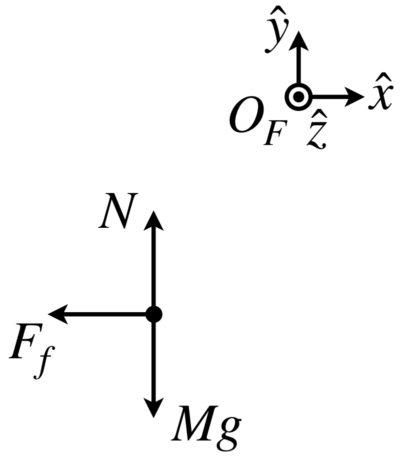
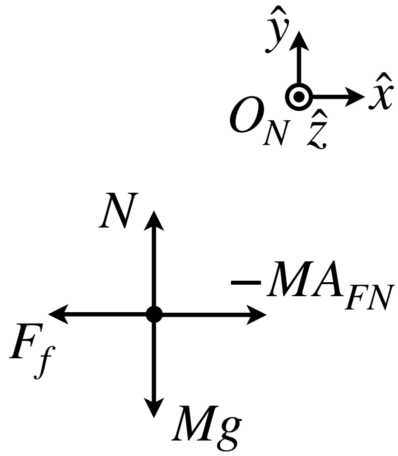
# 1. Box on the Metro (5 points)




a. (**1.0 point**) *Calculate the smallest magnitude of the deceleration $a_T$ for which the box will slide. Note that you may include the variables $\mu_s$ and $\mu_k$ in your answer.*

We will show how to solve this problem in each of the two reference frames: $O_F$ and $O_N$. Starting with the laboratory "fixed" frame $O_F$, we draw the free body diagram for the box shown above (left). From this we see that Newton's second law in the $\hat{y}$ direction is

$$N - Mg = Ma_y \quad \Rightarrow \quad N - Mg = 0 \quad \Rightarrow \quad N = Mg, \tag{1}$$

as we know that even if the box slides it will not accelerate vertically (i.e. $a_y = 0$). In the $\hat{x}$ direction Newton's second law is

$$-F_f = Ma_x \quad \Rightarrow \quad F_f = -Ma_x. \tag{2}$$

In order for the box not to slide, static friction must hold and the box must decelerate together with the train, meaning that $a_x = -a_T$. Substituting this, we find that

$$F_f = Ma_T. \tag{3}$$

We know that the limit on the magnitude of the static friction force is given by $F_f \leq \mu_s N$. Substituting equations (1) and (3) into this, we find that the maximum deceleration for which static friction can persist is

$$Ma_T \leq \mu_s Mg \quad \Rightarrow \quad a_T \leq \mu_s g. \tag{4}$$

Thus, $a_T = \mu_s g$ is the critical value above which the box will slide.

We can also solve this problem using the reference frame moving with the train $O_N$, which is non-inertial. In this frame, we must add the fictitious force associated with horizontal translation acceleration, $-MA_{FN}$, as shown in the free body diagram above (right). We can already tell that it will have to point in the $+\hat{x}$ direction in order to oppose the friction force and prevent slipping. In the non-inertial reference frame, Newton's second law in the $\hat{y}$ direction remains as equation (1) since the reference frame does not accelerate vertically. However, in the $\hat{x}$ direction Newton's second law changes to be

$$-MA_{FN} - F_f = Ma_x \quad \Rightarrow \quad -MA_{FN} - F_f = 0. \tag{5}$$

This is because the train is not accelerating when viewed from the reference frame $O_N$, so we require $a_x = 0$ for the box not to slide. The quantity $A_{FN}$ represents the horizontal acceleration of the origin of the reference frame $O_N$ (as seen in $O_F$), so it is simply the acceleration of the train $A_{FN} = -a_T$. Substituting this, we find that

$$F_f = Ma_T. \tag{6}$$

The limit on the magnitude of the static friction force is still $F_f \leq \mu_s N$. Substituting equations (1) and (6) into this gives the final answer of

$$Ma_T \leq \mu_s Mg \quad \Rightarrow \quad a_T \leq \mu_s g, \tag{7}$$

which is identical to what we found in the fixed reference frame (i.e. equation (4)).

b. (**1.0 point**) *If the box begins to slide, what will be the magnitude of the box's acceleration $a_b$ as viewed in the frame of reference of the train $O_N$? Note that you may include the variables $\mu_s$, $\mu_k$, and $a_T$ in your answer.*

As before, we will show how to solve this problem in both reference frames. In both frames the free body diagram will remains the same as in part a. Instead the only difference is that we have kinetic instead of static friction. Thus, in the fixed reference frame $O_F$, Newton's second law is still

$$N - Mg = 0 \quad \Rightarrow \quad N = Mg \tag{8}$$

in the $\hat{y}$ direction and

$$-F_f = Ma_x \quad \Rightarrow \quad F_f = -Ma_x \tag{9}$$

in the $\hat{x}$ direction. However, in this part we must allow the box to slide, meaning that it is free to have its own acceleration $a_x$ that is different than that of the train. Next, we use the form of the kinetic friction force $F_f = \mu_k N$ and substitute in equations (8) and (9) to find

$$-Ma_x = \mu_k Mg \quad \Rightarrow \quad a_x = -\mu_k g. \tag{10}$$

Here we have found the acceleration of the box, but we must remember that this is still in the fixed laboratory frame $O_F$. The problem asks for the acceleration in the frame moving with the train $a_b$, which is non-inertial. Since the non-inertial coordinate system is simply accelerating in a straight line, we know from lecture 5 (and our intuition!) that

$$\vec{a}_N = \vec{a}_F - \vec{A}_{FN}, \tag{11}$$

where $\vec{a}_N$ is the acceleration perceived in the non-inertial reference frame $O_N$ (which we are looking for), $\vec{a}_F$ is the acceleration perceived in the fixed reference frame $O_F$ (which we have just calculated), and $\vec{A}_{FN}$ is the acceleration of the origin of $O_N$ as seen in $O_F$. We can identify that $\vec{a}_N = a_b\hat{x}$, $\vec{a}_F = a_x\hat{x}$, and $\vec{A}_{FN} = -a_T\hat{x}$. Substituting these values as well as equation (10) into the $\hat{x}$ component of equation (11) gives

$$a_b = -\mu_k g + a_T \quad \Rightarrow \quad a_b = a_T - \mu_k g. \tag{12}$$

This is the final answer.

The calculation is easier in the non-inertial reference frame $O_N$. Since the free body diagram remains the same as in part a, Newton's second law is still

$$N - Mg = 0 \quad \Rightarrow \quad N = Mg \tag{13}$$

in the $\hat{y}$ direction and

$$-MA_{FN} - F_f = Ma_x \tag{14}$$

in the $\hat{x}$ direction. Since we are already in the non-inertial reference frame $O_N$ moving with the train, we can identify that $a_x = a_b$. Substituting this and the form of the kinetic friction force $F_f = \mu_k N$ into equation (14) gives

$$-MA_{FN} - \mu_k N = Ma_b. \tag{15}$$

Lastly, substituting equation (13) and the fact that $A_{FN} = -a_T$ gives

$$Ma_T - \mu_k Mg = Ma_b \quad \Rightarrow \quad a_b = a_T - \mu_k g. \tag{16}$$

This is identical to equation (12), the answer we obtained when performing the analysis in the fixed frame.

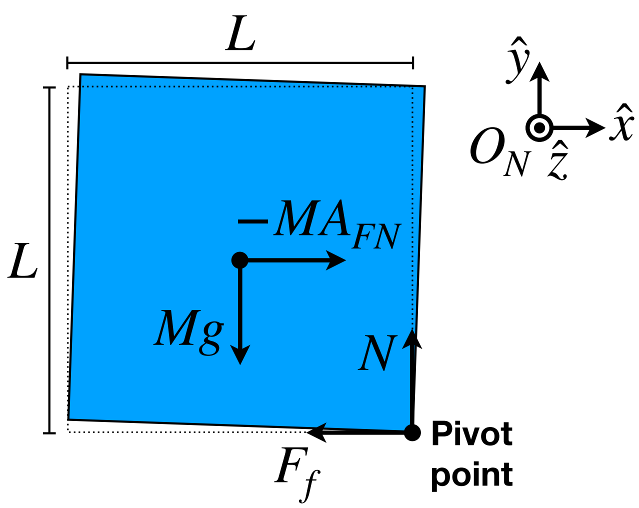
c. (**0.5 points**) *What is the smallest possible value of $a_b$? Note that you may include the variables $\mu_s$, $\mu_k$, and $a_T$ in your answer.*

To solve this problem, we combine the solutions to parts a and b. The value of $a_b$ is given by equation (16) and equation (7) shows the minimum possible value of $a_T$. Substituting equation (7) into equation (16) yields the final answer of

$$a_b \geq \mu_s g - \mu_k g = (\mu_s - \mu_k)g. \tag{17}$$

This is a positive number since the coefficient of static friction is larger than the coefficient of kinetic friction.

d. (**1.5 points**) *Calculate the smallest magnitude of the acceleration $a_T$ for which the box will start to tip over. Note that you may include the variables $\mu_s$ and $\mu_k$ in your answer.*




For this problem we will use the non-inertial reference frame $O_N$. This is because, to analyze rotation, it is easiest to take the reference frame moving with the pivot point of the object (i.e. the point around which the object is rotating). Since static friction is strong enough to prevent slipping, the pivot point will be at rest relative to the train and, hence, at rest in the non-inertial reference frame $O_N$. Since we are interested in tipping, we will start by drawing a free body diagram to illustrate the torques (shown above). The forces are the same as in previous parts, but we must carefully decide where they are applied. Here we are drawing the picture when the box has just barely starting tipping as this is what the problem asks about. Moreover, it is clear from this state where the normal force and friction forces should be applied. They must be at the pivot point as this is the only point of contact between the box and the seat. Next, we apply Newton's second law for rotation about the pivot point

$$\sum \vec{\tau} = I\vec{\alpha}, \tag{18}$$

where $\vec{\tau}$ are the torques created by the various forces, $I$ is the moment of inertia around the pivot point, and $\vec{\alpha}$ is the angular acceleration. Given the forces present, the torques are

$$\vec{r}_g \times \vec{F}_g + \vec{r}_A \times \vec{F}_A + \vec{r}_N \times \vec{F}_N + \vec{r}_f \times \vec{F}_f = I\vec{\alpha}. \tag{19}$$

Since the friction force $\vec{F}_f$ and the normal force $\vec{F}_N$ are applied at the pivot point, their displacement vectors are $\vec{r}_f = 0$ and $\vec{r}_N = 0$. However, the gravitational force $\vec{F}_g = -Mg\hat{y}$ and the translational fictitious force $\vec{F}_A = -MA_{FN}\hat{x}$ are both is applied at the center of mass of the box. Since the box is uniform and square, we have $\vec{r}_g = \vec{r}_A = -(L/2)\hat{x} + (L/2)\hat{y}$. Substituting this information and simplifying gives

$$\begin{aligned} \left(-\frac{L}{2}\hat{x} + \frac{L}{2}\hat{y}\right) \times (-Mg\hat{y}) + \left(-\frac{L}{2}\hat{x} + \frac{L}{2}\hat{y}\right) \times (-MA_{FN}\hat{x}) = I\vec{\alpha} \\ \Rightarrow \quad \frac{L}{2}Mg\left(\hat{x} \times \hat{y}\right) - \frac{L}{2}MA_{FN}\left(\hat{y} \times \hat{x}\right) = I\vec{\alpha} \\ \Rightarrow \quad \vec{\alpha} = \frac{ML}{2I}\left(g + A_{FN}\right)\hat{z}. \end{aligned} \tag{20}$$

This shows that the angular acceleration will be in the $\pm\hat{z}$ direction, so we only need to consider this component. Additionally, as above we know that $A_{FN} = -a_T$, which means that equation (20) becomes

$$\alpha = \frac{ML}{2I}\left(g - a_T\right). \tag{21}$$

Using the right hand rule, we see that the box will tip over only if $\alpha < 0$ (a positive angular acceleration will rotate the box counter-clockwise down into the seat). Applying this condition gives

$$\frac{ML}{2I}\left(g - a_T\right) < 0 \quad \Rightarrow \quad g - a_T < 0 \quad \Rightarrow \quad a_T > g. \tag{22}$$

Fortunately, we do not need to actually calculate the moment of inertia because we don't care about the magnitude of the angular acceleration, just if it is greater or less than zero. Thus, the smallest magnitude of deceleration that can cause the box to start to tip is $a_T = g$.

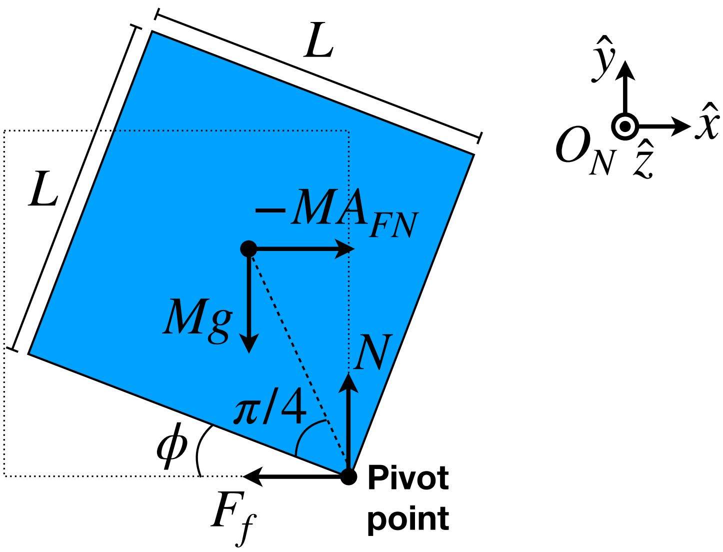
e. (**0.5 points**) *What is the critical value of the coefficient of static friction $\mu_s$ that separates when the box tips versus slides? Note that you may include the variables $a_T$ and $\mu_k$ in your answer.*

This problem can be solved by considering the solutions to parts a and d. From part a, we know that the box will start to *slide* if $a_T > \mu_s g$. From part d, we know that the box will start to *tip* if $a_T > g$. If these two accelerations are equal, then the box will start to slide at the exact same acceleration that it will start to tip. This is represents the critical case between sliding and tipping. Equating the two accelerations gives

$$\mu_s g = g \quad \Rightarrow \quad \mu_s = 1. \tag{23}$$

Note that it is possible to have a coefficient of static friction greater than 1 (e.g. the rubber used in rock climbing shoes has a value of $\mu_s \approx 1.2$ when in contact with rock). It just means that the object is more difficult to slide than it is to pick up.

f. (**0.5 points**) *Calculate the smallest magnitude of the acceleration $a_T$ for which the box will tip all the way over (i.e. rotate from flat on the seat to past 45°). Note that you may include the variables $\mu_s$ and $\mu_k$ in your answer.*




To rigorously solve this problem, one must consider the box at all tilt angles $\phi$ (defined in the figure above) between 0 and $\pi/4$. However, if one is in a hurry, you might consider the specific case of $\phi = \pi/4$, which makes the trigonometry easier. Then you can compare it with the $\phi \approx 0$ analyzed in part d and assume that solution for intermediate values of $\phi$ will be in between these two solutions. This will lead to the correct answer and is probably preferable, given the time constraints of an exam. Here we will present the full rigorous solution, but you can evaluate the expressions below at $\phi = \pi/4$ to find the expressions for that case.

We start in the same way as in part d (and also use the non-inertial frame $O_N$), but we will consider the box at an arbitrary tilt angle $\phi$. Here equation (19) still applies and the torques from the normal force and the friction force remain zero (as they are applied at the pivot point). Thus, we have

$$\vec{r}_g \times \vec{F}_g + \vec{r}_A \times \vec{F}_A = I\vec{\alpha}. \tag{24}$$

Finding the displacement vector from the pivot point to the center of mass is a bit challenging. The key is to first recognize that, as the box rotates the center of mass always remains the same distance away from the pivot point. From the Pythagorean theorem, we can see that this distance is

$$\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^2 = r_g^2 = r_A^2 \quad \Rightarrow \quad r_g = r_A = \frac{L}{\sqrt{2}}. \tag{25}$$

To find the direction of the displacement, it is helpful to consider the angle $\phi + \pi/4$, which represents the angle between the $-\hat{x}$ direction and the center of mass. Considering this angle, we can see that the direction of the displacement is

$$\vec{r}_g = \vec{r}_A = r_g \left(-\cos\left(\phi + \frac{\pi}{4}\right)\hat{x} + \sin\left(\phi + \frac{\pi}{4}\right)\hat{y}\right). \tag{26}$$

Combining this with equation (25), we find that the displacement vector for both the gravitational and fictitious forces is given by

$$\vec{r}_g = \vec{r}_A = \frac{L}{\sqrt{2}}\left(-\cos\left(\phi + \frac{\pi}{4}\right)\hat{x} + \sin\left(\phi + \frac{\pi}{4}\right)\hat{y}\right). \tag{27}$$

Substituting this, along the with expressions for the forces, into equation (24) gives

$$\begin{aligned} &\frac{L}{\sqrt{2}}\left(-\cos\left(\phi + \frac{\pi}{4}\right)\hat{x} + \sin\left(\phi + \frac{\pi}{4}\right)\hat{y}\right) \times (-Mg\hat{y}) \\ &\quad + \frac{L}{\sqrt{2}}\left(-\cos\left(\phi + \frac{\pi}{4}\right)\hat{x} + \sin\left(\phi + \frac{\pi}{4}\right)\hat{y}\right) \times (-MA_{FN}\hat{x}) = I\vec{\alpha}. \end{aligned} \tag{28}$$

Simplifying and using that $A_{FN} = -a_T$ produces

$$\frac{ML}{\sqrt{2}} g \cos\left(\phi + \frac{\pi}{4}\right) \hat{x} \times \hat{y} + \frac{ML}{\sqrt{2}} a_T \sin\left(\phi + \frac{\pi}{4}\right) \hat{y} \times \hat{x} = I\vec{\alpha} \tag{29}$$

$$\Rightarrow \quad \vec{\alpha} = \frac{ML}{\sqrt{2}I} \left(g \cos\left(\phi + \frac{\pi}{4}\right) - a_T \sin\left(\phi + \frac{\pi}{4}\right)\right) \hat{z}. \tag{30}$$

This shows that the angular acceleration will be in the $\pm\hat{z}$ direction, so we only need to consider this component. This gives

$$\alpha = \frac{ML}{\sqrt{2}I} \left(g \cos\left(\phi + \frac{\pi}{4}\right) - a_T \sin\left(\phi + \frac{\pi}{4}\right)\right). \tag{31}$$

We see that if we take $\phi = 0$, we recover equation (21) as expected (because $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$).

Ultimately, the problem asks us to find the condition for which the box tips all the way over (i.e. from $\phi = 0$ past $\phi = \pi/4$). In principle, one could determine this by recognizing that, since $\alpha = d^2\phi/dt^2$, equation (31) represents a differential equation that can be solved for the angular position $\phi$. If one knew $\phi(t)$, you could determine the conditions for which the box passes $\phi = \pi/4$. However, equation (31) is very difficult to solve. Instead there is a much easier way. From part d, we know that we need negative values of angular acceleration for the box to tip and we require $a_T > g$ for the box to start to tip at all. Studying equation (31), we realize that the angular acceleration only gets more negative as the box rotates (because the cosine function decreases from $\pi/4$ to $\pi/2$, while the sine function increases from $\pi/4$ to $\pi/2$). This means that the angular speed of the box will get faster and faster as it moves. Thus, the most challenging thing is to get the box to start rotating from rest at $\phi = 0$. To see this more clearly, we can use equation (31) to determine the condition for which the angular acceleration $\alpha < 0$ is negative, which is

$$\frac{ML}{\sqrt{2}I} \left(g \cos\left(\phi + \frac{\pi}{4}\right) - a_T \sin\left(\phi + \frac{\pi}{4}\right)\right) < 0 \quad \Rightarrow \quad a_T > g \frac{\cos\left(\phi + \frac{\pi}{4}\right)}{\sin\left(\phi + \frac{\pi}{4}\right)}. \tag{32}$$

This condition is hardest to satisfy at $\phi = 0$. For example, evaluating equation (32) at $\phi = \pi/4$ gives $a_T > 0$. Thus, our final answer is found by evaluating equation (32) at the most limiting angular position of $\phi = 0$ (or simply copying equation (22)), which yields

$$a_T > g. \tag{33}$$