Solutions to Problem Set 10

Collisions

PHYS-101(en)

1. A collision

1. In an elastic collision, both the mechanical energy and momentum are conserved. Conservation of momentum in one dimension is

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}, \tag{1}$$

where the subscripts i and f indicate the state just before and just after the collision respectively and the subscripts 1 and 2 indicate the first or second ball respectively. Since the first ball has an initial speed of $v_{1i} = v_0$ and the second ball starts at rest $v_{2i} = 0$, equation (1) becomes

$$m_1 v_0 = m_1 v_{1f} + m_2 v_{2f}. \tag{2}$$

Solving this equation for the final velocity of the first ball gives

$$v_{1f} = v_0 - \frac{m_2}{m_1} v_{2f}. \tag{3}$$

We chose to solve for v_{1f}, rather than v_{2f} because we are searching for v_{2f} and substituting equation (3) into the conservation of energy equation will allow us to eliminate v_{1f}.

Since the collision is elastic, mechanical energy is conserved during the collision and we can write

$$\frac{m_1}{2} v_{1i}^2 + \frac{m_2}{2} v_{2i}^2 = \frac{m_1}{2} v_{1f}^2 + \frac{m_2}{2} v_{2f}^2. \tag{4}$$

Given the initial velocities of the two balls, this becomes

$$\frac{m_1}{2} v_0^2 = \frac{m_1}{2} v_{1f}^2 + \frac{m_2}{2} v_{2f}^2. \tag{5}$$

Substituting equation (3) into equation (5) allows us to eliminate v_{1f} and gives

$$\frac{m_1}{2} v_0^2 = \frac{m_1}{2}\left(v_0 - \frac{m_2}{m_1} v_{2f}\right)^2 + \frac{m_2}{2} v_{2f}^2 \tag{6}$$

$$\frac{m_1}{2} v_0^2 = \frac{m_1}{2}\left(v_0^2 - 2v_0 \frac{m_2}{m_1} v_{2f} + \frac{m_2^2}{m_1^2} v_{2f}^2\right) + \frac{m_2}{2} v_{2f}^2 \tag{7}$$

$$\frac{m_1}{2} v_0^2 = \frac{m_1}{2} v_0^2 - 2v_0 \frac{m_2}{2} v_{2f} + \frac{m_2}{2}\frac{m_2}{m_1} v_{2f}^2 + \frac{m_2}{2} v_{2f}^2 \tag{8}$$

$$0 = -2v_0 \frac{m_2}{2} v_{2f} + \left(\frac{m_2}{m_1} + 1\right)\frac{m_2}{2} v_{2f}^2 \tag{9}$$

$$\left(\frac{m_1 + m_2}{m_1}\right) v_{2f} = 2v_0 \tag{10}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_0. \tag{11}$$

To calculate the maximum height attained by the second ball after the collision, we can use conservation of mechanical energy. Since there are no non-conservative forces doing work, we have

$$K_{2f} + U_{2f} = K_{2max} + U_{2max}, \tag{12}$$

where K_{2max} and U_{2max} are the kinetic and gravitational potential energies at the maximum height. Given that it is the maximum height, we know that $K_{2max} = 0$. Additionally, we can define the reference point for the gravitational potential to be at the height of the collision. Thus, $U_{2f} = 0$, implying that equation (12) becomes

$$K_{2f} + 0 = 0 + U_{2max} \quad \Rightarrow \quad \frac{m_2}{2} v_{2f}^2 = m_2 g h_{max} \quad \Rightarrow \quad h_{max} = \frac{1}{2g} v_{2f}^2. \tag{13}$$

Substituting equation (11) gives

$$h_{max} = \frac{1}{2g} \left(\frac{2m_1}{m_1 + m_2} v_0 \right)^2 = \frac{2}{g} \left(\frac{m_1}{m_1 + m_2} \right)^2 v_0^2. \tag{14}$$

Plugging in numbers, we find that

$$h_{max} = 2.22 \text{ cm}. \tag{15}$$

2. In this case momentum is still conserved, but mechanical energy is not. Thus, equation (2) still remains valid. However, since the objects stick together after the collision, we know that $v_{1f} = v_{2f} = v_f$. Substituting this constraint into equation (2) gives

$$m_1 v_0 = m_1 v_f + m_2 v_f \quad \Rightarrow \quad v_f = \frac{m_1}{m_1 + m_2} v_0. \tag{16}$$

Even though mechanical energy is not conserved by the collision (due to the non-conservative frictional forces involved in two objects sticking together), it is conserved *after* the collision. Thus, we have

$$K_f + U_f = K_{max} + U_{max} \tag{17}$$

for the combined object with mass $m_1 + m_2$. As before, we can define the reference point for the gravitational potential to be at the height of the collision and find

$$K_f + 0 = 0 + U_{max} \quad \Rightarrow \quad \frac{m_1 + m_2}{2} v_f^2 = (m_1 + m_2)\, g h_{max} \quad \Rightarrow \quad h_{max} = \frac{1}{2g} v_f^2. \tag{18}$$

Substituting equation (16) produces

$$h_{max} = \frac{1}{2g} \left(\frac{m_1}{m_1 + m_2} \right)^2 v_0^2. \tag{19}$$

Plugging in numbers gives

$$h_{max} = 0.56 \text{ cm}, \tag{20}$$

which is a factor of 4 lower than for the elastic collision. This is due to the negative work that non-conservative forces do on the system during the inelastic collision. Ultimately, this lost energy becomes heat.

2. Bouncing balls

For this problem, we will decompose the situation into four successive parts: the projectile motion of the descent, then the collision between ball 1 and the ground, then the collision between ball 1 and ball 2, and finally the projectile motion of the ascent.

1.a) **Projectile motion of the descent:** Before the first collision, both balls experience projectile motion. By conservation of mechanical energy during the entire fall, we have

$$0 + m_1 gh = \frac{m_1}{2} v_{1f}^2 + 0 \quad \text{and} \quad 0 + m_2 gh = \frac{m_2}{2} v_{2f}^2 + 0, \tag{1}$$

where we will define $y = 0$ to be the ground and let $\hat{y}$ point upwards (as shown in the problem statement). Solving both these equations for the velocity just before impact with the ground

$$v_{1f} = v_{2f} = \pm\sqrt{2gh} = -\sqrt{2gh} = -v \quad \Rightarrow \quad v = \sqrt{2gh} \tag{2}$$

reveals that it is the same for both balls (which is expected as we know that objects fall at the same rate regardless of their mass). We chose the negative sign as we know that both balls are falling downwards in the $-\hat{y}$ direction. Additionally, we chose to define a new parameter v such that it is positive.

Collision between ball 1 and the ground: Next, ball 1 experiences an elastic collision with the ground. We have the intuition that the ball will maintain its speed and reverse its direction as a result of the collision, but this can be demonstrated rigorously. We start by writing requiring conservation of momentum

$$m_1 v_{1i} + M_E v_{Ei} = m_1 v_{1f} + M_E v_{Ef}, \tag{3}$$

where the subscript E refers to the earth. We know that $v_{Ei} = 0$ (given that we have taken a laboratory reference frame) and the velocity of ball 1 just before the collision is equal to its final velocity after projectile motion (i.e. equation (2)). Thus, we can find

$$-m_1 v + 0 = m_1 v_{1f} + M_E v_{Ef} \quad \Rightarrow \quad v_{Ef} = -\frac{m_1}{M_E}\left(v + v_{1f}\right). \tag{4}$$

Since the collision is elastic, we can also enforce conservation of mechanical energy

$$\frac{m_1}{2} v^2 + 0 = \frac{m_1}{2} v_{1f}^2 + \frac{M_E}{2} v_{Ef}^2. \tag{5}$$

Substituting equation (4) shows that

$$\frac{m_1}{2} v^2 = \frac{m_1}{2} v_{1f}^2 + \frac{M_E}{2}\left(\frac{m_1}{M_E}\left(v + v_{1f}\right)\right)^2 \quad \Rightarrow \quad \frac{m_1}{2}\left(v^2 - v_{1f}^2\right) = \frac{m_1}{M_E}\frac{m_1}{2}\left(v + v_{1f}\right)^2. \tag{6}$$

Crucially, we see that the right-hand side of the equation has a factor of m_1/M_E, while the left-hand side does not. Since the earth is *much* more massive than the ball, we know that $m_1/M_E \ll 1$ and we can neglect the right-hand side of the equation. Thus, we find that ball 1 has a velocity of

$$\frac{m_1}{2}\left(v^2 - v_{1f}^2\right) = 0 \quad \Rightarrow \quad v_{1f}^2 = v^2 = 2gh \quad \Rightarrow \quad v_{1f} = \pm\sqrt{2gh} = \sqrt{2gh} = v \tag{7}$$

after colliding with the ground, where we've used equation (2) and chosen the plus sign (as the ball must change direction to avoid passing through the earth). Thus, we've found that, before the two balls collide, ball 1 has a velocity of $v = \sqrt{2gh}$, while ball 2 still has a velocity of $-v = -\sqrt{2gh}$.

Collision between ball 1 and ball 2: Now the two balls collide elastically with one another. In class, we derived the general formulas for the velocities that result from an elastic collision between two objects in one dimension. This derivation was streamlined to minimize the math, but was not intuitive. You are free to simply use the formulas we found. Here we present a more intuitive derivation that, as a result, is considerably more messy (even for the case considered here of identical initial speeds).

Enforcing conservation of momentum and using the final velocities from the previous part gives

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \quad \Rightarrow \quad m_1 v - m_2 v = m_1 v_{1f} + m_2 v_{2f} \quad \Rightarrow \quad v_{1f} = \frac{m_1 - m_2}{m_1} v - \frac{m_2}{m_1} v_{2f}. \tag{8}$$

Since the collision is elastic, we enforce conservation of mechanical energy

$$\frac{m_1}{2} v_{1i}^2 + \frac{m_2}{2} v_{2i}^2 = \frac{m_1}{2} v_{1f}^2 + \frac{m_2}{2} v_{2f}^2 \quad \Rightarrow \quad \frac{m_1 + m_2}{2} v^2 = \frac{m_1}{2} v_{1f}^2 + \frac{m_2}{2} v_{2f}^2. \tag{9}$$

Substituting equation (8) and simplifying produces

$$\frac{m_1 + m_2}{2} v^2 = \frac{m_1}{2} \left(\frac{m_1 - m_2}{m_1} v - \frac{m_2}{m_1} v_{2f} \right)^2 + \frac{m_2}{2} v_{2f}^2 \tag{10}$$

$$(m_1 + m_2) v^2 = m_1 \left(\frac{(m_1 - m_2)^2}{m_1^2} v^2 - 2 \frac{m_1 m_2 - m_2^2}{m_1^2} v v_{2f} + \frac{m_2^2}{m_1^2} v_{2f}^2 \right) + m_2 v_{2f}^2 \tag{11}$$

$$\left(m_1^2 + m_1 m_2\right) v^2 = \left(m_1^2 - 2 m_1 m_2 + m_2^2\right) v^2 - 2 \left(m_1 m_2 - m_2^2\right) v v_{2f} + m_2^2 v_{2f}^2 + m_1 m_2 v_{2f}^2 \tag{12}$$

$$0 = \left(m_2^2 - 3 m_1 m_2\right) v^2 - 2 \left(m_1 m_2 - m_2^2\right) v v_{2f} + \left(m_2^2 + m_1 m_2\right) v_{2f}^2. \tag{13}$$

Applying the quadratic formula gives

$$v_{2f} = \frac{2 \left(m_1 m_2 - m_2^2\right) v \pm \sqrt{4 \left(m_1 m_2 - m_2^2\right)^2 v^2 - 4 \left(m_2^2 + m_1 m_2\right) \left(m_2^2 - 3 m_1 m_2\right) v^2}}{2 \left(m_2^2 + m_1 m_2\right)} \tag{14}$$

$$v_{2f} = v \frac{m_1 m_2 - m_2^2 \pm \sqrt{\left(m_1^2 m_2^2 - 2 m_1 m_2^3 + m_2^4\right) - \left(m_2^4 + m_1 m_2^3 - 3 m_1 m_2^3 - 3 m_1^2 m_2^2\right)}}{m_2^2 + m_1 m_2} \tag{15}$$

$$v_{2f} = v \frac{m_1 m_2 - m_2^2 \pm 2 m_1 m_2}{m_2^2 + m_1 m_2} = v \frac{m_1 - m_2 \pm 2 m_1}{m_1 + m_2} = v \frac{(1 \pm 2) m_1 - m_2}{m_1 + m_2}. \tag{16}$$

We see that the solution with the minus sign gives $v_{2f} = -v$, which is identical to the initial velocity of ball 2. Thus, the physical solution must be the plus sign, which gives

$$v_{2f} = v \frac{3 m_1 - m_2}{m_1 + m_2} = \sqrt{2gh} \frac{3 m_1 - m_2}{m_1 + m_2} \tag{17}$$

using equation (2). Inserting this result into equation (8) allows us to find the velocity of ball 1 to be

$$v_{1f} = \frac{m_1 - m_2}{m_1} v - \frac{m_2}{m_1} \left(v \frac{3 m_1 - m_2}{m_1 + m_2} \right) = \frac{v}{m_1} \left(m_1 - m_2 + \frac{m_2^2 - 3 m_1 m_2}{m_1 + m_2} \right) \tag{18}$$

$$v_{1f} = \frac{v}{m_1} \left(\frac{(m_1 - m_2)(m_1 + m_2)}{m_1 + m_2} + \frac{m_2^2 - 3 m_1 m_2}{m_1 + m_2} \right) = \frac{v}{m_1} \frac{m_1^2 - 3 m_1 m_2}{m_1 + m_2} \tag{19}$$

$$v_{1f} = v \frac{m_1 - 3 m_2}{m_1 + m_2} = \sqrt{2gh} \frac{m_1 - 3 m_2}{m_1 + m_2}. \tag{20}$$

using equation (2).

1.b) In order for both balls to bounce off the ground upwards, we require that

$$v_{f1} > 0 \quad \text{and} \quad v_{f2} > 0. \tag{21}$$

Using equations (17) and (20), this gives the conditions that

$$m_1 > 3m_2 \text{ and } 3m_1 > m_2 \tag{22}$$

respectively. If the first is satisfied, the second will also be. Thus, for both balls to bounce upwards we require that

$$m_1 > 3m_2. \tag{23}$$

1.c) Ball 1 will end up immobile on the ground if $v_{1f} = 0$. From equation (20), we that this will occur when $m_1 = 3m_2$. If $m_1 < 3m_2$ the first ball will travel downwards and collide with the ground again.

1.d) **Projectile motion of the ascent:** After the balls collide, they again experience projectile motion. In the limit that $m_1 \gg m_2$, equation (17) shows that the initial velocity of ball 2 during this stage is

$$v_{2i} = 3v. \tag{24}$$

There are no non-conservative forces, so we can apply conservation of mechanical energy. At its maximum, ball 2 will have no kinetic energy and we can define the reference point for the gravitational potential energy to be the ground. Thus, we have

$$\frac{m_2}{2}v_{2i}^2 + 0 = 0 + m_2 g h_{2max} \quad \Rightarrow \quad h_{2max} = \frac{1}{2g}v_{2i}^2. \tag{25}$$

Substituting equations (2) and (24) give the final answer of

$$h_{2max} = \frac{9}{2g}v^2 = 9h. \tag{26}$$

2.a) The **projectile motion of the descent** and the **collision between ball 1 and the ground** remain unchanged.

Collision between ball 1 and ball 2: The difference only arises when the two balls collide as the collision is now purely inelastic. This means that kinetic energy is not conserved. Momentum is still conserved, but, since the two balls stick together, we know that $v_{1f} = v_{2f} = v_f$. Thus, equation (8) is replaced by

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_f + m_2 v_f \quad \Rightarrow \quad m_1 v - m_2 v = (m_1 + m_2)\, v_f \quad \Rightarrow \quad v_f = \frac{m_1 - m_2}{m_1 + m_2}v. \tag{27}$$

Substituting equation (2) gives the final answer of

$$v_f = \frac{m_1 - m_2}{m_1 + m_2}\sqrt{2gh} \tag{28}$$

2.b) For the two balls to go upwards, the final velocity $v_f > 0$ must be positive. Using equation (28), we see that this occurs when $m_1 > m_2$.

2.c) Ball 1 will end up immobile on the ground if $v_{1f} = v_f = 0$. From equation (28), we that this will occur when $m_1 = m_2$. If $m_1 < m_2$ the first ball will travel downwards and collide with the ground again.

2.d) **Projectile motion of the ascent:** In the limit that $m_1 \gg m_2$, equation (28) becomes

$$v_f = \sqrt{2gh}. \tag{29}$$

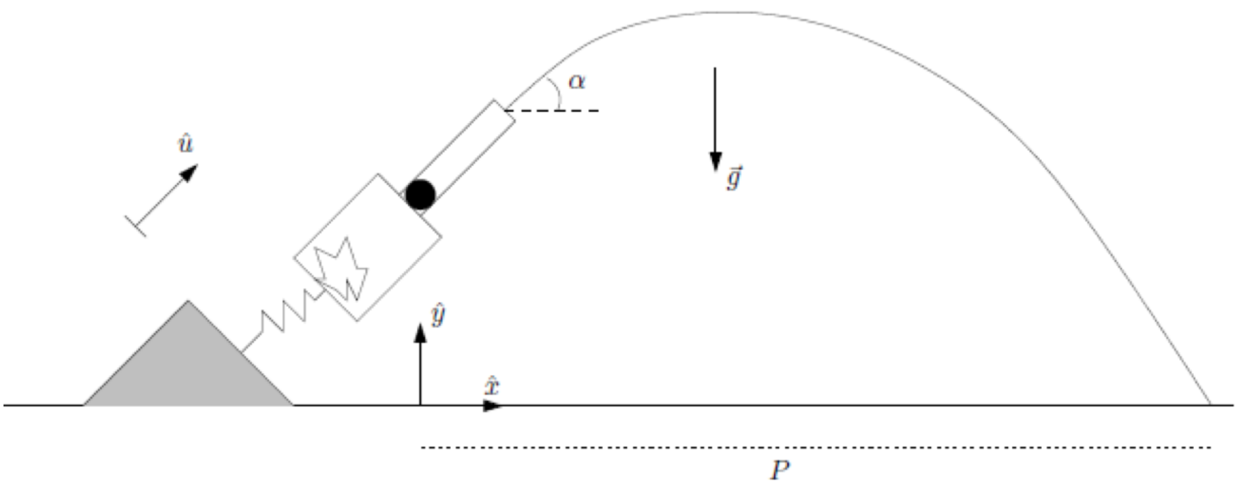
Since the two balls undergo projectile motion together after the collision. We can apply conservation of energy between the ground and the point of maximum height according to

$$\frac{m_1 + m_2}{2}v_f^2 + 0 = 0 + (m_1 + m_2) g h_{2max}. \tag{30}$$

Substituting equation (29) gives the final answer of

$$h_{2max} = \frac{v_f^2}{2g} = h. \tag{31}$$

3. Damped cannon



1. As a result of the explosion, the entire cannon moves backwards and the spring compresses. The energy of the system is comprised of kinetic K and spring potential energy U_s. We will ignore any small change in the elevation of the cannon, thereby neglecting the gravitational potential energy U_g. Given that energy is always conserved, we have

$$K_i + U_{si} = K_f + U_{sf}. \tag{1}$$

We will take the initial state to be the moment immediately after the explosion, when the cannon is moving at its maximum speed V_0, but the spring is still at its equilibrium position. For the final state, we will take the moment when the spring has its maximum displacement Δx_{max} and the cannon has come to a temporary rest. Thus, equation (1) becomes

$$\frac{M}{2}V_0^2 + 0 = 0 + \frac{k}{2}\Delta x_{max}^2 \quad \Rightarrow \quad \Delta x_{max} = \sqrt{\frac{M}{k}}V_0. \tag{2}$$

Unfortunately, we do not know the initial speed of the cannon V_0. To find this, we must use the information given in the problem statement about the dynamics of the cannonball. To relate the dynamics of the cannon to that of the cannonball, we can use conservation of momentum. Given that the cannon-cannonball system is initially at rest, conservation of momentum is

$$0 = mv_0\hat{u} - MV_0\hat{u} \quad \Rightarrow \quad V_0 = \frac{m}{M}v_0, \tag{3}$$

where v_0 is the initial velocity of the cannonball and the $\hat{u}$ direction is defined in the diagram above. Substituting this into equation (2) yields

$$\Delta x_{max} = \frac{m}{\sqrt{kM}}v_0. \tag{4}$$

To find the velocity v_0, we can use the equations of projectile motion

$$x(t) = v_{0x}t + x_0 \tag{5}$$
$$y(t) = -\frac{g}{2}t^2 + v_{0y}t + y_0, \tag{6}$$

where we have defined a Cartesian coordinate system as shown in the diagram above. We will define the origin of the coordinate system to be the cannon, such that the cannonball is at $x_0 = 0$ and $y_0 = 0$ when the cannon fires at $t = 0$. Given the information in the problem statement, we know that $v_{0x} = v_0 \cos\alpha$ and $v_{0y} = v_0 \sin\alpha$. Thus, equations (5) and (6) become

$$x(t) = v_0 t \cos\alpha \tag{7}$$
$$y(t) = -\frac{g}{2}t^2 + v_0 t \sin\alpha. \tag{8}$$

At some time $t = t_f$, we know that the cannonball will land at the location

$$x(t_f) = P = v_0 t_f \cos\alpha \quad \Rightarrow \quad t_f = \frac{P}{v_0 \cos\alpha} \tag{9}$$
$$y(t_f) = 0 = -\frac{g}{2}t_f^2 + v_0 t_f \sin\alpha \quad \Rightarrow \quad 0 = -\frac{g}{2}t_f + v_0 \sin\alpha \quad \Rightarrow \quad t_f = \frac{2}{g}v_0 \sin\alpha. \tag{10}$$

Equating equations (9) and (10) gives

$$\frac{P}{v_0 \cos\alpha} = \frac{2}{g}v_0 \sin\alpha \quad \Rightarrow \quad v_0 = \sqrt{\frac{gP}{2\sin\alpha\cos\alpha}}. \tag{11}$$

Substituting this into equation (4) gives the final answer of

$$\Delta x_{max} = \frac{m}{\sqrt{kM}}\sqrt{\frac{gP}{2\sin\alpha\cos\alpha}}. \tag{12}$$

Plugging in the numerical values from the problem statement gives

$$\Delta x_{max} = 0.2 \text{ m}, \tag{13}$$

where we note that $M = 10$ tons $= 10^4$ kg.

2. The energy delivered by the explosion is equal to the energy imparted to the cannon and the cannonball combined. From equation (2), we see that we can represent the total energy of the cannon by its maximum spring potential energy $U_{sf} = (k/2)\Delta x_{max}^2$. On the other hand, the total energy of the cannonball is just its initial kinetic energy $K_i = (m/2)v_0^2$. Thus, we have

$$E_{tot} = \frac{k}{2}\Delta x_{max}^2 + \frac{m}{2}v_0^2 = \frac{k}{2}\Delta x_{max}^2 + \frac{m}{2}\frac{gP}{2\sin\alpha\cos\alpha}, \tag{14}$$

where we have used equation (11). Plugging in numerical values (including equation (13)) gives

$$E_{tot} = 2 \times 10^6 \text{ J}. \tag{15}$$

4. Space collision

1. The initial speed of the projectile is equal to the magnitude of its escape velocity. Given the definition of escape velocity, this can be found from the condition that

$$K_0 + U_{G0} = 0, \tag{1}$$

where the subscript 0 indicates the initial value and U_G indicates the general gravitational potential (which is distinct from the gravitational potential near the earth's surface $U_g = mgy$). We can calculate U_G from the form of the gravitational force between the two objects

$$\vec{F}_G(r) = -\frac{GmM_e}{r^2}\hat{r}. \tag{2}$$

Choosing to use a spherical coordinate system, the change in the potential energy due to the force $\vec{F}_G$ is

$$\begin{aligned}\Delta U_G = U_G(R) - U_G(\infty) = -\int_C \vec{F}_G \cdot d\vec{l} = -\int_C \left(-\frac{GmM_e}{r^2}\hat{r}\right) \cdot \left(dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}\right) \\ = \int_C \left(\frac{GmM_e}{r^2}\right) dr, \end{aligned}\tag{3}$$

where the integration path C is along any path from $r = \infty$ to $r = R$. However, we see that, since the force is purely radial, only the change in radial position matters. Thus, we can write

$$\Delta U_G = U_G(R) - U_G(\infty) = \int_\infty^R \frac{GmM_e}{r^2} dr. \tag{4}$$

Taking the integral gives a potential energy difference of

$$\Delta U_G = U_G(R) - U_G(\infty) = \left(-\frac{GmM_e}{r}\right|_\infty^R = -\frac{GmM_e}{R}. \tag{5}$$

Since the reference point for the potential energy is at $R = \infty$ (i.e. we define the potential energy such that $U(\infty) = 0$), equation (5) implies that

$$U_G(R) = -\frac{GmM_e}{R}. \tag{6}$$

Substituting this into equation (1) allows us to find the initial speed of the projectile when it is launched

$$\frac{m}{2}v_0^2 - \frac{GmM_e}{R_e} = 0 \quad \Rightarrow \quad v_0 = \sqrt{\frac{2GM_e}{R_e}}, \tag{7}$$

where $R = R_e$ is the initial position.

2. Since all the forces are conservative, mechanical energy is conserved and we have

$$K_c + U_{Gc} = K_0 + U_{G0}, \tag{8}$$

where the subscript c indicates the value just before the collision. Therefore, we can use equations (1) and (6) to write equation (8) as

$$K_c + U_{Gc} = 0 \quad \Rightarrow \quad \frac{m}{2}v_c^2 - \frac{GmM_e}{2R_e} = 0 \quad \Rightarrow \quad v_c = \sqrt{\frac{GM_e}{R_e}} \tag{9}$$

at the location just before the collision $R_c = 2R_e$.

3. Drawing a free body diagram for the satellite and using the gravitational force given by equation (2), we see that Newton's second law in the $\hat{r}$ direction is

$$-\frac{GmM_e}{(2R_e)^2} = -m\left(\frac{v_s^2}{2R_e}\right), \tag{10}$$

where we have used the form of the centripetal acceleration and v_s is the speed of the satellite. Solving this equation for the speed gives

$$v_s = \sqrt{\frac{GM_e}{2R_e}}. \tag{11}$$

4. Given the the collision happens quickly, we can use the impulse approximation to ignore the effect of gravity *during* the collision. Thus, momentum is conserved throughout the collision, so we can write

$$m\vec{v}_{pi} + m\vec{v}_{si} = m\vec{v}_{pf} + m\vec{v}_{sf} \quad \Rightarrow \quad \vec{v}_{pi} + \vec{v}_{si} = \vec{v}_{pf} + \vec{v}_{sf}, \tag{12}$$

where the subscript p indicates the projectile, the subscript s indicates the satellite, the subscript i indicates the value just before the collision, and the subscript f indicates the value just after the collision. Since the collision is purely inelastic and the objects stick together, instead of conservation of energy, we require that the final velocities of the projectile and satellite be the same

$$\vec{v}_{pf} = \vec{v}_{sf} = \vec{v}_f. \tag{13}$$

We will adopt a polar coordinate system, such that the projectile is moving in the $\hat{r}$ direction before the collision and the satellite is moving in the $\hat{\phi}$ direction before the collision. From our solutions to parts 2 and 3 of this problem, we see that $\vec{v}_{pi} = v_c\hat{r} = \sqrt{GM_e/R_e}\hat{r}$ and $\vec{v}_{si} = v_s\hat{\phi} = \sqrt{GM_e/(2R_e)}\hat{\phi}$. Substituting these initial values and equation (13) into equation (12), we find the final velocity of the combined projectile-satellite object to be

$$\sqrt{\frac{GM_e}{R_e}}\hat{r} + \sqrt{\frac{GM_e}{2R_e}}\hat{\phi} = 2\vec{v}_f \quad \Rightarrow \quad \vec{v}_f = \frac{1}{2}\left(\sqrt{\frac{GM_e}{R_e}}\hat{r} + \sqrt{\frac{GM_e}{2R_e}}\hat{\phi}\right). \tag{14}$$

To get the speed, we simply take the magnitude according to

$$\begin{aligned} v_f &= \sqrt{\vec{v}_f \cdot \vec{v}_f} = \sqrt{\frac{1}{4}\left(\sqrt{\frac{GM_e}{R_e}}\hat{r} + \sqrt{\frac{GM_e}{2R_e}}\hat{\phi}\right)\cdot\left(\sqrt{\frac{GM_e}{R_e}}\hat{r} + \sqrt{\frac{GM_e}{2R_e}}\hat{\phi}\right)} = \frac{1}{2}\sqrt{\frac{GM_e}{R_e} + \frac{GM_e}{2R_e}} \\ v_f &= \frac{1}{2}\sqrt{\frac{3GM_e}{2R_e}}. \end{aligned} \tag{15}$$

5. Stream bouncing off a wall

We start by considering a single particle as it collides with the surface. The change in its momentum due to the collision is given by

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = -mv\hat{x} - mv\hat{x} = -2mv\hat{x}, \tag{1}$$

where we have defined the $\hat{x}$ direction to point to the right. This change in momentum is related to the force of the wall on the particle $\vec{F}_{wp}$ through the impulse, which is given by

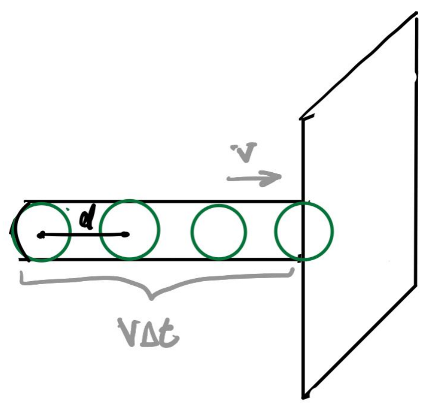
$$\Delta\vec{p} = \vec{I} = \int_{t_i}^{t_f} \vec{F}_{wp}dt. \tag{2}$$

Since we only care about the average force (and not the details about how it changes with time), we can model the impulse as an average force applied over the same time interval according to

$$\vec{I} = \int_{t_i}^{t_f} \vec{F}_{wp} dt = \vec{F}_{wp}^{avg}(t_f - t_i) = \vec{F}_{wp}^{avg}\Delta t, \tag{3}$$

where the time interval $\Delta t = t_f - t_i$ is the time between successive particles hitting the wall. This time interval Δt is straightforward to calculate as the particles are a distance d apart and travel at a constant velocity of v. Thus, immediately after one hits, the time before the next one hits will be equal to the time it takes a particle to travel a distance d. This gives

$$d = v\Delta t \quad \Rightarrow \quad \Delta t = \frac{d}{v}. \tag{4}$$



Substituting equations (1), (3), and (4) into equation (2) gives

$$-2mv\hat{x} = \vec{F}_{wp}^{avg}\Delta t \quad \Rightarrow \quad -2mv\hat{x} = \vec{F}_{wp}^{avg}\left(\frac{d}{v}\right) \quad \Rightarrow \quad \vec{F}_{wp}^{avg} = -\frac{2mv^2}{d}\hat{x}. \tag{5}$$

By Newton's third law, the force exerted by the wall on the particles has the same magnitude as the force of the particles on the wall. Thus, we find that

$$F_{pw}^{avg} = \frac{2mv^2}{d}. \tag{6}$$