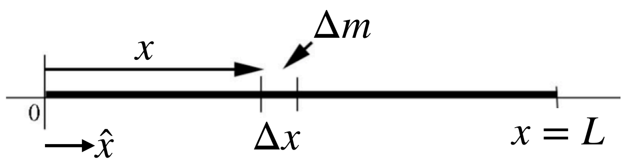
Solutions to Problem Set 6

Momentum, impulse, center of mass

PHYS-101(en)

1. Center of mass of a rod



1. We start by choosing a coordinate system with the rod aligned along the x-axis and the origin located at the left end of the rod. Next, we'll decompose the rods into differential elements of mass Δm and consider the element located a distance x from the origin. Since the rod is uniform, the linear mass density is

$$\lambda = \frac{M}{L} \tag{1}$$

as well as

$$\lambda = \frac{\Delta m}{\Delta x}, \tag{2}$$

where Δx is the length of the differential element. Taking the limit of equation (2) as $\Delta x \to 0$ gives

$$\lambda = \frac{dm}{dx}. \tag{3}$$

The center of mass for a continuous system is defined by

$$\vec{R}_{CM} = \frac{\int_M \vec{r} dm}{\int_M dm}, \tag{4}$$

where $\int_M (\ldots) dm$ indicates an integral over the entire mass distribution of the whole object. We can perform a change of variables from mass to position to find

$$\vec{R}_{CM} = \frac{\int_0^L \vec{r} \frac{dm}{dx} dx}{\int_0^L \frac{dm}{dx} dx}, \tag{5}$$

where $\int_0^L (\ldots) dx$ indicates an integral over the entire length of the object. Then substituting equation (3) we find

$$\vec{R}_{CM} = \frac{\int_0^L \vec{r} \lambda dx}{\int_0^L \lambda dx} = \frac{\int_0^L (x\hat{x}) \lambda dx}{\lambda(L-0)} = \frac{\int_0^L x dx \hat{x}}{L} = \frac{(x^2/2]_{x=0}^{x=L}}{L}\hat{x} = \frac{L^2/2 - 0}{L}\hat{x} = \frac{L}{2}\hat{x}. \tag{6}$$

As expected from the symmetry of a uniform rod, the center of mass is exactly in the middle.

2. We will use the same coordinate system as in part 1 and again consider a differential mass element Δm located a distance x from the origin. The linear mass density $\lambda(x)$ is no longer uniform along the length of the rod, so $\lambda \neq M/L$ at every point. However, the definition of the local linear mass density is
$$\lambda(x) = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x}, \tag{7}$$
where Δx is the length of the differential element. Since the right side is the definition of the derivative, we have
$$\lambda(x) = \frac{dm}{dx}. \tag{8}$$
Substituting the functional form of the density from the problem statement gives
$$\frac{dm}{dx} = \lambda_0 \frac{x}{L}. \tag{9}$$
The total mass is found by summing the mass elements over the entire length of the rod according to
$$M = \int_M dm. \tag{10}$$
Performing a change of variables to position and substituting equation (9) gives
$$M = \int_0^L \frac{dm}{dx} dx = \int_0^L \lambda_0 \frac{x}{L} dx = \frac{\lambda_0}{L} \int_0^L x dx = \frac{\lambda_0}{L} \left(\frac{x^2}{2} \right]_{x=0}^{x=L} = \frac{\lambda_0}{2L}(L^2 - 0) = \frac{\lambda_0}{2} L. \tag{11}$$
Since the problem gives us both M and L as known quantities, we can solve this equation for λ_0 to find
$$\lambda_0 = \frac{2M}{L}. \tag{12}$$
The definition of the center of mass is
$$\vec{R}_{CM} = \frac{\int_M \vec{r} dm}{\int_M dm}. \tag{13}$$
We can calculate the numerator in a similar way as above to find
$$\vec{R}_{CM} = \frac{\int_0^L \vec{r} \frac{dm}{dx} dx}{M} = \frac{1}{M} \int_0^L (x\hat{x}) \lambda_0 \frac{x}{L} dx = \frac{\lambda_0}{ML} \int_0^L x^2 dx \hat{x} = \frac{\lambda_0}{ML} \left(\frac{x^3}{3} \right]_{x=0}^{x=L} \hat{x} = \frac{\lambda_0 L^2}{3M} \hat{x}, \tag{14}$$
where we have used equation (9). Using equation (12), we can simplify further to find
$$\vec{R}_{CM} = \frac{2M}{L} \frac{L^2}{3M} \hat{x} = \frac{2}{3} L \hat{x}. \tag{15}$$

2. Center of mass of the particle-rod system

In this problem, we are tasked with finding the center of mass dynamics of the system that includes both the particle and the rod. Thus, we will begin by writing down the expressions for the center of mass position and velocity
$$\vec{R}_{CM}(t) = \frac{\sum_{i=1}^N m_i \vec{r}_i}{\sum_{i=1}^N m_i} = \frac{m_1 \vec{r}_1(t) + m_2 \vec{r}_2(t)}{m_1 + m_2} = \frac{M\vec{r}_1(t) + M\vec{r}_2(t)}{2M} = \frac{\vec{r}_1(t) + \vec{r}_2(t)}{2} \tag{1}$$
$$\vec{V}_{CM}(t) = \frac{\sum_{i=1}^N m_i \vec{v}_i}{\sum_{i=1}^N m_i} = \frac{m_1 \vec{v}_1(t) + m_2 \vec{v}_2(t)}{m_1 + m_2} = \frac{M\vec{v}_1(t) + M\vec{v}_2(t)}{2M} = \frac{\vec{v}_1(t) + \vec{v}_2(t)}{2} \tag{2}$$

respectively, where $\vec{r}_1(t)$ is the position of the center of mass of the rod, $\vec{r}_2(t)$ is the position of the particle, $\vec{v}_1(t)$ is the velocity of the center of mass of the rod, $\vec{v}_2(t)$ is the velocity of the particle. As for the center of mass acceleration, since the net external force on the system is zero we know that the center of mass acceleration is zero

$$\vec{F}^{ext}_{net} = M\vec{A}_{CM} = 0 \quad \Rightarrow \quad \vec{A}_{CM} = 0. \tag{3}$$

Thus, due to the relationship between position, velocity, and acceleration, we know that the center of mass velocity is constant in time

$$\vec{V}_{CM}(t) = \vec{V}_{CM}(0). \tag{4}$$

As an aside, for systems with constant mass, having a constant center of mass velocity ($\vec{V}_{CM}(t) = \text{constant}$) is equivalent to conservation of momentum ($\vec{p}_{sys}(t) = \sum_{i=1}^{N} m_i\vec{v}_i(t) = \text{constant}$) as equation (2) shows that $\vec{V}_{CM}(t) = \vec{p}_{sys}(t)/\sum_{i=1}^{N} m_i$.

Next, if the center of mass velocity is constant, then we know that the center of mass position is linear in time

$$\vec{R}_{CM}(t) = \vec{V}_{CM}(0)t + \vec{R}_{CM}(0). \tag{5}$$

Thus, we just need to find the initial center of mass position $\vec{R}_{CM}(0)$ and initial center of mass velocity $\vec{V}_{CM}(0)$. Since the problem statement gives us the state of the system at $t = 0$, we can compute these quantities using equations (1) and (2).

We will first calculate the center of mass position. At $t = 0$, we can deduce that the center of mass of the rod is

$$\vec{r}_1(0) = \frac{L}{2}\hat{y} \tag{6}$$

by symmetry. Alternatively, as in part 1 of problem 1 above, this could be found by integrating over the constant mass distribution $\lambda = M/L = dm/dy$ of the rod according to

$$\vec{r}_1(0) = \frac{\int_M \vec{r}dm}{\int_M dm} = \frac{\int_0^L \vec{r}\lambda dy}{\int_0^L \lambda dy} = \frac{\lambda\int_0^L y\hat{y}dy}{\lambda\int_0^L dy} = \frac{\int_0^L ydy}{\int_0^L dy}\hat{y} = \frac{(y^2/2]_{y=0}^{y=L}}{(y]_0^L}\hat{y} = \frac{L^2/2-0}{L-0}\hat{y} = \frac{L}{2}\hat{y}, \tag{7}$$

where $\int_M(\ldots)dm$ indicates an integral over the entire mass distribution of the whole object. At time $t = 0$, the particle is at the origin, so its center of mass is at

$$\vec{r}_2(0) = 0. \tag{8}$$

Thus, substituting equations (6) and (8) into equation (1), we see that the center of mass of the whole system at time $t = 0$ is

$$\vec{R}_{CM}(0) = \frac{(L/2)\hat{y} + 0}{2} = \frac{L}{4}\hat{y}. \tag{9}$$

The initial center of mass velocity can be found in a similar manner. The rod is at rest at $t = 0$, so $\vec{v}_1(0) = 0$. At time $t = 0$, the particle is moving in the positive x-direction with a velocity $\vec{v}_2(0) = V_0\hat{x}$. Thus, equation (2) tells us that the velocity of the center of mass of the system at $t = 0$ is

$$\vec{V}_{CM}(0) = \frac{0 + V_0\hat{x}}{2} = \frac{V_0}{2}\hat{x}. \tag{10}$$

Thus, substituting equation (10) into equation 4 gives

$$\vec{V}_{CM}(t) = \frac{V_0}{2}\hat{x} \tag{11}$$

and substituting equations (9) and (10) into equation 5 gives

$$\vec{R}_{CM}(t) = \frac{V_0}{2}t\hat{x} + \frac{L}{4}\hat{y}. \tag{12}$$

3. Two particles colliding

We will start by choosing a coordinate system with the origin defined to be the location of m_1 at $t = t_0$. This choice is sensible as we will then know that

$$L = x_1(t_1), \tag{1}$$

i.e. the position of m_1 will be identical to the distance traveled since $t = t_0$ (which is what the problem asks us to calculate). Additionally we note that the problem is one-dimensional, so we will only need to consider the $\hat{x}$ direction.

Since there is no external force on the two-particle system, we know that the acceleration of the center of mass is zero, regardless of the force between the objects. Thus, we know that the center of mass velocity is constant in time

$$V_{CM}(t) = \frac{\sum_{i=1}^{N} m_i v_i}{\sum_{i=1}^{N} m_i} = \frac{m_1 v_1(t) + m_2 v_2(t)}{m_1 + m_2} = V_{CM}(t_0) \tag{2}$$

and, thus, doesn't change from its value at $t = t_0$. This is significant because the problem statement gives us information about the system at $t = t_0$.

Since the center of mass velocity is constant, we know that the center of mass position is linear in time. We can write it as

$$X_{CM}(t) = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} = \frac{m_1 x_1(t) + m_2 x_2(t)}{m_1 + m_2} = V_{CM}(t_0)(t - t_0) + X_{CM}(t_0), \tag{3}$$

where we have chosen the form of our linear function to depend only on quantities at $t = t_0$, again because that is the time that the problem statement gives us information about.

To determine the center of mass velocity at $t = t_0$, we substitute the facts from the problem statement that $v_1(t_0) = v_0$ and $v_2(t_0) = 0$ into equation (2) to find

$$V_{CM}(t_0) = \frac{m_1 v_0 + m_2(0)}{m_1 + m_2} = \frac{m_1}{m_1 + m_2} v_0. \tag{4}$$

To determine the center of mass position at $t = t_0$, we substitute the facts from the problem statement that $x_1(t_0) = 0$ (due to the definition of our origin) and $x_2(t_0) = d$ into equation (3) to find

$$X_{CM}(t_0) = \frac{m_1(0) + m_2 d}{m_1 + m_2} = V_{CM}(t_0)(t_0 - t_0) + X_{CM}(t_0) \quad \Rightarrow \quad X_{CM}(t_0) = \frac{m_2}{m_1 + m_2} d. \tag{5}$$

Thus, substituting equations (4) and (5) into (3), we find the center of mass position as a function of time to be

$$X_{CM}(t) = \frac{m_1}{m_1 + m_2} v_0 (t - t_0) + \frac{m_2}{m_1 + m_2} d. \tag{6}$$

The final crucial insight is to recognize that the center of mass position at $t = t_1$ is position of the collision. Since the particles are colliding (and are small), both particles are at the same location, which also must be the center of mass. Thus, we have

$$x_1(t_1) = x_2(t_1) = X_{CM}(t_1). \tag{7}$$

Evaluating equation (6) at $t = t_1$ and combining it with equations (1) and (7) gives the final answer

$$X_{CM}(t_1) = L = \frac{m_1}{m_1 + m_2} v_0 \left(t_1 - t_0\right) + \frac{m_2}{m_1 + m_2} d = \frac{m_1 v_0 \left(t_1 - t_0\right) + m_2 d}{m_1 + m_2}. \tag{8}$$

Note that we never needed to use the form of the force between the objects. It was enough to know that there was no external force on the two particles and that they eventually collide.

4. Drag force at low speeds

1. We start by applying Newton's second law to the ball in the $\hat{x}$ direction. The only force is drag, so

$$-\beta v_x = ma_x. \tag{1}$$

Thus, we find the acceleration in the $\hat{x}$ direction is

$$\frac{dv_x}{dt} = -\frac{\beta}{m} v_x. \tag{2}$$

2. Next, we apply Newton's second law in the $\hat{y}$ direction. Given the coordinate system, gravity will accelerate the ball downwards, so the drag force is upwards. Thus, in the $\hat{y}$ direction we have

$$mg - \beta v_y = ma_y, \tag{3}$$

which can be rearranged to find

$$\frac{dv_y}{dt} + \frac{\beta}{m} v_y = g. \tag{4}$$

3. To obtain $v_x(t)$, we must solve the differential equation given by equation (2). To do so, we can use the technique of separation of variables to find

$$\int \frac{1}{v_x} dv_x = -\frac{\beta}{m} \int dt \quad \Rightarrow \quad \ln(v_x(t)) = -\frac{\beta}{m} t + C_x \tag{5}$$

$$\Rightarrow \quad v_x(t) = \exp\left(-\frac{\beta}{m} t + C_x\right) \quad \Rightarrow \quad v_x(t) = \exp(C_x) \exp\left(-\frac{\beta}{m} t\right). \tag{6}$$

Making use of the initial condition that $v_x(0) = u$, we see that the integration constant must satisfy $\exp(C_x) = u$, so

$$v_x(t) = u \exp\left(-\frac{\beta}{m} t\right). \tag{7}$$

4. To obtain $v_y(t)$, we must solve the differential equation given by equation (4). This is a first-order inhomogeneous differential equation with constant coefficients. We can solve this equation by substitution. If we define a new function $V_y(t)$ such that

$$v_y(t) = V_y(t) + \frac{mg}{\beta}, \tag{8}$$

we can substitute it to get the homogeneous differential equation

$$\frac{dV_y}{dt} + \frac{\beta}{m} V_y(t) + \frac{\beta}{m}\frac{mg}{\beta} = g \quad \Rightarrow \quad \frac{dV_y}{dt} + \frac{\beta}{m} V_y(t) = 0. \tag{9}$$

This can be solved analogously to part 3 to find

$$\int \frac{1}{V_y} dV_y = -\frac{\beta}{m} \int dt \quad \Rightarrow \quad \ln(V_y(t)) = -\frac{\beta}{m} t + C_y \tag{10}$$

$$\Rightarrow \quad V_y(t) = \exp\left(-\frac{\beta}{m} t + C_y\right) \quad \Rightarrow \quad V_y(t) = \exp(C_y) \exp\left(-\frac{\beta}{m} t\right). \tag{11}$$

Substituting this solution back into equation (8) gives

$$v_y(t) = \exp\left(C_y\right)\exp\left(-\frac{\beta}{m}t\right) + \frac{mg}{\beta}, \tag{12}$$

where we still must determine the integration constant C_y. This is done using the initial condition $v_y(0) = 0$, which implies that

$$v_y(0) = 0 = \exp\left(C_y\right) + \frac{mg}{\beta} \quad \Rightarrow \quad \exp\left(C_y\right) = -\frac{mg}{\beta}. \tag{13}$$

Substituting this into equation (12) gives the final answer of

$$v_y(t) = -\frac{mg}{\beta}\exp\left(-\frac{\beta}{m}t\right) + \frac{mg}{\beta} = \frac{mg}{\beta}\left(1 - \exp\left(-\frac{\beta}{m}t\right)\right). \tag{14}$$

5. To calculate $v_{x\infty}$, we simply take the long time limit (i.e. $t \to \infty$) of equation (7) and find

$$v_{x\infty} = \lim_{t\to\infty} v_x(t) = 0. \tag{15}$$

6. To calculate $v_{y\infty}$, we simply take the long time limit (i.e. $t \to \infty$) of equation (14) and find

$$v_{y\infty} = \lim_{t\to\infty} v_y(t) = \frac{mg}{\beta}. \tag{16}$$