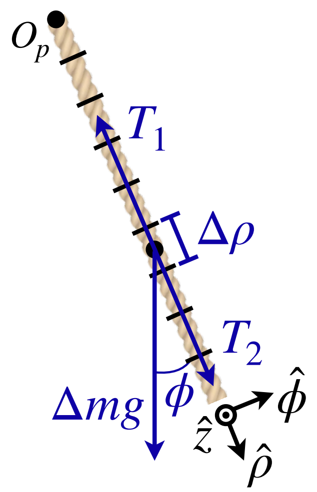
1. Rope swing (10 points)



a. (**2.0 points**) *Calculate the tension in the rope $T(\rho)$ as a function of ρ (the radial distance from the pivot point O_p) as well as ϕ (the angle of the rope) and ω (the instantaneous angular speed of the rope). Note that in this part you may assume the* **tension is zero at the end of the rope** *and include the variables ρ, ϕ, and ω in your answer.*

To calculate the tension in a massive rope, we start by dividing it into differential elements and then draw the free body diagram for an arbitrary element in the middle (shown above). We will draw it at an arbitrary angular position ϕ. From this we see that there are three forces acting on the differential element: gravity (Δmg), the tension force from the differential element above (T_1), and the tension force from the differential element below (T_2). To find how the tension changes with location, we only need to consider the forces acting *along* the rope, which is in the $\hat{\rho}$ direction. Thus, from our free body diagram, we see that the radial component of Newton's second law is

$$T_2 - T_1 + \Delta mg\cos\phi = \Delta m a_\rho, \tag{1}$$

where a_ρ is the radial component of the acceleration of the differential element. Since the rope stays straight and simply pivots around the point O_p, it is executing circular motion (though the circular motion is not uniform as ω is changing). We know that the radial component of the acceleration of an object executing circular motion is the centripetal acceleration $\vec{a}_{cent} = -\rho\omega^2\hat{\rho}$, where ρ is the radial location of the differential element that we are considering. We can substitute this to find

$$\Delta T + \Delta mg\cos\phi = -\Delta m\rho\omega^2, \tag{2}$$

where we have identified the change in tension across the differential element as $\Delta T = T_2 - T_1$. Note that there will also be a tangential component of the acceleration a_ϕ as the rope is experiencing angular acceleration due to gravity, but this does not appear in the *radial* component of Newton's second law.

Next, we can use the density to convert the differential mass to a differential length. Since the rope has a uniform linear mass density, we know that

$$\lambda = \frac{m}{\ell} = \frac{\Delta m}{\Delta\rho} \quad \Rightarrow \quad \Delta m = \frac{m}{\ell}\Delta\rho. \tag{3}$$

Substituting this into equation (2) and rearranging gives

$$\Delta T = -\Delta m \left(\rho\omega^2 + g\cos\phi\right) = -\frac{m}{\ell}\Delta\rho \left(\rho\omega^2 + g\cos\phi\right) \quad \Rightarrow \quad \frac{\Delta T}{\Delta\rho} = -\frac{m\omega^2}{\ell}\rho - \frac{mg}{\ell}\cos\phi. \tag{4}$$

Now we can take the limit as $\Delta\rho \to 0$ to find the differential equation

$$\frac{dT}{d\rho} = -\frac{m\omega^2}{\ell}\rho - \frac{mg}{\ell}\cos\phi. \tag{5}$$

Integrating in ρ once, we find

$$T(\rho) = -\frac{m\omega^2}{2\ell}\rho^2 - \frac{mg}{\ell}\rho\cos\phi + C, \tag{6}$$

where C is the integration constant. To find it we apply the boundary condition given in the problem statement, $T(\ell) = 0$, and see that

$$T(\ell) = 0 = -\frac{m\omega^2}{2\ell}\ell^2 - \frac{mg}{\ell}\ell\cos\phi + C \quad \Rightarrow \quad C = \frac{m\omega^2\ell}{2} + mg\cos\phi. \tag{7}$$

Substituting this into equation (6) gives

$$T(\rho) = -\frac{m\omega^2}{2\ell}\rho^2 - \frac{mg}{\ell}\rho\cos\phi + \frac{m\omega^2\ell}{2} + mg\cos\phi. \tag{8}$$

Rearranging we find the final answer of

$$T(\rho) = \frac{m\omega^2\ell}{2}\left(1 - \frac{\rho^2}{\ell^2}\right) + mg\cos\phi\left(1 - \frac{\rho}{\ell}\right). \tag{9}$$

b. (**0.5 points**) *At what value of ρ is the rope most likely to break? Note that you should not need to perform any additional calculations.*

The rope is mostly likely to break at the location where the tension is the highest. Given that m, ω^2, ℓ, g, and $\cos\phi$ are all positive numbers, we see from our solution to part a that the tension is greatest at $\rho = 0$. This result is also intuitive – the highest point of the rope should have the largest tension since it is supporting the entirety of the rope below it. Thus, $\rho = 0$ is the location where the rope is most likely to break.

c. (**0.5 points**) *At what value of ϕ is the rope most likely to break? Note that you should not need to perform any additional calculations.*

Since the rope is most likely to break at $\rho = 0$, we can evaluate our answer from part a (i.e. equation (9)) to see that

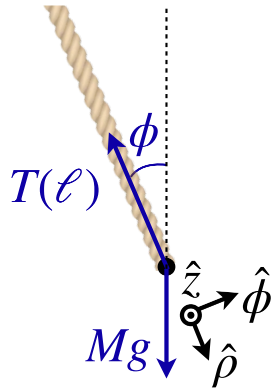
$$T(0) = \frac{m\omega^2\ell}{2} + mg\cos\phi. \tag{10}$$

This shows how the tension at $\rho = 0$ depends on ϕ. To solve the problem, we want to determine the value of ϕ that maximizes the tension.

First we note that both of the two terms are positive. The second, which is the contribution from the gravitational acceleration, clearly has its maximum at $\phi = 0$. The first term, which is the contribution from the centripetal acceleration, looks like a constant, but we have to remember that ω will change as rope swings through different angles. However, we have the intuition that the object will speed up as the rope is descending towards $\phi = 0$ and then slow down as the rope ascends after moving past $\phi = 0$. Thus, ω is maximum at $\phi = 0$, which means that the first term is maximal at $\phi = 0$. Since both terms are largest at $\phi = 0$, this is the location where the rope is most likely to break.

d. (**1.0 point**) *Draw the free body diagram for the person, labeling all the forces involved.*

There are two forces acting on the person: tension from the end of the rope ($T(\ell)$) and gravity (Mg).



e. (**1.0 point**) *Find the differential equation for $\phi(t)$ (the angular position of the person as a function of time). Note that you may include the variable t in your answer.*

From the free body diagram in part d, we can immediately write down Newton's second law for the person

$$-T(\ell)\hat{\rho} + Mg\cos\phi\hat{\rho} - Mg\sin\phi\hat{\phi} = M\vec{a}. \tag{11}$$

In particular, note that there should be a negative sign in front of the $\hat{\phi}$ component of the gravitational force. In the drawing above, ϕ is greater than zero, which means that this term will be point in the $-\hat{\phi}$ direction as is the case physically. For angles $\phi < 0$, this term changes direction to point in the $+\hat{\phi}$ direction because $\sin\phi$ becomes negative.

The radial component of equation (11) can be used to determine the tension at the end of the rope (as it will be different than zero when there is a person hanging on it). However, this information is not needed. Instead we will consider the azimuthal component

$$-Mg\sin\phi = Ma_\phi \quad \Rightarrow \quad a_\phi + g\sin\phi = 0, \tag{12}$$

where a_ϕ is the azimuthal component of the acceleration. We know that this relates to the angular acceleration α according to

$$a_\phi = \ell\alpha, \tag{13}$$

where $\rho = \ell$ is the distance from the center of the circular motion. Additionally, we know that the definition of the angular acceleration is

$$\alpha = \frac{d^2\phi}{dt^2}. \tag{14}$$

Substitution equations (13) and (14) into equation (12) gives

$$\ell\frac{d^2\phi}{dt^2} + g\sin\phi = 0 \quad \Rightarrow \quad \frac{d^2\phi}{dt^2} + \frac{g}{\ell}\sin\phi = 0. \tag{15}$$

This is the differential equation that can be solved to find $\phi(t)$.

f. (**1.0 point**) *Assume that the initial angular position is small (i.e. $|\phi_0| \ll 1$), so that $\sin\phi \approx \phi$. Then, solve the differential equation to find the form of $\phi(t)$ as well as $\vec{\omega}(t)$ (the angular velocity of the person as a function of time). Note that you may include the variable t as well as the integration constants A and φ in your answer.*

Using the small angle approximation, we can replace $\sin\phi \to \phi$, so that equation (15) becomes

$$\frac{d^2\phi}{dt^2} + \frac{g}{\ell}\phi = 0. \tag{16}$$

This matches the form of the differential equation in the problem statement, where $x = \phi$, $t = t$, and $\omega_0^2 = g/\ell$. Thus, $\omega_0 = \pm\sqrt{g/\ell}$ and the problem statement tells us that the form of the solution is

$$\phi(t) = A\cos\left(\pm\sqrt{\frac{g}{\ell}}t + \varphi\right). \tag{17}$$

Note that you may be unsure about the significance of the plus or minus sign in the cosine. However, we can eliminate it by using the property of cosine that $\cos(x) = \cos(-x)$ to transform equation (17) into

$$\phi(t) = A\cos\left(\sqrt{\frac{g}{\ell}}t \pm \varphi\right). \tag{18}$$

Then we can redefine the constant $\varphi \to \pm\varphi$ so that it absorbs the plus or minus sign (as a constant times negative one is still a constant). Thus, ultimately we have

$$\phi(t) = A\cos\left(\sqrt{\frac{g}{\ell}}t + \varphi\right). \tag{19}$$

To find the angular velocity, we first note that the angular speed is $\omega = d\phi/dt$. Thus, we can differentiate equation (19) to find

$$\omega(t) = -A\sqrt{\frac{g}{\ell}}\sin\left(\sqrt{\frac{g}{\ell}}t + \varphi\right). \tag{20}$$

To find the angular velocity we must consider the direction of rotation. From the drawing the problem statement, we see that the rotation is counter-clockwise. Using the right hand rule, we curl the fingers of our right hand in the direction of the rotation and see that our thumb points out of the page in the $+\hat{z}$ direction. Thus, the angular velocity is

$$\vec{\omega}(t) = \pm A\sqrt{\frac{g}{\ell}}\sin\left(\sqrt{\frac{g}{\ell}}t + \varphi\right)\hat{z}. \tag{21}$$

Note that we have added the plus or minus sign out in front as we are not sure if integration constants A and φ will be positive or negative. However, after we determine them we can choose the sign such that $\vec{\omega}$ is in the $+\hat{z}$ direction.

g. (**1.0 point**) *Use the initial conditions to determine the integration constants and find $\phi(t)$ and $\vec{\omega}(t)$. Note that you may include the variable t in your answer (but not A nor φ).*

The problem statement directly gives us the two initial conditions we need. It says that at $t = 0$ the rope is at an angle ϕ_0 (which is less than zero) and you have no initial speed at $t = 0$. Thus, we can enforce $\phi(0) = \phi_0$ and $\vec{\omega}(0) = 0$ to determine the two integration constants.

Applying initial condition on the angular position (using equation (19)) gives

$$\phi(0) = \phi_0 = A\cos\varphi, \tag{22}$$

while the initial condition on the angular velocity (using equation (21)) gives

$$\vec{\omega}(0) = 0 = \pm A\sqrt{\frac{g}{\ell}}\sin\varphi\hat{z} \quad\Rightarrow\quad 0 = A\sin\varphi. \tag{23}$$

From equation (22), we see that $A \neq 0$, which means that equation (23) becomes

$$0 = \sin\varphi \quad\Rightarrow\quad \varphi = 0. \tag{24}$$

Substituting this result into equation (22) gives

$$A = \phi_0. \tag{25}$$

Substituting our solutions for the two integration constants into equations (19) and (21), we see that the angular position and angular velocity are

$$\phi(t) = \phi_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right) \tag{26}$$

and

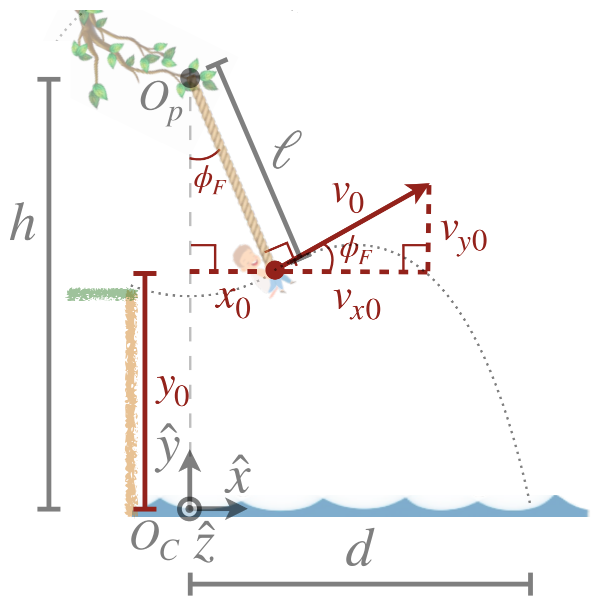
$$\vec{\omega}(t) = \pm\phi_0\sqrt{\frac{g}{\ell}} \sin\left(\sqrt{\frac{g}{\ell}}t\right)\hat{z} \tag{27}$$

respectively. Given that $\phi_0 < 0$, we must choose the minus sign in the angular velocity to find

$$\vec{\omega}(t) = -\phi_0\sqrt{\frac{g}{\ell}} \sin\left(\sqrt{\frac{g}{\ell}}t\right)\hat{z}. \tag{28}$$

This choice ensures that $\vec{\omega}$ is in the $+\hat{z}$ direction. From this expression we see that our intuition in part c was correct – the angular speed is maximum when the angular position is $\phi = 0$ (i.e. at $t = (\pi/2)\sqrt{\ell/g}$).

h. (**1.5 points**) *Find x_0 and y_0 (the horizontal and vertical locations where you release the rope respectively) as well as v_{x0}, and v_{y0} (the horizontal and vertical speeds at which you release the rope respectively) in the Cartesian coordinate system shown in the figure.*



To solve this problem it is best to draw a large picture of the situation at $t = t_F$ (as shown above), when the person releases the rope. We will call $\phi_F = \phi(t_F)$ the angular position at this time. From the drawing and trigonometry we immediately see that

$$x_0 = \ell \sin\phi_F \tag{29}$$

and

$$h = y_0 + \ell\cos\phi_F \quad \Rightarrow \quad y_0 = h - \ell\cos\phi_F. \tag{30}$$

Using equation (26) we can find

$$\phi_F = \phi_0 \cos\left(\sqrt{\frac{g}{\ell}}t_F\right). \tag{31}$$

Substituting this into equation (29) and (30) gives

$$x_0 = \ell \sin\left(\phi_0 \cos\left(\sqrt{\frac{g}{\ell}} t_F\right)\right) \tag{32}$$

and

$$y_0 = h - \ell \cos\left(\phi_0 \cos\left(\sqrt{\frac{g}{\ell}} t_F\right)\right) \tag{33}$$

respectively.

To find the initial velocities when the person begins free fall, we must remember that the instantaneous velocity is always tangent to the circular trajectory. Thus, the person's initial velocity vector will be at a right angle to the rope. From the drawing above we see that

$$v_{x0} = v_0 \cos\phi_F \tag{34}$$

and

$$v_{y0} = v_0 \sin\phi_F. \tag{35}$$

We have already found that ϕ_F is given by equation (31), so we can plug it in to get

$$v_{x0} = v_0 \cos\left(\phi_0 \cos\left(\sqrt{\frac{g}{\ell}} t_F\right)\right) \tag{36}$$

and

$$v_{y0} = v_0 \sin\left(\phi_0 \cos\left(\sqrt{\frac{g}{\ell}} t_F\right)\right). \tag{37}$$

However, we must still determine v_0, which is the tangential speed at the moment the rope is released $v_0 = v_\phi(t_F)$. We know that the tangential speed is related to the angular speed through $v_\phi(t) = \ell\omega(t)$ (as the person is a distance $\rho = \ell$ away from the center of the circular motion), so we have

$$v_0 = \ell\omega(t_F). \tag{38}$$

The angular speed $\omega(t_F)$ can be found from the angular velocity $\vec{\omega}(t)$, which we determined in part g. Evaluating equation (28) at $t = t_F$ gives

$$\vec{\omega}(t_F) = -\phi_0\sqrt{\frac{g}{\ell}} \sin\left(\sqrt{\frac{g}{\ell}} t_F\right)\hat{z}. \tag{39}$$

Next we take the magnitude to find

$$\omega(t_F) = -\phi_0\sqrt{\frac{g}{\ell}} \sin\left(\sqrt{\frac{g}{\ell}} t_F\right). \tag{40}$$

Note that the minus sign in front might be alarming as we know the speed must be a positive number. However, we know that is a positive quantity as we already ensured it such that $\vec{\omega}(t)$ in equation (28) was in the $+\hat{z}$ direction. Then, substituting equation (40) into equation (38) gives

$$v_0 = -\ell\phi_0\sqrt{\frac{g}{\ell}} \sin\left(\sqrt{\frac{g}{\ell}} t_F\right) = -\phi_0\sqrt{g\ell} \sin\left(\sqrt{\frac{g}{\ell}} t_F\right). \tag{41}$$

Lastly, we can substitute this into equations (36) and (37) to find

$$v_{x0} = -\phi_0\sqrt{g\ell} \sin\left(\sqrt{\frac{g}{\ell}} t_F\right) \cos\left(\phi_0 \cos\left(\sqrt{\frac{g}{\ell}} t_F\right)\right) \tag{42}$$

and

$$v_{y0} = -\phi_0 \sqrt{g\ell} \sin\left(\sqrt{\frac{g}{\ell}} t_F\right) \sin\left(\phi_0 \cos\left(\sqrt{\frac{g}{\ell}} t_F\right)\right). \tag{43}$$

Thus, the final answers for this part are equations (32), (33), (42), and (43).

i. (**1.5 points**) *Assume that you are given x_0, y_0, v_{x0}, and v_{y0}. Find d, the horizontal distance from O_p at which you land in the water (shown in the figure above). Note that you may include x_0, y_0, v_{x0}, and v_{y0} in your answer.*

This part is conceptually straightforward, but the mathematics require considerable attention to detail. Additionally, we are given the initial position and velocity, which means that this part is independent of the rest of the problem. This means that, even if you got stuck on a previous part, you could still attempt this part.

From the time the person releases the rope until they hit the water, they are in free fall under the influence of gravity. Thus, they are executing ballistic motion, which we know is governed by the equations

$$x(t) = v_{x0} t + x_0 \tag{44}$$

and

$$y(t) = -\frac{g}{2} t^2 + v_{y0} t + y_0. \tag{45}$$

We are only interested in the position at which the person lands in the water (and not the time), so we can solve equation (44) for time then substitute it into equation (45) to find the trajectory. Doing so, equation (44) gives

$$t(x) = \frac{x - x_0}{v_{x0}}, \tag{46}$$

so the trajectory is

$$y(x) = -\frac{g}{2}\left(\frac{x - x_0}{v_{x0}}\right)^2 + v_{y0}\left(\frac{x - x_0}{v_{x0}}\right) + y_0 = -\frac{g}{2v_{x0}^2}(x - x_0)^2 + \frac{v_{y0}}{v_{x0}}(x - x_0) + y_0. \tag{47}$$

We are seeking the horizontal location d at which the person lands in the water. Given the Cartesian coordinate system shown in the problem statement, we know that $y(d) = 0$, so equation (47) becomes

$$-\frac{g}{2v_{x0}^2}(d - x_0)^2 + \frac{v_{y0}}{v_{x0}}(d - x_0) + y_0 = 0 \quad \Rightarrow \quad -g(d - x_0)^2 + 2v_{x0}v_{y0}(d - x_0) + 2v_{x0}^2 y_0 = 0. \tag{48}$$

This is a quadratic equation which can be solved for $d - x_0$. We chose this form to eliminate the fractions, which will simplify the algebra involved in solving the quadratic formula. Additionally, note that we could factor out the x_0 terms and solve for d (instead of $d - x_0$), but this would be more complicated and take longer. Thus, we apply the quadratic formula to equation (48) and find the solution

$$d - x_0 = \frac{-2v_{x0}v_{y0} \pm \sqrt{4v_{x0}^2 v_{y0}^2 + 8g v_{x0}^2 y_0}}{-2g} = \frac{v_{x0}v_{y0} \mp \sqrt{v_{x0}^2 v_{y0}^2 + 2g v_{x0}^2 y_0}}{g}. \tag{49}$$

Rearranging further gives

$$d - x_0 = \frac{v_{x0}}{g}\left(v_{y0} \mp \sqrt{v_{y0}^2 + 2g y_0}\right). \tag{50}$$

As with all solutions to quadratic equations, we see two solutions embodied by the plus and minus signs. However, when written in this form, it is straightforward to see which solution we are seeking. As shown in the figure above, we know that $d - x_0$ will be a positive number. Thus, since $v_{y0} < \sqrt{v_{y0}^2 + 2gy_0}$ we should chose the plus sign because the minus sign will lead a negative value of $d - x_0$. This gives

$$d - x_0 = \frac{v_{x0}}{g}\left(v_{y0} + \sqrt{v_{y0}^2 + 2gy_0}\right), \tag{51}$$

which we can solve for d to find

$$d = x_0 + \frac{v_{x0}}{g}\left(v_{y0} + \sqrt{v_{y0}^2 + 2gy_0}\right). \tag{52}$$

This is the final answer. If you wanted to link the two parts of the problem (i.e. rotational motion and ballistic motion) completely, you could substitute the results from part h into this expression. This would tell you, given the geometry (i.e. h, ℓ, ϕ_0) and release time t_F, how far you would travel before landing in the water.