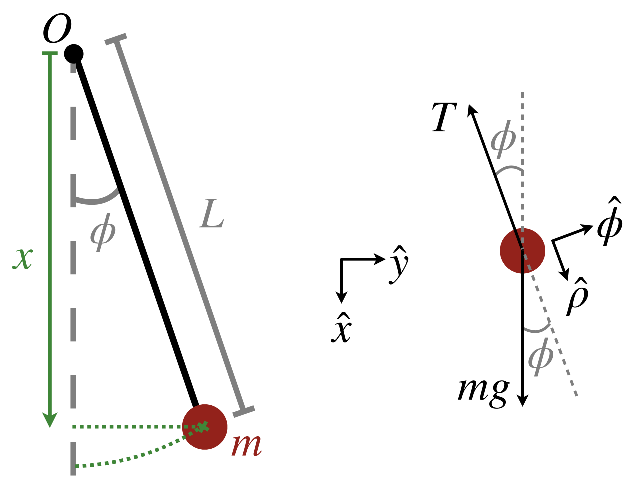
Solutions to Problem Set 13

Harmonic motion and gyroscopes

PHYS-101(en)

1. Simple pendulum

Below is the free body diagram for the point mass of the simple pendulum in polar coordinates at an arbitrary angular position of ϕ. We define a polar coordinate system, which is natural for describing the rotational motion of the mass. Additionally, we define a Cartesian coordinate system such that the polar angle ϕ measures the angle from the x axis towards the positive y axis (as is conventional).



1. We can solve part one in two ways: using Newton's second law or conservation of energy.

   Using Newton's second law: At a given angular position, the gravitational force on the point mass is given by

   $$\vec{F}_g = mg\hat{x} = mg\left(\cos\phi\hat{\rho} - \sin\phi\hat{\phi}\right), \tag{1}$$

   where we have related the Cartesian and polar unit vectors using the formula $\hat{x} = \cos\phi\hat{\rho} - \sin\phi\hat{\phi}$ from lecture 4a. We can calculate the tension from the radial component of Newton's second law using the centripetal acceleration $\vec{a}_{cent} = -(v_\phi^2/L)\hat{\rho}$ to get

   $$mg\cos\phi - T = ma_{cent} = -m\frac{v_\phi^2}{L} \quad \Rightarrow \quad T = mg\cos\phi + m\frac{v_\phi^2}{L}, \tag{2}$$

   where v_ϕ is the speed of the point mass (which is purely tangential). However, this will not be needed for to solve the problem. Instead, the tangential component of Newton's second law is useful

   $$-mg\sin\phi = ma_\phi \quad \Rightarrow \quad -g\sin\phi = a_\phi. \tag{3}$$

   The form of the acceleration in polar coordinates is given by

   $$\vec{a} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\rho} + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{\phi}. \tag{4}$$

Since $\rho = L$ is a constant, we have $\dot{\rho} = 0$ and we can substitute the tangential component of equation (4) into equation (3) to find

$$-g \sin\phi = L\frac{d^2\phi}{dt^2} \quad \Rightarrow \quad \frac{d^2\phi}{dt^2} + \frac{g}{L}\sin\phi = 0. \tag{5}$$

Using the small angle approximation $\sin\phi \approx \phi$ gives the differential equation of a simple harmonic oscillator

$$\frac{d^2\phi}{dt^2} + \frac{g}{L}\phi = 0. \tag{6}$$

Using conservation of energy: Since all of the forces acting on the pendulum are conservative, we can impose conservation of mechanical energy

$$E_{m0} = E_{mf} \tag{7}$$

between the initial state described in the problem (denoted by the subscript 0) and the final state when the pendulum is at any arbitrary angular position ϕ (denoted by the subscript f). The only forces involved are gravity, so equation (7) is

$$K_0 + U_{g0} = K_f + U_{gf}. \tag{8}$$

The system is released from rest, so $K_0 = 0$. Additionally, we will define the reference point for the gravitational potential energy to be the origin O of our polar coordinate system (i.e. the location of the pivot point at the top of the string). Given the Cartesian coordinate system shown above, the gravitational potential energy of the point mass is given by

$$U_g = -mgx = -mg\rho\cos\phi = -mgL\cos\phi, \tag{9}$$

where we have expressed the Cartesian coordinate x in polar coordinates using the formula $x = \rho\cos\phi$ from lecture 4a. Substituting this, $K_0 = 0$, and the form of the rotational kinetic energy $K = (I/2)\omega^2$ into equation (8) gives

$$0 - mgL\cos\phi_0 = \frac{I}{2}\omega^2 - mgL\cos\phi, \tag{10}$$

where ω is the angular speed. For a point mass located a distance $\rho = L$ away from the axis of rotation, the moment of inertia is $I = mL^2$. Employing this and taking the time derivative of equation (10) using the chain rule yields

$$0 = \frac{mL^2}{2}\left(2\omega\frac{d\omega}{dt}\right) + mgL\sin\phi\frac{d\phi}{dt} \quad \Rightarrow \quad 0 = L\omega\frac{d\omega}{dt} + g\sin\phi\frac{d\phi}{dt}. \tag{11}$$

Identifying that $\omega = d\phi/dt$ produces

$$0 = L\omega\frac{d^2\phi}{dt^2} + g\omega\sin\phi \quad \Rightarrow \quad \frac{d^2\phi}{dt^2} + \frac{g}{L}\sin\phi = 0, \tag{12}$$

which is identical to equation (5). Thus, energy conservation gives the same answer as Newton's second law.

2. The differential equation for a simple harmonic oscillator has the form

$$\frac{d^2\phi}{dt^2} + \omega_0^2\phi = 0, \tag{13}$$

where ω_0 is the angular frequency of oscillation. Thus, by comparison with equation (6), we see that

$$\omega_0 = \sqrt{\frac{g}{L}}. \tag{14}$$

To get the frequency f_0 (i.e. the number of oscillations per second) from the angular frequency (i.e. the average number of radians the object completes in its oscillation per second), we use the fact that one oscillation corresponds to 2π radians. This means that $\omega_0 = 2\pi f_0$, so

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{g}{L}}. \tag{15}$$

3. The period is the time it takes for the object to complete a full oscillation (i.e. the number of seconds per oscillation). This is simply the inverse of the number of oscillations per second (i.e. the frequency). Thus, using $T_0 = 1/f_0$ we find the period to be

$$T_0 = 2\pi\sqrt{\frac{L}{g}}. \tag{16}$$

4. The angular velocity of the point mass at a given location is easiest to calculate from conservation of energy. We can take equation (10) and let $\phi = 0$ to choose the final state to be when the pendulum is at the bottom of its swing

$$-mgL\cos\phi_0 = \frac{I}{2}\omega^2 - mgL. \tag{17}$$

Given that $I = mL^2$, we can rewrite this as

$$-mgL\cos\phi_0 = \frac{mL^2}{2}\omega^2 - mgL \quad \Rightarrow \quad \omega = \sqrt{2\frac{g}{L}\left(1-\cos\phi_0\right)}. \tag{18}$$

The translational speed $v = v_\phi$ is purely tangential, which is related to the angular speed ω through

$$v_\phi = \rho\omega. \tag{19}$$

Substituting equation (18) and the fact that $\rho = L$ gives

$$v_\phi = L\sqrt{2\frac{g}{L}\left(1-\cos\phi_0\right)} \quad \Rightarrow \quad v_\phi = \sqrt{2gL\left(1-\cos\phi_0\right)}. \tag{20}$$

5. No, these are not the same. The angular speed $\omega = d\phi/dt$ is the rate of change in the angular position of the pendulum (where 2π radians corresponds to a *full circle* in our polar coordinate ϕ). The angular frequency $\omega_0 = \sqrt{g/L}$ represents the average number of radians the object completes per second (where 2π radians corresponds to a *full oscillation*). Moreover, the angular speed ω will change throughout the pendulum's motion, while the angular frequency ω_0 is defined such that it is always a constant.

6. No, as shown by equation (16). Since the mass appears on both sides in Newton's second law (see equation (3)), it cancels out.

2. Gyroscope

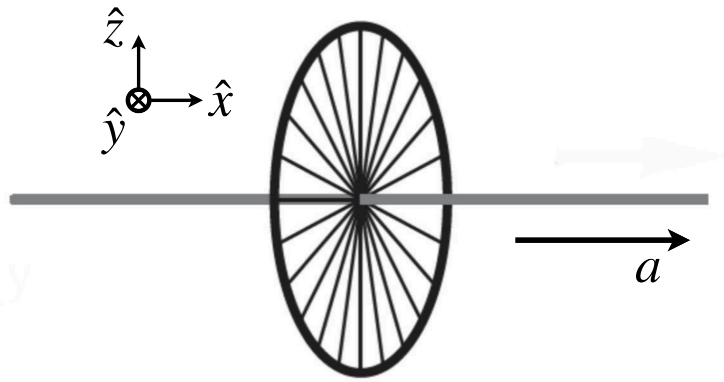
1. Since we are interesting in finding the *additional* force required to achieve particular motion, we can ignore the force of gravity. The only forces we must consider are the additional force from the demonstrator's hand on the *right* end of the bar $\vec{F}_r$ and the additional force from the demonstrator's hand on the *left* end of the bar $\vec{F}_l$. Note that here the subscript r indicate the right end of the bar, while the subscript l indicates the left. If the gyroscope is displaced in a direction parallel to its axis of rotation, there is no change in angular momentum. Since $d\vec{L}/dt = 0$, the net torque can be found to be

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} = 0 \tag{1}$$

from Newton's second law for rotation, where we recall that a torque is defined to be $\vec{\tau} = \vec{r} \times \vec{F}$. Thus, since $\vec{F}_r$ is applied at $\vec{r}_r = \ell\hat{x}$ and $\vec{F}_l$ is applied at $\vec{r}_l = -\ell\hat{x}$, they must satisfy

$$\vec{\tau}_{net} = \vec{\tau}_r + \vec{\tau}_l = \vec{r}_r \times \vec{F}_r + \vec{r}_l \times \vec{F}_l = \ell\hat{x} \times \vec{F}_r - \ell\hat{x} \times \vec{F}_l = 0 \quad \Rightarrow \quad \hat{x} \times \left(\vec{F}_r - \vec{F}_l\right) = 0. \tag{2}$$

Note that we have defined the origin to be the initial position of the center of the wheel.



To accelerate a mass M, we need a net force $\vec{F}_{net}$ given by Newton's second law of

$$\vec{F}_{net} = M\vec{a} = Ma\hat{x}. \tag{3}$$

Thus, the additional forces from the demonstrator's hands must also satisfy

$$\vec{F}_r + \vec{F}_l = Ma\hat{x} \quad \Rightarrow \quad \vec{F}_l = Ma\hat{x} - \vec{F}_r. \tag{4}$$

Substituting equation (4) into equation (2) gives

$$\hat{x} \times \left(\vec{F}_r - \left(Ma\hat{x} - \vec{F}_r\right)\right) = 0 \quad \Rightarrow \quad \hat{x} \times \vec{F}_r = 0. \tag{5}$$

as $\hat{x} \times \hat{x} = 0$. Writing out $\vec{F}_r = F_{rx}\hat{x} + F_{ry}\hat{y} + F_{rz}\hat{z}$ component-by-component in Cartesian coordinates allows us to reformulate the equation as

$$\hat{x} \times (F_{rx}\hat{x} + F_{ry}\hat{y} + F_{rz}\hat{z}) = 0 \quad \Rightarrow \quad F_{ry}\hat{z} - F_{rz}\hat{y} = 0. \tag{6}$$

If we take the dot product of this equation with $\hat{z}$, we see that $F_{ry} = 0$ and if we take the dot product of this equation with $\hat{y}$, we see that $F_{rz} = 0$. This implies that

$$\vec{F}_r = F_{rx}\hat{x} = F_r\hat{x} \tag{7}$$

must be purely in the $\hat{x}$ direction. Through substitution into equation (4), we see that the same is true for

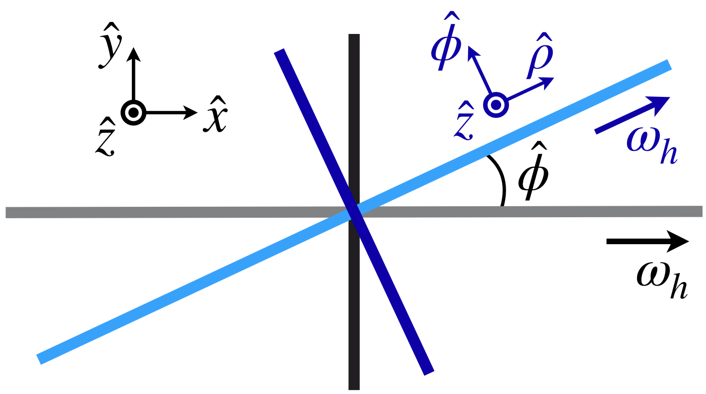
$$\vec{F}_l = Ma\hat{x} - F_r\hat{x} = (Ma - F_r)\,\hat{x} = F_l\hat{x}. \tag{8}$$

Taking the dot product of both sides of this equation with $\hat{x}$, we find

$$Ma - F_r = F_l \quad \Rightarrow \quad F_r + F_l = Ma. \tag{9}$$

Thus, the additional forces from each hand must only have an $\hat{x}$ component and satisfy $F_r + F_l = Ma$.

2. This problem is tricky because there are two different types of rotation. There is the rotation of the hoop about the axis, which has a constant magnitude ω_h that starts out in the $\hat{x}$ direction at $t = 0$. However, it changes direction because the bar is made to rotate about its center with a constant angular velocity $\omega_b \hat{z}$. The second type of rotation (i.e. the rotation of the bar in the x-y plane) causes the angular momentum to change. To quantify this second type of rotation, we will use the cylindrical coordinate system shown in the figure below (which shows the wheel from a top view relative to the figure in the problem statement).



We can write the total angular momentum of the system (including both types of rotation) as

$$\vec{L}_{sys} = \vec{L}_h + \vec{L}_b = I_h \omega_h \hat{\rho} + I_b \omega_b \hat{z}, \tag{10}$$

where $\hat{\rho}$ is changing with time and the subscripts h and b indicate the hoop and bar respectively. The hoop is thin and rotating about its center, so its moment of inertia is $I_h = MR^2$. The bar is massless, but its rotation also rotates the hoop about its diameter. Thus, it has a moment of inertia of $I_b = MR^2/2$, which we have found from a table noting that the hoop is thin (meaning it has zero width). Substituting these values into equation (10), we find

$$\vec{L}_{sys} = MR^2 \omega_h \hat{\rho} + \frac{M}{2} R^2 \omega_b \hat{z}. \tag{11}$$

A change in angular momentum is always caused by a net torque, which can be found from Newton's second law for rotation

$$\vec{\tau}_{net} = \frac{d\vec{L}_{sys}}{dt}. \tag{12}$$

Substituting equation (11) gives

$$\vec{\tau}_{net} = \frac{d}{dt}\left(MR^2 \omega_h \hat{\rho} + \frac{M}{2} R^2 \omega_b \hat{z} \right) = MR^2 \omega_h \frac{d\hat{\rho}}{dt}. \tag{13}$$

In lecture 4a, we calculated the derivatives of the cylindrical unit vectors, finding that $d\hat{\rho}/dt = (d\phi/dt)\hat{\phi}$. Since the bar is changing angular position ϕ at a constant angular speed $\omega_b = d\phi/dt$, we can write $d\hat{\rho}/dt = \omega_b \hat{\phi}$. Substituting this into equation (13) produces

$$\vec{\tau}_{net} = MR^2 \omega_h \omega_b \hat{\phi}. \tag{14}$$

This torque must arise from the forces applied at the end of the bar by the demonstrator's hands. Thus, analogously to equation 2, we have

$$\vec{\tau}_{net} = \vec{\tau}_r + \vec{\tau}_l = \vec{r}_r \times \vec{F}_r + \vec{r}_l \times \vec{F}_l = \ell \hat{\rho} \times \vec{F}_r - \ell \hat{\rho} \times \vec{F}_l = MR^2 \omega_h \omega_b \hat{\phi} \tag{15}$$

$$\Rightarrow \quad \hat{\rho} \times \left(\vec{F}_r - \vec{F}_l \right) = \frac{MR^2}{\ell} \omega_h \omega_b \hat{\phi}, \tag{16}$$

where we must take into account the rotation of the bar into our expression for the radial position of the demonstrator's hands by using $\hat{\rho}$ instead of $\hat{x}$.

Additionally, we know that the center of the bar (i.e. the center of mass of the system) has no translational motion. Thus, by Newton's first law, the net force must be equal to

$$\vec{F}_{net} = \vec{F}_r + \vec{F}_l = 0 \quad \Rightarrow \quad \vec{F}_l = -\vec{F}_r. \tag{17}$$

Substituting this into equation (16) gives

$$\hat{\rho} \times \left(\vec{F}_r + \vec{F}_r \right) = \frac{MR^2}{\ell} \omega_h \omega_b \hat{\phi} \quad \Rightarrow \quad \hat{\rho} \times \vec{F}_r = \frac{MR^2}{2\ell} \omega_h \omega_b \hat{\phi}. \tag{18}$$

Writing out $\vec{F}_r = F_{r\rho}\hat{\rho} + F_{r\phi}\hat{\phi} + F_{rz}\hat{z}$ component-by-component in cylindrical coordinates shows that

$$\hat{\rho} \times \left(F_{r\rho}\hat{\rho} + F_{r\phi}\hat{\phi} + F_{rz}\hat{z} \right) = \frac{MR^2}{2\ell} \omega_h \omega_b \hat{\phi} \quad \Rightarrow \quad F_{r\phi}\hat{z} - F_{rz}\hat{\phi} = \frac{MR^2}{2\ell} \omega_h \omega_b \hat{\phi}. \tag{19}$$

Looking at each component of this equation, we see that we must have $F_{r\phi} = 0$ and

$$-F_{rz} = \frac{MR^2}{2\ell} \omega_h \omega_b \quad \Rightarrow \quad F_{rz} = -\frac{MR^2}{2\ell} \omega_h \omega_b. \tag{20}$$

Plugging this into equation (17) shows that $F_{l\phi} = 0$ and

$$F_{lz} = \frac{MR^2}{2\ell} \omega_h \omega_b. \tag{21}$$

Thus, our final answer is

$$\vec{F}_r = F_{\rho r}\hat{\rho} - \frac{MR^2}{2\ell} \omega_h \omega_b \hat{z} \tag{22}$$

$$\vec{F}_l = -F_{\rho r}\hat{\rho} + \frac{MR^2}{2\ell} \omega_h \omega_b \hat{z}. \tag{23}$$

We see that the radial component of the force in unconstrained as long as it is equal and opposite according to equation (17). This is intuitive as the tension in the bar will transmit the force along the bar, while not affecting its motion. Less intuitive is the fact that, to get the bar to rotate in the $\hat{\phi}$ direction, you must apply forces in the $\hat{z}$ direction. This is the nature of a gyroscope — applying a force creates motion in a perpendicular direction.

3. Bumpy road

We choose to use a Cartesian coordinate system with $\hat{x}$ in the direction of travel of the car and $\hat{y}$ pointing upwards.

1. We are told that the road has a cosine height profile given by

$$y_r(x) = C_1 \cos(C_2 x), \tag{1}$$

where C_1 and C_2 are unknown constants that we must determine from the geometry of the road. Note that, by taking the form of equation (1), we are implicitly choosing the y location of the origin to be halfway between the top and bottom of the bumps. C_1 quantifies the height of the bumps. Since the maximum and minimum of the cosine function are 1 and -1 respectively, the total change in elevation of equation (1) is $2C_1$. Since the physical change in elevation (from trough to peak) is H, we have $H = 2C_1$, which implies that

$$C_1 = \frac{H}{2}. \tag{2}$$

Next, we see that C_2 quantifies the length of the bump. From the problem statement, we know that, if you are at the peak of a bump, you will arrive at the next peak by traveling a horizontal distance L. Similarly, if you are at a maximum of the cosine function, you will find the next maximum by increasing the argument of the cosine function by 2π (as the cosine function is 2π periodic). We must choose C_2 such that these two periodic behaviors correspond to one another. Thus, if $x = L$ the argument of the cosine function should be 2π. Looking at equation (1), this means that $2\pi = C_2 L$ and we find

$$C_2 = \frac{2\pi}{L}. \tag{3}$$

Substituting equations (2) and (3) into equation (1) gives the shape of the road in terms of known parameters

$$y_r(x) = \frac{H}{2}\cos\left(\frac{2\pi}{L}x\right). \tag{4}$$

The wheel has a negligibly small radius, so its trajectory is the same as the road's

$$y_w(x) = y_r(x) = \frac{H}{2}\cos\left(\frac{2\pi}{L}x\right). \tag{5}$$

We know that the body of the car moves forwards with a constant horizontal velocity of v_x, so the wheel must as well (otherwise the car would be torn apart). Since the velocity is constant we know that the horizontal position of the wheel is just

$$x(t) = v_x t + x_0 = v_x t, \tag{6}$$

where we have taken the x loction of the origin to be the location of the wheel at $t = 0$ so that $x_0 = 0$. This allows us to replace the x coordinate in equation (5) to find the vertical position of the wheel as a function of time

$$y_w(t) = \frac{H}{2}\cos\left(\frac{2\pi}{L}v_x t\right). \tag{7}$$

2. The equation of motion for the car comes from Newton's second law

$$\vec{F}_{net} = \vec{F}_g + \vec{F}_s = m\vec{a}_c, \tag{8}$$

where

$$\vec{F}_g = -mg\hat{y} \tag{9}$$

is the gravitational force,

$$\vec{F}_s = -k\Delta y\hat{y} \tag{10}$$

is the spring force, and the c subscript indicates it refers to the body of the car. The challenge is to determine Δy, the extension of the spring relative to its equilibrium length. First, we will let y_c denote the height of the car, which is a function of time. From the picture in the problem statement, we see that, at any instant, the length of the spring will be $\ell = y_c - y_w$. Given that the equilibrium length is ℓ_0, we have that

$$\Delta y = \ell - \ell_0 = y_c - y_w - \ell_0. \tag{11}$$

Substituting equations (9), (10), and (11) into the $\hat{y}$ component of equation (8) gives

$$-mg - k\Delta y = ma_c \quad \Rightarrow \quad -mg - k\left(y_c - y_w - \ell_0\right) = m\frac{d^2 y_c}{dt^2}. \tag{12}$$

Substituting equation (7) from part 1 yields

$$-mg - k\left(y_c - \frac{H}{2}\cos\left(\frac{2\pi}{L}v_x t\right) - \ell_0\right) = m\frac{d^2 y_c}{dt^2}. \tag{13}$$

Rearranging we can isolate the inhomogeneous terms on the right side and arrive at the equation of motion

$$\frac{d^2 y_c}{dt^2} + \frac{k}{m}y_c = \frac{k}{m}\frac{H}{2}\cos\left(\frac{2\pi}{L}v_x t\right) + \frac{k}{m}\ell_0 - g. \tag{14}$$

3. Equation (14) is a complicated differential equation, but we can recognize it as the simple harmonic oscillator equation with the addition of forcing terms (i.e. all the terms on the right side of the equation). To attack this, we first let

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{15}$$

$$\omega_d = \frac{2\pi v_x}{L} \tag{16}$$

to get the simpler and more familiar form of

$$\frac{d^2 y_c}{dt^2} + \omega_0^2 y_c = \omega_0^2\frac{H}{2}\cos(\omega_d t) + \omega_0^2\ell_0 - g. \tag{17}$$

You can solve this equation by looking it up in the Math Review document on the course Moodle. However, here we will show how to solve it directly. This differential equation is similar to an equation we solved in lecture 13a, but there are extra constant terms. Because these terms are constant, we can eliminate them by defining a new vertical coordinate Y_c such that

$$\omega_0^2 Y_c = \omega_0^2 y_c - \omega_0^2\ell_0 + g \quad \Rightarrow \quad \omega_0^2 y_c = \omega_0^2 Y_c + \omega_0^2\ell_0 - g. \tag{18}$$

Taking two derivatives of this equation yields

$$\omega_0^2\frac{d^2 y_c}{dt^2} = \omega_0^2\frac{d^2 Y_c}{dt^2} \quad \Rightarrow \quad \frac{d^2 y_c}{dt^2} = \frac{d^2 Y_c}{dt^2}. \tag{19}$$

Substituting equations (18) and (19) into equation (17) gives

$$\frac{d^2 Y_c}{dt^2} + \omega_0^2 Y_c + \omega_0^2\ell_0 - g = \omega_0^2\frac{H}{2}\cos(\omega_d t) + \omega_0^2\ell_0 - g \quad \Rightarrow \quad \frac{d^2 Y_c}{dt^2} + \omega_0^2 Y_c = \omega_0^2\frac{H}{2}\cos(\omega_d t), \tag{20}$$

where we see that the two offending terms have been eliminated. This is now identical to the differential equation solved in lecture 13a, so long as we identify that the notation in class corresponds to $x = Y_c$, $\lambda = 0$, and $F_d/m = \omega_0^2 H/2$. Thus, the solution is

$$Y_c(t) = A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t} + \frac{H}{2}\frac{\omega_0^2}{\omega_0^2 - \omega_d^2}\cos(\omega_d t). \tag{21}$$

We can find the solution for the original vertical coordinate by substituting equations (15) and (21) into equation (18) to get

$$\begin{aligned} \omega_0^2 y_c &= \omega_0^2\left(A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t} + \frac{H}{2}\frac{\omega_0^2}{\omega_0^2 - \omega_d^2}\cos(\omega_d t)\right) + \omega_0^2\ell_0 - g \\ \Rightarrow \quad y_c(t) &= A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t} + \frac{H}{2}\frac{\omega_0^2}{\omega_0^2 - \omega_d^2}\cos(\omega_d t) + \ell_0 - \frac{mg}{k}, \end{aligned} \tag{22}$$

where A_1 and A_2 are integration constants that are determined by the initial conditions. This is the full solution to the differential equation of equation (14). Note that the first two terms correspond to oscillation, which can be seen through applying Euler's formula $\exp(ix) = \cos(x) + i\sin(x)$. However, this oscillation occurs at the natural angular frequency ω_0 of the system and is independent of the characteristics of the bumps. This oscillation would occur on flat ground, if the body of the car was displaced a bit vertically. The problem statement asks us about the oscillations caused by the bumps. These are represented by the third term, which we see depends on the height of the bumps as well as their length (through equation (16)). Studying this term, we can see that the amplitude of the vertical oscillations caused by the bumps is

$$\Delta y_c = \frac{H}{2}\frac{\omega_0^2}{\omega_0^2 - \omega_d^2}. \tag{23}$$

Thus, the ride would be least comfortable when $\omega_d^2 = \omega_0^2$ because the amplitude of the vertical oscillation would become infinite, which represents resonance. This condition is equivalent to

$$\omega_0^2 = \omega_d^2 \quad \Rightarrow \quad \frac{k}{m} = \left(\frac{2\pi v_x}{L}\right)^2 \quad \Rightarrow \quad v_x = \frac{L}{2\pi}\sqrt{\frac{k}{m}} \tag{24}$$

and satisfying it for an extended period of time might cause damage to the car.

4. Tuning fork

1. We are given that the solution has the form

$$x(t) = A_0 e^{-\gamma t}\cos(\omega_1 t + \varphi). \tag{1}$$

We can determine the coefficients by substitution into the equation that it is supposed to solve. Using the product rule, we can find that the first derivative of equation (1) is

$$\dot{x}(t) = -\gamma A_0 e^{-\gamma t}\cos(\omega_1 t + \varphi) - \omega_1 A_0 e^{-\gamma t}\sin(\omega_1 t + \varphi) \tag{2}$$

$$= -\gamma x - \omega_1 A_0 e^{-\gamma t}\sin(\omega_1 t + \varphi). \tag{3}$$

Again using the product rule, the second derivative is

$$\ddot{x}(t) = -\gamma\dot{x} + \gamma\omega_1 A_0 e^{-\gamma t}\sin(\omega_1 t + \varphi) - \omega_1^2 A_0 e^{-\gamma t}\cos(\omega_1 t + \varphi) \tag{4}$$

$$= -\gamma\dot{x} + \gamma\omega_1 A_0 e^{-\gamma t}\sin(\omega_1 t + \varphi) - \omega_1^2 x. \tag{5}$$

By substituting equation (5) into the equation for the damped harmonic oscillator given in the problem statement

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \tag{6}$$

we find

$$-\gamma\dot{x} + \gamma\omega_1 A_0 e^{-\gamma t}\sin(\omega_1 t + \varphi) - \omega_1^2 x + 2\gamma\dot{x} + \omega_0^2 x = 0 \tag{7}$$

$$\Rightarrow \quad \gamma\omega_1 A_0 e^{-\gamma t}\sin(\omega_1 t + \varphi) + \gamma\dot{x} + \left(\omega_0^2 - \omega_1^2\right)x = 0. \tag{8}$$

Then, substituting equation (3) into this gives

$$\gamma\omega_1 A_0 e^{-\gamma t}\sin(\omega_1 t + \varphi) + \gamma\left(-\gamma x - \omega_1 A_0 e^{-\gamma t}\sin(\omega_1 t + \varphi)\right) + \left(\omega_0^2 - \omega_1^2\right)x = 0 \tag{9}$$

$$\Rightarrow \quad -\gamma^2 x + \left(\omega_0^2 - \omega_1^2\right)x = 0 \tag{10}$$

$$\Rightarrow \quad \left(\omega_0^2 - \omega_1^2 - \gamma^2\right)x = 0. \tag{11}$$

Thus, the damped harmonic oscillator equation is satisfied if

$$\omega_0^2 - \omega_1^2 - \gamma^2 = 0 \quad \Rightarrow \quad \omega_1 = \pm\sqrt{\omega_0^2 - \gamma^2} \quad \Rightarrow \quad \omega_1 = \sqrt{\omega_0^2 - \gamma^2} \tag{12}$$

as we know that the frequency is positive quantity.

2. Therefore, the solution to the equation of motion for the damped harmonic oscillator can be written as

$$x(t) = A_0 e^{-\gamma t} \cos\left(\omega_1 t + \varphi\right), \tag{13}$$

where $\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$ and A_0 and φ are determined by initial conditions. To solve this problem, we first need to determine how the amplitude of the oscillation changes with time. From studying equation (13), we see that it is simply $A(t) = C_0 e^{-\gamma t}$. Evaluating it at two different times t_0 and t_1 and taking the ratio allows us to determine γ to be

$$\frac{A(t_1)}{A(t_0)} = e^{-\gamma(t_1 - t_0)} \quad \Rightarrow \quad \gamma = \frac{1}{t_1 - t_0} \ln\left(\frac{A(t_0)}{A(t_1)}\right). \tag{14}$$

We can substitute this into equation (12) to find

$$\omega_0^2 - \omega_1^2 - \left[\frac{1}{t_1 - t_0} \ln\left(\frac{A(t_0)}{A(t_1)}\right)\right]^2 = 0 \quad \Rightarrow \quad \omega_0 = \sqrt{\omega_1^2 + \left[\frac{1}{t_1 - t_0} \ln\left(\frac{A(t_0)}{A(t_1)}\right)\right]^2}. \tag{15}$$

Using the relationship between the frequency and angular frequency $\omega = 2\pi f$, this becomes

$$2\pi f_0 = \sqrt{(2\pi f_1)^2 + \left[\frac{1}{t_1 - t_0} \ln\left(\frac{A(t_0)}{A(t_1)}\right)\right]^2} \quad \Rightarrow \quad f_0 = f_1 \sqrt{1 + \left[\frac{1}{2\pi f_1} \frac{1}{t_1 - t_0} \ln\left(\frac{A(t_0)}{A(t_1)}\right)\right]^2}. \tag{16}$$

Plugging in numbers, we find that

$$f_0 = (400 \text{ Hz}) \sqrt{1 + \left[\frac{1}{2\pi(400 \text{ Hz})} \frac{1}{12 \text{ s}} \ln\left(\frac{1}{0.9}\right)\right]^2} = 400.0000000024 \text{ Hz}. \tag{17}$$

Thus, given the number of significant digits in the numerical input quantities, we find that air has a negligible effect on the frequency of the tunning fork

$$f_0 = 400 \text{ Hz}. \tag{18}$$