

## Example 2: Bayesian Linear Regression

### Bayesian linear regression model

We have the model

$$\begin{aligned}y_i \mid \boldsymbol{\beta} &\sim \mathcal{N}(\mathbf{x}_i^T \boldsymbol{\beta}, \phi^{-1}), \quad i = 1, \dots, n; \\ \boldsymbol{\beta} \mid \kappa &\sim \mathcal{N}(0, \kappa^{-1} \mathbf{I}); \\ \kappa &\sim \text{Gamma}(a_0, b_0);\end{aligned}$$

where  $\phi = 1/\sigma^2$  is the precision parameter, which we assume is known,  $\mathbf{x}_i$ ,  $i = 1, \dots, n$  are known covariates,  $\boldsymbol{\beta} \in \mathbb{R}^p$  includes the intercept, and is unknown, and  $\mathbf{I}$  is the identity matrix. Assume  $a_0$  and  $b_0$  are known.

We want a mean-field approximation to the posterior

$$p(\boldsymbol{\beta}, \kappa \mid \mathbf{y})$$

on the form

$$q(\boldsymbol{\beta}, \kappa) = q(\boldsymbol{\beta})q(\kappa).$$

### Solution

For  $\kappa$ , the variational approximation is on the form

$$q(\kappa) \propto \text{Gamma}(a_n, b_n)$$

with

$$a_n = a_0 + \frac{p}{2}, \quad b_n = b_0 + \mathbb{E}[\boldsymbol{\beta}^T \boldsymbol{\beta}]$$

For  $\boldsymbol{\beta}$ , the variational approximation is on the form

$$q(\boldsymbol{\beta}) \propto \mathcal{N}(\mathbf{m}_n, \mathbf{S}_n)$$

with

$$\mathbf{S}_n = (\mathbb{E}[\kappa] \mathbf{I} + \phi \mathbf{X}^T \mathbf{X})^{-1}, \quad \mathbf{m}_n = \phi \mathbf{S}_n \mathbf{X}^T \mathbf{y}$$

where

$$\mathbb{E}[\kappa] = \frac{a_n}{b_n}, \quad \mathbb{E}[\boldsymbol{\beta}^T \boldsymbol{\beta}] = \mathbf{m}_n \mathbf{m}_n^T + \mathbf{S}_n$$