Solutions to Problem Set 8

Work and energy

PHYS-101(en)

1. Throwing a ball in the wind

We start by choosing a coordinate system with the $\hat{x}$ pointing to the east and the $\hat{y}$ direction pointing upwards. The ball is thrown straight up, so normally its motion would be one-dimensional, along the y-axis. However, in this case the force of the wind F is constant and in the $\hat{x}$ direction, so the motion is not one-dimensional. The work done by the wind ΔW as the ball undergoes a small displacement $\Delta\vec{l}$ is given by

$$\Delta W = \vec{F} \cdot \Delta\vec{l},$$

where $\Delta\vec{l}$ is the displacement vector. Thus, $\vec{F} = F\hat{x}$ and $\Delta\vec{l} = \Delta x\hat{x} + \Delta y\hat{y}$. From the definition of the dot product, only the x component of the displacement contributes, so integrating over the trajectory gives

$$W = \int_0^L \vec{F} \cdot d\vec{l} = \int_0^L F\hat{x} \cdot d\vec{l} = F\int_0^D dx = FD, \tag{1}$$

where L is the total distance travel by the ball.

2. Work-kinetic energy theorem and Newton's 2^{nd} law: Tetherball

In this scenario, we are being asked to check the work-kinetic energy theorem. To calculate the net work done on the body, we first need to identify the forces applied. We start by defining a cylindrical coordinate system with the origin located at the fixed ring in the center of the circular motion. We are told that the string is pulled downward with constant velocity, so the body moves inward with $\dot{\rho} = -V = \text{constant}$ and $\ddot{\rho} = 0$.

In order to calculate the work done by the string, we need to know the tension force and the trajectory of the ball. Thus, we want to apply Newton's second law to the situation, but first we must recall the formula for acceleration in cylindrical coordinates

$$\vec{a} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\rho} + \left(2\dot{\rho}\dot{\phi} + \rho\ddot{\phi}\right)\hat{\phi} + \ddot{z}\hat{z}. \tag{1}$$

We identify that $\dot{\phi} = \omega$, $\ddot{z} = 0$, so Newton's 2nd law can be written as

$$-T = -m\rho\omega^2 \tag{2}$$

$$0 = m(2\dot{\rho}\omega + \rho\dot{\omega}) \tag{3}$$

in the radial and tangential directions respectively. The first equation gives us the tension as a function of radius, but we do not know how ω changes with radius. This can be found from the second equation, which can be rewritten as

$$\frac{1}{\omega}\frac{d\omega}{dt} = -\frac{2}{\rho}\frac{d\rho}{dt} \quad \Rightarrow \quad \frac{1}{\omega}d\omega = -\frac{2}{\rho}d\rho. \tag{4}$$

Integrating this gives

$$\int \frac{1}{\omega}d\omega = -2\int \frac{1}{\rho}d\rho \quad \Rightarrow \quad \ln(\omega) = -2\ln(\rho) + C \quad \Rightarrow \quad \ln(\omega) = \ln\left(\rho^{-2}\right) + C \quad \Rightarrow \quad \omega(\rho) = \rho^{-2}\exp(C), \tag{5}$$

where C is an integration constant and we have used identities that $A\ln(B) = \ln(B^A)$ and $\exp(A+B) = \exp(A)\exp(B)$. Substituting the initial condition that $\omega(\rho_0) = \omega_0$ allows us to calculate that the integration constant is

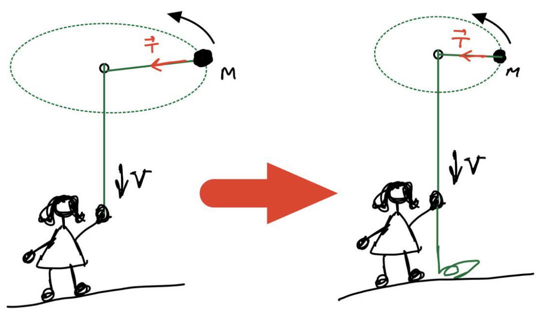
$$\omega(\rho_0) = \omega_0 = \rho_0^{-2}\exp(C) \quad \Rightarrow \quad \exp(C) = \omega_0\rho_0^2. \tag{6}$$

Substituting this into equation (5) gives

$$\omega(\rho) = \frac{\rho_0^2}{\rho^2}\omega_0. \tag{7}$$

Together with equation 2, this allows us to calculate the tension

$$T = m\frac{\rho_0^4}{\rho^3}\omega_0^2. \tag{8}$$



We can now consider the work done by the tension on the ball, as it is the only force in the problem. From the above figure, we see that the tension force is always pointed inwards. Thus, the work done by the string in moving the ball from a radius ρ_0 to a radius ρ_f is given by

$$W = \int \vec{T}\cdot d\vec{l} = -\int T\hat{\rho}\cdot d\vec{l} = -\int_{\rho_0}^{\rho_f} T d\rho. \tag{9}$$

Substituting equation (8) gives

$$W = -\int_{\rho_0}^{\rho_f} m\frac{\rho_0^4}{\rho^3}\omega_0^2 d\rho = -m\rho_0^4\omega_0^2\int_{\rho_0}^{\rho_f}\rho^{-3}d\rho = -m\rho_0^4\omega_0^2\left(-\frac{1}{2}\rho^{-2}\Big|_{\rho_0}^{\rho_f}\right) = \frac{m\rho_0^4\omega_0^2}{2}\left(\frac{1}{\rho_f^2} - \frac{1}{\rho_0^2}\right), \tag{10}$$

which is our final solution for the work.

To calculate the change in kinetic energy, we can use the formula

$$\Delta K = K_f - K_0 = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2, \tag{11}$$

where the subscripts f and 0 indicate the final and initial values respectively. To calculate the velocity, we can use its formula in cylindrical coordinates

$$\vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z} \tag{12}$$

and note that $\dot{\rho} = 0$, $\dot{z} = 0$ in the initial and final states. Thus, the initial and final speeds are

$$v_0 = \rho_0\omega_0 \tag{13}$$
$$v_f = \rho_f\omega_f \tag{14}$$

respectively. Plugging these into equation 11 reveals that

$$\Delta K = \frac{1}{2}m\rho_f^2\omega_f^2 - \frac{1}{2}m\rho_0^2\omega_0^2 = \frac{m\rho_0^4\omega_0^2}{2}\left(\frac{\rho_f^2\omega_f^2}{\rho_0^4\omega_0^2} - \frac{1}{\rho_0^2}\right), \tag{15}$$

but we still must determine ω_f. This can be done by evaluating equation 7 at $\rho = \rho_f$ to find

$$\omega(\rho_f) = \omega_f = \frac{\rho_0^2}{\rho_f^2}\omega_0. \tag{16}$$

Substituting this into equation 15 gives

$$\Delta K = \frac{m\rho_0^4\omega_0^2}{2}\left(\frac{\rho_f^2}{\rho_0^4\omega_0^2}\left(\frac{\rho_0^2}{\rho_f^2}\omega_0\right)^2 - \frac{1}{\rho_0^2}\right) = \frac{m\rho_0^4\omega_0^2}{2}\left(\frac{1}{\rho_f^2} - \frac{1}{\rho_0^2}\right), \tag{17}$$

which is equal to the work (i.e. equation 10) as expected. Thus, the work-kinetic energy theorem holds.

3. Fragmenting projectile (former exam problem)

1. **At what time does the explosion occur?**

   We start by defining a Cartesian coordinate system with the origin at the cannon, a vertical axis z pointing upwards, and a horizontal axis x such that it contains the horizontal component of $\vec{v}_0$. The projectile undergoes motion under a constant acceleration of $\vec{g} = -g\hat{z}$, so its vertical velocity and position are given by

   $$v_z(t) = -gt + v_0 \sin\alpha \tag{1}$$
   $$z(t) = -\frac{g}{2}t^2 + v_0 t \sin\alpha \tag{2}$$

   respectively, where we have used the initial conditions that $v_z(0) = v_0 \sin\alpha$ and $z(0) = 0$. The peak of the trajectory is reached when $v_z(t_e) = 0$, so we can use equation (1) to find the time of the explosion t_e to be

   $$v_z(t_e) = 0 = -gt_e + v_0 \sin\alpha \quad \Rightarrow \quad t_e = \frac{v_0 \sin\alpha}{g}. \tag{3}$$

   At what time do the pieces hit the ground?

   From equations (2) and (3), we see that the vertical position of the projectile at the time of the explosion is

   $$z(t_e) = z_e = -\frac{g}{2}\left(\frac{v_0 \sin\alpha}{g}\right)^2 + v_0\left(\frac{v_0 \sin\alpha}{g}\right)\sin\alpha = \frac{(v_0 \sin\alpha)^2}{2g}. \tag{4}$$

   At this time, the projectile explodes into a thousand pieces, which each still undergo projectile motion. Thus, their height evolves according to

   $$z(t) = -\frac{g}{2}(t - t_e)^2 + z_e \tag{5}$$

   because their vertical velocity is $v_z(t_e) = 0$ and their position is $z(t_e) = z_e$ at time $t = t_e$. From this we can calculate the time t_g at which they hit the ground $z(t_g) = 0$ to be

   $$z(t_g) = 0 = -\frac{g}{2}(t_g - t_e)^2 + z_e \quad \Rightarrow \quad t_g = t_e + \sqrt{\frac{2z_e}{g}}. \tag{6}$$

Substituting equations (3) and (4) gives

$$t_g = \frac{v_0 \sin\alpha}{g} + \sqrt{\frac{2}{g}\frac{(v_0 \sin\alpha)^2}{2g}} = 2\frac{v_0 \sin\alpha}{g}. \tag{7}$$

This is a sensible result as the fragments all have zero vertical velocity just before and just after the explosion. Thus, it will take them the same amount of time to fall to the ground as it did for them to ascend. Thus, we'd expect $t_g = 2t_e$, which we see is the case by comparing equations (3) and (7).

Why do they all hit the ground at the same time?

Just before the explosion, the projectile has a purely horizontal velocity (since it is at the peak of its trajectory). As a result of the explosion, the fragments don't gain any vertical velocity because their velocity relative to the original projectile is purely horizontal. Thus, their velocities relative to the ground are also horizontal just after time t_e. This means that the pieces all have the same height and vertical velocity at t_e. Because gravity accelerates objects the same regardless of mass, they all hit the ground at the same time.

2. **The pieces land on the ground in a circle. Why?**

Given that we know the time at which the fragments hit the ground, in this part of the problem we will only consider the horizontal motion of the objects and ignore the vertical motion. It is not needed.

First, we view the dynamics from the frame of reference moving with the horizontal location of the center of mass of the system (i.e. the projectile). In this frame, we will use cylindrical coordinates with the origin defined to be the ground directly below the location of the center of mass. Immediately after the explosion, all the pieces have the same speed u with respect to the origin of our moving frame and are moving purely horizontally. This means their velocity is entirely in the outwards radial direction. In this direction, they experience no acceleration, so their horizontal velocity and position are given by

$$v_\rho(t) = u \tag{8}$$

$$\rho(t) = u(t - t_e) \tag{9}$$

for any time $t > t_e$. Here we used the fact that at $t = t_e$ the horizontal position of every piece is at the origin and the initial velocity is u. Thus, after a time interval $t - t_e$ the pieces lie on a circle with radius $u(t - t_e)$ centered around the origin of the moving center of mass frame. To see what shape they make when they land on the ground, we need to convert to the stationary frame of the ground. To do so, we use equations (8) and (9) and the formulas to convert between references frames to write

$$\vec{v}_i(t) = \vec{V}_{CM}(t) + u\hat{r}_i \tag{10}$$

$$\vec{r}_i(t) = \vec{R}_{CM}(t) + u(t - t_e)\hat{r}_i, \tag{11}$$

where $\vec{v}_i(t)$ and $\vec{r}_i(t)$ are the horizontal velocity and position of the i^{th} fragment in the ground reference frame, $\vec{V}_{CM}(t)$ and $\vec{R}_{CM}(t)$ are the horizontal velocity and position of the origin of the center of mass frame, and $\hat{r}_i$ is the radial unit vector pointing at the i^{th} fragment from the origin of the center of mass frame. Due to the fact that there is no net external horizontal force on the system, the horizontal momentum of the projectile does not change. Thus, $\vec{V}_{CM}(t)$ is a constant and equal to its initial value when the projectile is launched $\vec{V}_{CM}(t) = v_0 \cos\alpha\hat{x}$. This implies that the center of mass position is $\vec{R}_{CM}(t) = v_0 t \cos\alpha\hat{x}$ because the center of mass position at $t = 0$ is at the origin in the ground reference frame (i.e. the integration constant that arises in going from velocity to position is zero). Plugging this result into equation (11), we see that

$$\vec{r}_i(t) = v_0 t \cos\alpha\hat{x} + u(t - t_e)\hat{r}_i \tag{12}$$

at any time $t > t_e$. The last term tell us that, in the center of mass reference frame, all the pieces are flying apart outwards from each other with the same speed u, but in different directions $\hat{r}_i$. Thus, they form a circle of radius

$$R(t) = u(t - t_e). \tag{13}$$

The first term on the right side of equation (12) tells us how this motion changes when we convert to the reference frame of the ground. Since the subscript i doesn't appear, we know that this term is the same for all fragments. Specifically, all fragments are moving together in the $\hat{x}$ direction with time. This corresponds to a displacement in the center of the circle by

$$\vec{d}(t) = v_0 t \cos\alpha \hat{x}. \tag{14}$$

However, since all pieces are moving together, it doesn't change the shape from being circular. In conclusion, in the reference frame of the ground the pieces form a circle that shifts in the $\hat{x}$ direction with time.

What is the distance between the center of the circle and the cannon?

This can be found by taking the magnitude of equation (14) at the time when the fragments hit the ground $t = t_g$ using equation (7), giving

$$d(t_g) = 2\frac{v_0^2}{g}\sin\alpha\cos\alpha. \tag{15}$$

This is the same location that the projectile would have hit had it not exploded.

Calculate the speed u and then the radius of the circle.

To calculate u it is most convienant to return the reference frame moving with the horizontal location of the center of mass of the system. Just before the explosion the total kinetic energy of the system is zero as there is no vertical motion and we are in a reference frame moving with the horizontal motion of the projectile. From the problem statement, we know that just after the explosion the kinetic energy of the system increases by W. This must be contained in the speed u possessed by each of the 1000 fragments. Using this we can calculate u to be

$$W = \sum_{i=1}^{1000} K_i = \sum_{i=1}^{1000} \frac{1}{2} m_i u^2 = \frac{1}{2}u^2 \sum_{i=1}^{1000} m_i = \frac{1}{2}u^2 M \quad \Rightarrow \quad u = \sqrt{\frac{2W}{M}}. \tag{16}$$

To find the radius of the circle we evaluate equation (13) at the time when the fragments hit the ground $t = t_g$ and substitute equations (3), (7), and (16). This gives

$$R(t_g) = \sqrt{\frac{2W}{M}}\frac{v_0}{g}\sin\alpha. \tag{17}$$

3. If a fragment hits the cannon, then we know the radius of the circle must be equal to the distance between the cannon and the center of the circle at $t = t_g$,

$$R(t_g) = d(t_g). \tag{18}$$

Substituting equations (15) and (17) gives

$$\sqrt{\frac{2W}{M}}\frac{v_0}{g}\sin\alpha = 2\frac{v_0^2}{g}\sin\alpha\cos\alpha \quad \Rightarrow \quad W = 2Mv_0^2\cos^2\alpha \quad \Rightarrow \quad W = 4K_0\cos^2\alpha \tag{19}$$

as the initial kinetic energy of the projectile is $K_0 = Mv_0^2/2$.

4. Travel on surface/loop

This problem may seem complicated at first (all those parameters!), but the work-kinetic energy theorem makes it tractable, even simple. The work-kinetic energy theorem tells us that the work done by the net force on the object is equal to the change in its kinetic energy. So let's take the initial state to be when the spring is compressed by a distance x_0 and the final state to be when the object is at its maximum height. In both of these states, the velocity is equal to zero, so the kinetic energy is zero as well. Thus, as a consequence of the work-kinetic energy theorem, we know that the total work done on the object during its path must be equal to zero.

We choose the positive $\hat{x}$ direction to point to the left since it is the direction of motion. The motion has four stages and we need to calculate the work on the object during each:

1. Using the form of the spring force, we can calculate the work done by the spring to be

$$W_{spring} = \int_{-x_0}^{0} F_{spring} dx = \int_{-x_0}^{0} (-kx) dx = \frac{1}{2} k x_0^2. \tag{1}$$

2. Using the form of the friction force, we can calculate the work done by the horizontal track to be

$$W_{fric} = \int_0^d F_{fric} dx = -\int_0^d (mg\mu(x)) dx = -mg \int_0^d \mu_0 + \mu_1 \left(\frac{x}{d}\right) dx = -mgd \left(\mu_0 + \frac{\mu_1}{2}\right). \tag{2}$$

3. The work done by the normal force of the surface of the loop is zero because the normal force is always perpendicular to the surface and the object is always moving along the surface. Thus, the normal force is perpendicular to the trajectory, so the dot product of the force and the displacement is zero.

4. Using the form of the gravitational force, we can calculate the work done by gravity as the object moves to a height h to be

$$W_{grav} = -\int_0^h F_{grav} dy = -\int_0^h (mg) dy = -mgh. \tag{3}$$

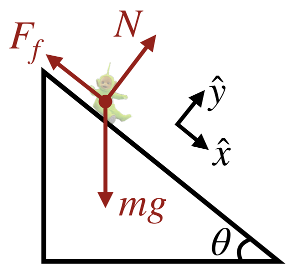
The total work done on the object is the sum of all these contributions, which must be equal to zero by the work-kinetic energy theorem. Thus, we find

$$W = W_{spring} + W_{fric} + W_{grav} = \frac{1}{2} k x_0^2 - mgd \left(\mu_0 + \frac{\mu_1}{2}\right) - mgh = 0. \tag{4}$$

Solving this for h gives the final answer of

$$h = \frac{k x_0^2}{2mg} - d \left(\mu_0 + \frac{\mu_1}{2}\right). \tag{5}$$

5. Homework: Slide



1. First, we take a coordinate system with the $\hat{x}$ and $\hat{y}$ unit vectors defined as shown in the figure above. In the $\hat{x}$ and $\hat{y}$ directions, Newton's second law is

$$mg\sin\theta - F_f = ma_x \tag{1}$$
$$N - mg\cos\theta = ma_y = 0 \quad \Rightarrow \quad N = mg\cos\theta \tag{2}$$

respectively, where F_f is the magnitude of the kinetic friction force, N is the magnitude of the normal force, and we know that there is no acceleration in the $\hat{y}$ direction. Using the form of the friction force and equation (2), we find

$$F_f = \mu_k N = \mu_k mg\cos\theta. \tag{3}$$

This allows us to calculate the total work done by friction to be

$$W_f = \int_0^d \vec{F}_f \cdot \hat{x}dx = -\int_0^d F_f dx = -F_f d = -\mu_k mgd\cos\theta. \tag{4}$$

Plugging in the numerical values, we find

$$W_f = -(0.2)(20\text{ kg})(9.8\text{ m/s}^2)(5\text{ m})(\cos(20^\circ)) = -180\text{ J}. \tag{5}$$

2. In this part, we will calculate the total work performed on the child and then use the work-kinetic energy theorem to determine the kinetic energy, which allows us to find the final speed. We already know the work done by friction from part 1. The only other force in the problem is gravity, for which we can calculate the work as

$$W_g = \int_{h_0}^0 \vec{F}_g \cdot \hat{y}dy = \int_{h_0}^0 (-mg\hat{y}) \cdot \hat{y}dy = \int_0^{h_0} mg\hat{y} \cdot \hat{y}dy = mg(h_0 - 0) = mg(d\sin\theta) = mgd\sin\theta, \tag{6}$$

where we have adopted a new coordinate system with $\hat{y}$ pointing straight upwards and note that the child is traveling *from* a height h_0 *to* a height 0. We also used trigonometry to calculate the height of the slide $h_0 = d\sin\theta$ in terms of known quantities. Thus, the total work performed on the child is

$$W = W_f + W_g = -\mu_k mgd\cos\theta + mgd\sin\theta. \tag{7}$$

By the work-kinetic energy theorem, this work must be equal to the change in kinetic energy according to

$$W = \Delta K = K_f - K_i. \tag{8}$$

Since the child starts at rest, the initial kinetic energy is

$$K_i = 0. \tag{9}$$

At the bottom, they are moving at some speed v_f (which we want to calculate), so

$$K_f = \frac{1}{2}mv_f^2. \tag{10}$$

Plugging equations (7), (9), and (10) into equation (8) gives

$$-\mu_k mgd\cos\theta + mgd\sin\theta = \frac{1}{2}mv_f^2 - 0. \tag{11}$$

We can solve this to find that the speed of the child at the bottom of the slide is

$$v_f = \sqrt{2gd\left(\sin\theta - \mu_k\cos\theta\right)}. \tag{12}$$

Plugging in the numbers gives

$$v_f = \sqrt{2(9.8\text{ m/s}^2)(5.0\text{ m})(\sin(20^\circ) - (0.2)\cos(20^\circ))} = 3.9\text{ m/s}. \tag{13}$$

3. To calculate the time it takes for the child to slide down the slide, we use Newton's second law in the $\hat{x}$ direction (in the coordinate system of part 1). Substituting equation (3) into equation (1) gives

$$a_x(t) = g\sin\theta - \mu_k g\cos\theta = g\left(\sin\theta - \mu_k\cos\theta\right). \tag{14}$$

Integrating this once in time yields the velocity

$$v_x(t) = gt\left(\sin\theta - \mu_k\cos\theta\right), \tag{15}$$

where the integration constant is zero because the child starts at rest. Since we know the final speed from part 2, we can use it to calculate the final time t_f as

$$v_x(t_f) = v_f = gt_f\left(\sin\theta - \mu_k\cos\theta\right) \quad\Rightarrow\quad t_f = \frac{v_f}{g\left(\sin\theta - \mu_k\cos\theta\right)}. \tag{16}$$

Substituting equation (12) gives

$$t_f = \frac{\sqrt{2gd\left(\sin\theta - \mu_k\cos\theta\right)}}{g\left(\sin\theta - \mu_k\cos\theta\right)} = \sqrt{\frac{2d}{g\left(\sin\theta - \mu_k\cos\theta\right)}}. \tag{17}$$

Plugging in the numbers, we find

$$t_f = \sqrt{\frac{2(5.0\text{ m})}{(9.8\text{ m/s}^2)(\sin(20^\circ) - (0.2)\cos(20^\circ))}} = 2.6\text{ s}. \tag{18}$$

4. Since we can assume that the children start and finish at rest, by the work-kinetic energy theorem we know that the total work must be zero. As a child ascends, it experiences two forces. The normal force from the ladder, which the child uses to push itself up, and the gravitational force, which is pulling downwards on the child. Thus, we have

$$W = W_g + W_N = \Delta K = 0, \tag{19}$$

where we note that W_N represents the work done by the child via the normal force. Defining a coordinate system with the origin at the ground and the $\hat{y}$ direction pointing upwards, we see that the work done by gravity as the child ascends is

$$W_g = \int_0^{h_0} \vec{F}_g \cdot \hat{y}dy = \int_0^{h_0} (-mg\hat{y}) \cdot \hat{y}dy = -mgh_0. \tag{20}$$

Substituting this into equation (19) allows us to calculate the work done by the child

$$W_N = mgh_0. \tag{21}$$

We see that this doesn't depend on time, only on the mass of the child. Since the two children have equal masses, they do the same amount of work.