Solutions to Problem Set 12

Angular momentum

PHYS-101(en)

1. Planetary survey

After the instrument is launched, the only force it will experience is its gravitational attraction to the planet. This force is given by

$$\vec{F}_G = -\frac{Gm_pm_i}{r^2}\hat{r}, \tag{1}$$

where r is the distance between the instrument and the center of the planet, $\hat{r}$ is the radial unit vector pointing from the center of the planet towards the instrument, and G is the universal gravitational constant. Since this is the only force acting on the instrument, it experiences a total external torque about the center of the planet of

$$\sum \vec{\tau}_{ext} = \vec{\tau}_G = \vec{r} \times \vec{F}_G = \vec{r} \times \left(-\frac{Gm_pm_i}{r^2}\hat{r}\right) = -\frac{Gm_pm_i}{r^2}\vec{r} \times \hat{r} = 0, \tag{2}$$

where we have used equation (1) and the fact that the cross product of parallel vectors is zero. Thus, since the total external torque on the instrument about the center of the planet is zero, its angular momentum about the center of the planet must be conserved throughout its motion. Conservation of angular momentum is expressed as

$$\vec{L}_i = \vec{L}_f, \tag{3}$$

where the angular momentum is

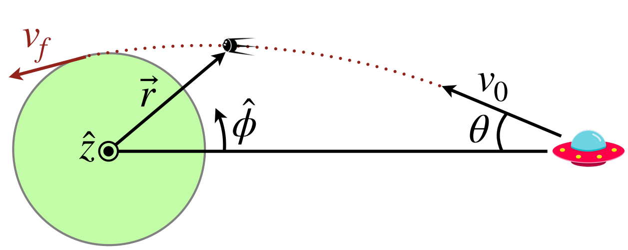
$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m_i\vec{v}. \tag{4}$$

Here $\vec{p}$ and $\vec{v}$ are the momentum and velocity of the instrument respectively. It is natural to define a cylindrical coordinate system with its origin at the center of the planet. The initial velocity can be found from trigonometry and the figure below to be

$$\vec{v}_i = -v_0\cos\theta\hat{r} + v_0\sin\theta\hat{\phi}. \tag{5}$$

Substituting this and the initial position vector of the instrument into equation (4) gives

$$\vec{L}_i = (5r_p\hat{r}) \times m_i\left(-v_0\cos\theta\hat{r} + v_0\sin\theta\hat{\phi}\right) = 5r_pm_iv_0\sin\theta\left(\hat{r}\times\hat{\phi}\right) = 5r_pm_iv_0\sin\theta\hat{z}. \tag{6}$$



We will consider the final state to occur when the instrument just grazes the surface of the planet. At this instant, the position vector is $\vec{r}_f = r_p\hat{r}$ and the final velocity $\vec{v}_f = v_f\hat{\phi}$ is in the direction exactly tangent to the surface. Thus, the final angular momentum is

$$\vec{L}_f = (r_p\hat{r}) \times m_i\left(v_f\hat{\phi}\right) = r_pm_iv_f\left(\hat{r}\times\hat{\phi}\right) = r_pm_iv_f\hat{z}. \tag{7}$$

Plugging equations (6) and (7) into equation (3) yields

$$5r_p m_i v_0 \sin\theta \hat{z} = r_p m_i v_f \hat{z} \quad \Rightarrow \quad v_f = 5v_0 \sin\theta. \tag{8}$$

However, this equation still has two unknowns v_f and θ, so we require another condition.

To determine the final velocity of the instrument, we can think about the situation physically. We realize that the instrument will be accelerated as it falls into the gravitational potential of the planet. The change in speed of the instrument can be found from conservation of mechanical energy because there are no nonconservative forces acting on the instrument. Thus, we have

$$E_{mi} = E_{mf} \quad \Rightarrow \quad K_i + U_{Gi} = K_f + U_{Gf}. \tag{9}$$

In previous problem sets, we've found the universal gravitational potential (with a reference point infinitely far away) to be

$$U_G(R) = -\frac{Gm_i m_p}{R}. \tag{10}$$

Plugging this and the form of the kinetic energy into equation (9) gives

$$\frac{m_i}{2}v_0^2 - \frac{Gm_i m_p}{5r_p} = \frac{m_i}{2}v_f^2 - \frac{Gm_i m_p}{r_p}. \tag{11}$$

Substituting equation (8) allows us to find the final answer of

$$\frac{m_i}{2}v_0^2 - \frac{Gm_i m_p}{5r_p} = \frac{m_i}{2}\left(5v_0\sin\theta\right)^2 - \frac{Gm_i m_p}{r_p} \quad \Rightarrow \quad (5v_0\sin\theta)^2 = v_0^2 + \frac{8Gm_p}{5r_p} \tag{12}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{5}\sqrt{1 + \frac{8Gm_p}{5v_0^2 r_p}}\right). \tag{13}$$

2. Toy locomotive

We begin by choosing our system to consist of the locomotive and the track. Because there are no *external* torques about the central vertical axis, the angular momentum of our system must remain constant about that axis

$$\vec{L}^i_{sys} = \vec{L}^f_{sys}. \tag{1}$$

The initial angular momentum of the system is

$$\vec{L}^i_{sys} = 0 \tag{2}$$

because both the locomotive and the track are at rest. The final angular momentum will be composed of the angular momentum of the locomotive and the track according to

$$\vec{L}^f_{sys} = \vec{L}^f_L + \vec{L}^f_T, \tag{3}$$

where the subscripts L and T refer to the locomotive and track respectively.

The final angular momentum of the locomotive (which can be considered to be a point mass) is given by

$$\vec{L}^f_L = \vec{R}_L \times \vec{p}_L = R_T \hat{r} \times m_L \vec{v}_L, \tag{4}$$

where $\vec{r}_L$ is the position vector from the pivot point at the top of the vertical axis to the locomotive, $\vec{p}_L$ is momentum of the locomotive in the ground reference frame, and $\vec{v}_L$ is the velocity of the locomotive relative to the ground. From the figure we see that the locomotive is moving only in the $\hat{\phi}$ direction, so $\vec{v}_L = v_f\hat{\phi}$

(where v_f is the final speed relative to the floor that we are trying to determine). Plugging this into equation (4) produces

$$\vec{L}_L^f = R_T\hat{r} \times m_L v_f \hat{\phi} = R_T m_L v_f \hat{z}, \tag{5}$$

where the $\hat{\phi}$ unit vector points counter-clockwise (when viewed from above) and the $\hat{z}$ unit vector points upwards.

Next we must determine the final angular momentum of the track (which is a continuous system). We know that the moment of inertia of a thin uniform ring about an axis passing through its center is

$$I_T = m_T R_T^2. \tag{6}$$

This can also be calculated from the definition of the moment of inertia according to

$$I_T = \int_{ring} \rho^2 dm = \int_{ring} \rho^2 \lambda dl_\phi = \int_{ring} \rho^2 \lambda \rho d\phi = \lambda \rho^3 \int_0^{2\pi} d\phi = \lambda R_T^3 (2\pi - 0) = \left(\frac{m_T}{2\pi R_T}\right) R_T^3 2\pi = m_T R_T^2 \tag{7}$$

using the linear mass density $\lambda = m_T/(2\pi R_T) = dm/dl_\phi$ and the arc length along the track $l_\phi = \rho\phi$. Using the definition of the angular momentum of a continuous system, we know that the final angular momentum of the track is

$$\vec{L}_T^f = I_T \vec{\omega}_f, \tag{8}$$

where $\vec{\omega}_f$ is the final angular velocity of the track. The angular velocity can be related to the tangential velocity of any point on the track through

$$\vec{\omega}_f = \frac{\vec{\rho} \times \vec{v}_\phi}{\rho^2} = \frac{R_T\hat{\rho} \times \left(-v_T\hat{\phi}\right)}{R_T^2} = -\frac{v_T\hat{\rho} \times \hat{\phi}}{R_T} = -\frac{v_T}{R_T}\hat{z}, \tag{9}$$

where v_T is the final tangential speed of the track relative to the ground and we have deduced its direction from imagining the physical situation (i.e. by Newton's third law, if the locomotive goes one way, the track must go the other). Substituting this and equation (6) into equation (8) gives

$$\vec{L}_T^f = \left(m_T R_T^2\right)\left(-\frac{v_T}{R_T}\hat{z}\right) = -m_T R_T v_T \hat{z}. \tag{10}$$

Unfortunately, we do not know the tangential speed of the track v_T. However, we can find it from information given in the problem statement. Specifically, we know the speed of the locomotive relative to the track v. The formula for converting velocities between different reference frames is

$$\vec{v}_{gL} = \vec{v}_{gT} + \vec{v}_{TL} \quad \Rightarrow \quad v_f\hat{\phi} = -v_T\hat{\phi} + v\hat{\phi} \quad \Rightarrow \quad v_T = v - v_f, \tag{11}$$

where $\vec{v}_{gL} = v_f\hat{\phi}$ is the velocity of the locomotive in the reference frame of the ground, $\vec{v}_{gT} = -v_T\hat{\phi}$ is the velocity of the track in the reference frame of the ground, and $\vec{v}_{TL} = v\hat{\phi}$ is the velocity of the locomotive in the reference frame of the *track*. Plugging this into equation (10) gives

$$\vec{L}_T^f = -m_T R_T (v - v_f)\hat{z}, \tag{12}$$

which is now composed exclusively of known quantities and the parameter we are trying to find v_f.

Finally, we can substitute equations (2), (3), (5), and (12) into equation (1) to find the final answer of

$$0 = \vec{L}_L^f + \vec{L}_T^f \quad \Rightarrow \quad 0 = R_T m_L v_f \hat{z} - m_T R_T (v - v_f)\hat{z} \quad \Rightarrow \quad v_f = \frac{m_T}{m_L + m_T} v. \tag{13}$$

3. Particle-rod collision revisited

1. The motion of any rigid body can be represented as the motion of the center of mass, plus a rotation about the center of mass. In problem set 6, we found the position of the center of mass after the collision to be

$$\vec{R}_{CM}(t) = \frac{V_0}{2}t\hat{x} + \frac{\ell}{4}\hat{y}. \tag{1}$$

However, to completely specify the motion of the particle-rod system, we must also calculate the angular velocity of rotation of the system *about its center of mass.*

To calculate the rotation after the collision from the information just before the collision, we will use conservation of angular momentum about the center of mass (as there are no external torques acting on the particle-rod system). This is expressed as

$$\vec{L}_b = \vec{L}_a. \tag{2}$$

where the subscript "b" indicates that the quantity is evaluated just before the collision and the subscript "a" indicates just after. Just before the collision, the angular momentum of the system about the center of mass is the sum of the angular momenta of all the objects i in the system. This is

$$\vec{L}_b = \sum_i \vec{L}_{ib} = \sum_i \vec{r}_{ib} \times m_i\vec{v}_{ib} = \vec{r}_{pb} \times M\vec{v}_{pb}, \tag{3}$$

where the subscript "p" indicates the particle. Note that there is no contribution to the angular momentum from the rod as it is completely stationary before the collision. From inspecting the problem statement we see that, just before the collision, the particle is moving with $\vec{v}_{pb} = V_0\hat{x}$ at a position $\vec{r}_{pb} = -(\ell/4)\hat{y}$ relative to the center of mass of the particle-rod system at $t = 0$. Substituting these values, equation (3) becomes

$$\vec{L}_b = \frac{M}{4}\ell V_0\hat{z}. \tag{4}$$

After the collision, the rod and particle form a combined object that rotates at a common angular velocity $\vec{\omega}$ about its center of mass. The angular momentum of such an rotating extended object is

$$\vec{L}_a = I_{CM}\vec{\omega}, \tag{5}$$

where I_{CM} is the momentum of inertia of the particle-rod system about its center of mass. Substituting this and equation (4) into equation (2) allows us to find

$$\vec{\omega} = \frac{M}{4I_{CM}}\ell V_0\hat{z}. \tag{6}$$

This is almost the final solution, but we don't yet know I_{CM}. To calculate it, we start from the definition of the center of mass

$$I_{CM} = \int_M \rho^2 dm, \tag{7}$$

where the integral is taken over the entire mass of the combined object. Because integrals are just summations of infinitesimally small differential elements, we can separate it into the contributions from the two objects

$$I_{CM} = \int_{rod} \rho^2 dm + \int_{particle} \rho^2 dm = I_{rod} + I_{particle}. \tag{8}$$

Since the particle is well represented by a point mass, all of its mass is located at the same distance $\rho = \ell/4$ from the center of mass. Thus,

$$I_{particle} = \int_{particle} \rho^2 dm = \int_{particle} \left(\frac{\ell}{4}\right)^2 dm = \frac{\ell^2}{16} \int_{particle} dm = \frac{1}{16} M\ell^2. \tag{9}$$

There are two ways to find the contribution to the moment of inertia from the rod. This first is simpler and uses the parallel axis theorem. From the table of moments of inertia presented in lecture, we know that a uniform thin rod rotated about its center of mass (i.e. its geometric center) has a moment of inertia of $I_{rod}^{center} = M\ell^2/12$. However, we are interested in the rotation of the rod about the center of mass of the particle-rod system, not the center of mass of the rod alone. From equation (1), we see that the center of mass of the rod (which is at $(\ell/2)\hat{y}$) is a distance of $h = \ell/4$ away from the center of mass of the particle-rod system at $t = 0$. Thus, we will use the parallel axis theorem to see that

$$I_{rod} = I_{rod}^{center} + Mh^2 = \frac{1}{12} M\ell^2 + \frac{1}{16} M\ell^2 = \frac{7}{48} M\ell^2. \tag{10}$$

The second way to find the moment of inertia of the rod is to directly evaluate the integral in equation (8). This approach is more challenging, but applies to a wider variety of situations. To convert from an integral over mass to an integral in space, we use the linear mass density $\lambda = dm/d\rho$ and the fact that the density is uniform $\lambda = M/\ell$ to see that $dm = (M/\ell)d\rho$. Substituting this we see that

$$I_{rod} = \int_{rod} \rho^2 dm = \frac{M}{\ell} \int_{rod} \rho^2 d\rho. \tag{11}$$

To determine the bounds of the integral, we must think about the geometry of the problem. Here ρ represents the distance from the center of mass of the particle-rod system, which is at $y = \ell/4$. Thus, to integrate over the full object we must consider the part of the rod above and below the center of mass, which is tricky as some of these points have the same value of ρ. This can be handled by splitting the integral into the contributions above and below, which are given by

$$I_{rod} = \frac{M}{\ell} \int_0^{3\ell/4} \rho^2 d\rho + \frac{M}{\ell} \int_0^{\ell/4} \rho^2 d\rho. \tag{12}$$

respectively. Evaluting the integrals is straightforward and yields

$$I_{rod} = \frac{M}{\ell} \left(\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=3\ell/4} + \frac{M}{\ell} \left(\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=\ell/4} = \frac{M}{3\ell} \left(\frac{3}{4}\ell\right)^3 + \frac{M}{3\ell} \left(\frac{1}{4}\ell\right)^3 = \frac{1}{3} \left(\frac{27}{64} + \frac{1}{64}\right) M\ell^2 = \frac{7}{48} M\ell^2, \tag{13}$$

which is identical to the solution using the parallel axis theorem (i.e. equation (10)).

Substituting equation (9) and (13) into equation (8) gives the total moment of inertia of the particle-rod system around its center of mass, which is

$$I_{CM} = \frac{7}{48} M\ell^2 + \frac{1}{16} M\ell^2 = \frac{5}{24} M\ell^2. \tag{14}$$

Substituting this into equation (6) gives the final answer of

$$\vec{\omega} = \frac{6}{5} \frac{V_0}{\ell} \hat{z}. \tag{15}$$

Importantly, since there are no additional forces acting at later times, we have conservation of angular momentum. Thus, the angular velocity of the particle-rod system remains the same at all times $t \geq 0$.

2. We know that the particle-rod system moves based on the combination of two types of motion. Its center of mass translates, which has been calculated in equation (1). Additionally, in the center of mass reference frame, all points in the system rotate about the center of mass with a constant angular velocity $\vec{\omega} = (6/5)(V_0/\ell)\hat{z}$. This rotation is uniform circular motion, so the angular velocity corresponds to a velocity of

$$\vec{v} = \vec{\omega} \times \vec{\rho} = \omega\hat{z} \times \rho\hat{\rho} = \rho\omega\hat{\phi}. \tag{16}$$

We are asked about the position of the particle, which is located at a distance $\rho = \ell/4$ away from the center of mass. Thus, it has a velocity of

$$\vec{v} = \frac{\ell}{4}\omega\hat{\phi} \tag{17}$$

after the collision. Note that if we substitute the value for ω, we find $\vec{v} = (6/20)V_0\hat{\phi}$, which shows that the particle is slowed down substantially as a result of the collision.

Ultimately, we want to express the position in Cartesian coordinates, so we will convert the cylindrical unit vector $\hat{\phi}$ according to

$$\vec{v} = \frac{\ell}{4}\omega\left(-\sin\phi\hat{x} + \cos\phi\hat{y}\right). \tag{18}$$

Given that ω is constant, we can integrate the definition of the angular speed $\dot{\phi} = \omega$ to find

$$\phi(t) = \omega t + C, \tag{19}$$

where C is an integration constant. Since ϕ is the angle from the $+x$-axis and increases towards the $+y$-axis, at $t = 0$ the particle is at $\phi(0) = -\pi/2$. Using this initial condition, we find that $C = -\pi/2$. Substituting this and equation (19) into equation (18) gives

$$\vec{v}(t) = \frac{\ell}{4}\omega\left(-\sin\left(\omega t - \frac{\pi}{2}\right)\hat{x} + \cos\left(\omega t - \frac{\pi}{2}\right)\hat{y}\right) = \frac{\ell}{4}\omega\left(\cos\left(\omega t\right)\hat{x} + \sin\left(\omega t\right)\hat{y}\right), \tag{20}$$

where in the second step we have used trigonometric identities that one can find in a table. This result is consistent with our intuition – the bottom of the rod should start rotating to the right, in the same direction the particle strikes it.

To find the position, we simply integrate equation (20) to find

$$\vec{r}(t) = \frac{\ell}{4}\left(\sin\left(\omega t\right)\hat{x} - \cos\left(\omega t\right)\hat{y}\right). \tag{21}$$

However, we must remember that the position $\vec{r}(t)$ is in the reference frame moving with the center of mass of the particle-rod system. Thus, we must change back to the reference frame given in the problem statement using $\vec{R}_p(t) = \vec{R}_{CM}(t) + \vec{r}(t)$. This yields the final answer of

$$\vec{R}_p(t) = \vec{R}_{CM}(t) + \frac{\ell}{4}\left(\sin(\omega t)\hat{x} - \cos(\omega t)\hat{y}\right). \tag{22}$$

4. Former exam question: The ringmaster

1. (4 points) In the vertical direction, the ring experiences only gravity and the normal force and does not accelerate. Thus, the vertical component of Newton's second law for the ring tells us that

$$N - mg = 0 \quad \Rightarrow \quad N = mg. \tag{1}$$

In the horizontal direction, kinetic friction causes a force

$$\vec{F}_f = -\mu N\hat{x} = -\mu mg\hat{x}, \tag{2}$$

where we have defined the direction of motion of the ring to be the $\hat{x}$ direction. Since this is the only horizontal force on the ring, Newton's second law in the $\hat{x}$ direction tells us that the corresponding acceleration is constant and has a value of

$$F_{fx} = ma \quad \Rightarrow \quad -\mu mg = ma \quad \Rightarrow \quad a = -\mu g. \tag{3}$$

According to the work-kinetic energy theorem, the change in kinetic energy is equal to the work done by friction. Since the disk starts out with kinetic energy $K_i = E_0$ and finishes at rest with $K_f = 0$, we know that

$$\Delta K = W \quad \Rightarrow \quad K_f - K_i = \int \vec{F}_f \cdot d\vec{l} \quad \Rightarrow \quad 0 - E_0 = \int_0^{\Delta x} -\mu mg\hat{x} \cdot (\hat{x}dx) \quad \Rightarrow \quad E_0 = \mu mg\Delta x, \tag{4}$$

where Δx is the total distance traveled by the ring and we have used equation (2). The ring undergoes motion under constant acceleration, so we can immediately write down its velocity as

$$v(t) = at + v_0 = -\mu gt + v_0, \tag{5}$$

where we have used equation (3). We can use this to find the time at which the ring will stop, which gives

$$v(t_1) = 0 = -\mu gt_1 + v_0 \quad \Rightarrow \quad t_1 = \frac{v_0}{\mu g}. \tag{6}$$

To calculate the initial velocity, we can use the formula for the translational kinetic energy and find

$$K = \frac{m}{2}v^2 \quad \Rightarrow \quad K_i = E_0 = \frac{m}{2}v_0^2 \quad \Rightarrow \quad v_0 = \sqrt{\frac{2E_0}{m}}. \tag{7}$$

Substituting this into equation (6) gives the final answer of

$$t_1 = \frac{1}{\mu g}\sqrt{\frac{2E_0}{m}}. \tag{8}$$

Alternative solution:

Kinetic friction causes a horizontal force, which will reduce the momentum of the ring. Let the initial velocity of the ring be $\vec{v}_0 = v_0\hat{x}$, directed along the horizontal axis x. The initial momentum and kinetic energy of the ring are

$$\vec{p}_0 = mv_0\hat{x} \tag{9}$$

$$K_i = E_0 = \frac{m}{2}v_0^2 \tag{10}$$

respectively. Solving equation (10) for v_0 and plugging it into equation (9) gives

$$\vec{p}_0 = m\sqrt{\frac{2E_0}{m}}\hat{x} = \sqrt{2mE_0}\hat{x}. \tag{11}$$

The only net horizontal force on the ring arises from kinetic friction, which is given by equation (2). This force $\vec{F}_f$ is constant. Thus, the generalized version of Newton's second law, $d\vec{p}/dt = \vec{F}$, can be integrated to show that

$$\vec{p}(t) = \vec{F}_f t + \vec{p}_0 = \left(-\mu mgt + \sqrt{2mE_0}\right)\hat{x} \tag{12}$$

using equations (2) and (11). From this we can find the time at which the ring comes to rest to be

$$\vec{p}(t_1) = 0 = \left(-\mu m g t_1 + \sqrt{2mE_0}\right)\hat{x} \quad \Rightarrow \quad t_1 = \frac{1}{\mu m g}\sqrt{2mE_0} \quad \Rightarrow \quad t_1 = \frac{1}{\mu g}\sqrt{\frac{2E_0}{m}}, \tag{13}$$

which is consistent with equation (8) .

2. (5 points) For the rest of this problem, we use a cylindrical coordinate system (ρ, ϕ, z) with its origin O at the center of the ring and the vertical $\hat{z}$ direction pointing *downwards*. Friction results in a vertical torque, which will reduce the angular momentum of the ring with time. We will first calculate the initial angular momentum and then find the strength of the torque to determine how quickly it will come to rest.

   Let $\vec{\omega}_0 = \omega_0 \hat{z}$ be the initial angular velocity with $\hat{z}$ defined to be downwards such that $\omega_0 > 0$. The initial angular momentum about pivot point O is then

$$\vec{L}_0 = I\vec{\omega}_0 = I\omega_0\hat{z}, \tag{14}$$

   where I is the moment of inertia about the $\hat{z}$ axis. The initial kinetic energy of the ring is its rotational kinetic energy, which can be used to find the initial angular velocity to be

$$K_i = E_0 = \frac{I}{2}\omega_0^2 \quad \Rightarrow \quad \omega_0 = \pm\sqrt{\frac{2E_0}{I}} \quad \Rightarrow \quad \omega_0 = \sqrt{\frac{2E_0}{I}}. \tag{15}$$

   Here we have used the fact that $\omega_0 > 0$ to choose the positive sign in front of the square root. We can plug equation (15) into equation (14) to obtain

$$\vec{L}_0 = I\sqrt{\frac{2E_0}{I}}\hat{z} = \sqrt{2IE_0}\hat{z}. \tag{16}$$

   Given that the moment of inertia of a thin horizontal ring around the z-axis, this becomes

$$\vec{L}_0 = \sqrt{2\left(m\rho_0^2\right)E_0}\hat{z} = \rho_0\sqrt{2mE_0}\hat{z}. \tag{17}$$

   To calculate the torque exerted by the kinetic friction force, we need to integrate around the circumference of the ring. A given infinitesimal element of the ring has a mass Δm_i, where the subscript denotes which differential element is being considered. Drawing a free body diagram for the differential element i shows that the only forces in the vertical direction are gravity $\Delta m_i g$ and the normal force ΔN_i. Since no part of the ring experiences a vertical acceleration, the vertical component of Newton's second law gives

$$\Delta N_i - \Delta m_i g = 0 \quad \Rightarrow \quad \Delta N_i = \Delta m_i g. \tag{18}$$

   In the tangential $\hat{\phi}$ direction, the differential element will experience a kinetic frictional force given by

$$\Delta\vec{F}_{fi} = -\mu\Delta N_i\hat{\phi} = -\mu\Delta m_i g\hat{\phi}, \tag{19}$$

   which points in the opposite direction to the rotation as to slow the object down. Note that $\hat{\phi}$ must point clockwise when viewed from above for the (ρ, ϕ, z) coordinate system to be right-handed. Since the differential element has a position vector of $\vec{r}_i = \rho_i\hat{\rho} = \rho_0\hat{\rho}$, we can calculate the torque from this force to be

$$\Delta\vec{\tau}_i = \vec{r}_i \times \Delta\vec{F}_{fi} = (\rho_0\hat{\rho}) \times \left(-\mu\Delta m_i g\hat{\phi}\right) = -\rho_0\mu\Delta m_i g\hat{\rho}\times\hat{\phi} = -\rho_0\mu\Delta m_i g\hat{z}. \tag{20}$$

   To find the overall torque on the ring $\vec{\tau}$, we add up the contributions from every differential element, which, in the limit of $\Delta m_i \to 0$, becomes the integral

$$\vec{\tau} = \lim_{\Delta m_i \to 0}\sum_i \Delta\vec{\tau}_i = \lim_{\Delta m_i \to 0}\sum_i \left(-\rho_0\mu\Delta m_i g\hat{z}\right) = \int_{ring}\left(-\rho_0\mu g\hat{z}\right)dm = -\rho_0\mu g\left(\int_{ring} dm\right)\hat{z}. \tag{21}$$

Given that the integral of 1 over the entire mass of the object is just the total mass m, we find

$$\vec{\tau} = -\rho_0 \mu g \,(m)\, \hat{z}. \tag{22}$$

We see that this torque is in the direction opposite to the ring's angular velocity as is intuitive. Since this torque is constant, integrating the formula $d\vec{L}/dt = \vec{\tau}$ in time shows that

$$\vec{L}(t) = \vec{\tau} t + \vec{L}_0. \tag{23}$$

Plugging in equations (17) and (22) gives

$$\vec{L}(t) = -\rho_0 \mu g m t \hat{z} + \rho_0 \sqrt{2mE_0}\hat{z} = \rho_0 \left(\sqrt{2mE_0} - \mu g m t\right)\hat{z}. \tag{24}$$

This can be used to calculate the time t_2 at which the ring stops spinning according to

$$\vec{L}(t_2) = 0 = \rho_0 \left(\sqrt{2mE_0} - \mu g m t_2\right)\hat{z} \quad \Rightarrow \quad t_2 = \frac{1}{\mu g m}\sqrt{2mE_0} = \frac{1}{\mu g}\sqrt{\frac{2E_0}{m}}. \tag{25}$$

Dividing this by equation (8) shows that the ratio $t_2/t_1 = 1$.

Alternative solution:

Since friction does not depend on the surface area, we can consider the torque due to the friction of the ring as equivalent to the torque from a single point mass at the same radius

$$\vec{\tau}_f = \vec{r} \times \vec{F}_f = (\rho_0 \hat{\rho}) \times \left(-\mu N \hat{\phi}\right) = -\mu N \rho_0 \hat{\rho} \times \hat{\phi} = -\mu m g \hat{z}, \tag{26}$$

where we have used $N = mg$ from Newton's law in the vertical direction. We can then apply Newton's second law for rotational motion in the $\hat{z}$ direction and substitute equation (26) to find

$$\Sigma\vec{\tau} = I\vec{\alpha} \quad \Rightarrow \quad \vec{\tau}_f = m\rho_0^2\vec{\alpha} \quad \Rightarrow \quad -\mu m g \rho_0 = m\rho_0^2 \alpha \quad \Rightarrow \quad \alpha = -\frac{\mu g}{\rho_0}, \tag{27}$$

where $I = m\rho_0^2$ is the moment of inertia of the ring around the z axis and the only net torque is from the friction force. Thus, the angular acceleration is constant in time. Integrating equation (27) in time gives

$$\omega(t) = -\frac{\mu g}{\rho_0} t + \omega_0, \tag{28}$$

where ω_0 is the initial angular velocity at $t = 0$. Substituting equation (15) and $I = m\rho_0^2$, this can be used to find the time t_2 at which the angular velocity becomes zero to be

$$\omega(t_2) = 0 = -\frac{\mu g}{\rho_0} t_2 + \sqrt{\frac{2E_0}{I}} = -\frac{\mu g}{\rho_0} t_2 + \sqrt{\frac{2E_0}{m\rho_0^2}} \quad \Rightarrow \quad t_2 = \frac{1}{\mu g}\sqrt{\frac{2E_0}{m}}. \tag{29}$$

This is consistent with equation (25).

3. (1 points) The calculation of the time t_1 stays the same because, for pure translation, the object is considered to be a point mass, so it doesn't matter if the object is a ring or a disk. To find t_2, we should first calculate the total torque arising from the frictional force. As in part 2, we can do this by integrating the force acting on all the differential elements that make up the object. The frictional force on a differential mass element is still given by equation (19). However, the torque must take into account that different mass elements can now be located at different radii $\vec{r}_i = \rho_i \hat{\rho}$. Thus, the torque expressed by equation (20) becomes modified to be

$$\Delta\vec{\tau}_i = \vec{r}_i \times \Delta\vec{F}_{fi} = (\rho_i \hat{\rho}) \times \left(-\mu \Delta m_i g \hat{\phi}\right) = -\rho_i \mu \Delta m_i g \hat{\rho} \times \hat{\phi} = -\rho_i \mu \Delta m_i g \hat{z}, \tag{30}$$

where we must remember that we've defined $\hat{\phi}$ to point clockwise and $\hat{z}$ to point down. To find the overall torque on the ring $\vec{\tau}$, we add up the contributions from every differential element, which, in the limit of $\Delta m_i \to 0$, becomes the integral

$$\vec{\tau} = \lim_{\Delta m_i \to 0} \sum_i \Delta\vec{\tau}_i = \lim_{\Delta m_i \to 0} \sum_i \left(-\rho_i \mu \Delta m_i g \hat{z}\right) = \int_{disk} \left(-\rho_i \mu g \hat{z}\right) dm. \tag{31}$$

Given that the disk is uniform, we can use the areal density to show

$$\sigma = \frac{m}{\pi\rho_0^2} = \frac{dm}{dA} \quad \Rightarrow \quad dm = \frac{m}{\pi\rho_0^2} dA, \tag{32}$$

where dA is the differential element of area. In polar coordinates, it is expressed as $dA = \rho_i d\phi d\rho_i$. Substituting this and equation (32) into equation (31) gives

$$\vec{\tau} = \int_{ring} \left(-\rho_i \mu g \hat{z}\right) \frac{m}{\pi\rho_0^2} \rho_i d\phi d\rho_i = -\mu g \frac{m}{\pi\rho_0^2} \int_0^{\rho_0} \int_0^{2\pi} \rho_i^2 d\phi d\rho_i \hat{z}. \tag{33}$$

Evaluating the integral in ϕ gives

$$\vec{\tau} = -\mu g \frac{m}{\pi\rho_0^2} \int_0^{\rho_0} \rho_i^2 \left(\int_0^{2\pi} d\phi\right) d\rho_i \hat{z} = -\mu g \frac{m}{\pi\rho_0^2} \int_0^{\rho_0} \rho_i^2 \left(2\pi - 0\right) d\rho_i \hat{z} = -2\mu g \frac{m}{\rho_0^2} \int_0^{\rho_0} \rho_i^2 d\rho_i \hat{z}. \tag{34}$$

Evaluating the integral in radius gives

$$\vec{\tau} = -2\mu g \frac{m}{\rho_0^2} \left(\frac{\rho_0^3}{3} - \frac{0^3}{3}\right) \hat{z} = -\frac{2}{3} \mu g m \rho_0 \hat{z}. \tag{35}$$

We see that this torque is in the direction opposite to the ring's angular velocity as is intuitive. Since this torque is constant, integrating the formula $d\vec{L}/dt = \vec{\tau}$ (i.e. the generalized version of Newton's second law for rotation) in time shows that

$$\vec{L}(t) = \vec{L}_0 + \vec{\tau} t. \tag{36}$$

To calculate L_0 we can use equation (16) from part 2, but we must use the moment of inertia of a disk $I = m\rho_0^2/2$ (rather than a ring) to find

$$\vec{L}_0 = \sqrt{2 \frac{m\rho_0^2}{2} E_0} \hat{z} = \rho_0 \sqrt{m E_0} \hat{z}. \tag{37}$$

Plugging in this and equation (35) into equation (36) gives

$$\vec{L}(t) = \rho_0 \sqrt{m E_0} \hat{z} - \frac{2}{3} \mu g m \rho_0 t \hat{z} = \rho_0 \left(\sqrt{m E_0} - \frac{2}{3} \mu g m t\right) \hat{z}. \tag{38}$$

This can be used to calculate the time t_2 at which the ring stops spinning according to

$$\vec{L}(t_2) = 0 = \rho_0 \left(\sqrt{m E_0} - \frac{2}{3} \mu g m t_2\right) \hat{z} \quad \Rightarrow \quad t_2 = \frac{3}{2} \frac{1}{\mu g m} \sqrt{m E_0} = \frac{3}{2} \frac{1}{\mu g} \sqrt{\frac{E_0}{m}}. \tag{39}$$

Dividing this by equation (8) shows that the ratio $t_2/t_1 = 3/(2\sqrt{2})$.

5. Optional: Elliptic Orbit

1. As in problem 1, the motion of the satellite will conserve both angular momentum and mechanical energy according to

$$\vec{L}_f = \vec{L}_c \tag{1}$$
$$E_{mf} = E_{mc}. \tag{2}$$

We will choose to evaluate angular momentum about the center of the planet and take the reference point for the gravitational potential energy to be infinitely far away. Thus, conservation of angular momentum and mechanical energy become

$$\vec{r}_f \times m_s\vec{v}_f = \vec{r}_c \times m_s\vec{v}_c \tag{3}$$
$$K_f + U_{Gf} = K_c + U_{Gc} \tag{4}$$

respectively. Taking a cylindrical coordinate system and substituting the forms of the kinetic and gravitational potential energy gives

$$(r_f\hat{r}) \times m_s\left(v_f\hat{\phi}\right) = (r_c\hat{r}) \times m_s\left(v_c\hat{\phi}\right) \quad\Rightarrow\quad m_s r_f v_f\hat{z} = m_s r_c v_c\hat{z} \quad\Rightarrow\quad r_c = \frac{v_f}{v_c}r_f \tag{5}$$
$$\frac{m_s}{2}v_f^2 - \frac{Gm_sm_p}{r_f} = \frac{m_s}{2}v_c^2 - \frac{Gm_sm_p}{r_c} \quad\Rightarrow\quad \frac{2Gm_p}{r_c} - \frac{2Gm_p}{r_f} = v_c^2 - v_f^2. \tag{6}$$

Substituting equation (5) into equation (6) gives

$$\frac{2Gm_p}{r_f}\frac{v_c}{v_f} - \frac{2Gm_p}{r_f} = v_c^2 - v_f^2 \quad\Rightarrow\quad \frac{2Gm_p}{r_fv_f}(v_c - v_f) = (v_c + v_f)(v_c - v_f) \quad\Rightarrow\quad v_c = \frac{2Gm_p}{r_fv_f} - v_f. \tag{7}$$

We can plug this into equation (5) to find

$$r_c = \frac{v_f}{\frac{2Gm_p}{r_fv_f} - v_f}r_f = r_f\left(\frac{2Gm_p}{r_fv_f^2} - 1\right)^{-1}. \tag{8}$$

2. Since the satellite is not burning any fuel, the gravitational attraction to the planet must be causing the centripetal acceleration enabling the uniform circular motion. This condition is expressed through Newton's second law for the satellite as

$$\vec{F}_G = m_s\vec{a}_{cent}. \tag{9}$$

The centripetal acceleration is given by $\vec{a}_{cent} = -r_0\omega^2\hat{r} = -r_0(v_0/r_0)^2\hat{r} = -(v_0^2/r_0)\hat{r}$, where ω is the angular speed of the satellite. Substituting this and the form of the gravitational force into equation (9) gives

$$-\frac{Gm_sm_p}{r_0^2}\hat{r} = -m_s\frac{v_0^2}{r_0}\hat{r} \quad\Rightarrow\quad \frac{Gm_p}{r_0} = v_0^2 \quad\Rightarrow\quad v_0 = \sqrt{\frac{Gm_p}{r_0}}. \tag{10}$$

This solution can be checked by taking the circular case in our solution to part 1. If we let $r_f = r_c = r_0$, $v_f = v_c = v_0$, and substitute equation (10), we find that both equations (7) and (8) are satisfied.

Now let's compare v_0 with v_c, given that $r_0 = r_c$. Even though we have a solution for both, given by equations (7) and (10), this turns out to be surprisingly tricky. To make the comparison easier, we want to eliminate the velocity v_f from equation (7). Thus, we rearrange equation (5) to find

$$v_f = \frac{r_c}{r_f}v_c. \tag{11}$$

We substitute this into equation (7) to get

$$v_c = \frac{2Gm_p}{r_c v_c} - \frac{r_c}{r_f} v_c \quad \Rightarrow \quad v_c^2 = \frac{2Gm_p}{r_c} - \frac{r_c}{r_f} v_c^2 \quad \Rightarrow \quad \left(\frac{r_f + r_c}{r_f}\right) v_c^2 = \frac{2Gm_p}{r_c} \quad \Rightarrow \quad v_c = \sqrt{\frac{Gm_p}{r_c}} \sqrt{\frac{2r_f}{r_f + r_c}}. \tag{12}$$

We can now evaluate equation (10) at $r_0 = r_c$ to get

$$v_0 = \sqrt{\frac{Gm_p}{r_c}} \tag{13}$$

and compare with equation (12). Since $r_c < r_f$, we know that $\sqrt{2r_f/(r_f + r_c)} > 1$. Thus, we find that

$$v_c > v_0 \tag{14}$$

and the speed throughout the circular orbit is less than the speed at the point of closest approach in an elliptical orbit.