [16] we define a (branched) double covering of M on which the square root of Q is well-defined:

$$\pi : \tilde{M} := \{\omega_x \in K_x | x \in M, \omega_x^2 = Q_x\} \to M.$$

We denote the involution $\omega_x \mapsto -\omega_x$ by $\sigma : \tilde{M} \to \tilde{M}$. There exists a tautological section

$$\omega \in H^0(\tilde{M}, \pi^* K_M)$$

satisfying

$$\omega^2 = \pi^* Q \quad \text{and} \quad \sigma^* \omega = -\omega.$$

As the Hopf differential is invariant under φ_2, φ_3 and τ these symmetries of M lift to symmetries of $\tilde{M}$ denoted by the same symbols. The tautological section is invariant under these symmetries

$$\varphi_2^* \omega = \omega, \ \varphi_3^* \omega = \omega, \ \tau^* \omega = \omega,$$

where we have naturally identified $\varphi_2^* \pi^* K_M = \pi^* \varphi_2^* K_M = \pi^* K_M$ and analogous for φ_3 and τ. On $\tilde{M}$ the eigenlines of $\pi^* \Psi$ are well-defined:

$$L_\pm := \ker \pi^* \Psi \mp \omega Id.$$

Clearly, $\sigma^* L_\pm = L_\mp$. As the zeros of $Q = \det \Psi$ are simple, Ψ has a one-dimensional kernel at these zeros. Therefore, the eigenline bundles $L_\pm$ intersect each other with order 1 in $\pi^* V$ at the branch points of π. Otherwise said, there is a holomorphic section

$$\wedge \in H^0(\tilde{M}, Hom(L_+ \otimes L_-, \Lambda^2 \pi^* V))$$

which has zeros of order 1 at the branch points of π. Thus, $\wedge$ can be considered as a constant multiple of $\omega \in H^0(\tilde{M}, \pi^* K_M)$ which has also simple zeros exactly at the branch points of π by construction. Because $\Lambda^2 V$ is the trivial holomorphic line bundle, the eigenline bundles satisfy

$$L_+ \otimes L_- = L_+ \otimes \sigma(L_+) = \pi^* K_M^*, \tag{3}$$

which means that $L_\pm$ lie in an affine Prym variety for π. Recall that the Prym variety of $\pi : \tilde{M} \to M$ is by definition

$$Prym(\pi) = \{L \in Jac(\tilde{M}) \mid \sigma^* L = L^*\}.$$

After fixing the line bundle $L = \pi^* S^*$, which clearly satisfies (3), every other line bundle L^+ satisfying (3) is given by $L^+ = \pi^* S^* \otimes E$ for some holomorphic line bundle $E \in Prym(\pi)$.

3.2 Reconstruction of holomorphic rank 2 bundles

We briefly describe how to reconstruct the bundle V from an eigenline bundle $L_+ \to \tilde{M}$ of a symmetric Higgs field $\Psi \in H^0(M, K End_0(V))$ with non-vanishing determinant $\det \Psi \neq 0$. This construction will be used later to study Lawson symmetric holomorphic connections on $\tilde{M}$. First consider an open subset $U \subset M$ which does not contain a branch value of π. The preimage $\pi^{-1}(U) \subset \tilde{M}$ consists of two disjoint copies $U_1 \cup U_2 \subset \tilde{M}$ of U. Because

$$\pi^* V_{|U_i} = (L_+ \oplus L_-)_{|U_i}$$

and $\sigma(L_\pm) = L_\mp$, we obtain a basis of holomorphic sections of V over U which is given by the non-vanishing sections

$$s_1 \in H^0(U_1, L_+) \quad \text{and} \quad s_2 \in H^0(U_1, L_-) \cong H^0(U_2, L_+).$$

This local basis of holomorphic sections in V is special linear if and only if

$$\wedge(s_1 \otimes s_2) = 1 \in H^0(U_1, \pi^* K_M \otimes L_+ \otimes L_-) = H^0(U_1, \mathbb{C})$$

in U_1.

Next we consider the case of a branch point p of π: Let $z : U \subset \tilde{M} \to \mathbb{C}$ be a local coordinate centered at p such that $\sigma(z) = -z$ and $\sigma(U) = U$. A local coordinate on $\pi(U)$ around $\pi(p) \in M$ is given by y with $y = z^2$. We may choose z in such a way that

$$\wedge = z dy + higher\ order\ terms \in H^0(U, \pi^* K_M),$$

where $dy \in H^0(U, \pi^* K_M)$ is the pull-back as a section and not as a 1-form. Let $t_1 \in H^0(U, L_+), t_2 = \sigma(t_1) \in H^0(U, L_-)$ be holomorphic sections without zeros such that

$$\wedge(t_1 \otimes t_2) = z \in H^0(U, \mathbb{C}).$$

Then there are local holomorphic basis fields s_1, s_2 of $V \to M$ with $s_1 \wedge s_2 = 1$ such that $\pi^* s_1(p) = t_1(p) = t_2(p)$ and

$$t_1 = \pi^* s_1 - \frac{z}{2} \pi^* s_2, \ \ t_2 = \pi^* s_1 + \frac{z}{2} \pi^* s_2 \qquad (4)$$

in $\pi^* V$, or equivalently

$$\pi^* s_1 = \frac{1}{2} t_1 + \frac{1}{2} t_2, \ \ \pi^* s_2 = \frac{1}{z} t_2 - \frac{1}{z} t_1.$$

As the last equation is invariant under σ this gives us a well-defined special linear holomorphic frame $\pi^* s_1, \pi^* s_2$ of V over $\pi(U) \subset M$.

By going through the above construction carefully without a priori knowing the existence of a rank 2 bundle one can construct a holomorphic rank 2 bundle $V \to M$ for any line bundle L in the affine Prym variety. Then one can show that this rank 2 bundle has trivial determinant and that there exists a Higgs field on V whose determinant is Q. See for example [16] for details on this.

Remark 2 The above reconstruction is the differential geometric formulation of the sheaf theoretic push-forward construction $\pi_* L_\pm^*$.

Remark 3 Because of Lemma 1 a generic Lawson symmetric stable bundle $V \to M$ corresponds via the above construction to exactly two different line bundles L_+ and $L_- = \sigma(L_+)$.

3.3 The torus parametrizing holomorphic structures

The Prym variety of the double covering $\pi : \tilde{M} \to M$ is complex 3-dimensional and the moduli space $\mathcal{S}$ of Lawson symmetric holomorphic structures is only 1-dimensional. We now determine which line bundles L_+ in the affine Prym variety correspond to Lawson symmetric holomorphic structures.

Let $\bar{\partial}$ be a Lawson symmetric holomorphic structure which admits a symmetric Higgs field Ψ whose determinant is the Hopf differential Q of the Lawson surface. By the definition of symmetric Higgs fields the eigenlines $L_\pm$ of Ψ are isomorphic to $\varphi_2^* L_\pm, \varphi_3^* L_\pm$ and $\tau^* L_\pm$. Recall that the same is true for our base point $\pi^* S^*$ in the affine Prym variety. Therefore, it remains to determine the connected component of those holomorphic line bundles $\tilde{E}$ of degree 0 on $\tilde{M}$ whose isomorphism class is invariant under φ_2, φ_3 and τ. The quotient

$$\tilde{\pi} : \tilde{M} \to \tilde{M}/\mathbb{Z}_3$$

of the $\mathbb{Z}_3$-action induced by φ_3 is a square torus. Moreover, φ_2 and τ induce fixed-point free holomorphic involutions on $\tilde{M}/\mathbb{Z}_3$ (denoted by the same symbols). They are given by translations. Therefore, the pull-back $\tilde{E} = \tilde{\pi}^* E$ of every line bundle $E \in Jac(\tilde{M}/\mathbb{Z}_3)$ is invariant under φ_2, φ_3 and τ.

In general one has to distinguish between those bundles which are pull-backs of bundles on the quotient of some automorphism on a Riemann surface and bundles whose isomorphism class is invariant under the automorphism. In our situation they turn out to be the same:

Lemma 2 *Let $\tilde{E}$ be a holomorphic line bundle of degree 0 on $\tilde{M}$. If its isomorphism class is invariant under φ_2, φ_3 and τ then $\tilde{E}$ is isomorphic to the pull-back $\tilde{\pi}^* E$ for some $E \in Jac(\tilde{M}/\mathbb{Z}_3)$.*

Proof We only sketch the proof of the lemma: Consider the corresponding flat unitary connection ∇ on $\tilde{E}$. As the isomorphism class of $\tilde{E}$ is invariant under φ_2, φ_3 and τ the gauge equivalence class of ∇ is also invariant under φ_2, φ_3 and τ. This gauge equivalence class is determined by its (abelian) monodromy representation

$$\pi_1(\tilde{M}) \to U(1) = S^1 \subset \mathbb{C}.$$

Using the symmetries φ_2, φ_3 and τ one can easily deduce that the connection is (gauge equivalent) to the pull-back of a flat connection on the torus $\tilde{M}/\mathbb{Z}_3$. $\qquad\square$

Lemma 3 *The connected component of the space of $\mathbb{Z}_3$-invariant line bundles in the Prym variety of $\pi : \tilde{M} \to M$ containing the trivial holomorphic line bundle is given by the (pull-back of the) Jacobian of the torus $\tilde{M}/\mathbb{Z}_3$.*

Proof Any line bundle on the torus is given by $E = L(x - p)$, where x is a suitable point on the torus and p is the image of the branch point $P_1 \in \tilde{M}$. The involution σ descends to an involution on $\tilde{M}/\mathbb{Z}_3$ with four fixed points which are exactly the images of the branch points of $\tilde{\pi}$. Therefore, the quotient of $\tilde{M}/\mathbb{Z}_3$ by σ is the projective line $\mathbb{P}^1$ and

$$E \otimes \sigma^* E = L(x - p + \sigma(x) - p) = \underline{\mathbb{C}}$$

which implies $\tilde{\pi}^* E \otimes \sigma^*(\tilde{\pi}^* E) = \tilde{\pi}^*(E \otimes \sigma^* E) = \underline{\mathbb{C}}.$ $\qquad\square$

These two lemmas enable us to define a double covering $\Pi : Jac(\tilde{M}/\mathbb{Z}_3) \to \mathcal{S} = \mathbb{P}^1$: Take a line bundle $L \in Jac(\tilde{M}/\mathbb{Z}_3)$ and consider

$$L_+ := \pi^* S^* \otimes \tilde{\pi}^* L \to \tilde{M}.$$

The isomorphism class of this line bundle is invariant under φ_2, φ_3 and τ and it satisfies

$$L_+ \otimes \sigma(L_+) = \pi^* K_M$$

by Lemma 3. As we have seen in Sect. 3.2, L_+ is an eigenline bundle of a symmetric Higgs field of the pullback $\pi^* V \to \tilde{M}$ of a holomorphic rank two bundle $V \to M$ with trivial determinant.

Proposition 4 *There exists an even holomorphic map*

$$\Pi \colon Jac(\tilde{M}/\mathbb{Z}_3) \to \mathcal{S} = \mathbb{P}^1$$

of degree 2 to the moduli space $\mathcal{S}$ of Lawson symmetric holomorphic bundles. This map is determined by $\Pi(L) = [\bar{\partial}]$ for $L \neq \mathbb{C} \in Jac(\tilde{M}/\mathbb{Z}_3)$ such that $\pi^*S^* \otimes \tilde{\pi}^*L$ is isomorphic to an eigenline bundle of a symmetric Higgs field of the Lawson symmetric holomorphic rank two bundle $(V, \bar{\partial})$, and by $\Pi(\mathbb{C}) = [\bar{\partial}^0] \in \mathcal{S}$ (see Lemma 1). The branch points are the spin bundles of $\tilde{M}/\mathbb{Z}_3$ and the branch images of the non-trivial spin bundles are exactly the isomorphism classes of the semi-stable non-stable holomorphic bundles.

Proof First we show that for $L \neq \mathbb{C}$, the corresponding rank two bundle is semi-stable: Assume that E is a holomorphic line subbundle of a Lawson symmetric holomorphic bundle $V \to M$ of degree greater than 0. If E is not a spin bundle of the genus 2 surface M the rank two bundle V would be isomorphic to the holomorphic direct sum $E \oplus E^*$. In this case one easily sees that there does not exist a Higgs field whose determinant has simple zeros. If E is a spin bundle it must be isomorphic to the spin bundle S of the Lawson immersion because of the symmetries. Let the rank two holomorphic structure be given by

$$\bar{\partial} = \begin{pmatrix} \bar{\partial}^S & \alpha \\ 0 & \bar{\partial}^{S^*} \end{pmatrix}$$

on the topological direct sum $V = S \oplus S^*$ for some $\alpha \in \Gamma(M, \bar{K}K)$. The eigenline bundle $\pi^*S^* \otimes \tilde{\pi}^*L \subset V$ would be given by a map

$$\begin{pmatrix} a \\ b \end{pmatrix} \colon \pi^*S^* \otimes \tilde{\pi}^*L \to \pi^*S \oplus \pi^*S^*$$

satisfying $\bar{\partial}a + \alpha b = 0$ and $\bar{\partial}b = 0$. For $L \neq \mathbb{C} \in Jac(\tilde{M}/\mathbb{Z}_3)$ there does not exists a holomorphic map from $\pi^*S^* \otimes \tilde{\pi}^*L$ to π^*S^*. Therefore the eigenline bundle $\pi^*S^* \otimes \tilde{\pi}^*L$ would be π^*S, which is impossible because of the degree. Moreover one easily sees that the corresponding holomorphic rank two bundle for $L = \mathbb{C}$ must be isomorphic to the holomorphic direct sum $V = \pi^*S \oplus \pi^*S^*$. The orbit of this holomorphic structure under the gauge group is infinitesimally near to the one of the holomorphic structure $\bar{\partial}^0$ of Lemma 1. Therefore we can map $\mathbb{C} \in Jac(\tilde{M}/\mathbb{Z}_3)$ to the equivalence class of the stable holomorphic structure $\bar{\partial}^0$ in $\mathcal{S} = \mathbb{P}^1$ in order to obtain a well-defined holomorphic map $\Pi \colon Jac(\tilde{M}/\mathbb{Z}_3) \to \mathcal{S}$.

Because of Lemma 1 and Remark 3 the degree of the map Π is 2. Clearly $\Pi(L) = \Pi(L^*)$ for all $L \in Jac(\tilde{M}/\mathbb{Z}_3)$. Therefore the spin bundles of $\tilde{M}/\mathbb{Z}_3$ are the only branch points of Π. It remains to show that the non-trivial spin bundles in $Jac(\tilde{M}/\mathbb{Z}_3)$ correspond to the strictly semi-stable bundles $V \to M$. This can either be seen by analogous methods as in [16] used for the computation of the unstable locus in the Prym variety, or more directly as follows: Consider for example the non stable semi-stable bundle $V = \mathbb{C} \oplus \mathbb{C}$. Then, a symmetric Higgs field is given by

$$\Psi = \begin{pmatrix} 0 & \omega_1 \\ \omega_2 & 0 \end{pmatrix},$$

where ω_1 and ω_2 are holomorphic differentials with simple zeros at P_1 and P_3 respectively P_2 and P_4 such that $Q = \omega_1\omega_2$. Then the eigenlines $\ker(\Psi \pm \alpha)$ are both isomorphic to $L(-P_1 - P_3) = \pi^*S^* \otimes L(3P_1 - 3P_3)$. Clearly, $L(3P_1 - 3P_3) = \tilde{\pi}^*L(\tilde{\pi}(P_3) - \tilde{\pi}(P_1))$, and $L(\tilde{\pi}(P_3) - \tilde{\pi}(P_1))$ is a non-trivial spin bundle of $\tilde{M}/\mathbb{Z}_3$. Therefore, the gauge orbit of the trivial holomorphic rank 2 bundle $\underline{\mathbb{C}}^2 \to M$ is a branch image of Π, and similarly one can show that the same is true for the remaining two semi-stable non-stable holomorphic structures. $\qquad\square$

Remark 4 This double covering of the moduli space $\mathcal{S}$ of Lawson symmetric holomorphic rank two bundles is very similar to the one of the moduli space of holomorphic rank two bundles with trivial determinant on a Riemann surface Σ of genus 1. The latter space consist of all bundles of the form $L \oplus L^*$ where $L \in Jac(\Sigma)$ together with the non-trivial extensions of the spin bundles of Σ with itself, see [2].

4 Flat Lawson symmetric $SL(2, \mathbb{C})$-connections

We use the results of the previous chapter to study the moduli space of flat Lawson symmetric connections on M as an affine bundle over the moduli space of Lawson symmetric holomorphic structures. A similar approach was used by Donagi and Pantev [9] in their study of the geometric Langlands correspondence.

The underlying holomorphic structure ∇'' of a flat Lawson symmetric connection ∇ is determined by a holomorphic line bundle $L \in Jac(\tilde{M}/\mathbb{Z}_3)$ (Proposition 4). Conversely, for all non-trivial holomorphic line bundles $L \in Jac(\tilde{M}/\mathbb{Z}_3)$ there exists a Lawson symmetric holomorphic structure which is semi-stable. Because of the Theorem of Narasimhan and Seshadri [23], these holomorphic structures admit flat unitary connections, and, because of the uniqueness part in [23], the gauge equivalence class of the flat unitary connection is also invariant under φ_2, φ_3 and τ. In order to obtain all flat Lawson symmetric connections we only need to add symmetric Higgs fields to the unitary connections. We will see in Theorem 1 that flat Lawson symmetric connections on M are uniquely and explicitly determined by a flat connection on the corresponding line bundle $L \in Jac(\tilde{M}/\mathbb{Z}_3)$ as long as L is not isomorphic the trivial holomorphic bundle $\underline{\mathbb{C}}$. Adding a symmetric Higgs field on the Lawson symmetric connection on M is equivalent to adding a holomorphic 1-form to the line bundle connection on $L \to \tilde{M}/\mathbb{Z}_3$. Therefore the affine bundle structure of the space of gauge equivalence classes of flat Lawson symmetric connections on M is determined by the affine bundle structure of the moduli space of flat line bundle connections over the Jacobian of the torus $\tilde{M}/\mathbb{Z}_3$. The case of the remaining flat Lawson symmetric connections (corresponding to the holomorphic structures which are either isomorphic to $\bar{\partial}^0$ or to the non-trivial extension $0 \to S \to V \to S^* \to 0$) is dealt with in the next chapter. We will see that they occur as special limits as L converges to the trivial holomorphic line bundle.

Let ∇ be a flat Lawson symmetric connection such that its underlying holomorphic structure ∇'' admits a symmetric Higgs field $\Psi \in H^0(K, End_0(V))$ with $\det \Psi = Q$. Equivalently, there is a non-trivial holomorphic line bundle $L \in Jac(\tilde{M}/\mathbb{Z}_3)$

with $\Pi(L) = [\nabla'']$. Consider the pull-back connection $\pi^*\nabla$ on $\pi^*V \to \tilde{M}$, where $\pi : \tilde{M} \to M$ is as in the previous chapter. As the eigenline bundles $L_\pm \to \tilde{M}$ of $\pi^*\Psi$ are holomorphic subbundles of π^*V, which only intersect at the branch points of π, there exists a holomorphic homomorphism

$$f : L_+ \oplus L_- \to \pi^*V$$

which is an isomorphism away from the branch points of π. Therefore there exists a unique meromorphic flat connection $\tilde{\nabla}$ on $L_+ \oplus L_- \to \tilde{M}$ such that f is parallel. The poles of $\tilde{\nabla}$ are at the branch points of π. Let z be a holomorphic coordinate on $\tilde{M}$ centered at a branch point P_i of π such that $\sigma(z) = -z$. Let s_1, s_2 be a special linear frame of V and let t_1 and $t_2 = \sigma(t_1)$ be local holomorphic sections in L_+ and L_- satisfying (4). The connection ∇ on $V \to M$ is determined locally by

$$\nabla s_j = \omega_{1,j} s_1 + \omega_{2,j} s_2$$

for $j = 1, 2$, where $\omega_{i,j}$ are the locally defined holomorphic 1-forms. As ∇ and the frame are special linear $\omega_{1,1} = -\omega_{2,2}$ holds. Because π has a branch point at P_i, the connection 1-forms $\pi^*\omega_{i,j}$ (of π^∇ with respect to π^*s_1, π^*s_2) have zeros at P_i. Using (4) one can compute the connection 1-forms of $\tilde{\nabla}$ with respect to the frame $t_1, t_2 = \sigma(t_1)$ of $L_+ \oplus L_-$. It turns out that they have first order poles at P_i. Moreover, the residue of $\tilde{\nabla}$ at P_i is given by

$$res_{P_i}\tilde{\nabla} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{5}$$

with respect to the frame t_1, t_2. We need to interpret this formula more invariantly. With respect to the direct sum decomposition $L_+ \oplus L_-$ the connection $\tilde{\nabla}$ splits

$$\tilde{\nabla} = \begin{pmatrix} \nabla^+ & \beta^- \\ \beta^+ & \nabla^- \end{pmatrix}. \tag{6}$$

Here, $\nabla^\pm$ are meromorphic connections on $L_\pm$ with simple poles at the branch points of π, and $\beta^\pm \in \mathcal{M}(\tilde{M}, K_{\tilde{M}} Hom(L_\pm, L_\mp))$ are the meromorphic second fundamental forms of $L_\pm$ which also have simple poles at the branch points. Recall that the eigenline bundles are given by

$$L_\pm = \pi^*S^* \otimes \tilde{\pi}^*L^{\pm 1} \tag{7}$$

for holomorphic line bundles $L^{\pm 1} \in Jac(\tilde{M}/\mathbb{Z}_3)$ and $\tilde{\pi} : \tilde{M} \to \tilde{M}/\mathbb{Z}^3$. Consider the holomorphic section $\wedge \in H^0(\tilde{M}, \pi^*K_M)$ which has simple zeros at the branch points of π. There exists a unique meromorphic connection ∇^{K_M} on π^*K_M such that $\wedge$ is parallel. Then $res_{P_i}\nabla^{K_M} = -1$ at the branch points $P_1, .., P_4$. As $\pi^*S^2 = \pi^*K_M$ there exists a unique meromorphic connection ∇^{S^*} on π^*S^* which has simple poles at the branch points of π with residue $\frac{1}{2}$. Using (7), the description of ∇^{S^*} and (5) we obtain a unique holomorphic connection $\tilde{\nabla}^+$ on $\tilde{\pi}^*L^+$ (respectively $\tilde{\nabla}^-$ on $\tilde{\pi}^*L^{-1}$) satisfying the formula

$$\nabla^{\pm} = \nabla^{S^*} \otimes \tilde{\nabla}^{\pm}.$$

Moreover, $\tilde{\nabla}^{\pm}$ are dual to each other. As in the proof of Lemma 2 one can show that $\tilde{\nabla}^{\pm}$ are invariant under φ_2, φ_3 and τ and that there exists holomorphic connections $\nabla^{L^{\pm}}$ on $L^{\pm} \to \tilde{M}/\mathbb{Z}^3$ such that

$$\tilde{\nabla}^{\pm} = \tilde{\pi}^* \nabla^{L^{\pm}}. \tag{8}$$

Then, all holomorphic connections on $L = L^+ \to \tilde{M}/\mathbb{Z}^3$ with its fixed holomorphic structure are given by $\nabla^{L^+} + \alpha$ for a holomorphic 1-form $\alpha \in H^0(\tilde{M}/\mathbb{Z}^3, K_{\tilde{M}/\mathbb{Z}^3})$. Clearly, the corresponding effect on the connection ∇ on $V \to M$ is given by the addition of (a multiple of) the symmetric Higgs field Ψ which diagonalizes on $\tilde{M}$ with eigenlines $L_{\pm}$.

4.1 The second fundamental forms

Next, we compute the second fundamental forms $\beta^{\pm} \in \mathcal{M}(\tilde{M}, K_{\tilde{M}} Hom(L_{\pm}, L_{\mp}))$ of the eigenlines of the symmetric Higgs field. We fix some notations first: The symmetries φ_2 and τ of $\tilde{M}$ yield fixed-point free symmetries on the torus $\tilde{M}/\mathbb{Z}_3$ denoted by the same symbols. The quotient by these actions is again a square torus, denoted by T^2, which is fourfold covered by $\tilde{M}/\mathbb{Z}_3$ and the corresponding map is denoted by

$$\pi^T : \tilde{M}/\mathbb{Z}_3 \to T^2.$$

Each $L \in Jac(\tilde{M}/\mathbb{Z}_3)$ is the pull-back of a line bundle $\hat{L}$ of degree 0 on T^2. This line bundle is not unique. Actually, the pullback map defines a fourfold covering

$$Jac(T^2) \to Jac(\tilde{M}/\mathbb{Z}_3).$$

In particular, there are four different line bundles on T^2 which pull-back to the trivial one on $\tilde{M}/\mathbb{Z}_3$. These are exactly the spin bundles on T^2, so their square is the trivial holomorphic bundle. This implies, that $\hat{L}^{\pm 2}$ is independent of a choice $\hat{L} \in Jac(T^2)$ which pulls back to a given $L \in Jac(\tilde{M}/\mathbb{Z}_3)$. Let $0 \in T^2$ be the (common) image of the branch points P_i of π. Then every holomorphic line bundle $E \to T^2$ of degree 0 is (isomorphic to) the line bundle $L(y - 0)$ associated to an divisor of the form $D = y - 0$ for some $y \in T^2$. In particular, for $y \neq 0$ there exists a meromorphic section $s_{y-0} \in \mathcal{M}(T^2, E)$ with divisor $(s_{y-0}) = D$. Moreover, y is uniquely determined by E and s_{y-0} is unique up to a multiplicative constant.

Proposition 5 *Let ∇ be a flat Lawson symmetric connection on M. Let $L_+ = L \in Jac(\tilde{M}/\mathbb{Z}_3)$ be a non-trivial holomorphic line bundle which is given by $L = (\pi^T)^* L(x - 0)$ for some $x \in T^2$ such that $\Pi(L) = [\nabla'']$. Then the point $y := -2x \in T^2$ is not 0 and the second fundamental form of L_+ is*

$$\beta^+ = \tilde{\pi}^*(\pi^T)^* s_{y-0} \in \mathcal{M}(\tilde{M}, K_{\tilde{M}} Hom(L_{\pm}, L_{\mp})) = \mathcal{M}(\tilde{M}, K_{\tilde{M}} \tilde{\pi}^* L^{\mp 2})$$

where for

$$s_{y-0} \in \mathcal{M}(T^2, L(y-0)) = \mathcal{M}(T^2, K_{T^2} L(y-0))$$

the multiplicative constant is chosen appropriately and the pullbacks are considered as pullbacks of (bundle-valued) 1-forms. If we denote $y^- = -y = 2x \in T^2$, then the second fundamental form β^- is given by $\beta^- = \tilde{\pi}^*(\pi^T)^* s_{y^--0}$.

Proof By assumption $L = (\pi^T)^* L(x-0)$ is not the trivial holomorphic line bundle. Therefore, $L(x-0)$ cannot be a spin bundle of T^2. Equivalently, $L(x-0)^{-2} = L(y-0)$ is not the trivial holomorphic line bundle which implies that $y \neq 0$.

The gauge equivalence class of the connection ∇ is invariant under the symmetries. Therefore, the set of poles and the set of zeros of the second fundamental forms $\beta^\pm$ of the eigenlines of the symmetric Higgs field are fixed under the symmetries, too. There are exactly 4 simple poles of β^+ and because $\tilde{M}$ has genus 5 and the degree of $Hom(L_\pm, L_\mp)) = \tilde{\pi}^* L^{\mp 2}$ is 0 there are 12 zeros of β^+ counted with multiplicity. The only fixed points of φ_3 are the branch points P_i of π and φ_2 and τ are fixed-point free on $\tilde{M}$. Therefore, the orbit of a zero of β^+ under the actions of φ_2, φ_3 and τ consists of exactly 12 points. This implies that the zeros of β^+ are simple. Moreover, these 12 points are mapped via $\pi^T \circ \tilde{\pi}$ to a single point $\tilde{y}$ in T^2. We claim that $\tilde{y} = y \in T^2$. To see this, we consider the (bundle-valued) meromorphic 1-form $\tilde{\pi}^*(\pi^T)^* s_{\tilde{y}-0}$ on $\tilde{M}$, which has simple poles exactly at the branch points P_i of π and simple zeros at the preimages of $\tilde{y}$. Therefore, $\tilde{\pi}^*(\pi^T)^* s_{\tilde{y}-0}$ is (up to a multiplicative constant) the second fundamental form β^+. As the bundle $L(y-0)$ is uniquely determined by L_+ we also get $\tilde{y} = y$. $\qquad\qquad\square$

Remark 5 In the case of $y = 0 \in T^2$ there is no meromorphic section in the trivial line bundle $L(y-0) = \mathbb{C}$ with a simple pole at 0. But $y = 0$ holds exactly for the trivial bundle $\mathbb{C} \in Jac(\tilde{M}/\mathbb{Z}_3)$. This line bundle corresponds to the non-stable holomorphic direct sum bundle $S^* \oplus S \to M$, see the proof of Proposition 4. As we have seen in Sect. 2.1 there does not exists a holomorphic connection on $S^* \oplus S \to M$.

By now, we have determined the second fundamental forms up to a constant. It remains to determine the exact multiplicative constant of

$$\hat{\gamma}^\pm := s_{y^\pm - 0}.$$

Note that the involution σ on $\tilde{M}$ gives rise to involutions on $\tilde{M}/\mathbb{Z}_3$ and T^2, denoted by the same symbol. Then, $\sigma(\beta^\pm) = \beta^\mp$ and $\sigma(\hat{\gamma}^\pm) = \hat{\gamma}^\mp$. From Eqs. 5 and 6 one sees that

$$\beta^+ \beta^- \in \mathcal{M}(\tilde{M}, K_{\tilde{M}}^2)$$

is a well-defined meromorphic quadratic differential with double poles at the branch points $P_1, .., P_4$ and with residue

$$res_{P_i}(\beta^+ \beta^-) = \frac{1}{4}.$$

As the branch order of $\tilde{\pi}$ at P_i is 2 we have

$$res_0(\hat{\gamma}^+\hat{\gamma}^-) = \frac{1}{36}. \tag{9}$$

Together with $\sigma(\hat{\gamma}^\pm) = \hat{\gamma}^\mp$ this completely determines $\hat{\gamma}^\pm$ and therefore also $\beta^\pm$ up to sign. Note that the sign has no invariant meaning as the sign of the off-diagonal terms of the connection can be changed by applying a diagonal gauge with entries i and $-i$.

4.2 Explicit formulas

We are now going to write down explicit formulas for a flat Lawson symmetric connection ∇ whose underlying holomorphic structure admits a symmetric Higgs field Ψ with $\det \Psi = Q$. To be precise, we compute the connection 1-form of $\pi^*\nabla \otimes \nabla^S$ with respect to some frame, where ∇^S is defined as above by the equation $(\nabla^S \otimes \nabla^S)\omega = 0$ for the tautological section $\omega \in H^0(\tilde{M}, \pi^*K_M)$. Then $\pi^*\nabla \otimes \nabla^S$ is a meromorphic connection on $\tilde{\pi}^*L^+ \oplus \tilde{\pi}^*L^- \to \tilde{M}$ with simple, off-diagonal poles at the branch points $P_1, .., P_4$ of π, where L^+ and L^- are holomorphic line bundles of degree 0 on the torus $\tilde{M}/\mathbb{Z}_3$ which are dual to each other and correspond to the eigenlines of the symmetric Higgs field via Proposition 4.

Recall that $\tilde{M}/\mathbb{Z}_3$ is a square torus, and we identify it as

$$\tilde{M}/\mathbb{Z}_3 \cong \mathbb{C}/(2\mathbb{Z} + 2i\mathbb{Z}).$$

We may assume without loss of generality that the half lattice points are exactly the images of the branch points P_i. The fourfold (unbranched) covering map π^T gets into the natural quotient map

$$\pi^T : \tilde{M}/\mathbb{Z}_3 \cong \mathbb{C}/(2\mathbb{Z} + 2i\mathbb{Z}) \to \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \cong T^2.$$

Let E be one choice of a holomorphic line bundle on T^2 which pulls back to $L^+ \to \tilde{M}/\mathbb{Z}_3$. As before, it is given by $E = L([x] - [0])$ for some $[x] \in T^2$, where $[0] \in \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \cong T^2$ is the common image of the points P_i.

The following lemma is of course well-known. We include it as it produces the trivializing sections which we use to write down the connection 1-form.

Lemma 4 *Consider the square torus* $T^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ *and the holomorphic line bundle* $E = L([x] - [0])$ *for some* $x \in \mathbb{C}$. *Then there exists a smooth section* $\underline{1} \in \Gamma(T^2, E)$ *such that the holomorphic structure* $\bar{\partial}^E$ *of* E *is given by*

$$\bar{\partial}^E \underline{1} = -\pi x d\bar{z} \underline{1}.$$

Proof The proof is merely included to fix our notations about the Θ-function of $T^2 = \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$, see [13] for details. There exists an even entire function $\theta : \mathbb{C} \to \mathbb{C}$ which has simple zeros exactly at the lattice points $\mathbb{Z} + i\mathbb{Z}$ and which satisfies

$$\theta(z+1) = \theta(z) \tag{10}$$

$$\theta(z+i) = \theta(z) \exp\left(-2\pi i \left(z - \frac{1+i}{2}\right) + \pi\right).$$

Then the function

$$s(z) := \frac{\theta(z-x)}{\theta(z)} \exp(\pi x(\bar{z} - z))$$

is doubly periodic and has simple poles at the lattice points $\mathbb{Z} + i\mathbb{Z}$ and simple zeros at $x + \mathbb{Z} + i\mathbb{Z}$. Moreover it satisfies $\bar{\partial}s = \pi x s$. Therefore, s can be considered as a meromorphic section with respect to the holomorphic structure $\bar{\partial} - \pi x d\bar{z}$ on $T^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ with simple poles at $[0] \in T^2$ and simple zeros at $[x] \in T^2$. This implies that the holomorphic structure $\bar{\partial} - \pi x d\bar{z}$ is isomorphic to the holomorphic structure of $E = L([x] - [0])$. The image $\underline{1}$ of the constant function 1 under this isomorphism satisfies the required equation $\bar{\partial}^E \underline{1} = -\pi x d\bar{z} \underline{1}$. □

The second fundamental forms $\beta^{\pm} = \tilde{\pi}^*(\pi^T)^* \hat{\gamma}^{\pm}$ can be written down in terms of Θ-functions as follows: From Proposition 5 and the proof of Lemma 4 one obtains that (with respect to the smooth trivializing section $\underline{1}$ of $E = L([x] - [0])$ and its dual section $\underline{1}^* \in \Gamma(T^2, E^*)$) $\hat{\gamma}^{\pm}$ are given by

$$\hat{\gamma}^+(z)\underline{1} = c \frac{\theta(z-y)}{\theta(z)} e^{-2\pi i y \Im(z)} \underline{1}^* dz \tag{11}$$

$$\hat{\gamma}^-(z)\underline{1}^* = c \frac{\theta(z+y)}{\theta(z)} e^{2\pi i y \Im(z)} \underline{1} dz$$

for some $c \in \mathbb{C}$, where θ is as in the proof of 4 and $y = -2x$. The constant $c \in \mathbb{C}$ is given by (9) as a choice of a square root

$$c = \frac{1}{6} \sqrt{\frac{\theta'(0)^2}{\theta(y)\theta(-y)}}, \tag{12}$$

where $'$ denotes the derivative with respect to z.

Remark 6 Note that c can be considered as a single-valued meromorphic function depending on $y \in \mathbb{C}$ with simple poles at the integer lattice points by choosing the sign of the square root at some given point $y \notin \mathbb{Z} + i\mathbb{Z}$.

Altogether, the connection $\pi^* \nabla \otimes \nabla^S$ is given on $T^2 = \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$ with respect to the frame $\underline{1}$, $\underline{1}^*$ by the connection 1-form

$$\begin{pmatrix} \pi a dz - \pi x d\bar{z} & c\frac{\theta(z+y)}{\theta(z)} e^{2\pi i y \Im(z)} dz \\ c\frac{\theta(z-y)}{\theta(z)} e^{-2\pi i y \Im(z)} dz & -\pi a dz + \pi x d\bar{z} \end{pmatrix} \tag{13}$$

for some $a \in \mathbb{C}$. The connection 1-form (13) is only meromorphic, but the corresponding connection ∇ on the rank 2 bundle over M has no singularities. Varying $a \in \mathbb{C}$ corresponds to adding a multiple of the symmetric Higgs field on the connection ∇.

Remark 7 In (13) we have written down the connection 1-forms on the torus $T^2 \cong \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$. But as the fourfold covering $\tilde{M}/\mathbb{Z}_3 \cong \mathbb{C}/(2\mathbb{Z}+2i\mathbb{Z}) \to T^2 \cong \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ is simply given by

$$z \, mod \, 2\mathbb{Z} + 2i\mathbb{Z} \longmapsto z \, mod \, \mathbb{Z} + i\mathbb{Z}$$

(13) gives also the connection 1-form for the connection $\pi^*\nabla \otimes \nabla^S$ on $\tilde{M}/\mathbb{Z}_3$ with respect to the frame $(\pi^T)^*\underline{1}$, $(\pi^T)^*\underline{1}^*$.

We summarize our discussion:

Theorem 1 (The abelianization of flat $SL(2,\mathbb{C})$-connections) *Let $\bar{\partial}$ be a Lawson symmetric semi-stable holomorphic structure on a rank 2 vector bundle over M. Assume that $\bar{\partial}$ is determined by the non-trivial holomorphic line bundle $L \in Jac(\tilde{M}/\mathbb{Z}_3)$, i.e., $\Pi(L) = [\bar{\partial}]$. Then there is a one-to-one correspondence between holomorphic connections on $L \to \tilde{M}/\mathbb{Z}_3$ and flat Lawson symmetric connections ∇ with $\nabla'' = \bar{\partial}$. The correspondence is given explicitly by the connection 1-form (13).*

4.3 Flat unitary connections

From the uniqueness part of the Narasimhan-Seshadri theorem one obtains an equivariance corollary: If the isomorphism class of a stable holomorphic structure $\bar{\partial}$ is invariant under some automorphisms of the Riemann surface then the gauge equivalence class of the unitary flat connection ∇ with $\bar{\partial} = \nabla''$ is also invariant under the same automorphisms. We apply this to the situation of Theorem 1.

Theorem 2 *Consider a Lawson symmetric holomorphic structure $\bar{\partial}$ of rank 2 on M whose isomorphism class is given by a non-trivial holomorphic line bundle $L \in Jac(\tilde{M}/\mathbb{Z}_3)$, i.e., $\Pi(L) = [\bar{\partial}]$. Let $x \in \mathbb{C} \setminus (\frac{1}{2}\mathbb{Z} + \frac{1}{2}i\mathbb{Z})$ such that the holomorphic structure of V is given by*

$$\bar{\partial}^E = \bar{\partial}^0 - \pi x d\bar{z}$$

on $\mathbb{C} \to \tilde{M}/\mathbb{Z}_3 \cong \mathbb{C}/(2\mathbb{Z} + 2i\mathbb{Z})$. Then there exists a unique $a^u = a^u(x) \in \mathbb{C}$ such that the flat Lawson symmetric connection ∇ on M which is given by the connection 1-form (13) is unitary with respect to a suitable chosen metric. The function

$$x \mapsto a^u(x)$$

is real analytic and odd in x. It satisfies

$$a^u \left(x + \frac{1}{2} \right) = a^u(x) + \frac{1}{2}$$

and

$$a^u\left(x + \frac{i}{2}\right) = a^u(x) - \frac{i}{2}$$

which means that it gives rise to a well-defined real analytic section $\mathcal{U}$ of the affine bundle of (the moduli space of) flat $\mathbb{C}^*$-connections over the Jacobian of $\tilde{M}/\mathbb{Z}_3$ away from the origin.

Remark 8 We show in Theorem 3 below that the section $\mathcal{U}$ has a first order pole at the origin.

Proof As the unitary flat connections depend (real) analytically on the underlying holomorphic structure, the function $x \mapsto a^u(x)$ is also real analytic. Moreover, it must be odd in x as the flat connection induced on $L^+ \to \tilde{M}/\mathbb{Z}_3$ is dual to the one induced on $L^- \to \tilde{M}/\mathbb{Z}_3$. The functional equations are simply a consequence of the gauge invariance of our discussion: On $\tilde{M}/\mathbb{Z}_3 = \mathbb{C}/(2\mathbb{Z} + 2i\mathbb{Z})$ the flat connections $d + \pi a\, dz - \pi x\, d\bar{z}, d + \pi(a - \frac{1}{2})dz - \pi(x - \frac{1}{2})d\bar{z}$ and $d + \pi(a + \frac{i}{2})dz - \pi(x - \frac{i}{2})d\bar{z}$ are gauge equivalent as well as the corresponding flat $SL(2, \mathbb{C})$-connections on M. $\qquad\square$

Remark 9 The Narasimhan–Seshadri section which maps an isomorphism class of stable holomorphic structures to its corresponding gauge class of unitary flat connections is a real analytic section in the holomorphic affine bundle of the moduli space of flat $SL(2, \mathbb{C})$ connections to the moduli space of stable holomorphic structures. The latter space is equipped with a natural symplectic structure. Then, the natural (complex anti-linear) derivative of the Narasimhan–Seshadri section can be interpreted as the symplectic form, see for example [6].

5 The exceptional flat $SL(2, \mathbb{C})$-connections

In the previous chapter we have studied all flat Lawson symmetric connections on M whose underlying holomorphic structures admit symmetric Higgs fields Ψ such that $\det \Psi = Q$. The holomorphic structures are determined by non-trivial holomorphic line bundles $L \in Jac(\tilde{M}/\mathbb{Z}_3)$, see Proposition 4. The construction of a connection 1-form as in (13) breaks down for the trivial holomorphic line bundle $\mathbb{C} \to \tilde{M}/\mathbb{Z}_3$, because the trivial line bundle corresponds to the holomorphic direct sum bundle $S \oplus S^* \to M$ which does not admit a holomorphic connection. But as we have already mentioned above, the gauge orbits of the remaining holomorphic structures which admit Lawson symmetric holomorphic connections are infinitesimally near to the gauge orbit of $S \oplus S^* \to M$ (see for example the proof of Proposition 4). We use this observation to construct the remaining flat Lawson symmetric connections as limits of the connections studied in Theorem 1 when L tends to the trivial holomorphic line bundle. Even more important for our purpose, we exactly determine for which meromorphic family of flat line bundle connections on $\tilde{M}/\mathbb{Z}_3$ the corresponding family of flat $SL(2, \mathbb{C})$-connections on M extends holomorphically through the points where the holomorphic line bundle is the trivial one, see Theorems 3 and 4 below.

5.1 The case of the stable holomorphic structure

We start our discussion with the case of a Lawson symmetric stable holomorphic structure which does not admit a symmetric Higgs field with non-trivial determinant. As we have seen, this holomorphic structure is isomorphic to $\bar\partial^0$.

Let ∇ be a flat unitary Lawson symmetric connection such that $(\nabla)'' = \bar\partial^0$. As we have seen in the proof of Lemma 1, $\bar\partial^0$ admits a nowhere vanishing symmetric Higgs field $\Psi \in H^0(M, K\,End_0(V, \bar\partial^0))$ with $\det\Psi = 0$. The kernel of Ψ is the dual of the spin bundle S of the Lawson surface. We split the connection

$$\nabla = \begin{pmatrix} \bar\partial^{S^*} & \bar q \\ 0 & \bar\partial^S \end{pmatrix} + \begin{pmatrix} \partial^{S^*} & 0 \\ -q & \partial^S \end{pmatrix}$$

with respect to the unitary decomposition $V = S^* \oplus S \to M$. Note that q is a multiple of the Hopf differential Q of the Lawson surface and that $\bar q \in \Gamma(M, \bar K K^{-1})$ is its adjoint with respect to the unitary metric. As explained above, we want to study $\nabla = \nabla^0$ as a limit of a family of flat Lawson symmetric connections

$$t \mapsto \nabla^t,$$

such that the holomorphic structures vary non-trivially in t. We restrict to the case where a choice of a corresponding line bundle $L_t^+ \in Jac(\tilde M/\mathbb{Z}_3)$ with $\Pi(L_t) = [(\nabla^t)'']$ is given by the holomorphic structure

$$\bar\partial_0 + t\,d\bar z,$$

where $\bar\partial_0 = d''$ is the trivial holomorphic structure on $\mathbb{C} \to \tilde M/\mathbb{Z}_3$. As Π branches at $\mathbb{C}$ (Proposition 4) this can always be achieved by rescaling the family as long as the map $t \mapsto [(\nabla^t)''] \in \mathcal{S}$ has a branch point of order 1 at 0. Pulling the family of connections back to $\tilde M$ (and applying gauge transformations to them which depend holomorphically on t on a disc containing $t = 0$) the holomorphic structures of the connections take the following form

$$(\pi^*\nabla^t)'' = \begin{pmatrix} \bar\partial^{S^*} + t\bar\eta & \bar q \\ 0 & \bar\partial^S - t\bar\eta \end{pmatrix},$$

where $\bar\eta = \tilde\pi^* d\bar z$. A family of Lawson symmetric Higgs fields

$$t \mapsto \Psi_t \in H^0(M, K_M End_0(V, (\nabla^t)''))$$

is given by

$$\pi^*\Psi_t = \begin{pmatrix} tc\eta & \omega + t\beta(t) \\ 0 & -tc\eta \end{pmatrix} \tag{14}$$

after pulling them back as 1-forms to $\tilde M$. Here $\beta(t)$ is a t-dependent section of $\pi^*K_M = K_{\tilde M}Hom(\pi^*S, \pi^*S^*)$, and $\omega \in H^0(\tilde M, K_{\tilde M}Hom(\pi^*S, \pi^*S^*))$ is the canonical section which has zeros at the branch points of π and c is a some non-

zero constant. Note that ω can be considered as the pull-back of the bundle-valued 1-form $1 \in H^0(M, K\,Hom(S, S^*))$, or as a square root of η. With respect to the fixed (non-holomorphic) background decomposition $\pi^*V = \pi^*S^* \oplus \pi^*S$ the eigenlines $L^t_\pm$ of $\pi^*\Psi_t$ on $\tilde{M}$ are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 \\ -2c\omega t + t^2(\ldots) \end{pmatrix}.$$

Therefore the expansion in t of the singular gauge transformation $f_t \colon L^t_+ \oplus L^t_- \to \pi^*V = \pi^*S^* \oplus \pi^*S$ is given by

$$\begin{pmatrix} 1 & 1 \\ 0 & -2c\omega t + t^2(\ldots) \end{pmatrix}.$$

The expansion of $\pi^*\nabla^t$ is of the form

$$\pi^*\nabla^t = \pi^*\nabla + t\begin{pmatrix} \bar{\eta} & 0 \\ 0 & -\bar{\eta} \end{pmatrix} + t\Gamma(t),$$

where $\Gamma(t) \in \Gamma(\tilde{M}, K_{\tilde{M}}End_0(V))$ depends holomorphically on t. Applying the gauge f_t we obtain the following asymptotic behavior

$$\nabla^t \cdot f_t = \frac{1}{t}\begin{pmatrix} -\frac{\pi^*q}{2c\omega} & 0 \\ 0 & \frac{\pi^*q}{2c\omega} \end{pmatrix} + \cdots$$

The pullback $\pi^*q \in H^0(K_{\tilde{M}}K_M)$ has zeros of order 3 at the branch points of π and therefore it is a constant multiple of $\eta\omega$. Hence, the holomorphic line bundle connections on $L^+_t = \pi^*S \otimes L^t_+ \to \tilde{M}/\mathbb{Z}_3$ given by the one-to-one correspondence in Theorem 1 (see also (8)) have the following expansion

$$\nabla^{L_t+} = d + td\bar{z} + \frac{\tilde{c}}{t}dz + \hat{e}(t)dz \tag{15}$$

for some holomorphic function $\hat{e}(t)$. In order to determine $\tilde{c}$, we expand the family of equations $(\nabla^t)''\Psi_t = 0$ as follows:

$$0 = (\pi^*\nabla^t)''\pi^*\Psi_t = t\begin{pmatrix} 0 & -2\pi^*\bar{q}c\eta + 2\omega\bar{\eta} + \bar{\partial}^{\pi^*K_M}\beta(0) \\ 0 & 0 \end{pmatrix} + t^2(\ldots).$$

As we have fixed $\omega \in H^0(\tilde{M}, K_{\tilde{M}}Hom(S, S^*)) = H^0(\tilde{M}, \pi^*K_M)$ up to sign by $\omega^2 = \eta = \tilde{\pi}^*dz$ we obtain from Serre duality applied to the bundle π^*K_M

$$\int\limits_{\tilde{M}} \pi^* \bar{q} c \eta \omega = \int\limits_{\tilde{M}} \bar{\eta} \omega^2 = 3 \int\limits_{\tilde{M}/\mathbb{Z}_3} d\bar{z} \wedge dz = 24i. \tag{16}$$

Recall that we have identified $\tilde{M}/\mathbb{Z}_3 \cong \mathbb{C}/(2\mathbb{Z} + 2i\mathbb{Z})$ and dz is the corresponding differential. The degree of $\pi^* S^* \to \tilde{M}$ is -2 and we obtain from the flatness of ∇ that

$$4\pi i = \int\limits_{\tilde{M}} \pi^* \bar{q} \wedge \pi^* q. \tag{17}$$

Combining (16) and (17) we obtain

$$-\frac{\pi^* q}{2c\omega} = -\frac{\pi}{12} \eta,$$

which exactly tells us the asymptotic of the family (15).

Theorem 3 *Let ∇^t be a holomorphic family of flat Lawson symmetric connections on M such that $(\nabla^0)''$ is isomorphic to $\bar{\partial}^0$. If $t \mapsto [(\nabla^t)''] \in \mathcal{S}$ branches of order 1 at $t = 0$, then, after reparametrization the family, ∇^t induces by means of Theorem 1 and (13) a meromorphic family of flat connections of the form*

$$\tilde{\nabla}^t = d + td\bar{z} - \frac{\pi}{12t}dz + te(t)dz \tag{18}$$

on $\mathbb{C} \to \tilde{M}/\mathbb{Z}_3$, where $e(t)$ is a holomorphic function in t.

Conversely, let $\tilde{\nabla}^t$ be a meromorphic family of flat connections on $\mathbb{C} \to \tilde{M}/\mathbb{Z}_3$ of the form (18). Then the induced family of flat Lawson symmetric connections ∇^t on the complex rank 2 bundle $V \to M$ extends (after a suitable t-dependent gauge) holomorphically to $t = 0$ such that ∇^0 is a flat Lawson symmetric connection and $(\nabla^0)''$ is isomorphic to $\bar{\partial}^0$.

Proof Our primarily discussion was restricted to the case where ∇^0 is unitary. In that case it remains to show that the function $\hat{e}(t)$ in (15) has a zero at $t = 0$. This follows from the fact that the function a^u in Theorem 2 is odd. For the general case we need to study the effect of adding a holomorphic family of Lawson symmetric Higgs fields

$$\Psi(t) \in H^0(M; K\,End_0(V, (\nabla^t)'')).$$

Such a holomorphic family of Higgs fields is given by

$$h(t) \begin{pmatrix} tc\eta & \omega + t\beta(t) \\ 0 & -tc\eta \end{pmatrix}$$

for some function $h(t)$ which is holomorphic in t, see (14). From this the first part easily follows. Moreover, by reversing the arguments one also obtains a proof of the converse direction. $\qquad\square$

Corollary 1 *The unitarizing function $a^u \colon \mathbb{C} \setminus \frac{1}{2}\mathbb{Z} + \frac{i}{2}\mathbb{Z} \to \mathbb{C}$ in Theorem 2 is given by*

$$a^u(x) = -\frac{1}{12\pi}\frac{\theta'(-2x)}{\theta(-2x)} + \frac{1}{12\pi}\frac{\theta'(2x)}{\theta(2x)} + \frac{1}{3}x + \frac{2}{3}\bar{x} + b(x),$$

where θ is the Θ-function as in (10), θ' is its derivative and $b(x) \colon \mathbb{C} \to \mathbb{C}$ is an odd smooth function which is doubly periodic with respect to the lattice $\frac{1}{2}\mathbb{Z} + \frac{i}{2}\mathbb{Z}$.

Proof The function $\tilde{a} \colon \mathbb{C} \setminus \frac{1}{2}\mathbb{Z} + \frac{i}{2}\mathbb{Z} \to \mathbb{C}$ defined by

$$\tilde{a}(x) = -\frac{1}{12\pi}\frac{\theta'(-2x)}{\theta(-2x)} + \frac{1}{12\pi}\frac{\theta'(2x)}{\theta(2x)} + \frac{1}{3}x + \frac{2}{3}\bar{x}$$

is an odd function in x which satisfies the same functional equations (see Theorem 2) as a^u. Note that the parametrization of the family of holomorphic rank 1 structures in Theorem 2 and in Theorem 3 differ by the multiplicative factor $-\pi$. Therefore, $\tilde{a}$ has the right asymptotic behavior at the lattice points $\frac{1}{2}\mathbb{Z} + \frac{i}{2}\mathbb{Z}$. So the difference $b = a^u - \tilde{a}$ is an odd, smooth and doubly periodic function. $\qquad\square$

5.2 The case of the non-stable holomorphic structure

We have already seen in Sect. 2.1 that every flat Lawson symmetric connection on M whose holomorphic structure is not semi-stable is gauge equivalent to

$$\nabla = \begin{pmatrix} \nabla^{spin*} & 1 \\ vol + cQ & \nabla^{spin} \end{pmatrix}$$

with respect to $V = S^* \oplus S \to M$. In this formula ∇^{spin} and vol are induced by the Riemannian metric of constant curvature -4, $c \in \mathbb{C}$ and Q is the Hopf differential of the Lawson surface. The gauge orbit of the holomorphic structure ∇'' is infinitesimally close to the gauge orbits of the holomorphic structures $\bar{\partial}^0$ and $\bar{\partial}^S \oplus \bar{\partial}^{S^*}$. As in Sect. 5.1, we approximate ∇ by a holomorphic family of flat Lawson symmetric connections $t \mapsto \nabla^t$ such that the isomorphism classes of the holomorphic structures $(\nabla^t)''$ vary in t. We obtain a similar result as Theorem 3.

Theorem 4 *Let ∇^t be a holomorphic family of flat Lawson symmetric connections on M such that $(\nabla^0)''$ is isomorphic to the non-trivial extension $S \to V \to S^*$ and such that $t \mapsto [(\nabla^t)''] \in \mathcal{S}$ branches of order 1 at $t = 0$. After reparametrization the family, ∇^t corresponds (via Theorem 1 and (13)) to a meromorphic family of flat connections $\tilde{\nabla}^t$ on $\underline{\mathbb{C}} \to \tilde{M}/\mathbb{Z}_3$ of the form*

$$\tilde{\nabla}^t = d + t d\bar{z} + \frac{\pi}{12t}dz + te(t)dz, \tag{19}$$

where $e(t)$ is holomorphic in t.

Conversely, let $\tilde{\nabla}^t$ be a meromorphic family of flat connections on $\mathbb{C} \to \tilde{M}/\mathbb{Z}_3$ of the form (19). Then the induced family of flat Lawson symmetric connections ∇^t on the complex rank 2 bundle $V \to M$ extends (after a suitable t-dependent gauge) holomorphically to $t = 0$ such that $(\nabla^0)''$ is isomorphic to the non-trivial extension $0 \to S \to V \to S^* \to 0$. Moreover, ∇^0 is gauge equivalent to the uniformization connection (see (2)) if the function e has a zero at $t = 0$.

Proof Consider a holomorphic family of flat Lawson symmetric connections $\hat{\nabla}^t$ such that $(\nabla^t)''$ is isomorphic to $(\hat{\nabla}^t)''$ for all t and such that $\hat{\nabla}^0$ is unitary. In particular, $(\hat{\nabla}^0)''$ is isomorphic to $\bar{\partial}^0$. Then, after applying the t-dependent gauge g_t the difference

$$\Psi_t := \hat{\nabla}^t - g_t^{-1} \nabla^t g \in H^0(M, K\,End_0(V, (\hat{\nabla}^t)''))$$

satisfies

$$\det \Psi_t = \frac{q}{t} + \text{higher order terms},$$

where q is a non-zero multiple of the Hopf differential. This implies, that the line bundle connections $\tilde{\nabla}^t$ have an expansion like

$$\tilde{\nabla}^t = d + t\,d\bar{z} + \frac{c}{t}\,dz + \text{higher order terms}$$

for some non-zero $c \in \mathbb{C}$. Then, analogous to the computation in Sect. 5.1, one obtains $c = \frac{1}{12\pi}$. Note that the reason for the different signs is because of the last sign in the degree formula for S^*:

$$-2\pi i \deg(S^*) = \int_M \bar{q} \wedge q = - \int_M 1 \wedge vol.$$

To show that the 0.-order term in the expansion of $\tilde{\nabla}^t$ vanishes we first observe that there exists an additional (holomorphic) symmetry $\tilde{\tau} \colon M \to M$ which induces the symmetry $z \mapsto iz$ on $\tilde{M}/\mathbb{Z}_3$. Note that $\tilde{\tau}^*Q = -Q$. Because the gauge equivalence class of the uniformization connection (2) is also invariant under $\tilde{\tau}$, one easily gets (as in the proof of Theorem 15) that the 0.-order term vanishes. Moreover one obtains that in the case of the uniformization connection also the first order term vanishes. $\qquad\square$

6 The spectral data

So far we have seen that the generic Lawson symmetric flat connection is determined (up to gauge equivalence), after the choice of one eigenline bundle of a symmetric Higgs field, by a flat line bundle connection on a square torus. Moreover, the remaining flat connections are explicitly given as limiting cases of the above construction. We now apply these results to the case of the family of flat connections ∇^λ associated to a minimal surface. We assume that the minimal surface is of genus 2 and has the conformal type and the symmetries φ_2, φ_3 and τ of the Lawson surface. The family

of flat connections induces a family of Lawson symmetric holomorphic structures $\bar{\partial}^\lambda = (\nabla^\lambda)''$ which extends to $\lambda = 0$. As it is impossible to make a consistent choice of the eigenline bundles of symmetric Higgs fields with respect to $\bar{\partial}^\lambda$ for all $\lambda \in \mathbb{C}^*$ (see Proposition 6) we need to introduce a so-called spectral curve which double covers the spectral plane $\mathbb{C}^*$ and enables us to parametrize the eigenline bundles. Then, the family of flat connections ∇^λ is determined (up to a λ-dependent gauge) by the corresponding family of flat line bundles over the torus. The behavior of this family of flat line bundles is very similar (at least around $\lambda = 0$) to the family of flat line bundles parametrized by the spectral curve of a minimal or CMC torus, compare with [15]. The main difference is that we have some kind of symmetry breaking between $\lambda = 0$ and $\lambda = \infty$: we do not treat the holomorphic and anti-holomorphic structures of a flat connection in the same way but consider the moduli space of flat connections as an affine bundle over the moduli space of holomorphic structures. As a consequence, we do not have an explicitly known reality condition, which seems to be the missing ingredient to explicitly determine the Lawson surface.

By taking the gauge equivalence classes of the associated family of holomorphic structures $\bar{\partial}^\lambda$ we obtain a holomorphic map

$$\mathcal{H}\colon \mathbb{C} \to \mathcal{S} \cong \mathbb{P}^1$$

to the moduli space of semi-stable Lawson symmetric holomorphic structures, see Proposition 1. This map is given by $\mathcal{H}(\lambda) = [\bar{\partial}^\lambda]$ for those λ where $\bar{\partial}^\lambda$ is semi-stable. By Remark 1 it extends holomorphically to the points λ where $\bar{\partial}^\lambda$ is not semi-stable.

Proposition 6 (The definition of the spectral curve) *There exists a holomorphic double covering $p\colon \Sigma \to \mathbb{C}$ defined on a Riemann surface Σ, the spectral curve, together with a holomorphic map $\mathcal{L}\colon \Sigma \to Jac(\tilde{M}/\mathbb{Z}_3)$ such that*

$$
\begin{array}{ccc}
\Sigma & \xrightarrow{\ \mathcal{L}\ } & Jac(\tilde{M}/\mathbb{Z}_3) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \Pi} \\
\mathbb{C} & \xrightarrow{\ \mathcal{H}\ } & \mathcal{S}
\end{array}
$$

commutes, where $\Pi\colon Jac(\tilde{M}/\mathbb{Z}_3) \to \mathcal{S}$ is as in Proposition 4. The map p branches over $0 \in \mathbb{C}$.

Proof We first define

$$\underline{\Sigma} = \{(\lambda, L) \in \mathbb{C} \times Jac(\tilde{M}/\mathbb{Z}_3) \mid \Pi(L) = \mathcal{H}(\lambda)\}$$

which is clearly a non-empty analytic subset of $\mathbb{C} \times Jac(\tilde{M}/\mathbb{Z}_3)$. Then, the spectral curve is given by the normalization

$$\Sigma \to \underline{\Sigma},$$

and $\mathcal{L}$ is the composition of the normalization with the projection onto the second factor. It remains to prove that Σ branches over 0. Because Π branches over $[\bar{\partial}^0]$

this follows if we can show that the map $\mathcal{H}$ is immersed at $\lambda = 0$. As $\bar{\partial}^0$ is stable, the tangent space at $[\bar{\partial}^0]$ of the moduli space of (stable) holomorphic structures with trivial determinant is given by $H^1(M, K End_0(V, \bar{\partial}^0))$. The cotangent space is given via trace and integration by $H^0(M, K End_0(V, \bar{\partial}^0))$. With

$$\frac{\partial}{\partial \lambda} \bar{\partial}^\lambda = \begin{pmatrix} 0 & 0 \\ vol & 0 \end{pmatrix}$$

and

$$\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in H^0(M, K End_0(V, \bar{\partial}^0))$$

we see that

$$\Phi \left(\frac{\partial}{\partial \lambda} \bar{\partial}^\lambda_{|\lambda=0} \right) = \int_M vol \neq 0$$

which implies that $\mathcal{H}$ is immersed at $\lambda = 0$. $\qquad\square$

In order to study the family of gauge equivalence classes $[\nabla^\lambda]$ we consider the moduli space of flat $\mathbb{C}^*$-connections on $\tilde{M}/\mathbb{Z}_3$ as an affine holomorphic bundle

$$\mathcal{A}^f \to Jac(\tilde{M}/\mathbb{Z}_3)$$

over the Jacobian by taking the complex anti-linear part of a connection. Then, as a consequence of Theorem 1 together with our discussion in Sect. 5, we obtain a meromorphic lift

$$
\begin{array}{ccc}
 & & \mathcal{A}^f \\
 & \overset{\mathcal{D}}{\nearrow} & \downarrow {\scriptstyle ''} \\
\Sigma & \underset{\mathcal{L}}{\longrightarrow} & Jac(\tilde{M}/\mathbb{Z}_3)
\end{array}
$$

of the map $\mathcal{L}$ which parametrizes the gauge equivalence classes $[\nabla^\lambda]$. The map $\mathcal{D}$ has poles at those points $p \in \Sigma$ where $\mathcal{L}(p) = \mathbb{C} \in Jac(\tilde{M}/\mathbb{Z}_3)$ is the trivial holomorphic bundle. Note that the (unique) preimage $0 \in \Sigma$ of $\lambda = 0$ always satisfies $\mathcal{L}(p) = \mathbb{C}$. The poles at $p \neq 0$ are generically simple, and the exact asymptotic behavior of $\mathcal{D}$ around p is determined by the results of Sect. 5.

Definition 3 The triple $(\Sigma, \mathcal{L}, \mathcal{D})$, which is determined by the associated family of flat connections of a compact minimal surface in S^3 with the symmetries φ_2, φ_3 and τ of the Lawson surface of genus 2, is called spectral data of the surface.

6.1 Asymptotic behavior of the family of flat connections

We have already seen that the spectral curve Σ of a compact minimal surface in S^3 with the symmetries of the Lawson surface branches over $\lambda = 0$ and that the map $\mathcal{L}$ is holomorphic. We claim that the asymptotic behavior of $\mathcal{D}$ around the preimage of $\lambda = 0$ is analogous to the case of minimal tori in S^3 [15].

In order to show this we consider a holomorphic family of flat Lawson symmetric $SL(2, \mathbb{C})$-connections

$$\lambda \mapsto \hat{\nabla}^{\lambda}$$

defined on an open neighborhood of $\lambda = 0$ such that $(\hat{\nabla}^{\lambda})'' = \bar{\partial}^{\lambda}$. This implies that for small $\lambda \neq 0$ the difference

$$\nabla^{\lambda} - \hat{\nabla}^{\lambda}$$

is a symmetric Higgs field $\Psi \in H^0(M, K\,End_0(V, \bar{\partial}^{\lambda}))$ whose determinant is a non-zero multiple of Q. An expansion of $\hat{\nabla}^{\lambda}$ around $\lambda = 0$ is given by

$$\hat{\nabla}^{\lambda} = \begin{pmatrix} \nabla^{spin^*} + \omega_0 & -\frac{i}{2}Q^* + \alpha \\ -\frac{i}{2}q & \nabla^{spin} - \omega_0 \end{pmatrix} + \lambda \begin{pmatrix} \omega_1 & \alpha_1 \\ -vol + \beta_1 & -\omega_1 \end{pmatrix} + \cdots, \qquad (20)$$

where q is a constant multiple of the Hopf differential, $\alpha_i \in \Gamma(M, KK^{-1})$, $\omega_i \in \Gamma(M, K)$ and $\beta_1 \in \Gamma(M, K^2)$. We claim that

$$Q - q \neq 0$$

is a non-zero constant multiple of the Hopf differential. To see this note that

$$\begin{aligned} F^{\nabla^{spin^*}} &= \frac{1}{4}Q^* \wedge Q + tr(\Phi \wedge \Phi^*) \\ &= \frac{1}{4}Q^* \wedge q - \bar{\partial}\omega_0 \end{aligned}$$

as a consequence of the flatness of ∇^{λ} as well as of $\hat{\nabla}^0$. The claim then follows from $\int_M \bar{\partial}\omega_0 = 0$ and $\int_M tr(\Phi \wedge \Phi^*) \neq 0$. Comparing (20) with Proposition 7 in Appendix A we obtain

$$\det(\nabla^{\lambda} - \hat{\nabla}^{\lambda}) = -\frac{i}{4}\lambda^{-1}(Q - q) + \cdots. \qquad (21)$$

This leads to the following theorem.

Theorem 5 *Let $(\Sigma, \mathcal{L}, \mathcal{D})$ be the spectral data associated to a compact minimal surface in S^3 with the symmetries φ_2, φ_3 and τ of the Lawson surface of genus 2. Let t be a coordinate of Σ around $p^{-1}(\{0\})$ such that locally*

$$\mathcal{L}(t) = \bar{\partial}_0 + t\,d\bar{z},$$

where z is the affine coordinate on $\tilde{M}/\mathbb{Z}_3 \cong \mathbb{C}/(2\mathbb{Z} + 2i\mathbb{Z})$. The asymptotic of the map $\mathcal{D}$ at 0 is given by

$$\mathcal{D}(t) = d + t\,d\bar{z} + \left(c_{-1}\frac{1}{t} + c_1 t + ..\right)dz$$

for some $c_{-1} \neq \pm\frac{\pi}{12}$ and with respect to the natural local trivialization of the affine bundle $\mathcal{A}^f \to Jac(\tilde{M}/\mathbb{Z}_3)$.

The covering $p\colon \Sigma \to \mathbb{C}$ branches at most over those points $\lambda \in \mathbb{C}$ where $\bar{\partial}^\lambda$ is one of the exceptional holomorphic structures, i.e., $\mathcal{L}(\mu) = \underline{\mathbb{C}}$ for $p(\mu) = \lambda$. Moreover $\mathcal{D}$ satisfies the reality condition

$$\mathcal{D}(\mu) = \mathcal{U}(\mathcal{L}(\mu))$$

for all $\mu \in p^{-1}(S^1) \subset \Sigma$ where $\mathcal{U}$ is the section given by Theorem 2, and the closing condition

$$\mathcal{D}(\mu) = \left[d + \frac{-1+i}{4}\pi\,dz + \frac{1+i}{4}\pi\,d\bar{z}\right]$$

for all $\mu \in p^{-1}(\{\pm 1\}) \subset \Sigma$.

Proof As in the proof of Theorem 3 we see that the effect of adding a family of Higgs fields $\nabla^\lambda - \hat{\nabla}^\lambda$ with asymptotic as in (21) on the corresponding $\mathbb{C}^*$-connections over $\tilde{M}/\mathbb{Z}_3$ is given by adding

$$\left(\frac{c_{-1}}{t} + c_0 + c_1 t + \cdots\right)dz$$

with $0 \neq c_1, c_0, c_1 \in \mathbb{C}$. As $\det(\nabla^\lambda - \hat{\nabla}^\lambda)$ is even in t by the definition of t, the constant c_0 vanishes. Together with Theorem 3 this implies the first statement.

The reality condition is a consequence of the fact that the connections ∇^λ are unitary for $\lambda \in S^1$ and of Theorem 2. The closing condition follows from the observation that the trivial connection of rank 2 on M corresponds to the connection

$$d + \frac{-1+i}{4}\pi\,dz + \frac{1+i}{4}\pi\,d\bar{z}$$

on $\tilde{M}/\mathbb{Z}_3$.

It remains to prove that the spectral curve cannot branch over the points $\lambda \in \mathbb{C}$ where $\bar{\partial}^\lambda$ is semi-stable and not stable. For this we consider the holomorphic line bundle $L \to \mathbb{C}$ whose fiber is at a generic point λ spanned by the 1-dimensional space symmetric Higgs fields of $\bar{\partial}^\lambda$. But the space of symmetric Higgs fields at the semi-stable points is also 1-dimensional, and the determinant of a non-zero symmetric Higgs field is a non-zero multiple of the Hopf differential Q. Therefore, the eigenlines of the

Higgs fields can be parametrized in λ as long as $\bar{\partial}^{\lambda}$ is not an exceptional holomorphic structure. □

6.2 Reconstruction

Conversely, a hyper-elliptic Riemann surface $\Sigma \to \mathbb{C}$ together with a map $\mathcal{L}: \Sigma \to Jac(\tilde{M}/\mathbb{Z}_3)$ and a lift $\mathcal{D}$ into the affine bundle of line bundle connections which satisfy the asymptotic condition, the reality and closing conditions of Theorem 5 give rise to a compact minimal surface of genus 2 in S^3. To prove this we first need some preparation:

Theorem 6 *Let $\lambda \in \mathbb{C}^* \mapsto \tilde{\nabla}^{\lambda}$ be a holomorphic family of flat $SL(2, \mathbb{C})$-connections on a rank 2 bundle $V \to M$ over a compact Riemann surface M of genus $g \geq 2$ such that*

– *the asymptotic at $\lambda = 0$ is given by*

$$\tilde{\nabla}^{\lambda} \sim \lambda^{-1}\Psi + \tilde{\nabla} + \cdots$$

where $\Psi \in \Gamma(M, K End_0(V))$ is nowhere vanishing and nilpotent;
– *for all $\lambda \in S^1 \subset \mathbb{C}$ there is a hermitian metric on V such that $\tilde{\nabla}^{\lambda}$ is unitary with respect to this metric;*
– *$\tilde{\nabla}^{\lambda}$ is trivial for $\lambda = \pm 1$.*

Then there exists a unique (up to spherical isometries) minimal surface $f: M \to S^3$ such that its associated family of flat connections ∇^{λ} and the family $\tilde{\nabla}^{\lambda}$ are gauge equivalent, i.e., there exists a λ-dependent holomorphic family of gauge transformations g which extends through $\lambda = 0$ such that $\nabla^{\lambda} \cdot g = \tilde{\nabla}^{\lambda}$.

Proof It is a consequence of the asymptotic of $\tilde{\nabla}^{\lambda}$ that $(\tilde{\nabla}^{\lambda})''$ is stable for generic $\lambda \in \mathbb{C}^*$, for more details see [19]. This implies that the generic connection $\tilde{\nabla}^{\lambda}$ is irreducible. Therefore the hermitian metric for which $\tilde{\nabla}^{\lambda}$ is unitary is unique up to constant multiples for generic $\lambda \in S^1 \subset \mathbb{C}^*$. For those $\lambda \in S^1$ the hermitian metric $(,)^{\lambda}$ is unique if we impose that it is compatible with the determinant on V, i.e., the determinant of an orthonormal basis is unimodular. The metric $(,)^{\lambda}$ depends real-analytically on $\lambda \in S^1 \setminus S$, where $S \subset S^1$ is the set of points where ∇^{λ} is not irreducible, and can be extended through the set S.

From now on we identify $V = M \times \mathbb{C}^2$ and fix a unitary metric $(,)$ on it. Hence, $(,)^{\lambda}$ can be identified with a section $[h] \in \Gamma(S^1 \times M, SL(2, C)/SU(2))$ which itself can be lifted to a section $h \in \Gamma(S^1 \times M, SL(2, C))$. Clearly, h is real analytic in λ and satisfies

$$h_{\lambda}^*(,) = (,)^{\lambda}.$$

We now apply the loop group Iwasawa decomposition to $g = h^{-1}$, i.e.,

$$g = BF,$$

where $B \in \Gamma(D^1 \times M, SL(2, \mathbb{C}))$ is holomorphic in λ on $D^1 = \{\lambda \in \mathbb{C} \mid \bar{\lambda}\lambda \leq 1\}$ and $F \in \Gamma(S^1 \times M, SU(2))$ is unitary, see [25] for details. Gauging

$$\nabla^\lambda = \tilde{\nabla}^\lambda \cdot B$$

we obtain a holomorphic family of flat connections ∇^λ on $D^1 \setminus \{0\}$ which is unitary with respect to $(,)$ on S^1 by construction. Applying the Schwarz reflection principle yields a holomorphic family of flat connections $\lambda \in \mathbb{C}^* \mapsto \nabla^\lambda$ which is unitary on S^1 and trivial for $\lambda = \pm 1$. Moreover, as B extends holomorphically to $\lambda = 0$, ∇^λ has the following asymptotic

$$\nabla^\lambda \sim \lambda^{-1}\Phi + \nabla + ..$$

where $\Phi = B_0^{-1}\Psi B_0$ is complex linear, nowhere vanishing and nilpotent. Using the Schwarzian reflection again, we obtain

$$\nabla^\lambda = \lambda^{-1}\Phi + \nabla - \lambda\Phi^*$$

for a unitary connection ∇. This proves the existence of an associated minimal surface $f \colon M \to S^3$.

Let $f_1, f_2 \colon M \to S^3$ be two minimal surface such that their associated families of flat connections ∇_1^λ and ∇_2^λ are gauge equivalent to $\tilde{\nabla}^\lambda$, where both families of gauge transformations extend holomorphically to $\lambda = 0$. Let $g \in \Gamma(\mathbb{C} \times M, SL(2, \mathbb{C}))$ be the gauge between these two families which, by assumption, also extends to $\lambda = 0$. We may assume that for all $\lambda \in S^1$ the connections ∇_1^λ and ∇_2^λ are unitary with respect to the same hermitian metric. As the connections are generically irreducible the gauge g is unitary along the unit circle. By the Schwarz reflection principle g extends to $\lambda = \infty$, and therefore g is constant in λ. Hence, the corresponding minimal surfaces f_1 and f_2 are the same up to spherical isometries. $\qquad\square$

Remark 10 There exists similar results as Theorems 6 and 8 for the DPW approach to minimal surface, see [11,26].

Remark 11 The above theorem is still true if the individual connections $\tilde{\nabla}^\lambda$ are only of class C^k for $k \geq 3$ and not necessarily smooth.

Similar to the case of tori, the knowledge of the gauge equivalence class of the associated family of flat connections $[\nabla^\lambda]$ for all λ is in general not enough to determine the minimal immersion uniquely. The freedom is given by λ-dependent meromorphic gauge transformations g which is unitary along the unit circle. Applying such a gauge transformation is known in the literature as dressing, see for example [8] or [27]. For tori, dressing is closely related to the isospectral deformations induced by changing the eigenline bundle of a minimal immersion. In fact, simple factor dressing with respect to special eigenlines of the connections ∇^λ (those which correspond to the eigenline bundle) generate the abelian group of isospectral deformations. The remaining eigenlines, which only occur at values of λ where the monodromy takes values in $\{\pm Id\}$, produce singularities in the spectral curve and therefore do not correspond

to isospectral deformations in the sense of Hitchin. Due to the fact that for minimal surfaces of higher genus the generic connection ∇^λ is irreducible there are in general no continuous families of dressing deformations:

Theorem 7 *Let $f, \tilde{f}\colon M \to S^3$ be two conformal minimal immersions from a compact Riemann surface of genus $g \geq 2$ together with their associated families of flat connections ∇^λ and $\hat{\nabla}^\lambda$. Assume that ∇^λ is gauge equivalent to $\tilde{\nabla}^\lambda$ for generic $\lambda \in \mathbb{C}^*$. Then there exists a meromorphic map*

$$g\colon \mathbb{CP}^1 \to \Gamma(M, End(V))$$

such that $\nabla^\lambda \cdot g = \tilde{\nabla}^\lambda$. This map g is holomorphic and takes values in the invertible endomorphisms away from those $\lambda_0 \in \mathbb{C}^*$ where ∇^{λ_0} or equivalently $\tilde{\nabla}^{\lambda_0}$ is reducible.

The space of such dressing deformations of surfaces $f \mapsto \tilde{f}$ is generated by simple factor dressing, i.e., by maps $d\colon \mathbb{CP}^1 \to \Gamma(M, End(V))$ of the form

$$d(\lambda) = \pi^L + \frac{1 - \bar{\lambda}_0^{-1}}{1 - \lambda_0} \frac{\lambda - \lambda_0}{\lambda - \bar{\lambda}_0^{-1}} \pi^{L^\perp},$$

where L is an eigenline bundle of the connection ∇^{λ_0} and $L^\perp$ is its orthogonal complement.

Proof We first show that ∇^λ and $\tilde{\nabla}^\lambda$ are gauge equivalent away from those $\lambda_0 \in \mathbb{C}^*$ where ∇^{λ_0} or $\tilde{\nabla}^{\lambda_0}$ is reducible. The gauge between two irreducible gauge equivalent connections ∇^λ and $\tilde{\nabla}^\lambda$ is unique up to a constant multiple of the identity. Moreover, multiples of this gauge are the only parallel endomorphisms with respect to the connection $\nabla^\lambda \otimes (\tilde{\nabla}^\lambda)^*$. As the connections depend holomorphically on λ there exists a holomorphic line bundle $\mathcal{G} \to \mathbb{C}^*$ whose line at $\lambda \in \mathbb{C}^*$ is a subset of the parallel endomorphisms (and coincides with it whenever ∇^λ or equivalently $\tilde{\nabla}^\lambda$ is irreducible). A non-vanishing section $g \in \Gamma(U, \mathcal{G})$ around $\lambda \in U \subset \mathbb{C}^*$ gives rise to the gauge between ∇^λ and $\tilde{\nabla}^\lambda$ as long as g^λ is an isomorphism. This can fail only in the case where ∇^λ or equivalently $\tilde{\nabla}^\lambda$ is reducible.

We need to prove that g extends holomorphically to an isomorphism at $\lambda = 0$. Then, as g is unitary along the unit circle, g also extends holomorphically to an isomorphism at $\lambda = \infty$ by the Schwarz reflection principle. Note that locally around $\lambda = 0$ all connections are irreducible and all holomorphic structures are stable away from $\lambda = 0$. Then, as above, there exists a family of gauge transformations g^λ which extend to a holomorphic endomorphism g^0 with respect to $\bar{\partial}^0 \otimes (\tilde{\bar{\partial}}^0)^*$. From the fact that the connections around $\lambda = 0$ are gauge equivalent and the expansions of the two families one deduces that g^0 is a holomorphic endomorphism between the stable pairs $(\tilde{\bar{\partial}}, \tilde{\Phi})$ and $(\bar{\partial}, \Phi)$, i.e., $\Phi \circ g^0 = g^0 \circ \tilde{\Phi}$. Therefore Proposition (3.15) of [16] implies that g^0 is an isomorphism.

In order to find the globally defined dressing $g\colon \mathbb{CP}^1 \to \Gamma(M, End(V))$ we first investigate the bundle $\mathcal{G} \to \mathbb{C}^*$. We have seen that it extends to $\lambda = 0$ holomorphically and by switching to anti-holomorphic structures, one can also show that it extends holomorphically to $\lambda = \infty$. Therefore there exists a meromorphic section

$\tilde{g} \in \mathcal{M}(\mathbb{CP}^1, \mathcal{G})$ whose only poles are at $\lambda = \infty$. As $\mathcal{G}$ is a holomorphic subbundle of $\mathbb{C} \times \Gamma(M; End(V))$ the determinant is a holomorphic map $\det \colon \mathcal{G} \to \mathbb{C}$. Consider the holomorphic function

$$h \colon \mathbb{C} \to \mathbb{C}, \; h(\lambda) = \det(\tilde{g}^\lambda).$$

Note that we may assume that h is non-vanishing along the unit circle as $\tilde{g}$ is a complex multiple of a unitary gauge there. The Iwasawa decomposition $h = h_+ h_u$ determines functions $h_+ \colon \{\lambda \mid \lambda\bar{\lambda} \leq 1\} \to \mathbb{C}^*$ and $h_u \colon \mathbb{C}^* \to \mathbb{C}$ which satisfies $\| h_u(\lambda) \| = 1$ for $\lambda \in S^1$. These are unique up to unimodular constants. The square root $\sqrt{h_+}$ is then well-defined on $\{\lambda \mid \lambda\bar{\lambda} \leq 1\}$ and we define

$$g = \frac{1}{\sqrt{h_+}}\tilde{g} \in H^0(\{\lambda \mid \lambda\bar{\lambda} \leq 1\}, \mathcal{G}).$$

The determinant $\det g$ is unimodular along the unit circle, and therefore, g is unitary along the unit circle. By the Schwarz reflection principle, we obtain a meromorphic map $g \in \mathcal{M}(\mathbb{CP}^1, \mathcal{G})$ which satisfies $\nabla^\lambda \cdot g = \tilde{\nabla}^\lambda$ by construction.

It is shown in [8] that a simple factor dressing $\nabla^\lambda \mapsto \nabla^\lambda \cdot d$ makes the associated family of a new minimal surface. We want to show by induction that any g as above is the product of simple factor dressings. Note that $\det g \colon \mathbb{CP}^1 \to \mathbb{CP}^1$ is a rational function. If its degree is 0, then $\det g$ is a non-zero constant, and g is constant in λ. As it is unitary on the unit circle, g acts as a spherical isometry on the surface. Assume that $\det g$ has a zero at λ_0. As we have seen $\lambda_0 \in \mathbb{C}^* \setminus S^1$. By multiplying with (a power of) $\frac{1-\bar{\lambda}_0^{-1}}{1-\lambda_0}\frac{\lambda-\lambda_0}{\lambda-\bar{\lambda}_0^{-1}} Id$ we can also assume that $g^{\lambda_0} \neq 0$. As g^{λ_0} is a non-zero parallel endomorphism with respect to $\nabla^{\lambda_0} \otimes (\tilde{\nabla}^{\lambda_0})$ we see that the line bundle $L \to M$, which is given by $L_p = \ker g_p^{\lambda_0}$ at generic points $p \in M$, is an eigenline bundle of $\tilde{\nabla}^{\lambda_0}$. As a consequence of the unitarity of $\tilde{\nabla}^\lambda$ along the unit circle, $L^\perp$ is an eigenline bundle of $\tilde{\nabla}^{\bar{\lambda}^{-1}}$. We can apply the simple factor dressing

$$d(\lambda) = \pi^{L^\perp} + \frac{1-\lambda_0}{1-\bar{\lambda}_0^{-1}}\frac{\lambda-\bar{\lambda}_0^{-1}}{\lambda-\lambda_0}\pi^L$$

to $\tilde{\nabla}^\lambda$. Then, the product gd is again a meromorphic family of gauge transformations which extends holomorphically through λ_0, and the degree of the rational function $\det(gd)$ is the degree of the rational function $\det(g)$ minus 1. $\qquad\square$

Lemma 5 *Let $f \colon M \to S^3$ be an immersed minimal surface of genus 2 having the symmetries φ_2, φ_3 and τ. Then a (non-trivial) dressing transformation of such a minimal immersion does not admit all symmetries φ_2, φ_3 and τ.*

Proof We have seen in the proof of Theorem 5 that there exists a local holomorphic family of Higgs fields $\Psi^\lambda \in H^0(M, K End_0(V, \bar{\partial}^\lambda))$ around every point λ_0 where ∇^{λ_0} is reducible such that $\det \Psi^\lambda$ is nowhere vanishing. Let $g \colon \mathbb{CP}^1 \to \Gamma(M, End(V))$ be a meromorphic family of gauges as in Theorem 7. Assume that $g(\lambda_0)$ exists but is

a non-zero endomorphism which is not invertible. It is easy to see that the family of Higgs fields

$$\hat{\Psi}^\lambda = g(\lambda)^{-1} \Psi^\lambda g(\lambda)$$

with respect to $\tilde{\nabla}^\lambda = \nabla^\lambda \cdot g$ has a pole at λ_0. Then, by resolving the pole by multiplying with an appropriate power of $(\lambda - \lambda_0)$, the (local) holomorphic nowhere vanishing family of Higgs fields

$$\tilde{\Psi}^\lambda = (\lambda - \lambda_0)^k \hat{\Psi}^\lambda$$

satisfies $\det(\tilde{\Psi}^{\lambda_0}) = 0$. This is not possible for a surface $\tilde{f}$ which has all three symmetries φ_2, φ_3 and τ.   □

Theorem 8 *Let Σ be a Riemann surface and $p \colon \Sigma \to \mathbb{C}$ be a double covering induced by the involution $\sigma \colon \Sigma \to \Sigma$ such that p branches over 0. Let $\mathcal{L} \colon \Sigma \to Jac(\tilde{M}/\mathbb{Z}_3)$ be a non-constant holomorphic map which is odd with respect to σ and satisfies $\mathcal{L}(0) = \underline{\mathbb{C}} \in Jac(\tilde{M}/\mathbb{Z}_3)$. Let $\mathcal{D} \colon \Sigma \setminus p^{-1}(0) \to \mathcal{A}^f$ be a meromorphic lift of $\mathcal{L}$ to the moduli space of flat $\mathbb{C}^*$- connections on $\tilde{M}/\mathbb{Z}_3$ which is odd with respect to σ and which satisfies the conditions of Theorems 3 and 4 at its poles, i.e., $\mathcal{D}$ defines a holomorphic map from $\mathbb{C}^*$ to the moduli space of flat $SL(2, \mathbb{C})$-connections on M. If $\mathcal{D}$ has a first order pole at 0 and satisfies the reality condition*

$$\mathcal{D}(\mu) = \mathcal{U}(\mathcal{L}(\mu))$$

for all $\mu \in p^{-1}(S^1) \subset \Sigma$, where $\mathcal{U}$ is the section given by Theorem 2, and the closing condition

$$\mathcal{D}(\mu) = \left[d + \frac{-1+i}{4} \pi \, dz + \frac{1+i}{4} \pi \, d\bar{z} \right]$$

for all $\mu \in p^{-1}(\{\pm 1\}) \subset \Sigma$ then there exists an immersed minimal surface $f \colon M \to S^3$ such that $(\Sigma, \mathcal{L}, \mathcal{D})$ are the spectral data of f. Let t be a holomorphic coordinate of Σ around $p^{-1}(0)$ such that $t^2 = \lambda$, and consider the expansion

$$\mathcal{D} \sim d - (x_1 t + ..) \pi \, d\bar{z} + \left(a_{-1} \frac{1}{t} + \cdots \right) \pi \, dz.$$

Then the area of f is given by

$$\text{Area}(f) = -12\pi \left(\frac{1}{6} - 2\pi x_1 a_{-1} \right).$$

If p only branches at those $\mu \in \Sigma$ where $\mathcal{L}(\mu) = \underline{\mathbb{C}} \in Jac(\tilde{M}/\mathbb{Z}_3)$ then there is a unique f which has the symmetries φ_2, φ_3 and τ.

Proof We first show that the spectral data give rise to a holomorphic $\mathbb{C}^*$-family of flat $SL(2,\mathbb{C})$-connections on M satisfying the conditions of Theorem 6. By assumption we obtain a holomorphic map into the moduli space of flat $SL(2,\mathbb{C})$-connections on M. Reversing the arguments of Sects. 4 and 5 we obtain locally on open subsets of $\mathbb{C}^*$ holomorphic families of flat $SL(2,\mathbb{C})$-connections on M which are lifts of the map to the moduli space. We cover $\mathbb{C}^*$ by these open sets U_i, $i \in \mathbb{N}$, such that for every U_i there exists at most one point where the corresponding connection is reducible. There also exist an open set U_0 containing 0 such that on $U_0 \setminus \{0\}$ there exists a lift ∇_0^λ of the map to the moduli space of flat $SL(2,\mathbb{C})$-connections on M which has at most a first order pole at $\lambda = 0$. Moreover, as $\mathcal{L}(0) = \mathbb{C}$, the residuum at 0 must be a complex linear 1-form $\Psi \in \Gamma(M, K End_0(V))$ which is nilpotent. We now fix such families of flat $SL(2,\mathbb{C})$-connections ∇_i^λ on every set U_i. Let $\mathcal{G}$ be the complex Banach Lie group of C^k gauges

$$\mathcal{G} = \{g \colon M \to SL(2,\mathbb{C}) \mid g \text{ is of class } C^k\},$$

where we have fixed a trivialization of the rank 2 bundle $V = M \times \mathbb{C}^2$ and $k \geq 4$. On $U_i \cap U_j$ we define a map $g_{i,j} \colon U_i \cap U_j \to \mathcal{G}$ by

$$\nabla_j^\lambda = \nabla_i^\lambda \cdot g_{i,j}.$$

Clearly, the maps $g_{i,j}$ are well-defined, and give rise to a 1-cocycle of $\mathbb{C} = \cup_{i \in \mathbb{N}_0} U_i$ with values in $\mathcal{G}$. As $\mathbb{C}$ is a Stein space the generalized Grauert theorem as proven in [7] shows the existence of maps $f_i \colon U_i \to \mathcal{G}$ satisfying $f_i f_j^{-1} = g_{i,j}$. Then

$$\nabla_i^\lambda \cdot f_i = \nabla_j^\lambda \cdot f_j$$

on $U_i \cap U_j$ and we obtain a well-defined $\mathbb{C}^*$-family of flat $SL(2,\mathbb{C})$-connections $\tilde{\nabla}^\lambda$ which satisfies the reality condition and the closing condition of Theorem 6. Applying the proof of Theorem 6 we see that the holomorphic structure $(\tilde{\nabla}^0)''$ is stable and therefore the residuum of $\tilde{\nabla}^\lambda$ at $\lambda = 0$ is a nowhere vanishing nilpotent complex linear 1-form. By Theorem 6 we obtain an immersed minimal surface $f \colon M \to S^3$. The formula for the energy of f can be computed by similar methods as used in Sect. 5.

Now assume that p only branches at those $\mu \in \Sigma$ where $\mathcal{L}(\mu) = \mathbb{C} \in Jac(\tilde{M}/\mathbb{Z}_3)$. Then the map $\mathcal{D}$ into the moduli space of flat $SL(2,\mathbb{C})$-connections can be locally lifted (denoted by ∇_i^λ) to the space of flat connections in such a way that a corresponding nowhere vanishing family of Higgs fields Ψ_λ^i has non-zero determinant whenever $\mathcal{L}(\mu) \neq \mathbb{C} \in Jac(\tilde{M}/\mathbb{Z}_3)$ for $p(\mu) = \lambda$. Arguing in the same lines as in the proof of Lemma 5 one sees that all families of connections ∇_i^λ are gauge equivalent to $\varphi^* \nabla_i^\lambda$ by holomorphic families of gauges for all symmetries $\varphi = \varphi_2,\ \varphi_3,\ \tau$. Then the uniqueness part of Theorem 6 proves that the corresponding minimal surface has the symmetries $\varphi_2,\ \varphi_3$ and τ. Moreover, Theorem 7 and Lemma 5 show the uniqueness of this minimal surface. $\qquad\square$
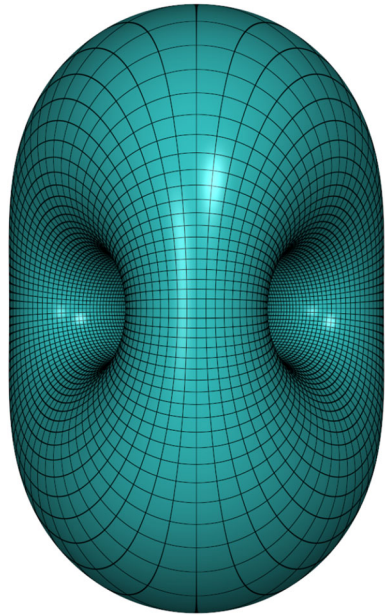
Remark 12 Computer experiments in [20] suggest that the spectral curve of the Lawson surface of genus 2 is not branched over the punctured unit disc $D = \{\lambda \in \mathbb{C} \mid 0 <\| \lambda \|\leq 1\}$. With these numerical spectral data the Lawson surface of genus 2 can be visualized as a conformal immersion from the Riemann surface M into S^3 by an implementation of Theorem 8 in the xlab software of Nicholas Schmitt (see Fig. 1).

7 Lawson symmetric CMC surface of genus 2

In [20] we found numerical evidence that there exist a deformation of the Lawson surface of genus 2 through compact CMC surface $f \colon M \to S^3$ of genus 2 which preserves the extrinsic symmetries φ_2, φ_3 and τ. We call these surfaces Lawson symmetric CMC surfaces. We briefly explain how to generalize our theory to Lawson symmetric CMC surfaces.

Due to the Lawson correspondence, one can treat CMC surfaces in S^3 in the same way as minimal surfaces, see for example [4]. Consequently, there also exists an associated family of flat $SL(2, \mathbb{C})$-connections $\lambda \in \mathbb{C}^* \mapsto \nabla^\lambda$ which are unitary along the unit circle. In contrast to the minimal case the Sym points $\lambda_1 \neq \lambda_2 \in S^1$, at which the connections ∇^{λ_i} are trivial, must not be the negative of each other. Then, the CMC surface is obtained as the gauge between these two flat connections, but the mean curvature is now given by $H = i \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}$. For $\lambda_1 = -\lambda_2$ we get a minimal surface.

As the extrinsic symmetries φ_2, φ_3 and τ are (assumed to be) holomorphic on the surface, the Riemann surface structure is almost fixed: It is given by the algebraic equation

$$y^3 = \frac{z^2 - a}{z^2 + 1}$$

for some $a \in \mathbb{C}^*$. The Lawson Riemann surface structure is then given by $a = 1$. Moreover, every individual connection ∇^λ of the associated family is equivariant with respect to the Lawson symmetries. All the theory developed for flat Lawson symmetric $SL(2, \mathbb{C})$-connections on the Lawson surface carries over to flat Lawson symmetric $SL(2, \mathbb{C})$-connections on M: The moduli space of Lawson symmetric holomorphic structures is double covered by the Jacobian of a complex 1-dimensional torus. This torus itself is given by the equation $y^2 = \frac{z^2-a}{z^2+1}$. There is a 2:1 correspondence between gauge equivalence classes of flat line bundle connections on the above mentioned torus and gauge equivalence classes of flat Lawson symmetric $SL(2, \mathbb{C})$-connections on M away from divisors in the corresponding moduli spaces. The correspondence extends to these divisors in the sense of Theorems 3 Theorem 4. The concrete formulas are analogous to the case of the Lawson surface.

From the observation that the moduli spaces of the flat Lawson symmetric $SL(2, \mathbb{C})$-connections can be described analogously to the case of the Lawson surface itself, it is clear that the definition and the basic properties of the spectral curve carries over to Lawson symmetric CMC surfaces of genus 2. Of course, the extrinsic closing condition changes, as well as the precise form of the energy formula.

Appendix A: The associated family of flat connections

In this appendix we briefly recall the gauge theoretic description of minimal surfaces in S^3 which is due to Hitchin [15]. For more details, one can also consult [18].

The Levi-Civita connection of the round S^3 is given with respect to the left trivialization $TS^3 = S^3 \times \Im\mathbb{H}$ as

$$\nabla = d + \frac{1}{2}\omega,$$

where ω is the Maurer–Cartan form of S^3 which acts via adjoint representation.

The hermitian complex rank 2 bundle $V = S^3 \times \mathbb{H}$ with complex structure given by right multiplication with $i \in \mathbb{H}$ is a spin bundle for S^3: The Clifford multiplication is given by $TS^3 \times V \to V; (\lambda, v) \mapsto \lambda v$ where $\lambda \in \Im\mathbb{H}$ and $v \in \mathbb{H}$, and this identifies TS^3 as the skew symmetric trace-free complex linear endomorphisms of V. There is a unique complex unitary connection on V which induces on $TS^3 \subset End(V)$ the Levi-Civita connection. It is given by

$$\nabla = \nabla^{spin} = d + \frac{1}{2}\omega,$$

where the $\Im\mathbb{H}$-valued Maurer-Cartan form acts by left multiplication in the quaternions.

Let M be a Riemann surface and $f \colon M \to S^3$ be a conformal immersion. Then the pullback $\phi = f^*\omega$ of the Maurer-Cartan form satisfies the structural equations

$$d^\nabla \phi = 0, \tag{22}$$

where $\nabla = f^*\nabla = d + \frac{1}{2}\phi$. The conformal map f is minimal if and only if it is harmonic, i.e., if

$$d^\nabla * \phi = 0. \tag{23}$$

holds. Let

$$\frac{1}{2}\phi = \Phi - \Phi^*$$

be the decomposition of $\phi \in \Omega^1(M; f^*TS^3) \subset \Omega^1(M; End_0(V))$ into the complex linear and complex anti-linear parts. As f is conformal

$$\det \Phi = 0.$$

Note that f is an immersion if and only if Φ is nowhere vanishing. In that case $\ker \Phi = S^*$ is the dual to the holomorphic spin bundle S associated to the immersion. The Eqs. 22 and 23 are equivalent to

$$\nabla''\Phi = 0, \tag{24}$$

where $\nabla'' = \frac{1}{2}(d^\nabla + i * d^\nabla)$ is the underlying holomorphic structure of the pullback of the spin connection on V. Of course (24) does not contain the property that $\nabla - \frac{1}{2}\phi = d$ is trivial. Locally this is equivalent to

$$F^\nabla = [\Phi \wedge \Phi^*] \tag{25}$$

as one easily computes.

From (24) and 25 one sees that the associated family of connections

$$\nabla^\lambda := \nabla + \lambda^{-1}\Phi - \lambda\Phi^* \tag{26}$$

is flat for all $\lambda \in \mathbb{C}^*$, unitary along $S^1 \subset \mathbb{C}^*$ and trivial for $\lambda = \pm 1$. This family contains all the informations about the surface, i.e., given such a family of flat connections one can reconstruct the surface as the gauge between ∇^1 and ∇^{-1}. Using Sym–Bobenko formulas one can also make CMC surfaces in S^3 and $\mathbb{R}^3$ out of the family of flat connections. These CMC surfaces do not close in general.

The family of flat connections can be written down in terms of the well-known geometric data associated to a minimal surface:

Proposition 7 *Let* $f\colon M \to S^3$ *be a conformal minimal immersion with associated complex unitary rank 2 bundle* (V, ∇) *and with induced spin bundle* S. *Let* $V = S^{-1}\oplus S$ *be the unitary decomposition, where* $S^{-1} = \ker \Phi \subset V$ *and* Φ *is the* K*-part of the differential of* f. *The Higgs field* $\Phi \in H^0(M, K\,End_0(V))$ *can be identified with*

$$\Phi = \frac{1}{2} \in H^0(M; K\,Hom(S, S^{-1})),$$

and its adjoint Φ^* *is given by* $i\,vol$ *where* vol *is the volume form of the induced Riemannian metric. The family of flat connections is given by*

$$\nabla^\lambda = \begin{pmatrix} \nabla^{spin^*} & -\frac{i}{2}Q^* \\ -\frac{i}{2}Q & \nabla^{spin} \end{pmatrix} + \lambda^{-1}\Phi - \lambda\Phi^*,$$

where ∇^{spin} *is the spin connection corresponding to the Levi-Civita connection on* M *and* Q *is the Hopf differential of* f.

Appendix B: Lawson's genus 2 surface

We recall the construction of Lawson's minimal surfaces of genus 2 in S^3, see [21]. Consider the round 3-sphere

$$S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \subset \mathbb{C} \oplus \mathbb{C}$$

and the geodesic circles $C_1 = S^3 \cap (\mathbb{C} \oplus \{0\})$ and $C_2 = S^3 \cap (\{0\} \oplus \mathbb{C})$ on it. Take the six points

$$Q_k = (e^{i\frac{\pi}{3}(k-1)}, 0) \in C_1$$

in equidistance on C_1, and the four points

$$P_k = (0, e^{i\frac{\pi}{2}(k-1)}) \in C_2$$

in equidistance on C_2. A fundamental piece of the Lawson surface is the solution to the Plateau problem for the closed geodesic convex polygon $\Gamma = P_1 Q_2 P_2 Q_1$ in S^3. This means that it is the smooth minimal surface which is area minimizing under all surfaces with boundary Γ. To obtain the Lawson surface one reflects the fundamental piece along the geodesic through P_1 and Q_1, then one rotates everything around the geodesic C_2 by $\frac{2}{3}\pi$ two times, and in the end one reflects the resulting surface across the geodesic C_1. Lawson has shown that the surface obtained in this way is smooth at all points. It is embedded, orientable and has genus 2. The umbilics, i.e., the zeros of the Hopf differential Q are exactly at the points $P_1, .., P_4$ of order 1.

A generating system of the symmetry group of the Lawson surface is given by

- the $\mathbb{Z}^2$-action generated by φ_2 with $(a, b) \mapsto (a, -b)$; it is orientation preserving on the surface and its fixed points are $Q_1, .. Q_6$;

- the $\mathbb{Z}_3$-action generated by the rotation φ_3 around $P_1 P_2$ by $\frac{2}{3}\pi$, i.e., $(a, b) \mapsto$ $(e^{i\frac{2}{3}\pi}a, b)$, which is holomorphic on M with fixed points $P_1, .., P_4$;
- the reflection at $P_1 Q_1$, which is antiholomorphic: $\gamma_{P_1 Q_1}(a, b) = (\bar{a}, \bar{b})$;
- the reflection at the sphere S_1 corresponding to the real hyperplane spanned by $(0, 1)$, $(0, i)$ and $(e^{\frac{1}{6}\pi i}, 0)$, with $\gamma_{S_1}(a, b) = (e^{\frac{\pi}{3}i}\bar{a}, b)$; it is antiholomorphic on the surface,
- the reflection at the sphere S_2 corresponding to the real hyperplane spanned by $(1, 0)$, $(i, 0)$ and $(0, e^{\frac{1}{4}\pi i})$, which is antiholomorphic on the surface and satisfies $\gamma_{S_2}(a, b) = (a, i\bar{b})$.

Note that all these actions commute with the $\mathbb{Z}_2$-action. The last two fix the polygon Γ. They and the first two map the oriented normal to itself. The third one maps the oriented normal to its negative.

Using the symmetries, one can determine the Riemann surface structure of the Lawson surface $f : M \to S^3$ as well as the other holomorphic data associated to it:

Proposition 8 *The Riemann surface M associated to the Lawson genus 2 surface is the threefold covering $\pi : M \to \mathbb{CP}^1$ of the Riemann sphere with branch points of order 2 over $\pm 1, \pm i \in \mathbb{CP}^1$, i.e., the compactification of the algebraic curve*

$$y^3 = \frac{z^2 - 1}{z^2 + 1}.$$

The hyper-elliptic involution is given by $(y, z) \mapsto (y, -z)$ and the Weierstrass points are $Q_1, .., Q_6$. The Hopf differential of the Lawson genus 2 surface is given by

$$Q = \pi^* \frac{ir}{z^4 - 1}(dz)^2$$

for a nonzero real constant $r \in \mathbb{R}$ and the spin bundle S of the immersion is

$$S = L(Q_1 + Q_3 - Q_5).$$

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