Example 2: Bayesian Linear Regression

Bayesian linear regression model

We have the model

$$y_i \mid \boldsymbol{\beta} \sim \mathcal{N}(\boldsymbol{x}_i^T \boldsymbol{\beta}, \phi^{-1}), \quad i = 1, \dots, n;$$

$$\boldsymbol{\beta} \mid \kappa \sim \mathcal{N}(0, \kappa^{-1} \mathbf{I});$$

$$\kappa \sim \operatorname{Gamma}(a_0, b_0);$$

where $\phi = 1/\sigma^2$ is the precision parameter, which we assume is known, x_i , i = 1, ..., n are known covariates, $\beta \in \mathbb{R}^p$ includes the intercept, and is unknown, and I is the identity matrix. Assume a_0 and b_0 are known.

We want a mean-field approximation to the posterior

$$p(\boldsymbol{\beta}, \kappa \mid \boldsymbol{y})$$

on the form

$$q(\boldsymbol{\beta}, \kappa) = q(\boldsymbol{\beta})q(\kappa).$$

Solution

For κ , the variational approximation is on the form

$$q(\kappa) \propto \text{Gamma}(a_n, b_n);$$

with

$$a_n = a_0 + \frac{p}{2}, \quad b_n = b_0 + \frac{\mathbb{E}_{q_{\boldsymbol{\beta}}}[\boldsymbol{\beta}^T \boldsymbol{\beta}]}{2}.$$

For β , the variational approximation is on the form

$$q(\boldsymbol{\beta}) \propto \mathcal{N}(\boldsymbol{m}_n, \boldsymbol{S}_n);$$

with

$$oldsymbol{S}_n = (\mathbb{E}_{q_\kappa}[\kappa] \mathrm{I} + \phi oldsymbol{X}^T oldsymbol{X})^{-1}, \quad oldsymbol{m}_n = \phi oldsymbol{S}_n oldsymbol{X}^T oldsymbol{y};$$

where

$$\mathbb{E}_{q_{\kappa}}[\kappa] = \frac{a_n}{b_n}, \quad \mathbb{E}_{q_{oldsymbol{eta}}}[oldsymbol{eta}^T oldsymbol{eta}] = oldsymbol{m}_n^T oldsymbol{m}_n + \mathrm{tr}(oldsymbol{S}_n).$$