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# The Total Variation Decreasing Property of a Conservative Front Tracking Technique

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**Abstract**—In [1–4], one of the authors developed a conservative front tracking technique. In this paper, we study the effect of the technique on the total variation of the numerical solution when the underlying scheme is total variation decreasing (TVD). We prove that the first order technique will retain the TVD property for the overall scheme. Numerical examples are presented to support the conclusion even for higher order front tracking techniques.

**Keywords**—Conservation laws, Conservative numerical scheme, Total variation decreasing.

## 1. INTRODUCTION

In this paper, we consider the numerical approximation to weak solutions of the initial value problem (IVP) for scalar hyperbolic conservation laws

$$u_t + f(u)_x = 0, \qquad u(x,0) = \phi(x), \qquad -\infty < x < \infty, \tag{1.1}$$

with $f$ convex and where $\phi(x)$ is assumed to be of bounded total variation. The weak solution satisfies (1.1) in the sense that for any test function $\varphi(x,t) \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$

$$\int_0^\infty \int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x)\,dt\,dx + \int_{-\infty}^\infty \phi(x)\varphi(x,0)\,dx = 0. \tag{1.2}$$

For a weak solution of the IVP (1.1), the total variation with respect to $x$, denoted $\mathrm{TV}(u(t))$, will not increase in $t$, i.e.,

$$\mathrm{TV}(u(t_2)) \le \mathrm{TV}(u(t_1)), \qquad \text{for } t_2 > t_1. \tag{1.3}$$

We consider now explicit $(2K+1)$-point finite difference schemes in conservation form approximating (1.1). That is,

$$u_j^{n+1} = u_j^n - \lambda\left(\hat{f}_{j+1/2}^n - \hat{f}_{j-1/2}^n\right), \tag{1.4a}$$

where

$$\hat{f}_{j+1/2}^n = \hat{f}_{j+1/2}^n\left(u_{j-K+1}^n, \dots, u_{j+K}^n\right). \tag{1.4b}$$

Here, $u_j^n$ is the numerical solution at grid point $(j\Delta x, n\Delta t)$ and $\hat{f}$ is the numerical flux, consistent with the flux in (1.1) in the sense that

$$\hat{f}(u,u,\dots,u) = f(u). \tag{1.5}$$

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Since the weak solution to (1.1) has the "total variation diminishing" (TVD) property described in (1.3), many schemes have been designed to keep it, i.e., for these schemes also

$$\mathrm{TV}\left(u^{n+1}\right) \leq \mathrm{TV}\left(u^{n}\right) \tag{1.6a}$$

holds where

$$\mathrm{TV}(u^n) = \sum_{j=-\infty}^{\infty} \left|\Delta_{j+1/2}^n u\right|, \qquad \text{with } \left|\Delta_{j+1/2}^n u\right| = \left|u_{j+1}^n - u_j^n\right|. \tag{1.6b}$$

Among them are monotone schemes (see [5,6]) and TVD schemes (see [6–8]).

In [1–4], one of the authors developed a conservative front tracking technique, which uses the conservation property of the numerical solution rather than the moving speeds to track discontinuities. This front tracking technique, which will be described in the following section, can be applied to any difference scheme of the form (1.4) and it will sharpen the discontinuities. Schemes with such a technique can be generally written by adding some extra terms on the right hand side (RHS) of (1.4):

$$u_j^{n+1} = u_j^n - \lambda\left(\hat{f}_{j+1/2}^n - \hat{f}_{j-1/2}^n\right) + p_{j+1/2}^n - p_{j-1/2}^n + q_j^{n+1} - q_j^n, \tag{1.7}$$

where $p_{j+1/2}^n$'s and $q_j^n$'s are related to the numerical solution. In particular, the $q_j^n$'s, which are in the $t$-direction, are called local conservation errors.

In this paper, we discuss how much this front tracking technique will effect the total variation of the numerical solutions. We will prove that if the underlying schemes are TVD, then under certain restrictions, the technique will retain the TVD property for the overall schemes.

This TVD preserving property is significant for the front tracking technique, since from the uniformly bounded total variation we can obtain uniform boundedness for all the numerical solution $p_{j+1/2}^n$'s and $q_j^n$'s. Thus, one can always pick a convergent subsequence from the numerical solution. Multiply (1.7) with a test function $\varphi_j^n = \varphi(jh, n\tau)$ and with $\tau h$, sum it over $-\infty < j < \infty$, and sum by parts, we obtain

$$\sum_{n=1}^{\infty}\sum_{j=-\infty}^{\infty}\left(\frac{\phi_j^n - \phi_j^{n-1}}{\tau} u_j^n + \frac{\phi_{j+1}^n - \phi_j^n}{h}\hat{f}_{j+1/2}^n\right)\tau h = \sum_{n=1}^{\infty}\sum_{j=-\infty}^{\infty}\left(\frac{\phi_j^n - \phi_j^{n-1}}{\tau} q_j^n + \frac{\phi_{j+1}^n - \phi_j^n}{h} p_{j+1/2}^n\right)\tau h, \tag{1.8}$$

where $h, \tau$ are space and time mesh sizes, respectively. The left hand side (LHS) of (1.8) tends to the LHS of (1.2) when $\tau, h \to 0$. If we track only a finite number of discontinuities, which means that only a finite number of $p_{j+1/2}^n$'s and $q_j^n$'s are nonzero at each time level, then from the uniform boundednesss of $p_{j+1/2}^n$'s and $q_j^n$'s, we see that the RHS of (1.8) tends to zero when $\tau, h \to 0$. This indicates that the limit function of the convergent subsequence of the numerical solution is a weak solution to (1.1). If the numerical solution also satisfies the entropy condition which guarantees the uniqueness of the weak solution, then the convergence of the overall scheme to the entropy solution is established.

The paper is organized as follows. In Section 2, we briefly introduce the front tracking technique developed in [1–4]. In Section 3, we prove the theorem that under certain restrictions, the front tracking technique retains TVD property. Numerical examples are presented at the end of the paper.

## 2. THE CONSERVATIVE FRONT TRACKING TECHNIQUE

In this section, we describe briefly the conservative front tracking technique for a scalar conservation law in one space dimension.

One of the features of the technique is that it does not need flow states on the tracked front; the computation is mainly kept in the regular grid. The discontinuities travel from cells to cells in the computation. A cell that contains a discontinuity is called a critical cell.

For a single critical cell $[x_{j_1}, x_{j_1+1}]$ at the $n^{\text{th}}$ level, the technique lets the computation on each side of it use information only from the same side. That is,

$$u_j^{n+1} = u_j^n - \lambda \left( \hat{f}_{j+1/2}^{n,-} - \hat{f}_{j-1/2}^{n,-} \right), \qquad \text{for } j \le j_1, \tag{2.1}$$

and

$$u_j^{n+1} = u_j^n - \lambda \left( \hat{f}_{j+1/2}^{n,+} - \hat{f}_{j-1/2}^{n,+} \right), \qquad \text{for } j \ge j_1 + 1, \tag{2.2}$$

where

$$\hat{f}_{j+1/2}^{n,-} = \begin{cases} \hat{f}\left(u_{j-k+1}^n, \dots, u_{j_1}^n, u_{j_1-k+1}^{n,-}, \dots, u_{j+k}^{n,-}\right), & \text{for } j \le j_1 - k, \\ \hat{f}_{j+1/2}^n, & \text{for } j > j_1 - k, \end{cases} \tag{2.3}$$

and

$$\hat{f}_{j+1/2}^{n,+} = \begin{cases} \hat{f}\left(u_{j-k+1}^{n,+}, \dots, u_{j_1}^{n,+}, u_{j_1+1}^n, \dots, u_{j+k}^n\right), & \text{for } j > j_1 + k - 1, \\ \hat{f}_{j+1/2}^n, & \text{for } j \ge j_1 + k - 1. \end{cases} \tag{2.4}$$

Here, $u_j^{n,-}$ for $j \ge j_1 + 1$ and $u_j^{n,+}$ for $j \le j_1$ are extrapolated data of the numerical solution on the two sides (see [1,2] and Figure 1). When the discontinuity moves to the left adjacent cell in a time step, then

$$u_{j_1}^{n+1} = u_{j_1}^{n,+} - \lambda \left( \hat{f}_{j_1+1/2}^{n,+} - \hat{f}_{j_1-1/2}^{n,+} \right), \tag{2.5}$$

or

$$u_{j_1}^{n+1} = u_{j_1}^{n+1,+}, \tag{2.6}$$

and when the discontinuity moves to the right adjacent cell,

$$u_{j_1+1}^{n+1} = u_{j_1+1}^{n,-} - \lambda \left( \hat{f}_{j_1+3/2}^{n,-} - \hat{f}_{j_1+1/2}^{n,-} \right), \tag{2.7}$$

or

$$u_{j_1+1}^{n+1} = u_{j_1+1}^{n+1,-}. \tag{2.8}$$

By defining

$$\hat{f}_{j+1/2}^n = \begin{cases} \hat{f}_{j+1/2}^{n,-}, & j \le j_1 - 1, \\ \hat{f}_{j+1/2}^{n,+}, & j \ge j_1, \end{cases} \tag{2.9}$$

we are able to write the overall scheme in a unified form (1.7). Explicit forms of $p_{j+1/2}^n$'s and $q_j^n$'s are given in [1–4]. For example, if the discontinuity moves to the left adjacent cell, then

$$\begin{aligned} p_{j+1/2}^n &= 0, \qquad \text{for all } j \ne j_1, \\ p_{j-1/2}^n &= -q_{j_1}^n + \left(u_{j_1}^n + u_{n_1}^{n,+}\right) + \lambda \left( \hat{f}_{j_1-1/2}^{n,-} - \hat{f}_{j_1-1/2}^{n,+} \right), \\ q_j^{n+1} &= 0, \qquad \text{for all } j \ne j_1 - 1, \\ q_{j_1-1}^{n+1} &= -p_{j_1-1/2}^n. \end{aligned}$$

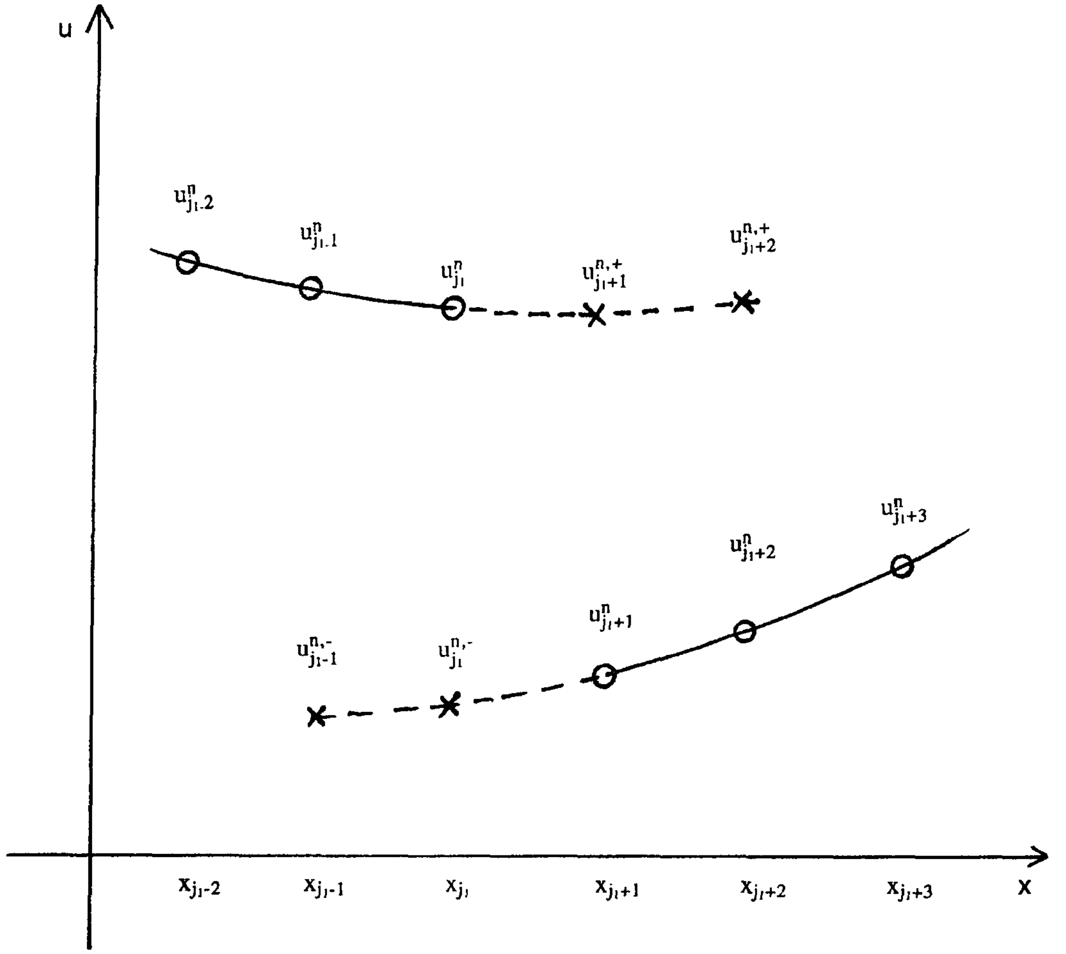



Figure 1. If the solution has a point of jump, then the numerical scheme extrapolates smoothly by values $u_j^{n,-}$ to the left and values $u_j^{n,+}$ to the right.

Obviously, $\sum_{j=-\infty}^{\infty} u_j^n$ is not conserved due the presence of $q_j^n$'s; however, $\sum_{j=-\infty}^{\infty} \left(u_j^n - q_j^n\right)$ is conserved.

The $q_j^n$, which are nonzero only at the left endpoints of critical cells, is called local conservation error. It represents how much the numerical solution is away from being conserved. First and second order formulas describing the relation between the local conservative error and the discontinuity position are given in [2]. In [1], whether the discontinuity remains in the previous cell or moves to the left or right adjacent cell is determined by choosing the smallest $|q_j^{n+1}|$. In [2,3], it is determined by the discontinuity position computed by the Hugoniot condition. And in [4], it is determined just as in [2,3]; however, the discontinuity position is recomputed by the local conservation error on the new level later.

When two discontinuities move into the same cell and their positions have not crossed over each other, we let the two critical cells overlap each other in the same cell and choose a middle state between them. In Cases (a) and (b) in Figure 2, we choose $u_{j_1}^n$ as the middle state and in Case (c), we choose $(1/2)(u_{j_1}^n + u_{j_1+1}^n)$ as the middle state.

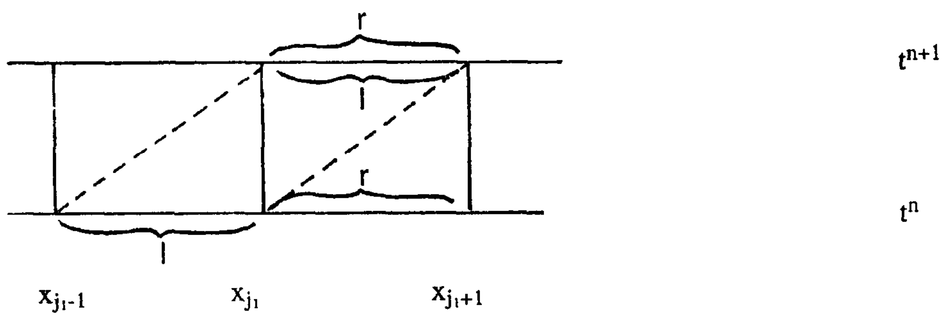
How do we define the artificial terms $q_j^{n+1}$ for overlapping intervals? Denote the local conservation errors coming from the left and right critical cells by $q_{j_1,l}^{n+1}$ and $q_{j_1,r}^{n+1}$. In Cases (a) and (b) in Figure 2, we merge the overlapping critical cells and define the local conservation error for the new cell as

$$q_{j_1}^{n+1} = q_{j_1,l}^{n+1} + q_{j_1,r}^{n+1}. \tag{2.10}$$

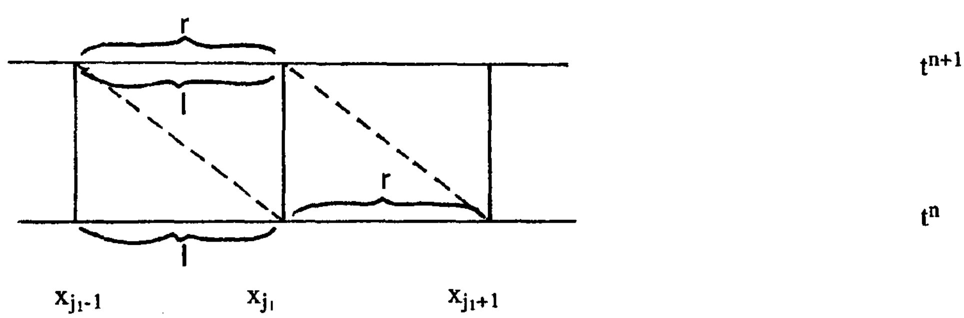
In Case (c) (Figure 2), algorithmically we hold one interval fixed and move only the other one.

Then, it gets treated as in Cases (a) and (b) above. In this case, the overall scheme still can be written in the form (1.7). This treatment can be extended to the case involving several critical cells overlapping each other.

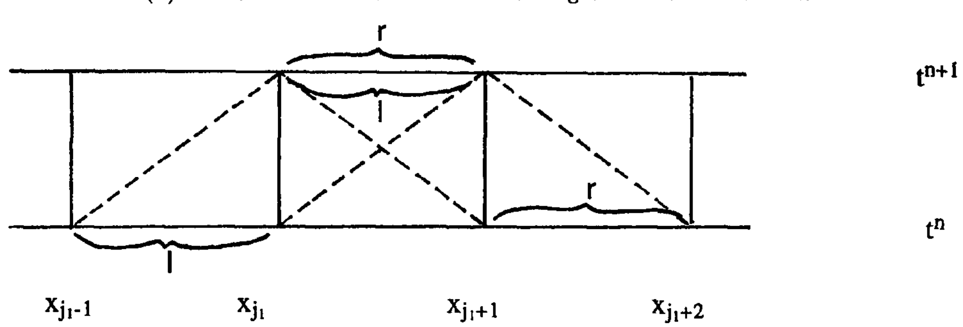
The technique is called $n^{\text{th}}$ order if the extrapolation used to calculate $u_{j_1}^{n,-}$ and $u_{j_1}^{n,+}$ is $(n-1)^{\text{th}}$ order.



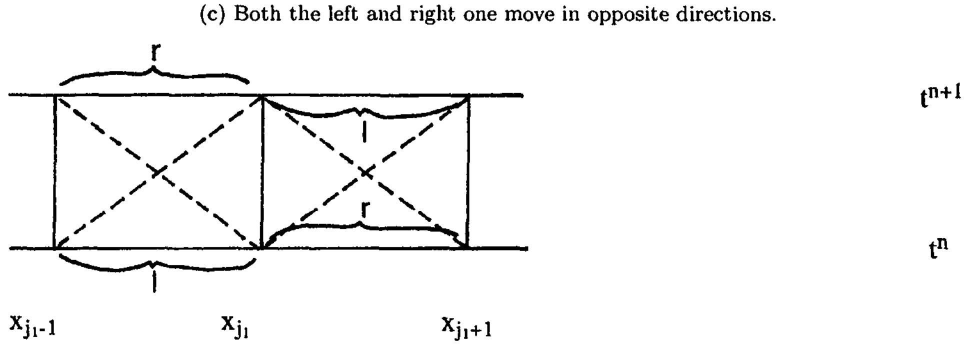

(a) The left critical cell moves to the right while the right one does not move.




(b) The left one does not move while the right one moves to the left.




(c) Both the left and right one move in opposite directions.




(d) Both the left and right one move in opposite directions.

Figure 2. The brackets with $l$ and $r$ represent the left and right critical cells. In (a), (b) and (c), the two critical cells are in the same cell on the $(n+1)^{\text{th}}$ level, but in (d), they cross over each other.

# 3. THE TVD PROPERTY OF THE FIRST ORDER FRONT TRACKING TECHNIQUE

For an overall scheme consisting of an underlying TVD scheme, together with the conservative front tracking technique, we define a new type of total variation.

DEFINITION. *For a numerical solution $u_j^n$ if $[x_{j_1}, x_{j_1+1}]$ is not a critical cell or only a single critical cell, we define*

$$\left|\tilde{\Delta}_{j+1/2}^n u\right| = \left|\Delta_{j+1/2}^n u\right|, \tag{3.1}$$

*and if there are several critical cells overlapping in $[x_{j_1}, x_{j_1+1}]$, we define*

$$\left|\tilde{\Delta}_{j+1/2}^n u\right| = \left|u_1^* - u_j^n\right| + \sum_{k=1}^{k_0-1} \left|u_{k+1}^* - u_k^*\right| + \left|u_{j+1}^n - u_{k_0}^*\right|, \tag{3.2}$$

*where $u_K^*$ is the $k^{\text{th}}$ middle state (see end of Section 2) and $k_0$ is the number of the middle states. Then, we define the total variation $\widetilde{TV}$ of $u^n$ as*

$$\widetilde{TV}(u^n) = \sum_{j=-\infty}^{\infty} \left|\tilde{\Delta}_{j+1/2}^n u\right|. \tag{3.3}$$

Obviously,

$$\mathrm{TV}(u^n) \leq \mathrm{TV}(u^n). \tag{3.4}$$

In this section, we will prove the following theorem.

THEOREM. *If the underlying scheme is TVD and the conservative front tracking technique is first order, then for all $u^n$ with bounded total variation $\widetilde{TV}$,*

$$\widetilde{TV}(u^{n+1}) \leq \widetilde{TV}(u^n). \tag{3.5}$$

This theorem shows that the first order technique retains the TVD property for a TVD underlying scheme.

The proof will be completed in two steps. First, we consider a single critical cell case; second, we consider the case of overlapping and emergence of two critical cells.

PROOF.

### Step 1

Suppose there is a single critical cell in $[x_{j_1}, x_{j_1+1}]$ on the $n^{\text{th}}$ level. We define the following two initial values on the $n^{\text{th}}$ level:

$$u_j^{n,-} = \begin{cases} u_j^n, & j \leq j_1, \\ u_{j_1}^n, & j > j_1, \end{cases} \quad \text{and} \quad u_j^{n,+} = \begin{cases} u_{j_1+1}^n, & j \leq j_1, \\ u_j^n, & j > j_1, \end{cases} \tag{3.6}$$

and denote the corresponding numerical solution computed by the underlying scheme on the $(n+1)^{\text{th}}$ level by $u_j^{n+1,-}$ and $u_j^{n+1,+}$, respectively. Since the underlying scheme is TVD, applying the underlying scheme to $u^-$ and $u^+$, respectively, we have

$$\sum_{j=-\infty}^{\infty} \left|u_{j+1}^{n+1,-} - u_j^{n+1,-}\right| \leq \sum_{j=-\infty}^{\infty} \left|u_{j+1}^{n,-} - u_j^{n,-}\right|, \tag{3.7}$$

and

$$\sum_{j=-\infty}^{\infty} \left|u_{j+1}^{n+1,+} - u_j^{n+1,+}\right| \leq \sum_{j=-\infty}^{\infty} \left|u_{j+1}^{n,+} - u_j^{n,+}\right|. \tag{3.8}$$

Since the extrapolation is $0^{\text{th}}$ order,

$$\sum_{j=-\infty}^{\infty} \left|u_{j+1}^{n,-} - u_j^{n,-}\right| = \sum_{j=-\infty}^{j_1-1} \left|u_{j+1}^{n} - u_j^{n}\right|, \tag{3.9}$$

and

$$\sum_{j=-\infty}^{\infty} \left|u_{j+1}^{n,+} - u_j^{n,+}\right| = \sum_{j=j_1+1}^{\infty} \left|u_{j+1}^{n} - u_j^{n}\right|. \tag{3.10}$$

From (1.4), we also have (the stencil has finite width from $-k$ to $k$)

$$\sum_{j=-\infty}^{\infty} \left|u_{j+1}^{n+1,-} - u_j^{n+1,-}\right| = \sum_{j=-\infty}^{j_1+k-1} \left|u_{j+1}^{n+1,-} - u_j^{n+1,-}\right|, \tag{3.11}$$

with $u_{j+1}^{n+1,+} = u_{j_1}^n$, and

$$\sum_{j=-\infty}^{\infty} \left|u_{j+1}^{n+1,+} - u_j^{n+1,+}\right| = \sum_{j=j_1-k+1}^{\infty} \left|u_{j+1}^{n+1,+} - u_j^{n+1,+}\right|, \tag{3.12}$$

with $u_{j_1-k+1}^{n+1,+} = u_{j_1+1}^n$. For simplicity, we will denote the sum of terms with " + " and " − " on the $(n+1)^{\text{th}}$ level by $\sum^{n+1,+}$ and $\sum^{n+1,-}$, respectively. For any $-k \le i \le k$, we have

$$\sum_{j=-\infty}^{j_1+k-1}{}^{n+1,-} = \sum_{j=-\infty}^{j_1-i-1}{}^{n+1,-} + \sum_{j=j_1-i}^{j_1+k-1}{}^{n+1,-}, \tag{3.13}$$

and

$$\sum_{j=j_1-k+1}^{\infty}{}^{n+1,+} = \sum_{j=j_1-k+1}^{j_1-i}{}^{n+1,+} + \sum_{j=j_1-i+1}^{\infty}{}^{n+1,+}. \tag{3.14}$$

Moreover, we have

$$\sum_{j=j_1-i}^{j_1+k-i} \left|u_{j+1}^{n+1,-} - u_j^{n+1,-}\right| + \left|u_{j_1+1}^{n} - u_{j_1}^{n}\right| + \sum_{j=j_1-k+i}^{j_1-i} \left|u_{j+1}^{n+1,+} - u_j^{n+1,+}\right| \ge \left|u_{j_1-i+1}^{n+1,+} - u_{j_1-i}^{n+1,-}\right|. \tag{3.15}$$

In the derivation of (3.15), we use $u_{j_1-i+1}^{n+1,+} = u_{j_1}^n$ and $u_{j_1-k+1}^{n+1,+} = u_{j_1+1}^n$.

When the discontinuity on the $(n+1)^{\text{th}}$ level remains in the same cell, we choose $i = 0$. At this moment,

$$u_j^{n+1,-} = u_j^{n+1}, \qquad \text{for } j \le j_1, \tag{3.16}$$

and

$$u_j^{n+1,+} = u_j^{n+1}, \qquad \text{for } j \ge j_1 + 1. \tag{3.17}$$

Then, from (3.9), (3.10), (3.15), (3.16) and (3.17), we obtain

$$\sum_{j=-\infty}^{\infty} \left|u_{j+1}^{n+1} - u_j^{n+1}\right| \le \sum_{j=-\infty}^{\infty} \left|u_{j+1}^{n} - u_j^{n}\right|. \tag{3.18}$$

When the discontinuity moves to the left or right adjacent cell, we choose $i = -1$ or $1$, respectively. Then, by the same argument, we obtain the conclusion for those cases.

**Step 2**

When two critical cells overlap each other in the same cell $[x_{j_1}, x_{j_1+1}]$ with a middle state $u^*$ in between, we define two initial values on the $n^{\text{th}}$ level,

$$u_j^{n,1} = \begin{cases} u_j^n, & j \le j_1, \\ u^*, & j \ge j_1+1, \end{cases} \quad \text{and} \quad u_j^{n,r} = \begin{cases} u^*, & j \le j_1, \\ u_j^n, & j > j_1+1, \end{cases} \tag{3.19}$$

each of which has a single critical cell in $[x_{j_1}, x_{j_1+1}]$. We denote the corresponding numerical solution on the $(n+1)^{\text{th}}$ level by $u_j^{n+1,1}$ and $u_j^{n+1,r}$, respectively.

If the two critical cells still overlap each other in the same cell $[x_{j_2}, x_{j_2+1}]$ on the $(n+1)^{\text{th}}$ level, ($j_2 = j_1 - 1, j_1$ or $j_1 + 1$), then from the discussion for a single critical cell, we have

$$\begin{aligned} \sum_{j=-\infty}^{j_2-1} \left|u_{j+1}^{n+1} - u_j^{n+1}\right| + \left|u^* - u_{j_2}^{n+1}\right| &= \widetilde{\mathrm{TV}}\left(u_j^{n+1,1}\right) \\ &\le \widetilde{\mathrm{TV}}\left(u_j^{n,1}\right) = \sum_{j=-\infty}^{j_1-1} \left|u_{j+1}^n - u_j^n\right| + \left|u^* - u_{j_1}^n\right| \end{aligned} \tag{3.20}$$

(here $u^* = u_{j_2+1}^{n+1}$, since $u_{j_2+k}^{n+1} = u^* \ \forall k \ge 1$), and

$$\begin{aligned} \sum_{j=j_2+1}^{\infty} \left|u_{j+1}^{n+1} - u_j^{n+1}\right| + \left|u_{j_1+1}^n - u^*\right| &= \widetilde{\mathrm{TV}}\left(u_j^{n+1,r}\right) \\ &\le \widetilde{\mathrm{TV}}\left(u_j^{n,r}\right) = \sum_{j=j_1+1}^{\infty} \left|u_{j+1}^n - u_j^n\right| + \left|u_{j_1}^n - u^*\right|. \end{aligned} \tag{3.21}$$

By combining (3.20) and (3.21), the conclusion follows for the case of two overlapping critical cells.

The extension of the discussion to the case involving finitely many overlapping critical cells (finitely many jumps) is straight forward.

When two separated critical cells overlap each other, the conclusion for Cases (a) and (b) in Figure 2 follows from the same discussion as for the case of two overlapping critical cells. When Case (c) in Figure 2 happens, defining two initial values

$$u_j^{n,1} = \begin{cases} u_j^n, & j \le j_1, \\ u_{j_1}^n, & j \ge j_1, \end{cases} \quad \text{and} \quad u_j^{n,r} = \begin{cases} u_{j_1+1}^n, & j \le j_1+1, \\ u_j^n, & j > j_1+1, \end{cases} \tag{3.22}$$

noticing that

$$u^* = \frac{1}{2}\left(u_{j_1}^n + u_{j_1+1}^n\right), \tag{3.23}$$

and therefore,

$$\left|u_{j_1}^n - u_{j_1-1}^{n+1}\right| + \left|u_{j_1+1}^n - u_{j_1}^n\right| + \left|u_{j_1+2}^{n+1} - u_{j_1+1}^n\right| \ge \left|u^* - u_{j_1-1}^{n+1}\right| + \left|u_{j_1+2}^{n+1} - u^*\right|. \tag{3.24}$$

We can obtain the conclusion by following the same discussion.

In the case of convex flux, where two shocks become one, the conclusion for the mergence of critical cells is quite straightforward, since

$$\left|u_{j_1+1}^n - u_{j_1}^n\right| \le \left|u_1^* - u_{j_1}^n\right| + \sum_{K=1}^{K_0-1} \left|u_{K+1}^* - u_K^*\right| + \left|u_{j_1+1}^n - u_{K_0}^*\right|, \tag{3.25}$$

where $u_K^*$ is the $K^{\text{th}}$ middle state and $K_0$ is the number of the middle states.

This ends the proof.

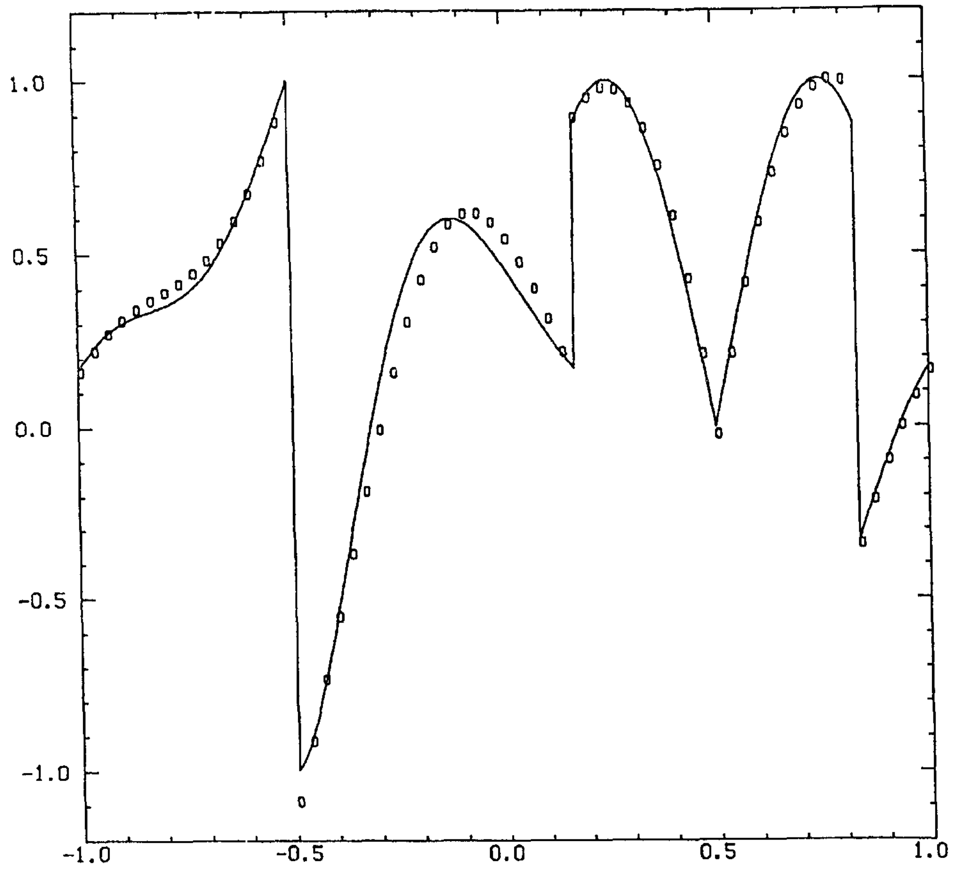



Figure 3. The solution to $u_t + u_x = 0$, with one period of the periodic initial data given by the solid line. The circles represent the numerical solution after some time, using the front tracking method.

## 4. TWO NUMERICAL EXAMPLES

EXAMPLE 1.

$$u_t + u_x = 0,$$
$$u(x+0.5,0) = \begin{cases} -x\sin\left(\frac{3}{2}\pi x^2\right), & -1 < x < -\frac{1}{3}, \\ |\sin(2\pi x)|, & |x| < \frac{1}{3}, \\ 2x - 1 - \frac{1}{6}\sin(3\pi x), & \frac{1}{3} < x < 1, \end{cases}$$

and

$$u(x+2,0) = u(x,0).$$

The solution to the problem is highly oscillatory, which contains three contact discontinuities and a jump in the first derivative. The underlying scheme is second order TVD with Runge-Kutta discretization for $t$-derivative. The second order technique is applied to all these discontinuities. Numerical solution at $t = 8$ with $h = 1/30$ is presented in Figure 3, where the solid line is the exact solution.

EXAMPLE 2.

$$u_t + u_x + u_y = 0, \qquad |x| \le 1, \quad |y| \le 1,$$
$$u(x,y,0) = \begin{cases} 0.75\cos((x+y)\pi)\cos((x-y)\pi), & x^2 + y^2 \le 0.6, \\ 0, & \text{otherwise}, \end{cases}$$

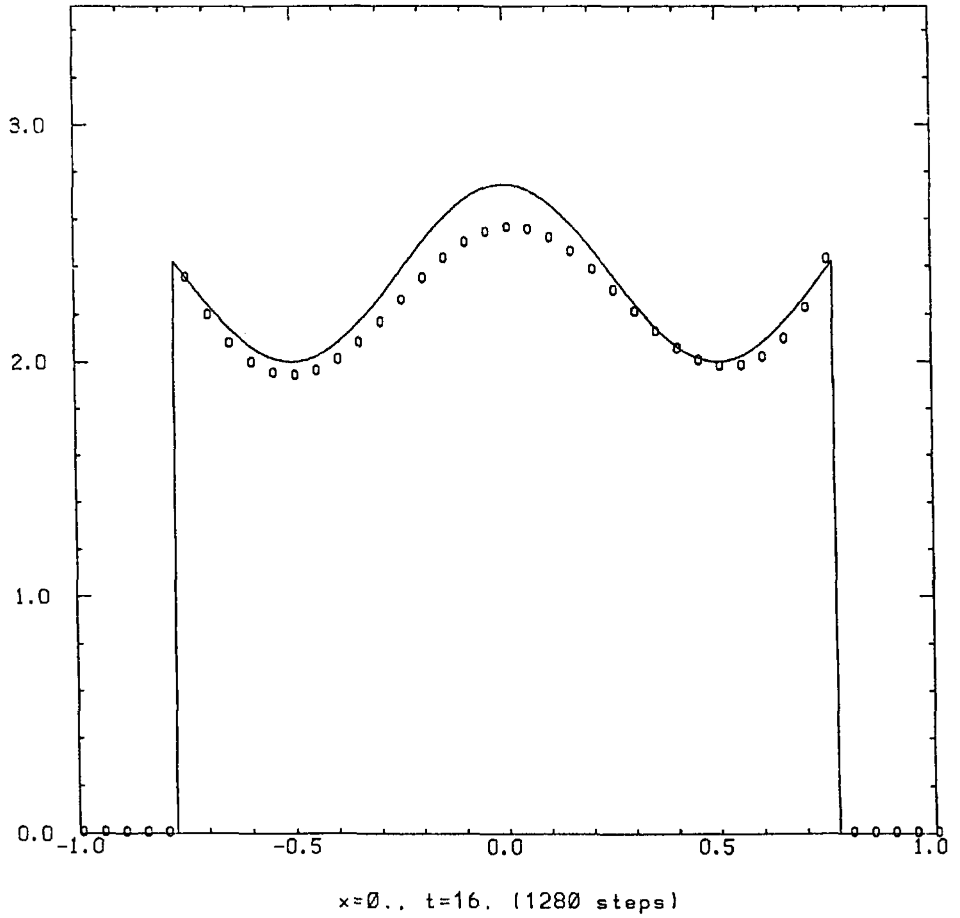



Figure 4. The solution to $u_t + u_x + u_y = 0$ on a square $|x| < 1, |y| < 1$, with periodic boundary conditions and discontinuous initial data. Here, a cross-section of the solution is presented, solid line is the exact solution, circles are computed using front tracking.

with periodic boundary condition. The exact solution has a circle discontinuity front. The underlying scheme is the same as in the previous example. The second order technique is applied to the front and $h = 1/20$. The cross-section of the numerical solution at $t = 16$, $y = 0$ is presented in Figure 4, where the solid line is the exact solution.

In the figures presented in the above two examples, we see that both the the numerical solutions are quite stable near discontinuities even after a long time computation, with no oscillation. This strongly supports our conclusion that even a higher order front tracking technique will not seriously affect the stability of the numerical solution.

## 5. CONCLUSION

We studied the effect of the conservative front tracking technique on the total variation of the numerical solution. We proved that when the underlying scheme is TVD and the technique is first order, the overall scheme is still TVD. Numerical experiments presented indicate this is also true for higher order front tracking techniques.

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