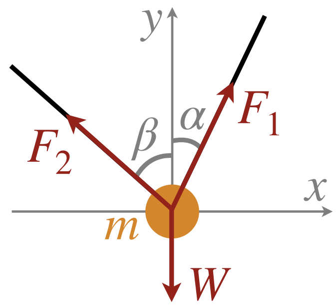
Solutions to Problem Set 3

Free body diagrams

PHYS-101(en)

1. Balancing forces

1. The forces exerted on the ball are the weight $W = mg$, the tension force exerted by the right cable F_1, and the tension force exerted by the left cable F_2.



2. The forces in the $\hat{x}$ direction are

$$W_x = 0$$
$$F_{1x} = F_1 \sin\alpha$$
$$F_{2x} = -F_2 \sin\beta$$

and the forces in the $\hat{y}$ direction are

$$W_y = -W = -mg$$
$$F_{1y} = F_1 \cos\alpha$$
$$F_{2y} = F_2 \cos\beta.$$

3. The ball undergoes no acceleration, so Newton's second law is $\Sigma\vec{F} = 0$ and we have

$$\vec{W} + \vec{F}_1 + \vec{F}_2 = 0. \tag{1}$$

We project this in the $\hat{x}$ direction to get

$$F_1 \sin\alpha - F_2 \sin\beta = 0.$$

Rearranging, we find

$$F_1 = F_2 \frac{\sin\beta}{\sin\alpha}. \tag{2}$$

We then project equation (1) in the $\hat{y}$ direction to get

$$F_1 \cos\alpha + F_2 \cos\beta - mg = 0. \tag{3}$$

Substituting (2) into (3) gives

$$F_2 \frac{\sin\beta}{\sin\alpha}\cos\alpha + F_2\cos\beta - mg = 0 \quad \Rightarrow \quad F_2\left(\frac{\sin\beta}{\sin\alpha}\cos\alpha + \cos\beta\right) = mg.$$

Solving for F_2 gives

$$F_2 = \frac{mg}{\frac{\sin\beta}{\sin\alpha}\cos\alpha + \cos\beta} = \frac{mg\sin\alpha}{\sin\beta\cos\alpha + \sin\alpha\cos\beta} = mg\frac{\sin\alpha}{\sin(\alpha+\beta)},$$

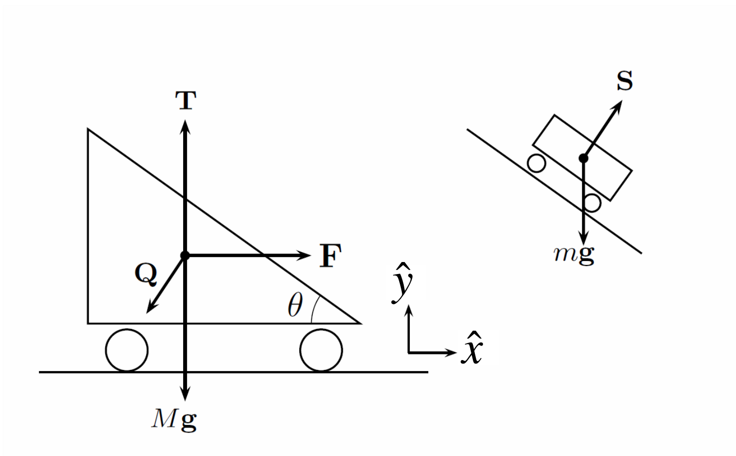
where in the last step we have used the sine angle sum trigonometric identity. By substituting this into equation (2) we find the final answer for F_1 of

$$F_1 = mg\frac{\sin\beta}{\sin(\alpha+\beta)}.$$

From equation (2), we see that, if $\alpha = \beta$, then $F_1 = F_2$ as would be expected from the symmetry of the problem.

2. Triangular trolley

1. The free body diagrams for both trolleys are shown below. The forces on the small trolley are the weight $m\vec{g}$ and the normal force from the triangular trolley $\vec{S}$. The forces on the triangular trolley are the weight $M\vec{g}$, the normal force from the small trolley $\vec{Q}$, the normal force from the ground $\vec{T}$, and the external force $\vec{F}$.



2. The forces on the small trolley are the weight $m\vec{g}$ and the normal force $\vec{S}$ from the triangular trolley acting on the small trolley. Thus, from Newton's second law the acceleration of the small trolley $\vec{a}$ is

$$m\vec{g} + \vec{S} = m\vec{a} \quad \Rightarrow \quad \vec{a} = \vec{g} + \frac{\vec{S}}{m}.$$

Projecting this in the $\hat{x}$ and $\hat{y}$ directions gives

$$a_x = \frac{S\sin\theta}{m}$$

and

$$a_y = \frac{S\cos\theta}{m} - g.$$

The forces on the triangular trolley are the weight $M\vec{g}$, the externally applied force $\vec{F}$, the normal force $\vec{T}$ from the ground acting on the triangular trolley, and the normal force $\vec{Q}$ from the small trolley acting on the triangular trolley.

We can recognize that $\vec{S}$ and $\vec{Q}$ are action-reaction pairs. Thus, from Newton's third law we know that

$$\vec{Q} = -\vec{S}.$$

Using this, Newton's second law for the triangular trolley becomes

$$M\vec{g} + \vec{T} + \vec{F} + \vec{Q} = M\vec{A} \quad \Rightarrow \quad \vec{A} = \vec{g} + \frac{\vec{T}}{M} + \frac{\vec{F}}{M} - \frac{\vec{S}}{M}.$$

Projecting this in the x and y directions gives

$$A_x = \frac{F}{M} - \frac{S\sin\theta}{M}$$

and

$$A_y = -g + \frac{T}{M} - \frac{S\cos\theta}{M},$$

where F and T are the norms of $\vec{F}$ and $\vec{T}$ respectively.

Since the triangular trolley is not accelerating vertically, we can take $A_y = 0$ to show that

$$T = Mg + S\cos\theta.$$

We want to find the force that leaves the small trolley immobile on the larger one, so we require

$$A_x = a_x$$

$$A_y = a_y,$$

which corresponds to

$$\frac{F}{M} - \frac{S\sin\theta}{M} = \frac{S\sin\theta}{m}$$

$$0 = \frac{S\cos\theta}{m} - g$$

respectively. From the second equation we see that

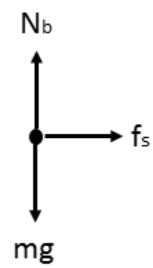
$$S = \frac{mg}{\cos\theta},$$

which can be substituted into the first to find the final answer,

$$F = g\,(M+m)\tan\theta.$$

3. Force with friction

The free body diagram for the books on their own is shown below, where we have the normal force of the table on the books $\vec{N}_b$, the static friction force from the table on the books $\vec{f}_s$, and the weight of the books $m\vec{g}$.



There is no motion in the $\hat{y}$ direction, so Newton's second law tells use that the weight is balanced by the normal force N_b according to

$$N_b - mg = 0 \quad \Rightarrow \quad N_b = mg. \tag{4}$$

The only horizontal force on the books is the static friction force f_s, which is equal to its maximum value of $f_s = \mu_s N_b$ when Carl is applying the maximum force for which the books do not slide. By applying Newton's second law in the $\hat{x}$ direction, we find

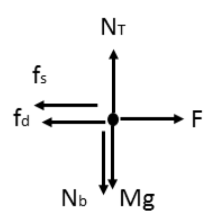
$$\sum F = ma \quad \Rightarrow \quad f_s = ma \quad \Rightarrow \quad \mu_s N_b = ma \quad \Rightarrow \quad a = \frac{\mu_s N_b}{m}.$$

Therefore, using equation (4) we see that the acceleration is

$$a = \mu_s g. \tag{5}$$

Now consider the table, whose free body diagram is shown below and includes a lot of forces. There is the normal force from the ground $\vec{N}_T$, the force applied by Carl $\vec{F}$, the kinetic friction force from the floor $\vec{f}_d$, the weight $m\vec{g}$, the normal force from the books on the table $-\vec{N}_b$, and the static friction force from the books $-\vec{f}_s$. Note that the static friction force on the table is an action-reaction pair with the static friction force in the free body diagram for the books, so it must be equal in magnitude and opposite in direction. Similarly the normal force from the books on the table is an action-reaction pair with the normal force in the free body diagram for the books.

In order to avoid sliding, the table and books must accelerate identically. The kinetic friction force $\vec{f}_d$ between the table and the floor has a magnitude of $f_d = \mu_d N_T$.



Since the table does not accelerate in the $\hat{y}$ direction, Newton's second law gives

$$N_T - N_b - Mg = 0 \quad \Rightarrow \quad N_T = N_b + Mg = (m + M)\, g, \tag{6}$$

where we have used equation (4). In the $\hat{x}$ direction, Newton's second law for the table is

$$F - f_d - f_s = Ma.$$

By substituting equations (4) through (6) and the forms of the friction forces (i.e. $f_s = \mu_s N_b$ and $f_d = \mu_d N_T$) from above, we obtain

$$F - \mu_d N_T - \mu_s N_b = M\mu_s g \quad \Rightarrow \quad F - \mu_d (m + M) g - \mu_s mg = M\mu_s g.$$

Solving for this equation for F gives the final answer of

$$F = \mu_d (m + M) g + \mu_s (m + M) g \quad \Rightarrow \quad F = (\mu_d + \mu_s)(M + m) g$$

and we can plug in numbers to find

$$F = 159 \text{ N}.$$

4. Challenge: Rugby up-and-under play

As indicated in the title, this problem is challenging. We start by defining the coordinate system such that y is upwards in the vertical direction and x is in the horizontal direction of the initial velocity of the ball. The origin is located at the position where the ball is kicked. We will denote the initial speed of the ball by v_{bi}, which we know must be less than v_{bi}^{max}. Using our general solution for projectile motion along with the initial position ($x_0 = 0$ and $y_0 = 0$) and velocity ($v_{x0} = v_{bi}\cos\alpha$ and $v_{y0} = v_{bi}\sin\alpha$), we can write the equations of motion for the ball as

$$\vec{a}_b(t) = -g\hat{y} \tag{7}$$

$$\vec{v}_b(t) = v_{x0}\hat{x} + (-gt + v_{y0})\hat{y} = v_{bi}\cos\alpha\hat{x} + (-gt + v_{bi}\sin\alpha)\hat{y} \tag{8}$$

$$\vec{r}_b(t) = (v_{x0}t + x_0)\hat{x} + \left(-\frac{g}{2}t^2 + v_{y0}t + y_0\right)\hat{y} = v_{bi}t\cos\alpha\hat{x} + \left(-\frac{g}{2}t^2 + v_{bi}t\sin\alpha\right)\hat{y}. \tag{9}$$

1. We want to find the distance at which the player catches the ball. To do so, we must first find the time at which the ball returns to the ground, which we will call t_1. The condition for the ball returning to the ground is $y_b(t_1) = 0$, so we can substitute the $\hat{y}$ component of equation (9) to find

$$y_b(t_1) = 0 = -\frac{g}{2}t_1^2 + v_{bi}t_1\sin\alpha. \tag{10}$$

This equation has two solutions, $t_1 = 0$ and

$$t_1 = \frac{2v_{bi}}{g}\sin\alpha. \tag{11}$$

The first solution corresponds to the time of the kick and the second corresponds to the catch, so the second solution is what we're looking for. By substituting this time into the equation for the horizontal position of the ball from equation (9), we can find the distance at which the ball lands to be

$$x_b(t_1) = v_{bi}t_1\cos\alpha = \frac{2v_{bi}^2}{g}\sin\alpha\cos\alpha. \tag{12}$$

Now we must analyze the player's motion. Since she runs at a constant velocity (that we will call v_p) and her initial position is at the origin, her position is given by

$$x_p(t) = v_p t. \tag{13}$$

Thus, at time $t = t_1$ her position is

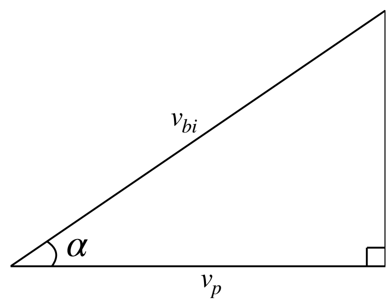
$$x_p(t_1) = v_p t_1 = \frac{2v_{bi}v_p}{g}\sin\alpha, \tag{14}$$

where we have made use of equation (11).

The ball and the player must be at the same location for a catch to occur, which we will call $\ell = x_b(t_1) = x_p(t_1)$. Thus, we require equations (12) and (14) to be equal, which allows us to determine the initial angle of the ball

$$v_{bi}\cos\alpha = v_p \quad \Rightarrow \quad \cos\alpha = \frac{v_p}{v_{bi}} \quad \Rightarrow \quad \alpha = \arccos\left(\frac{v_p}{v_{bi}}\right). \tag{15}$$

To use this information to find a simple expression for ℓ, we can draw the triangle implied by this equation (shown below). After using the Pythagorean theorem to find that the length of the missing



side is $\sqrt{v_{bi}^2 - v_p^2}$, we see that

$$\sin\alpha = \frac{\sqrt{v_{bi}^2 - v_p^2}}{v_{bi}}. \tag{16}$$

Substituting this result into equation (14) (or equation (12)) gives

$$\ell = \frac{2}{g} v_p \sqrt{v_{bi}^2 - v_p^2}. \tag{17}$$

This is the expression for the distance at which the ball lands, which we want to maximize. To do so, we can immediately see that we want to increase the initial velocity of the ball as much as possible by setting

$$v_{bi} = v_{bi}^{max}. \tag{18}$$

This is also intuitively obvious. The harder you kick the ball, the more time it will be in the air and the more time the player will have to run. The dependence on v_p is more complicated. We see that increasing it will increase the multiplying factor in front of the square root (thereby increasing ℓ), but it will also decrease the quantity in the square root (thereby decreasing ℓ). To find the maximum, we can remember our past analysis of projectile motion. The maximum vertical position occurred where the vertical velocity (which is the derivative of the vertical position) went to zero. This is a general technique to find the extrema (i.e. both maxima and minima) of functions: calculate the derivative and solve for the locations at which it is zero. Thus, we take equation (17) and calculate

$$\frac{d\ell}{dv_p} = 0 = \frac{2}{g}\sqrt{v_{bi}^2 - v_p^2} + \frac{2}{g}v_p\left(\frac{1}{2}\frac{-2v_p}{\sqrt{v_{bi}^2 - v_p^2}}\right) = \frac{2}{g}\sqrt{v_{bi}^2 - v_p^2} - \frac{2}{g}\frac{v_p^2}{\sqrt{v_{bi}^2 - v_p^2}} = \frac{2}{g\sqrt{v_{bi}^2 - v_p^2}}\left(v_{bi}^2 - 2v_p^2\right) \tag{19}$$

using the chain rule and product rules. Simplifying this expression, we find that there is only one extrema and it occurs at

$$v_p = \frac{v_{bi}}{\sqrt{2}}. \tag{20}$$

Substituting this result into equation (17) and comparing with any other choice of v_p (e.g. $v_p = 0$), we can verify that this extrema is, in fact, a maxima (as opposed to a minima). Thus, this is the optimal speed that the player would ideally run at. If this isn't possible because $v_p = v_{bi}/\sqrt{2} > v_p^{max}$, the player should run as close as possible to this value, namely at their maximum speed of v_p^{max}. Therefore, we have to explicitly distinguish these two possibilities by writing

$$v_p = \begin{cases} v_p^{max} & \text{if } \; v_{bi}/\sqrt{2} > v_p^{max} \\ v_{bi}/\sqrt{2} & \text{otherwise} \end{cases}. \tag{21}$$

Combining equations (17), (18), and (21), we find that the maximum distance to catch the ball is

$$\ell = \begin{cases} (2v_p^{max}/g)\sqrt{(v_{bi}^{max})^2 - (v_p^{max})^2} & \text{if } \; v_{bi}^{max}/\sqrt{2} > v_p^{max} \\ (v_{bi}^{max})^2/g & \text{otherwise} \end{cases}. \tag{22}$$

Combining equations (15), (18), and (21), we find that ideal angle to kick the ball is

$$\alpha = \begin{cases} \arccos\left(v_p^{max}/v_{bi}^{max}\right) & \text{if } \; v_{bi}^{max}/\sqrt{2} > v_p^{max} \\ \arccos\left(1/\sqrt{2}\right) = \pi/4 = 45^\circ & \text{otherwise} \end{cases}. \tag{23}$$

The interpretation of these results is that if you are sufficiently fast (i.e. v_p^{max} is sufficiently large), the second case in all three equations applies. In this case, you want to kick the ball at $\alpha = 45^\circ$, as this is the angle that maximizes the distance traveled by the ball, and run below your maximum speed, such that you arrive at the same time and place as the ball when it lands. However, the more realistic case is the first, that you can out-kick your running speed. In this case you want to run at your maximum speed and angle your kick higher (i.e. $\alpha > 45^\circ$) so that the ball stays in the air for longer and you have more time to run.

2. The position of the ball is given by equation (9). Using equations (15) and (16), we can write equation (9) as

$$\vec{r}_b(t) = v_p t\hat{x} + \left(-\frac{g}{2}t^2 + t\sqrt{v_{bi}^2 - v_p^2}\right)\hat{y}.$$

Solving the x component of this equation (i.e. $x_b(t) = v_p t$) for time gives $t = x_b/v_p$, which we can substitute into the y component to find

$$y_b(x_b) = -\frac{g}{2}\left(\frac{x_b}{v_p}\right)^2 + \frac{x_b}{v_p}\sqrt{v_{bi}^2 - v_p^2} = -\frac{g}{2v_p^2}x_b^2 + x_b\sqrt{\frac{v_{bi}^2}{v_p^2} - 1}.$$

This is the trajectory of the ball. To find where the defense should be placed, we need to determine at what x position the height of the ball is equal to that of the defense player's hand. Therefore, we set $y_b(x_b) = h$ to find

$$h = -\frac{g}{2v_p^2}x_b^2 + x_b\sqrt{\frac{v_{bi}^2}{v_p^2} - 1} \quad \Rightarrow \quad 0 = \frac{g}{2v_p^2}x_b^2 - x_b\sqrt{\frac{v_{bi}^2}{v_p^2} - 1} + h,$$

which we want to solve for x_b. This is a quadratic equation, which we can solve by first computing the discriminant

$$\Delta = \frac{v_{bi}^2}{v_p^2} - 1 - 4\frac{gh}{2v_p^2} = \frac{v_{bi}^2 - v_p^2 - 2gh}{v_p^2}$$

and then the solution

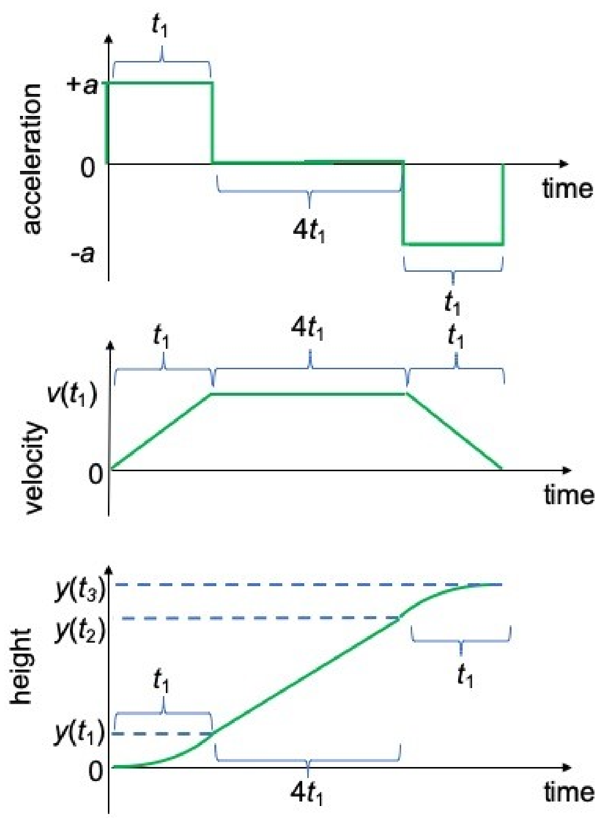
$$x_b = \frac{v_p}{g}\left(\sqrt{v_{bi}^2 - v_p^2} \pm \sqrt{v_{bi}^2 - v_p^2 - 2gh}\right).$$

We see there are 2 solutions — the shorter distance corresponds to the defense player catching the ball on its way up, and the longer distance corresponds to catching it on the way down. We arrive at the final answer by substituting equations (18) and (21) into the longer distance to get

$$x_b = \begin{cases} (v_p^{max}/g)\left(\sqrt{(v_{bi}^{max})^2 - (v_p^{max})^2} + \sqrt{(v_{bi}^{max})^2 - (v_p^{max})^2 - 2gh}\right) & \text{if } v_{bi}^{max}/\sqrt{2} > v_p^{max} \\ (v_{bi}^{max}/(2g))\left(\sqrt{(v_{bi}^{max})^2} + \sqrt{(v_{bi}^{max})^2 - 4gh}\right) & \text{otherwise} \end{cases}.$$

5. Homework: Elevator

1. The acceleration, velocity, and position as a function of time are plotted below.



2. There are three stages of motion given in the problem and we note that the problem is one-dimensional. The first and third stages are at constant acceleration and the second stage is at constant velocity. We have seen similar problems before. The connection between the stages is that the final speed after the first stage is the constant speed during the second stage and the initial speed for the third stage. Additionally, the position at the end of the first stage is the initial position for the second stage and the position at the end of the second stage is the initial position for the third stage.

3. From projectile motion, we know the general solution for the acceleration, velocity, and position during any constant acceleration $\overline{a}$ is

$$a(t) = \overline{a}$$

$$v(t) = \overline{a}t + v_0$$

$$y(t) = \frac{\overline{a}}{2}t^2 + v_0 t + y_0$$

respectively. During the first stage the acceleration is $\overline{a} = a$. Thus, after a time interval t_1 the elevator has an upward speed and displacement of

$$v_1 = v(t_1) = at_1$$

$$\Delta y_1 = y(t_1) - y_0 = \frac{a}{2}t_1^2$$

respectively, where we must remember that the acceleration a is positive and unknown.

During the second stage, the elevator has a constant acceleration of $\overline{a} = 0$, so the upward speed and displacement are

$$v(t) = v_0 = v_1 = at_1$$

$$y(t) - y_0 = v_0 t = v_1 t = at_1 t,$$

where we note that here t is the time since the second stage began. Thus, after a time interval $\Delta t_2 = 4t_1$ the elevator has a velocity and displacement of

$$v_2 = v(\Delta t_2) = at_1$$

$$\Delta y_2 = y(\Delta t_2) - y_0 = 4at_1^2.$$

During the third stage, we have constant acceleration of $\overline{a} = -a$, so the upward speed and displacement are

$$v(t) = -at + v_0 = -at + at_1$$

$$y(t) - y_0 = -\frac{a}{2}t^2 + v_0 t = -\frac{a}{2}t^2 + at_1 t.$$

After a time interval $\Delta t_3 = t_1$, the upward speed and displacement is

$$v_3 = v(\Delta t_3) = 0$$

$$\Delta y_3 = y(\Delta t_3) - y_0 = -\frac{a}{2}t_1^2 + at_1 t_1 = \frac{a}{2}t_1^2$$

respectively.

Thus, the total distance traveled is the sum of the displacements in the three stages and is also equal to the height of the building h, so

$$h = \Delta y_1 + \Delta y_2 + \Delta y_3 = \frac{a}{2}t_1^2 + 4at_1^2 + \frac{a}{2}t_1^2 = 5at_1^2.$$

Solving this equation for the acceleration gives the solution of

$$a = \frac{h}{5t_1^2}.$$

4. Let's assume that the sixth floor is about $h \approx 25$ m above the ground. This happens to be a slow elevator, taking approximately 30 s to reach the top so $t_1 \approx 5$ s. Therefore, the acceleration is:

$$a \approx \frac{25 \text{ m}}{5 \times (5 \text{ s})^2} \approx 0.2 \ \frac{\text{m}}{\text{s}^2}.$$

This number is reasonable as it is around 2% of the gravitational acceleration. In a slow elevator, one barely notices that the elevator is accelerating.