Problem Set 13

Harmonic motion and gyroscopes

PHYS-101(en)

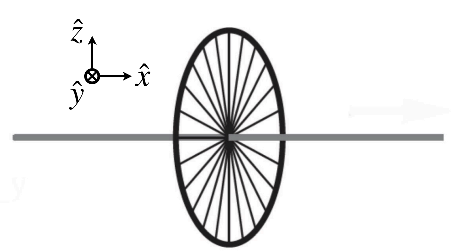
1. Simple pendulum

A simple pendulum consists of a massless string of length L hanging vertically and a point mass of mass m. The bottom of the string is attached to the mass, while the top position is fixed (but the string is free to pivot about this point). Suppose the string is initially pivoted by a small angle ϕ_0 from the vertical position and then released from rest.

1. Using the small angle approximation ($\sin\phi \approx \phi$) and either energy conservation or Newton's second law, show that the angle that the point mass makes with the vertical axis ϕ satisfies the differential equation of a simple harmonic oscillator.
2. What is the frequency and angular frequency of the oscillation?
3. How long will the pendulum take to return to its initial position, i.e. what is the period of its oscillation?
4. What are the angular and translational speeds of the point mass at the bottom of its swing?
5. Is the pendulum's angular speed the same as its angular frequency? Why or why not?
6. Does the period of the pendulum depend on m?

2. Gyroscope

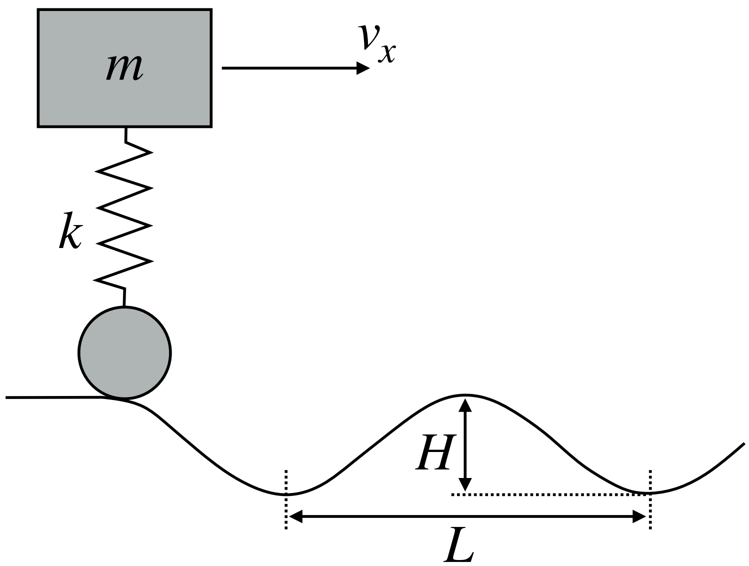
A gyroscope consists of a thin metal hoop of radius R and of mass M. Inside the metal hoop there are massless spokes, which radiate from the hoop's center. These connect to a massless bar of total length L, which is centered such that it extends a distance $\ell = L/2$ on each side of the hoop. The hoop and spokes are free to rotate about the bar. A demonstrator holds each end of the bar in a horizontal position, while the hoop is rotating at an angular velocity $\omega_h \hat{x}$. Find the magnitude and direction of the additional force exerted by each of the demonstrator's hands in order to cause the following motion. Note that you can neglect gravity in this problem as the demonstrator is preventing the gyroscope from falling and you are only concerned with finding the *additional* force (i.e. in addition to that needed to keep the bar at rest).



1. The bar is accelerated with a constant acceleration $a\hat{x}$ in the direction along the length of the bar.
2. The bar is rotated at a constant angular velocity $\omega_b \hat{z}$ in a horizontal plane around its center.

3. Bumpy road

We aim to model the trajectory of a car driving along a bumpy road in the following way. A point mass of mass m (representing the body of the car) moves forwards with a constant horizontal velocity v_x. The mass is connected to a spring with spring constant k and equilibrium length ℓ_0 (representing the shocks of the car). At the end of the spring, there is a massless wheel (representing a wheel of the car) with a negligibly small radius relative to the the other distances in the problem. The road has the shape of a cosine with bumps of height H and length L and at time $t = 0$ the wheel drops into the first bump (as shown below). Assume that the mass, spring, and wheel remain perfectly upright throughout the entire process.



1. Express the vertical position of the wheel as a function of time.
2. Using this information, deduce the equation of motion in the vertical direction for the body of the car.
3. Solve this equation to find the amplitude of vertical oscillations of the body of the car that are caused by the bumps. Under which conditions would your ride be the *least* comfortable?

4. Tuning fork

Consider the equation for a damped harmonic oscillator

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0,$$

where γ indicates the strength of the damping and ω_0 is the angular frequency if there was no damping.

1. If you are given that the solution to this differential equation is of the form

$$x(t) = A_0 e^{-\gamma t} \cos(\omega_1 t + \varphi),$$

   what must be the value of the angular frequency in the presence of damping ω_1? Here A_0 and φ are unknown integration constants.

2. We observe that *in air* the amplitude of the oscillations of a tuning fork with a frequency of $f_1 = \omega_1/(2\pi) = 400$ Hz are damped by 10% in 12 s. What is the frequency of this tuning fork in a vacuum $f_0 = \omega_0/(2\pi)$? Is the effect of air significant?