Solutions to Problem Set 1

Motion in one dimension

PHYS-101(en)

1. Review: Units

Note that, in the solutions below, we give our answers with the same number of significant digits as was provided in the question. This isn't absolutely crucial, but in physics (and science in general) there is always uncertainty. It doesn't make sense to take quantities that are only roughly known and use them to calculate a quantity that appears to have incredible precision. Typically, the number of significant digits in the results should be similar to the number of significant digits in the input quantities.

a. We convert the units as follows

$$5.00\ \frac{\text{m}}{\text{hr}} = 5.00\ \frac{\text{m}}{\text{hr}} \times \frac{1\ \text{mile}}{1609.31\ \text{m}} \times \frac{24\ \text{hr}}{1\ \text{day}} \times \frac{14\ \text{day}}{1\ \text{fortnight}} = 1.04\ \frac{\text{miles}}{\text{fortnight}}.$$

b. We apply same reasoning as above to find

$$1\ \frac{\text{g}}{\text{cm}^3} = 1\ \frac{\text{g}}{\text{cm}^3} \times \frac{1\ \text{kg}}{1000\ \text{g}} \times \left(\frac{100\ \text{cm}}{1\ \text{m}}\right)^3 = 10^3\ \frac{\text{kg}}{\text{m}^3}$$

$$1\ \frac{\text{g}}{\text{cm}^3} = 1\ \frac{\text{g}}{\text{cm}^3} \times \left(\frac{10\ \text{cm}}{1\ \text{dm}}\right)^3 = 10^3\ \frac{\text{g}}{\text{dm}^3}$$

$$1\ \frac{\text{g}}{\text{cm}^3} = 1\ \frac{\text{g}}{\text{cm}^3} \times \frac{1\ \text{kg}}{1000\ \text{g}} \times \left(\frac{1\ \text{cm}}{10\ \text{mm}}\right)^3 = 10^{-6}\ \frac{\text{kg}}{\text{mm}^3}.$$

c. The distance light travels during a certain amount of time can be calculated from

$$L = vt,$$

where L is the total distance traveled, $v = 3.0 \times 10^8$ m/s is the speed, and $t = 365.24$ days is the time elapsed. However, first we need to convert days into seconds according to

$$t = 365.24\ \text{days} = 365.24\ \text{days} \times \frac{24\ \text{hr}}{1\ \text{day}} \times \frac{60\ \text{min}}{1\ \text{hr}} \times \frac{60\ \text{s}}{1\ \text{min}} = 31556736\ \text{s}.$$

Now we can use the equation for the distance to find

$$L = \left(3.0 \times 10^8\ \frac{\text{m}}{\text{s}}\right) \times 31556736\ \text{s} = 9.5 \times 10^{15}\ \text{m}.$$

Converting from meters to kilometers, we find

$$L = 9.5 \times 10^{15}\ \text{m} \times \frac{1\ \text{km}}{1000\ \text{m}} = 9.5 \times 10^{12}\ \text{km}.$$

2. Review: Uncertainty and significant figures

a. The number of significant figures is

1. 1 significant figure
2. 5 significant figures
3. It could be either 1, 2, or 3 significant figures, but you can't tell which
4. 2 significant figures
5. 1 significant figure because the number of significant figures of the result corresponds to the minimum number of significant digits among the input quantities
6. 2 significant figures

b. The result is the mean, defined as

$$\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i,$$

where x_i is each individual measurement and N is the total number of measurements. A common estimation of error is the standard deviation σ, defined as

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \overline{x})^2}{N-1}}.$$

Plugging in values, we find the mean to be $\overline{x} = 0.346$ s and the standard deviation to be $\sigma = 0.077$ s. Thus, the solution is

$$0.35 \text{ s} \pm 0.08 \text{ s}.$$

Note that an alternative definition of the standard deviation uses N in the denominator, instead of $N-1$ and another way to estimate the error is to use the standard error of the mean, $SE = \frac{\sigma}{\sqrt{N}}$. These are both acceptable and will give similar (but not exactly the same) values.

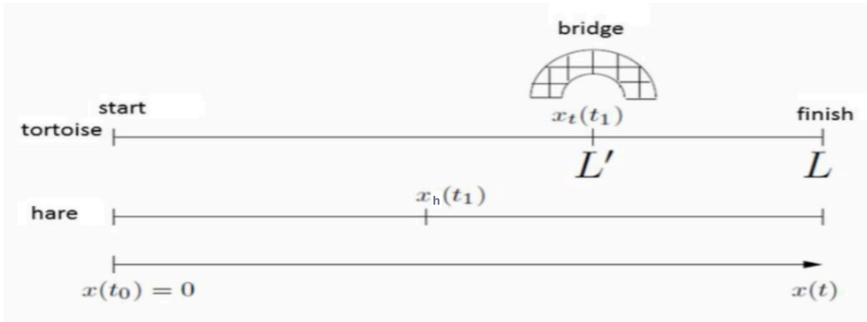
3. The tortoise and the hare

To help visualize the problem, we will start by drawing the situation, which we show in figure 1(a). Here x_t is the position of the tortoise, x_h is the position of the hare, and it is helpful to define the following times:

- $t_0 = 0$ (the time at which the race starts),
- t_1 (the time at which the tortoise reaches the bridge, which is when the hare starts to accelerate), and
- t_2 (the arrival time of the tortoise and hare).

To sketch the plot, we first consider the tortoise. It has a constant velocity throughout the entire race. A constant velocity corresponds to a position that is linear with time, so we can draw a straight line between the points $(t = t_0, x = 0)$ and $(t = t_2, x = L)$. The trajectory of the hare is a bit more complicated. The hare starts at a constant velocity that is slower than that of the tortoise and maintains the constant velocity until $t = t_1$. Thus, we can draw a straight line between the points $(t = t_0, x = 0)$ and $(t = t_1, x = L')$, where the slope of the line is less than that of the tortoise's. At this moment, the hare begins to accelerate with a constant positive acceleration. This means the slope of the line is constantly increasing. However, we know that the hare ties with the tortoise at the end of the race, so the hare's line must end at the point $(t = t_2, x = L)$. The final sketch is shown in figure 1(b), where the dashed line is trajectory of the tortoise and the solid line is trajectory of hare.

(a)



(b)

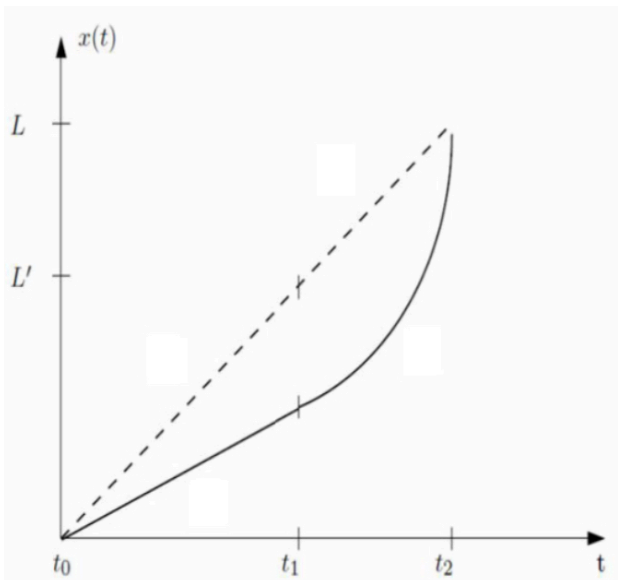


Figure 1: (a) A rough sketch of the problem as well as a plot of the positions of the tortoise (dashed) and the hare (solid) with time.

4. The jumping salmon

a. We start with the equation of motion

$$\ddot{x} = a,$$

where a is a constant. We integrate once in time to find the velocity

$$\dot{x} = at + C,$$

where C is an integration constant. To solve for C, we evaluate this equation at $t = 0$ and find

$$\dot{x} = 0 + C = C.$$

Thus, the constant C is equal to the velocity at $t = 0$, which we represent as $v_0 = C$. This shows that the velocity is given by

$$\dot{x} = v(t) = at + v_0. \tag{1}$$

We integrate a second time to find the position as a function of time

$$x(t) = \frac{1}{2}at^2 + v_0 t + C',$$

where C' is another integration constant. To solve for C', we evaluate this equation at $t = 0$ and find

$$x(t = 0) = 0 + 0 + C' = C'.$$

Thus, the constant C' is equal to the position at $t = 0$, which we represent as $x_0 = C'$. Finally, we can write the solution to the equation of motion $\ddot{x} = a$ as

$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0. \tag{2}$$

Here v_0 is the initial velocity at $t = 0$ and x_0 is the initial position at $t = 0$.

Note that a way to check that this solution is correct is by taking the second derivative with respect to time to ensure that $\ddot{x} = a$. This method produces

$$\ddot{x}(t) = \frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(\frac{1}{2}at^2 + v_0 t + x_0) = \frac{d}{dt}(at + v_0) = a.$$

b. The best way to understand the problem is to first make a drawing that displays all information given in the problem (e.g. figure 2). At $t = 0$, the salmon comes out of the lake with a vertical velocity v_0 upwards. Throughout the jump, the fish is subject to an acceleration $-g$ due to gravity. The salmon's motion as a function of time can be found from our solution to part (a) of this problem. Its position is given by equation (2) and its velocity is given by equation (1), where $a = -g$.

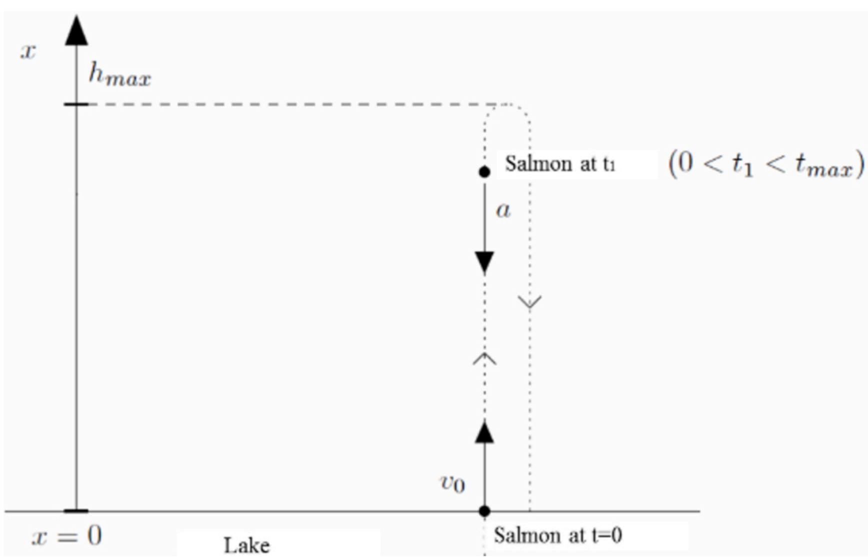
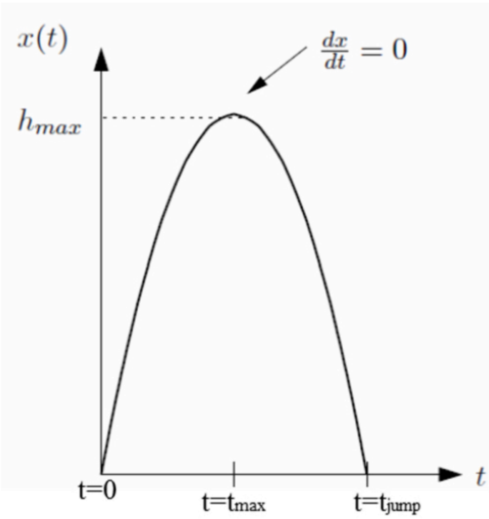


Figure 2: A sketch of problem, with important information indicated.

Graphically, the position and velocity as a function of time are shown in figure 3, from which we can note a few things:

- The position as a function of time is given by a second degree polynomial (i.e. equation (2)), which corresponds to a parabola. The maximum of the parabola is the maximum height reached by the salmon. We can denote the time at which this occurs as $t = t_{max}$.



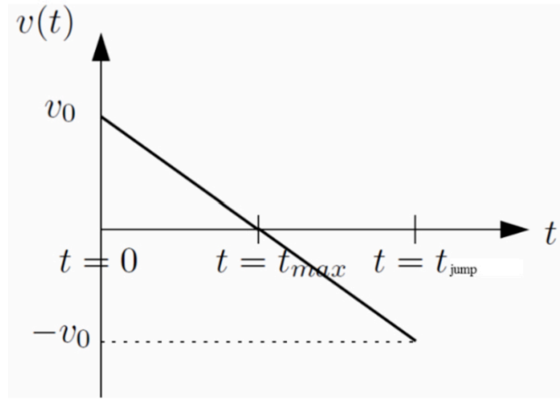


Figure 3: The position (left) and velocity (right), both as a function of time.

- The velocity decreases linearly. Graphically, this is a straight line and the slope of this line is the acceleration of the salmon.
- At $t = t_{max}$, the velocity intersects the horizontal axis, which means that the velocity of the salmon is 0 at the maximum height attained. In any case, the velocity corresponds to the derivative of the position. Thus, at $t = t_{max}$ a velocity of zero corresponds to the vertex of the parabola where the slope is 0.
- The parabola has reflectional symmetry about $t = t_{max}$. This implies that $t_{jump} = 2t_{max}$, where t_{jump} is the time at which the fish reenters the water (i.e. the second time when $x = 0$).

c. To find h_{max}, we start by using equations (1) and (2) from part (a), using $a = -g$. We also choose to define $x = 0$ as the surface of the lake, so $x_0 = 0$, and take the positive x direction to be upwards. Thus, equations (1) and (2) become

$$x(t) = -\frac{1}{2}gt^2 + v_0 t \tag{3}$$

and

$$v(t) = -gt + v_0$$

respectively. From part (b), we know that at the top of the parabola the salmon has zero velocity. Therefore, $v(t_{max}) = 0$, which we can solve to find t_{max}. We find that

$$v(t_{max}) = 0 = -gt_{max} + v_0 \quad \Rightarrow \quad t_{max} = \frac{v_0}{g}.$$

At this time the fish is at its maximum height $h_{max} = x(t_{max})$. Thus, by inserting our result for t_{max} into equation (3) for the position, we obtain

$$h_{max} = -\frac{1}{2}gt_{max}^2 + v_0 t_{max} = -\frac{1}{2}g\frac{v_0^2}{g^2} + v_0\frac{v_0}{g} \quad \Rightarrow \quad h_{max} = \frac{1}{2}\frac{v_0^2}{g}. \tag{4}$$

Checking the units for this equation gives

$$[\mathrm{m}] = \frac{\left[\frac{\mathrm{m}}{\mathrm{s}}\right]^2}{\left[\frac{\mathrm{m}}{\mathrm{s}^2}\right]} = [\mathrm{m}],$$

showing that our solution is plausible.

We can calculate t_{jump} using the final condition $x(t_{jump}) = 0$ when the fish reenters the water. We find

$$x(t_{jump}) = 0 = -\frac{1}{2}gt_{jump}^2 + v_0 t_{jump} = \left(v_0 - \frac{g}{2}t_{jump}\right) t_{jump}.$$

There are two possible solutions to this equation: $t_{jump} = 0$ and

$$t_{jump} = 2\frac{v_0}{g}. \tag{5}$$

We can discard the first one because it corresponds to the time that the fish first jumps out the water. The second result is the solution we are looking for and confirms that $t_{jump} = 2t_{max}$ (just as expected from the symmetry property mentioned in the solution to part (b)). We can also check the units to find

$$[\mathrm{s}] = \frac{\left[\frac{\mathrm{m}}{\mathrm{s}}\right]}{\left[\frac{\mathrm{m}}{\mathrm{s}^2}\right]} = [\mathrm{s}],$$

again showing that our solution is plausible.

Plugging the numerical values into our solutions given by equations (4) and (5), we find

$$h_{max} = \frac{1}{2}\frac{v_0^2}{g} = 0.45 \text{ m}$$

and

$$t_{jump} = 2\frac{v_0}{g} = 0.6 \text{ s}.$$

Lastly, we can ask ourselves "Are these results reasonable?" We see that both h_{max} and t_{jump} are positive as they should be (see figure 2). Moreover, the values of $h_{max} = 0.45$ m and $t_{jump} = 0.6$ s agree with our intuition for roughly how far and for how long a fish could jump out of the water.

5. The train

The motion of the train can be broken into three time intervals:

1. acceleration (for a duration of 60 s),
2. constant speed (for a duration of 300 s), and
3. deceleration (for a duration of 120 s).

Given the information from the problem, we can directly calculate the acceleration in each time interval to be

$$\begin{aligned}
a_1 &= \frac{\Delta v_1}{\Delta t_1} = \frac{72\ \frac{\text{km}}{\text{hr}} - 0\ \frac{\text{km}}{\text{hr}}}{1 \text{ min}} = 72\ \frac{\text{km}}{\text{hr-min}} \times \frac{1 \text{ hr}}{60 \text{ min}} \times \frac{1 \text{ min}}{60 \text{ s}} \times \frac{1 \text{ min}}{60 \text{ s}} \times \frac{1000 \text{ m}}{1 \text{ km}} = \frac{1}{3}\ \frac{\text{m}}{\text{s}^2} \\
a_2 &= \frac{\Delta v_2}{\Delta t_2} = \frac{72\ \frac{\text{km}}{\text{hr}} - 72\ \frac{\text{km}}{\text{hr}}}{5 \text{ min}} = 0 \\
a_3 &= \frac{\Delta v_3}{\Delta t_3} = \frac{-\Delta v_1}{2\Delta t_1} = -\frac{1}{2}a_1 = -\frac{1}{6}\ \frac{m}{s^2},
\end{aligned}$$

where the subscript refers to the first, second, and third time intervals respectively. In figure 4, we use these values to plot the acceleration as a function of time.

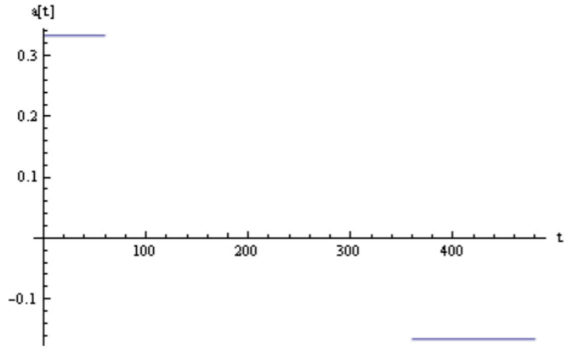


Figure 4: The acceleration of the train (in m/s^2) as a function of time (in s).

Given that we know the acceleration, it is natural to calculate the velocity next. We will denote the velocity throughout each of the time intervals by $v_i(t)$, where the subscript i equals 1, 2, or 3 and refers to the first, second, or third time interval respectively. Similarly, we will denote the initial velocity of the i^{th} time interval to be $v_{0,i}$. During all time intervals the acceleration is constant, so we can apply equations (1) and (2) from problem 4. Equation (1) becomes

$$v_i(t) = a_i t + v_{0,i}, \tag{6}$$

where a_i is the acceleration during the time interval that we have calculated above.

Importantly, using the form of equation (1) requires that we define the start of each time interval to be $t = 0$. This means that we are defining a separate time coordinate system for each time interval. Keeping track and correctly handling these different time coordinates is one of the most challenging parts of the problem. It is possible to calculate everything on a single time coordinate, but the mathematics become more complicated.

Since the train started with $v_{0,1} = 0$, during the first time interval equation (6) is

$$v_1(t) = a_1 t = \left(\frac{1}{3}\ \frac{\text{m}}{\text{s}^2}\right) t.$$

Therefore, at the interface between time intervals 1 and 2, the train is traveling at $v_1(60\ \text{s}) = 20\ \text{m/s} = v_{0,2}$. Substituting this initial velocity and $a_2 = 0$ into equation (6), we see that the velocity during the second interval is a constant value of

$$v_2(t) = v_{0,2} = 20\ \frac{\text{m}}{\text{s}}.$$

Thus, the initial condition for the third time interval is $v_2(300\ \text{s}) = 20\ \text{m/s} = v_{0,3}$, so

$$v_3(t) = a_3 t + v_{0,3} = \left(-\frac{1}{6}\ \frac{\text{m}}{\text{s}^2}\right) t + 20\ \frac{\text{m}}{\text{s}}.$$

To produce the plot for velocity we need to combine these three results. However, we must remember to account for the different time coordinate systems by shifting the second time interval by 60 s and the third time interval by 300 s + 60 s = 360 s. This gives figure 5.

Lastly, we can calculate the positions analogously to the velocities, using equation (2) instead of equation (1). We see that the position during each time interval changes quadratically as a function of time according to

$$x_i(t) = \frac{1}{2} a_i t^2 + v_{0,i} t + x_{0,i}, \tag{7}$$

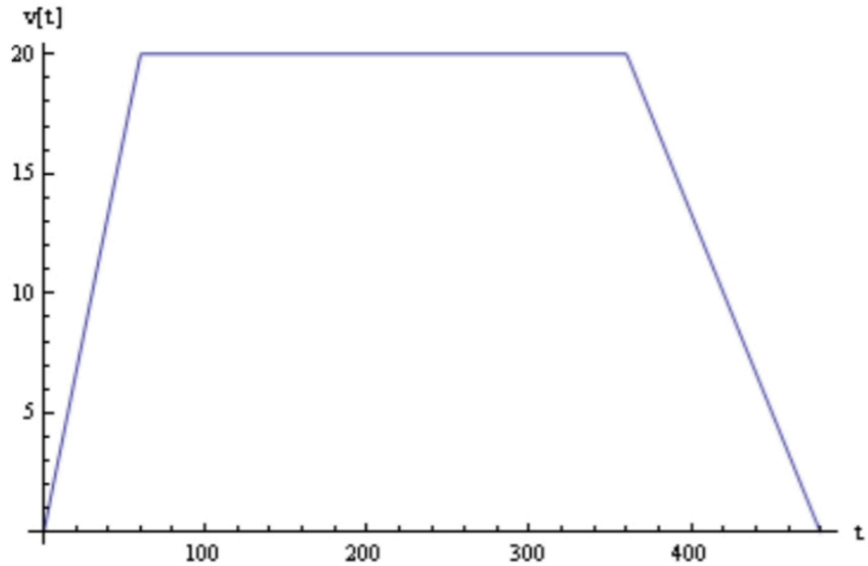


Figure 5: The velocity of the train (in m/s) as a function of time (in s).

where $x_{0,i}$ is the initial position of the train during the i^{th} time interval. Again, we are defining a separate time coordinate for each time interval such that the start of every time interval is $t = 0$. We will take $x = x_{0,1} = 0$ to be the initial location of the train at the start of the first time interval. We showed above that $v_{0,1} = 0$, so during the first time interval equation (7) becomes

$$x_1 = \frac{1}{2}\left(\frac{1}{3}\ \frac{\text{m}}{\text{s}^2}\right)t^2 = \left(\frac{1}{6}\ \frac{\text{m}}{\text{s}^2}\right)t^2.$$

At the interface between time intervals 1 and 2, the train is located at $x_1(60\text{ s}) = 600\text{ m} = x_{0,2}$. Given that we found $v_{0,2} = 20$ m/s above, the position throughout the second time interval is

$$x_2 = \left(20\ \frac{\text{m}}{\text{s}}\right)t + 600\text{ m}.$$

At the interface between time intervals 2 and 3, the train is located at $x_2(300\text{ s}) = 6600\text{ m} = x_{0,3}$. Given that $v_{0,3} = 20$ m/s from above, the position throughout the third time interval is

$$x_3 = \frac{1}{2}\left(-\frac{1}{6}\ \frac{\text{m}}{\text{s}^2}\right)t^2 + \left(20\ \frac{\text{m}}{\text{s}}\right)t + 6600\text{ m}.$$

Again combining the three time intervals, shifting the second and third in time by 60 s and 360 s respectively, we produce figure 6.

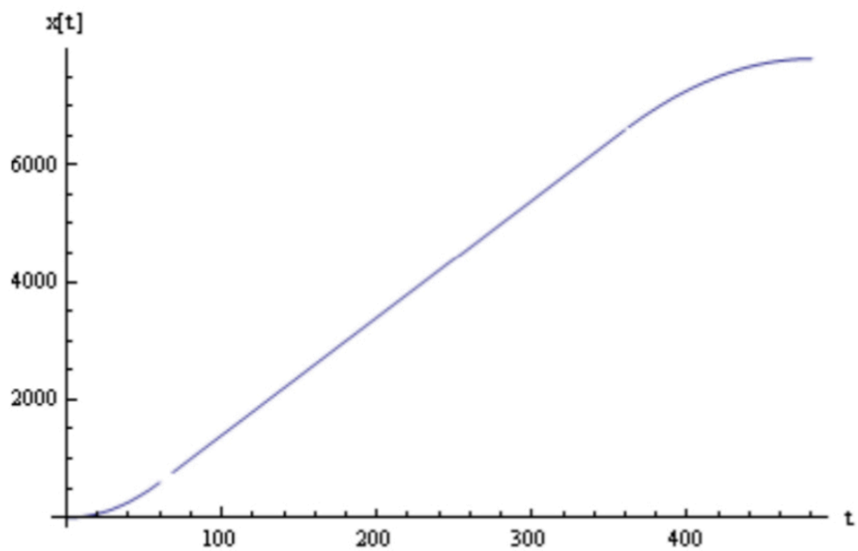


Figure 6: The position of the train (in m) as a function of time (in s).