Solutions to Problem Set 9

Potential energy, conservation of energy

PHYS-101(en)

1. Spring-propelled block going through a loop

1. As the block is initially at rest $v_0 = 0$, the initial kinetic energy before the latch is released is $K_0 = mv_0^2/2 = 0$. The initial potential energy of the compression of the spring is given by $U_{s0} = kx^2/2$. We will define the origin of our coordinate system to be at the height of the horizontal track, so the initial height of the block is $y_0 = 0$ and the initial gravitational potential energy is $U_{g0} = mgy_0 = 0$. Therefore, the total initial energy is

$$E_0 = K_0 + U_{s0} + U_{g0} = 0 + \frac{k}{2}x^2 + 0 = \frac{k}{2}x^2. \tag{1}$$

At the top of the loop $y_t = 2R$, the kinetic energy is $K_t = mv_t^2/2$, the gravitational potential energy is $U_{gt} = mgy_t = 2mgR$, and the spring is no longer in contact with the block so $U_{st} = 0$. Thus, the total energy at the top of the loop is given by

$$E_t = K_t + U_{st} + U_{gt} = \frac{m}{2}v_t^2 + 0 + 2mgR = \frac{m}{2}v_t^2 + 2mgR. \tag{2}$$

Since the track is frictionless and there is no air drag, mechanical energy is conserved and we can use equations (1) and (2) to write

$$E_0 = E_t \quad \Rightarrow \quad \frac{k}{2}x^2 = \frac{m}{2}v_t^2 + 2mgR \quad \Rightarrow \quad K_t = \frac{m}{2}v_t^2 = \frac{k}{2}x^2 - 2mgR. \tag{3}$$

2. From Newton's second law and the properties of uniform circular motion, we know that the net force in the radial direction is equal to the centripetal force. Thus, at the top of the loop Newton's second law in the $\hat{y}$ direction is

$$-mg - N = -\frac{mv_t^2}{R}, \tag{4}$$

where $\hat{y}$ is defined to point upwards. From the information given in the problem statement we know that $N = 2mg$, so equation (4) becomes

$$3mg = \frac{mv_t^2}{R} \quad \Rightarrow \quad v_t = \sqrt{3gR}. \tag{5}$$

3. We can substitute equation (5) into equation (3) and solve for x to find

$$\frac{m}{2}\left(\sqrt{3gR}\right)^2 = \frac{k}{2}x^2 - 2mgR \quad \Rightarrow \quad x = \sqrt{\frac{7mgR}{k}}. \tag{6}$$

2. Two-body interaction

1. Choosing to use a spherical coordinate system, the change in the potential energy due to the force $\vec{F}$ is

$$\begin{aligned}\Delta U = U(R) - U(\infty) = -\int_C \vec{F}\cdot d\vec{l} = -\int_C \left(-\frac{Gm_1m_2}{r^2} + \frac{C}{r^3}\right)\hat{r}\cdot\left(dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}\right)\\ = \int_C \left(\frac{Gm_1m_2}{r^2} - \frac{C}{r^3}\right)dr,\end{aligned} \tag{1}$$

where the integration path C is along the trajectory from $r = \infty$ to $r = R$. However, we see that, since the force is purely radial, only the change in the radial position matters. Thus, we can write

$$\Delta U = U(R) - U(\infty) = \int_\infty^R \frac{Gm_1m_2}{r^2} - \frac{C}{r^3}dr. \tag{2}$$

Taking the integral gives a potential energy difference of

$$\Delta U = U(R) - U(\infty) = \left(-\frac{Gm_1m_2}{r} + \frac{C}{2r^2}\right|_\infty^R = -\frac{Gm_1m_2}{R} + \frac{C}{2R^2}. \tag{3}$$

Since the reference point for the potential energy is at $r = \infty$ (i.e. we define the potential energy such that $U(\infty) = 0$), equation (3) implies that

$$U(R) = -\frac{Gm_1m_2}{R} + \frac{C}{2R^2}, \tag{4}$$

which will be useful in the next part of the problem.

2. Equilibrium occurs when the force on a particle is zero. Thus, we can find the locations $R = R_0$ that satisfy this by enforcing

$$\vec{F}(R_0) = 0 \quad \Rightarrow \quad \left(-\frac{Gm_1m_2}{R_0^2} + \frac{C}{R_0^3}\right)\hat{r} = 0 \quad \Rightarrow \quad \frac{Gm_1m_2}{R_0^2} = \frac{C}{R_0^3}. \tag{5}$$

Solving for R_0 shows that there is just one equilibrium point at

$$R_0 = \frac{C}{Gm_1m_2}. \tag{6}$$

Using this result and equation (4), we find that the value of the potential energy at $R = R_0$ is

$$U(R_0) = -\frac{Gm_1m_2}{R_0} + \frac{C}{2R_0^2} = -\frac{(Gm_1m_2)^2}{C} + \frac{(Gm_1m_2)^2}{2C} = -\frac{(Gm_1m_2)^2}{2C}. \tag{7}$$

To see if this equilibrium point is stable or not, we can calculate the second derivative of the potential energy and see if it is positive or negative. Using equation (4) to do so gives

$$\frac{dU}{dR} = \frac{Gm_1m_2}{R^2} - \frac{C}{R^3} \quad \Rightarrow \quad \frac{d^2U}{dR^2} = -2\frac{Gm_1m_2}{R^3} + 3\frac{C}{R^4}. \tag{8}$$

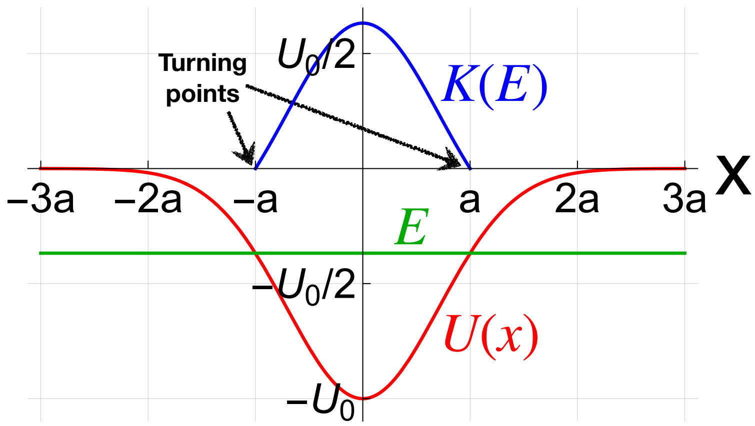
Evaluating this at $R = R_0$ using equation (6) and simplifying produces

$$\left.\frac{d^2U}{dR^2}\right|_{R=R_0} = -2\frac{Gm_1m_2}{R_0^3} + 3\frac{C}{R_0^4} = -2\frac{(Gm_1m_2)^4}{C^3} + 3\frac{(Gm_1m_2)^4}{C^3} = \frac{(Gm_1m_2)^4}{C^3}. \tag{9}$$

Given that C must be positive for the repulsive portion of the force in the problem statement to be repulsive, we see that $d^2U/dR^2|_{R=R_0} > 0$, so the equilibrium point is stable.

3. A particle in Gaussian potential

1. The energy diagram is shown below, where we note that the kinetic energy $K(x) = E - U(x)$ is the difference between the total energy E and the potential energy $U(x)$ due to conservation of mechanical energy.



2. The force on the particle in the x direction is calculated from

$$F(x) = -\frac{dU}{dx}. \tag{1}$$

Plugging in the form of $U(x)$ and using the chain rule gives

$$F(x) = -\frac{d}{dx}\left(-U_0 e^{-x^2/a^2}\right) = U_0 e^{-x^2/a^2}\frac{d}{dx}\left(\frac{-x^2}{a^2}\right) = -2\frac{U_0}{a^2} x e^{-x^2/a^2}. \tag{2}$$

3. To find the speed v_0 at $x = 0$, we can use conservation of mechanical energy between any two points x_1 and x_2 according to

$$E(x_1) = E(x_2) \quad \Rightarrow \quad K(x_1) + U(x_1) = K(x_2) + U(x_2). \tag{3}$$

Obviously we want to take one of the points to be the origin $x_1 = 0$ as it is where we want to find the speed. At this location the kinetic energy is related to the speed by $K(x_1) = K(0) = mv_0^2/2$. However, the second point requires more thought. It is best to choose one of the turning points because we know that the velocity is zero as the particle is reversing its direction. Thus, at $x_2 = a$ the kinetic energy is $K(x_2) = K(a) = 0$. Plugging in these facts and the form of $U(x)$ into equation (3) yields

$$\frac{m}{2}v_0^2 - U_0 e^{-0^2/a^2} = 0 - U_0 e^{-a^2/a^2} \quad \Rightarrow \quad \frac{m}{2}v_0^2 - U_0 = -U_0 e^{-1}. \tag{4}$$

Solving for v_0 gives

$$v_0 = \sqrt{\frac{2U_0}{m}(1 - e^{-1})}. \tag{5}$$

4. Circular loop

This problem is similar to problem 1. As the block is initially at rest $v_0 = 0$, the initial kinetic energy is $K_0 = mv_0^2/2 = 0$. The initial potential energy is due to gravity. We will define the reference point to be at the height of the table, so the initial height of the object is $y_0 = h$ and the initial gravitational potential energy is $U_{g0} = mgh$. Therefore, the total initial energy is

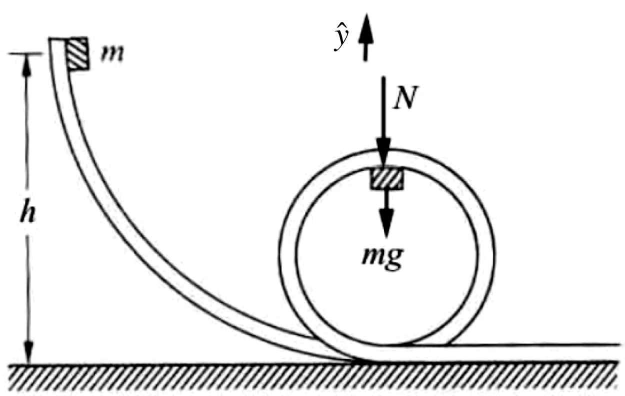
$$E_0 = K_0 + U_{g0} = 0 + mgh = mgh. \tag{1}$$

At the top of the loop $y_t = 2R$, the kinetic energy is $K_t = mv_t^2/2$ and the gravitational potential energy is $U_{gt} = mgy_t = 2mgR$. Thus, the total energy at the top of the loop is given by

$$E_t = K_t + U_{gt} = \frac{m}{2}v_t^2 + 2mgR. \tag{2}$$

Since the track is frictionless and there is no air drag, there is no non-conservative force and mechanical energy is conserved. We can use equations (1) and (2) to write

$$E_0 = E_t \quad \Rightarrow \quad mgh = \frac{m}{2}v_t^2 + 2mgR. \tag{3}$$



From the properties of circular motion, we know that the acceleration in the radial direction is equal to the centripetal acceleration $a_{cent} = -v_t^2/R$. Thus, at the top of the loop (shown in figure above) Newton's second law in the $\hat{y}$ direction is

$$-mg - N = -\frac{mv_t^2}{R}. \tag{4}$$

From the information given in the problem statement, we know that the mass just barely doesn't lose contact with the track. Thus, the normal force of the track on the mass must be effectively $N = 0$, so equation (4) becomes

$$mg = \frac{mv_t^2}{R} \quad \Rightarrow \quad v_t = \sqrt{gR}. \tag{5}$$

We can substitute equation (5) into equation (3) and solve for h to find

$$mgh = \frac{m}{2}\left(\sqrt{gR}\right)^2 + 2mgR \quad \Rightarrow \quad h = 2.5R. \tag{6}$$

5. Review: Tension in a massive rope

There are several methods to solve this problem. Below we show two of them.

1. In method 1, we will use differential elements. To calculate the tension, we start by considering a small piece of rope with length Δx and its left end located at an arbitrary position x along the rope. Here we define a coordinate system where $x = 0$ is the location at which the rope connects to the block and $\hat{x}$ points to the right. Since the rope has a uniform linear mass density λ, we can calculate the mass of the differential element Δm to be

$$\lambda = \frac{M}{L} = \frac{\Delta m}{\Delta x} \quad \Rightarrow \quad \Delta m = \frac{M}{L}\Delta x. \tag{1}$$

Drawing a free body diagram for the piece of rope, we see that Newton's second law in the $\hat{x}$ direction is given by

$$T(x + \Delta x) - T(x) = \Delta m a_r, \tag{2}$$

where a_r is the acceleration of the piece of rope. Substituting equation (1) and rearranging gives

$$T(x + \Delta x) - T(x) = \left(\frac{M}{L}\Delta x\right) a_r \quad \Rightarrow \quad \frac{T(x + \Delta x) - T(x)}{\Delta x} = \frac{M}{L} a_r. \tag{3}$$

Taking the limit that the differential element is infinitesimally small $\Delta x \to 0$ produces the differential equation

$$\frac{dT}{dx} = \frac{M}{L} a_r. \tag{4}$$

Importantly, since the rope does not stretch, we know that the entire rope must move together. This implies that all the different pieces of the rope accelerate together. In other words, $a_r = a$ does not vary with x location. This can be seen as a constraint condition. Thus, we can directly integrate equation (4) to find

$$T(x) = \frac{M}{L} a x + C, \tag{5}$$

where C is an integration constant.

To find the value of C we must apply a boundary condition. We can *either* use $T(L) = F_a$ or find a condition on $T(0)$ by analyzing the block. The first is quicker and allows us to directly calculate

$$T(L) = F_a = \frac{M}{L} a L + C \quad \Rightarrow \quad C = F_a - Ma \tag{6}$$

from equation (5). The second is more work. We draw a free body diagram for the block and see that Newton's second law in the $\hat{x}$ direction is

$$F_{rb} - F_f = m_b a, \tag{7}$$

where F_{rb} is the force of the rope on the block and we know that the block must have the same acceleration a as the rope. To calculate the kinetic friction force $F_f = \mu_k N$, we must know the normal force N. This can be found from the $\hat{y}$ component of Newton's second law for the block to be

$$N - m_b g = 0 \quad \Rightarrow \quad N = m_b g \tag{8}$$

since the block has no acceleration in the vertical direction. Thus, the kinetic friction force is given by

$$F_f = \mu_k m_b g. \tag{9}$$

Substituting this into equation (7) allows us to find the force of the rope on the block to be

$$F_{rb} = \mu_k m_b g + m_b a. \tag{10}$$

Newton's third law tells us that the magnitudes of the forces between the rope and block must be equal (i.e. $F_{rb} = F_{br}$). Additionally, we know that the force of the rope on the block is identical to the tension at the end of the rope (i.e. $T(0) = F_{rb}$). Thus, we can use these facts and equation (10) to show

$$T(0) = F_{rb} = F_{br} = \mu_k m_b g + m_b a. \tag{11}$$

Finally, we can solve for the integration constant C by evaluating equation (5) at $x = 0$ and using equation (11) to get

$$T(0) = \mu_k m_b g + m_b a = 0 + C \quad \Rightarrow \quad C = \mu_k m_b g + m_b a. \tag{12}$$

Here we have used two different ways to calculate the integration constant C. Equations (6) and (12) appear different, but we will soon see that they are identical.

Equation (5), together with either equation (6) or (12), gives the tension as a function of position. However, they contain the acceleration a, which we still do not know. To find it, we can consider the system of both the rope and the block together. Newton's second law in the $\hat{x}$ direction for the whole system is

$$F_a - F_f = (M + m_b)a. \tag{13}$$

Substituting equation (9) and solving for a gives

$$F_a - \mu_k m_b g = (M + m_b)a \quad \Rightarrow \quad a = \frac{F_a - \mu_k m_b g}{M + m_b}. \tag{14}$$

If we substitute this into either equation (6) or (12) and perform some algebra, we find that they both produce the same result of

$$C = \frac{m_b}{M + m_b} \left(F_a + \mu_k M g \right). \tag{15}$$

We can substitute this into equation (5) and use equation (14) to replace a to arrive at

$$T(x) = \frac{M}{L} \left(\frac{F_a - \mu_k m_b g}{M + m_b} \right) x + \frac{m_b}{M + m_b} \left(F_a + \mu_k M g \right). \tag{16}$$

After considerable algebra we can somewhat simplify this result to

$$T(x) = \frac{M}{M + m_b} \left(\left(1 - \frac{x}{L} \right) \mu_k m_b g + \left(\frac{m_b}{M} + \frac{x}{L} \right) F_a \right). \tag{17}$$

2. In method 2, we will simply divide the rope into two parts. Start by considering an arbitrary position x along the rope (where $x = 0$ is defined to be where the rope connects to the block). Since the rope has a uniform mass distribution, we can calculate the total mass of the rope to the *right* of that point to be

$$m_r = \frac{M}{L} \left(L - x \right), \tag{18}$$

while the total mass of the rope to the *left* of that point is

$$m_l = \frac{M}{L} x. \tag{19}$$

Now we can draw a free body diagram for the entire *right* side of the rope and see that Newton's second law in the $\hat{x}$ direction is

$$F_a - T\left(x\right) = m_r a_{rx} \quad \Rightarrow \quad F_a - T\left(x\right) = \frac{M}{L} \left(L - x \right) a_{rx} \quad \Rightarrow \quad a_{rx} = \frac{L}{M \left(L - x \right)} \left(F_a - T\left(x\right) \right), \tag{20}$$

where a_{rx} is the acceleration of the right side of the rope in the $\hat{x}$ direction and $T(x)$ is the tension in the rope at the position x. Also note that we have used equation (18) and the fact that the rope is completely horizontal. We can do the same for the *left* side of the rope and use equation (19) to find

$$T\left(x\right) - F_{br} = m_l a_{lx} \quad \Rightarrow \quad T\left(x\right) - F_{br} = \frac{M}{L} x a_{lx}, \tag{21}$$

where F_{br} is the force of the block on the rope and a_{lx} is the acceleration of the left side of the rope in the $\hat{x}$ direction.

From equation (21) we see that we must consider the block. Drawing a free body diagram, we can see that Newton's second law in the $\hat{x}$ direction is

$$F_{rb} - F_f = m_b a_{bx}, \tag{22}$$

where F_{rb} is the force of the rope on the block and a_{bx} is the acceleration of the block. To calculate the kinetic friction force $F_f = \mu_k N$, we must know the normal force N. This can be found from the $\hat{y}$ component of Newton's second law to be

$$N - m_b g = 0 \quad \Rightarrow \quad N = m_b g \tag{23}$$

since the block has no acceleration in the vertical direction. Substituting equation (23) and the form of the kinetic friction force into equation (22) allows us to find the force of the rope on the block to be

$$F_{rb} = \mu_k m_b g + m_b a_{bx}. \tag{24}$$

Newton's third law tells us that the magnitudes of the forces between the rope and block must be equal (i.e. $F_{rb} = F_{br}$). Thus, we can substitute equation (24) into equation (21) using $F_{rb} = F_{br}$ to find

$$T\left(x\right) - \left(\mu_k m_b g + m_b a_{bx}\right) = \frac{M}{L} x a_{lx}. \tag{25}$$

Now we must determine the constraint condition on the system to relate a_{bx}, a_{lx}, and a_{rx}. Since the entire block-rope system will move together in the positive $\hat{x}$ direction, we can directly see that

$$a_{bx} = a_{lx} = a_{rx}. \tag{26}$$

Thus, we can substitute equation (20) in for both a_{bx} and a_{lx} in equation (25) to find an equation for the tension

$$T\left(x\right) - \mu_k m_b g - m_b \left(\frac{L}{M\left(L-x\right)}\left(F_a - T\left(x\right)\right)\right) = \frac{M}{L} x \left(\frac{L}{M\left(L-x\right)}\left(F_a - T\left(x\right)\right)\right). \tag{27}$$

After a considerable amount of algebra, left as an exercise for the reader ;), we can solve this equation for the tension to find

$$T\left(x\right) = \frac{M}{M + m_b}\left(\left(1 - \frac{x}{L}\right)\mu_k m_b g + \left(\frac{m_b}{M} + \frac{x}{L}\right)F_a\right). \tag{28}$$