Solutions to Problem Set 5

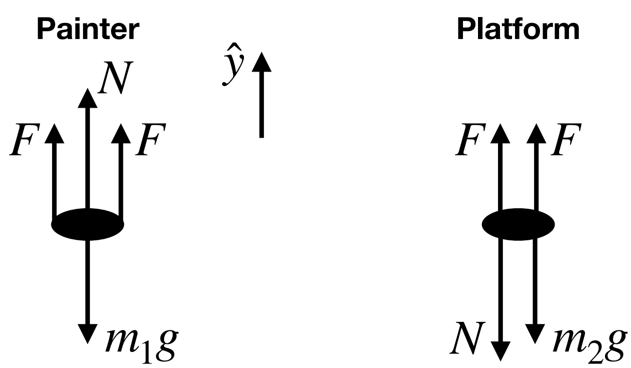
Applications of Newton's second law

PHYS-101(en)

1. Painter on a platform

We will explain two different, equally valid approaches to this question.

The first method is to draw two separate free body diagrams, one for the painter and one for the platform (both shown below). Since the pulleys are both massless and frictionless, the tension throughout each rope is F, the magnitude of the force the painter applies to each rope. The painter pulls the ropes downward, so the ropes exert an equal and opposite upward force on the painter by Newton's third law. Thus, the tension from the ropes pull upwards on both the painter and the platform. Additionally, the platform exerts an upward normal force on the painter, the magnitude of which we denote as N. By Newton's third law, there is a reaction normal force from the painter on the platform, which has an equal magnitude N and is in the opposite direction. Lastly, there is a downwards gravitational force $m_1 g$ on the painter and $m_2 g$ on the platform.



Let a_1 denote the magnitude of the acceleration of the painter and a_2 denote the magnitude of the acceleration of the platform. Since the painter is standing on the platform, they both have the same acceleration upwards. Thus, we know

$$a_1 = a_2 = a, \tag{1}$$

where we have denoted the common acceleration by the symbol a.

Applying Newton's second law to the painter in the vertical direction gives

$$2F + N - m_1 g = m_1 a, \tag{2}$$

while applying Newton's second law to the platform in the vertical direction gives

$$2F - N - m_2 g = m_2 a. \tag{3}$$

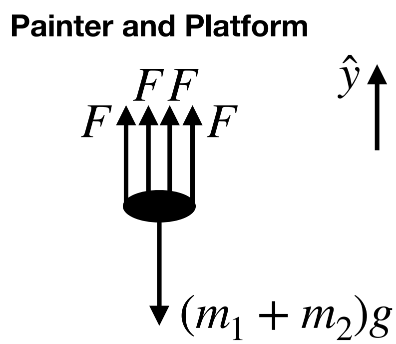
By adding equations (2) and (3), we can eliminate the normal force entirely and find

$$4F - (m_1 + m_2)g = (m_1 + m_2)a. \tag{4}$$

Solving for the acceleration gives the final answer

$$a = \frac{4F - (m_1 + m_2)g}{m_1 + m_2} = \frac{4F}{m_1 + m_2} - g. \tag{5}$$

A second method to find the same solution is to treat the painter and platform as a single system as we know that they will move together. In this case, the internal forces do not appear in the free body diagram (shown below). The four ropes still act on the combined painter-platform system with the same upward vertical force. The gravitational force still pulls downwards, but acts on the total mass of the system $(m_1 + m_2)$.



Applying Newton's second law to the combined painter-platform system in the vertical direction gives

$$4F - (m_1 + m_2)g = (m_1 + m_2)a. \tag{6}$$

This can be solved for the acceleration, yielding the same result as given in equation (5).

2. Blocks and pulleys

1. In this part, the vertical acceleration of block 3, $a_{3y} = a$, is a given quantity. Let T be the tension in the rope, which is constant throughout its length (as the rope is massless and the pulleys are massless and frictionless). The free body diagram on block 3 is shown below.

Defining a coordinate system such that $\hat{y}$ points up (antiparallel to gravity), we find Newton's second law becomes

$$2T - m_3 g = -m_3 a \tag{1}$$

for block 3. We can solve this to find the tension

$$T = \frac{1}{2} m_3 (g - a). \tag{2}$$

The problem tells us that the tension T exceeds the static friction force limit on one block, but not the other. To see which, we must calculate the static friction force, which requires us to know the normal forces. Thus, we draw the free body diagrams for both blocks (shown below), where f_1 and f_2 represent the friction force on each. We know that the friction force (whether it be static or kinetic) will point outwards as block 3 will tend to pull blocks 1 and 2 towards the pulleys.

Since blocks 1 and 2 do not accelerate vertically, Newton's second law in the vertical direction gives

$$N_1 - m_1 g = 0 \quad \Rightarrow \quad N_1 = m_1 g \tag{3}$$

and

$$N_2 - m_2 g = 0 \quad \Rightarrow \quad N_2 = m_2 g \tag{4}$$

for blocks 1 and 2 respectively. Substituting this into the equation for the static friction force gives

$$f_1 \leq \mu_s N_1 = \mu_s m_1 g \tag{5}$$

and

$$f_2 \leq \mu_s N_2 = \mu_s m_2 g, \tag{6}$$

where we have used the same coefficient of static friction in both equations as the two blocks (and the tables they sit on) are made of the same material. From this we conclude that block 1 does *not* move and block 2 *does* move because the maximum static friction force is higher on block 1 since $m_1 > m_2$. Since block 1 does not move, we can write the horizontal component of Newton's second law as

$$T - f_1 = 0, \tag{7}$$

where we have defined $\hat{x}$ to point towards the right. Solving this equation for the friction force and substituting equation (2) gives the final answer — block 1 does not move and the friction force holding it back points horizontally to the left with magnitude

$$f_1 = \frac{1}{2} m_3 (g - a). \tag{8}$$

Note we know that $g > a$ as block 3 would only accelerate at g if it were in free fall. Any upwards force (such as that from the rope) would only serve to slow its downwards acceleration.

2. Now consider the case where, after being released from rest, all three blocks begin to move. We are trying to solve for the accelerations a_{1x}, a_{2x}, and a_{3y} as well as the tension T in the rope. Thus, to solve for these 4 unknowns we need four equations. The free body diagrams for all three blocks remain the same (shown above), though all the friction forces are kinetic friction as every block is now moving. For all objects, we will choose to use a single coordinate system with $\hat{y}$ pointing upward and $\hat{x}$ pointing to the right, though other choices are possible. Given this coordinate system, Newton's second law for block 1 is

$$T - f_1 = m_1 a_{1x} \tag{9}$$

in the horizontal direction, while it is

$$N_1 - m_1 g = 0 \tag{10}$$

in the vertical direction (since there is no vertical acceleration). Therefore, as in part 1, we find

$$N_1 = m_1 g, \tag{11}$$

so the *kinetic* friction force is

$$f_1 = \mu_k N_1 = \mu_k m_1 g. \tag{12}$$

Substituting this into equation (9) gives

$$T - \mu_k m_1 g = m_1 a_{1x}. \tag{13}$$

Next, we consider block 2. Given its free body diagram (shown above), Newton's second law is

$$f_2 - T = m_2 a_{2x} \tag{14}$$

in the horizontal direction, while it is

$$N_2 - m_2 g = 0 \tag{15}$$

in the vertical direction (since there is no vertical acceleration). Therefore, as in part 1, we find

$$N_2 = m_2 g, \tag{16}$$

so the *kinetic* friction force is

$$f_2 = \mu_k N_2 = \mu_k m_2 g. \tag{17}$$

Substituting this into equation (14) gives

$$\mu_k m_2 g - T = m_2 a_{2x}. \tag{18}$$

Lastly, we turn to block 3. Given its free body diagram (shown above), Newton's second law is

$$2T - m_3 g = m_3 a_{3y} \tag{19}$$

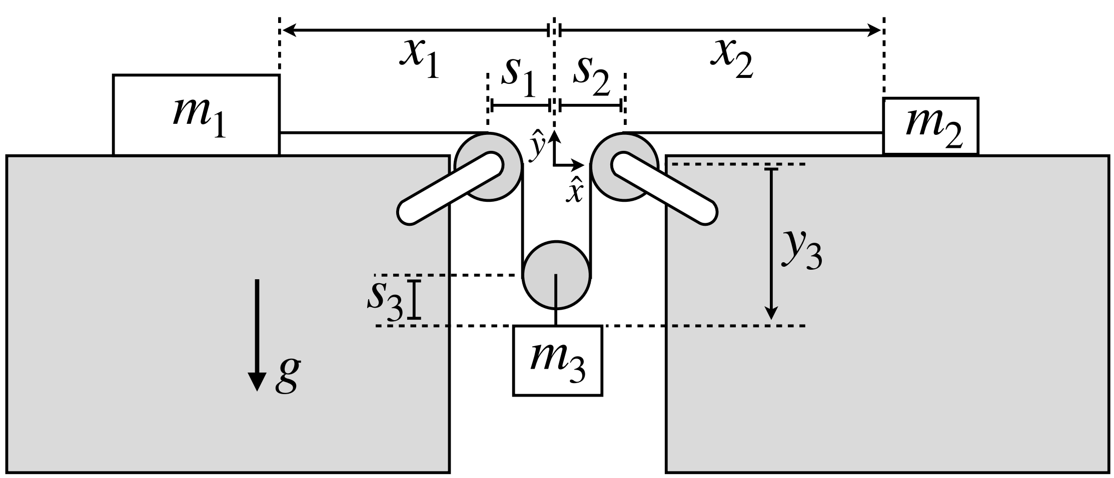
in the vertical direction and $0 = 0$ in the horizontal direction.

We can collect the equations for the three blocks (equations (13), (18), and (19)):

$$T - \mu_k m_1 g = m_1 a_{1x} \tag{20}$$

$$\mu_k m_2 g - T = m_2 a_{2x} \tag{21}$$

$$2T - m_3 g = m_3 a_{3y}. \tag{22}$$



However, we are still missing one equation (given that we have four unknowns and only three equations). This is the constraint condition. More specifically, studying the geometry of the system (see the figure above), we see that the motion of the three blocks is related by a geometrical constraint: because the rope is inextensible (i.e. does not stretch), its length L is constant. However, quantifying its length is complicated given that the blocks move. There are many different approaches to do this, which will give the same answer. We choose to define a single coordinate system for all three blocks and place its origin equidistant between the center of the two fixed pulleys (as shown above). Then we quantify some missing fixed distances (e.g. the radii of the pulleys R), which, as we will see, won't end up mattering as they don't change as the blocks move. Finally, we can add up all the sections of rope to express the total length of the rope as

$$L = (-x_1+s_1)+\frac{\pi}{2}R+(-y_3+s_3)+\pi R+(-y_3+s_3)+\frac{\pi}{2}R+(x_2-s_2) = -x_1+x_2-2y_3+s_1-s_2+2s_3+2\pi R. \tag{23}$$

Note that we have accounted for the fact that some of the positions (e.g. x_1 and y_3) will be negative, given our definitions of $\hat{x}$ and $\hat{y}$. If we differentiate equation (23) twice with respect to time, we find

$$0 = -a_{1x} + a_{2x} - 2a_{3y}. \tag{24}$$

Note that all of the unknown distances, like R and s_1, have disappeared as they don't change with time. Equation (24) is the equation that we were missing. We now have four equations (i.e. (20), (21), (22), and (24)) with which we can solve for the four unknowns (a_{1x}, a_{2x}, a_{3y}, and the tension T).

We solve equation (20) to find

$$a_{1x} = \frac{T}{m_1} - \mu_k g, \tag{25}$$

equation (21) to find

$$a_{2x} = \mu_k g - \frac{T}{m_2}, \tag{26}$$

and equation (22) to find

$$a_{3y} = \frac{2T}{m_3} - g. \tag{27}$$

Substituting these three results into equation (24) gives

$$0 = -\left(\frac{T}{m_1} - \mu_k g\right) + \left(\mu_k g - \frac{T}{m_2}\right) - 2\left(\frac{2T}{m_3} - g\right). \tag{28}$$

We rearrange this equation to find that the tension is

$$T = \frac{2g(\mu_k + 1)}{\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{4}{m_3}\right)}. \tag{29}$$

We can now substitute our solution for the tension to find the accelerations. Equation (25) becomes

$$a_{1x} = \frac{2g(\mu_k + 1)}{\left(1 + \frac{m_1}{m_2} + \frac{4m_1}{m_3}\right)} - \mu_k g, \tag{30}$$

equation (26) becomes

$$a_{2x} = \mu_k g - \frac{2g(\mu_k + 1)}{\left(\frac{m_2}{m_1} + 1 + 4\frac{m_2}{m_3}\right)}, \tag{31}$$

and equation (27) becomes

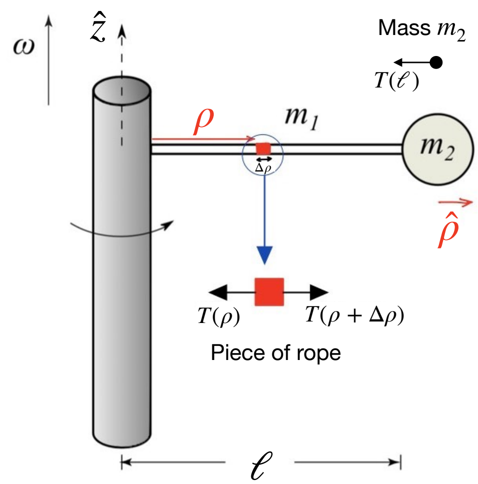
$$a_{3y} = \frac{4g(\mu_k + 1)}{\left(\frac{m_3}{m_1} + \frac{m_3}{m_2} + 4\right)} - g. \tag{32}$$

Note that the signs of these accelerations depend on the coordinate system used and so may be different in your solution. However, the tension given by equation (29) is a physical force and is independent of the choice of coordinates.

3. Tension in massive rotating rope

Given the circular motion of this problem, it is most convenient for us to use cylindrical coordinates. Since the rope has mass, the tension will *not* be constant along its length. However, since it is uniform, it has a constant linear mass density $\lambda = m_1/\ell$, which is everywhere equal to the total mass of the rope m_1 divided by its total length ℓ. We can divide the rope into very small pieces of length $\Delta\rho$ in the radial direction, which each have a mass of

$$\Delta m = \lambda \Delta\rho = \frac{m_1}{\ell}\Delta\rho. \tag{1}$$



We start by considering a piece of the rope that is located an arbitrary distance ρ from the shaft, whose free body diagram is shown above. The net radial force on the piece is equal to the difference between the tension on either side

$$\sum F_\rho = T(\rho + \Delta\rho) - T(\rho) = \Delta T, \tag{2}$$

which is simply the change in tension across the piece ΔT. Given that the piece is executing uniform circular motion, we know it is experiencing a centripetal acceleration of $\vec{a} = -\rho\omega^2\hat{\rho}$. Thus, the radial component of Newton's second law must be

$$\sum F_\rho = -\Delta m \omega^2 \rho. \tag{3}$$

Combining equations (1), (2), and (3) gives

$$\Delta T = -\Delta m \omega^2 \rho = -\frac{m_1}{\ell}\Delta\rho\omega^2\rho. \tag{4}$$

Dividing by $\Delta\rho$, we see that

$$\frac{\Delta T}{\Delta\rho} = -\frac{m_1}{\ell}\omega^2\rho. \tag{5}$$

In the limit that our very small pieces become infinitesimally small (i.e. $\Delta\rho \to 0$), equation (5) becomes the differential equation

$$\frac{dT}{d\rho} = -\frac{m_1}{\ell}\omega^2\rho. \tag{6}$$

From this, we see immediately that the tension decreases with increasing radius as m_1, ℓ, ω^2, and ρ are all positive quantities. We can solve this differential equation by direct integration of both sides of the equation according to

$$\int \frac{dT}{d\rho}d\rho = -\frac{m_1\omega^2}{\ell}\int \rho d\rho. \tag{7}$$

This simplifies to

$$T(\rho) = -\frac{m_1\omega^2}{2\ell}\rho^2 + C, \tag{8}$$

where C is an integration constant that we still need to determine. To find it, we need to determine the value of the tension at some location. To do so, we use the fact that the only radial force applied at the end of the rope is by the mass m_2. Using the free body diagram for mass m_2 (shown above), we see that Newton's second law in the radial direction is

$$-T(\ell)\hat{\rho} = -m_2\omega^2\ell\hat{\rho} \quad \Rightarrow \quad T(\ell) = m_2\omega^2\ell. \tag{9}$$

Evaluating equation (8) at $\rho = \ell$ and using equation (9) allows us to find

$$C = \frac{m_1\omega^2}{2\ell}\ell^2 + m_2\omega^2\ell. \tag{10}$$

Substituting this into equation (8), we arrive at the final answer of

$$T(\rho) = -\frac{m_1\omega^2}{2\ell}\rho^2 + \frac{m_1\omega^2}{2\ell}\ell^2 + m_2\omega^2\ell = \frac{m_1\omega^2}{2\ell}(\ell^2 - \rho^2) + m_2\omega^2\ell = \omega^2\left(m_1\frac{(\ell^2-\rho^2)}{2\ell} + m_2\ell\right). \tag{11}$$

4. Racing around a turn

The most challenging part of this problem is understanding and determining the physical meaning of each of the quantities from our derivation in class. In this solution we will simply state the physical meanings, but these can be seen and justified through the derivation presented in class. The first thing to note is that the mass of your own car m_1 does not matter at all in this problem. We already know the motion of your car (and importantly the motion of the reference frame it defines) and we are not interested in calculating the forces on your car.

Since the cars are executing circular motion, we can define a fixed cylindrical coordinate system F with the origin at the center of the circular curve and the $\hat{z}$ direction defined such that ω_1 and ω_2 are both in the positive $\hat{z}$ direction. Additionally, we observe that both cars are moving with the same angular velocity, which we will call $\vec{\omega} = \omega\hat{z} = \omega_1\hat{z} = v_1/R_1\hat{z} = \omega_2\hat{z} = v_2/R_2\hat{z}$. Next, we can write down the general coordinate system transformation for forces between the fixed inertial reference frame F and a non-inertial reference frame N rotating with an angular velocity $\vec{\omega}_1$. As derived in class, it is

$$\Sigma\vec{F}_N = \Sigma\vec{F}_F - m_2\vec{A}_{FN} - 2m_2\vec{\omega}_1 \times \vec{v}_N - m_2\vec{\alpha}_1 \times \vec{r}_N - m_2\vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_N). \tag{1}$$

Here $\Sigma\vec{F}_N$ is the sum of the forces acting on your friend's car, as perceived in the non-inertial reference frame N (in this case defined by the motion of your car). This is what we want to determine in order to solve the problem. The quantity $\Sigma\vec{F}_F$ is the sum of the standard forces acting on your friend's car in the fixed inertial reference frame F (i.e. someone standing at the center of the curve watching the cars drive by). In the inertial reference frame, your friend's car is moving with uniform circular motion of radius R_2, so the sum of the forces acting on them must be a centripetal force

$$\Sigma \vec{F}_F = -m_2\omega_2^2 R_2\hat{\rho}. \tag{2}$$

In practice this would be provided by the static friction force between the car tires and the road. Note that all the masses that appear are m_2 because we are only considering the forces acting on your friend's car, which has mass m_2.

The rest of the terms represent fictitious forces. The first is the translational acceleration fictitious force. The quantity $\vec{A}_{FN}$ is the acceleration of the reference frame N, as seen from the fixed frame F. Since the reference frame N is defined by your car and you are executing uniform circular motion in frame F, we know that the acceleration must be centripetal

$$\vec{A}_{FN} = -\omega_1^2 R_1\hat{\rho}. \tag{3}$$

Note that the angular speed and radius are that of *your* car (i.e. ω_1 and R_1).

The next term is the Coriolis force, which includes the velocity of your friend's car $\vec{v}_N$ as seen by you in frame N. Intuitively, since both cars are traveling with the same angular velocity $\omega_1 = \omega_2$, from your perspective your friend's car is staying stationary beside you. Her position relative to you would be a constant vector pointing outwards

$$\vec{r}_N = (R_2 - R_1)\hat{\rho}. \tag{4}$$

Then, taking a derivative in time justifies our intuitive argument that

$$\vec{v}_N = 0. \tag{5}$$

Note, in this step we have considered the unit vectors to be constant as we are determining how the position changes with time from the perspective of you within the reference frame N. In other words, from your perspective facing forwards in your car (i.e. continuously facing in the $\hat{\phi}$ direction) and believing yourself to be at rest, your friend's position is not changing with time. Thus, the Coriolis force is zero.

The next term is the Euler force, where $\vec{\alpha}_1 = \dot{\omega}_1\hat{z}$ is the angular acceleration of reference frame N. Since you are driving at a constant speed, $\dot{\omega}_1 = 0$, so $\vec{\alpha}_1 = 0$ and the Euler force disappears.

The final term is the centrifugal force, where we have already determined that the position of your friend's car from your perspective in frame N is given by equation (4). Thus, the centrifugal force is non-zero and equation (4) can be substituted into equation (1).

Collecting all of the results from above, we can rewrite equation (1) as

$$\Sigma \vec{F}_N = -m_2\omega_2^2 R_2\hat{\rho} + m_2\omega_1^2 R_1\hat{\rho} - m_2\omega_1\hat{z} \times (\omega_1\hat{z} \times (R_2 - R_1)\hat{\rho}). \tag{6}$$

Using the fact that $\omega = \omega_1 = \omega_2$ and taking the scalars out in front of the cross products allows us to simplify to

$$\Sigma \vec{F}_N = -m_2\omega^2(R_2 - R_1)\hat{\rho} - m_2\omega^2(R_2 - R_1)\hat{z} \times (\hat{z} \times \hat{\rho}). \tag{7}$$

In the right-handed (ρ, ϕ, z) coordinate system, we can use the right hand rule to see that $\hat{z} \times \hat{\rho} = \hat{\phi}$ and $\hat{z} \times (\hat{z} \times \hat{\rho}) = \hat{z} \times \hat{\phi} = -\hat{\rho}$, so

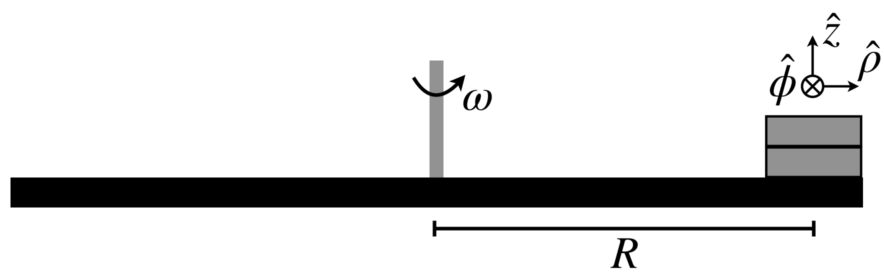
$$\Sigma \vec{F}_N = -m_2\omega^2(R_2 - R_1)\hat{\rho} + m_2\omega^2(R_2 - R_1)\hat{\rho} = 0. \tag{8}$$

Thus, from your perspective in the non-inertial reference frame N of your car, you see no net force acting on your friend's car. This differs from a fixed reference frame F, in which you would observe a physical centripetal force (provided by static friction between your friend's car and the road) as given by equation (2). The lack of a net force in frame N arises from a cancellation between the physical centripetal force, the translational acceleration fictitious force, and the centrifugal fictitious force.

Since the net force is zero, Newton's first or second laws tells us that the acceleration $\vec{a}_N$ is zero. Newton's first and second laws (but not the third) are still valid in non-inertial reference frames, as long as all of the fictitious forces are properly included. Alternatively, $\vec{a}_N = 0$ can be seen by taking the time derivative of equation (5).

5. Homework: Angular speed of coins

1. We choose to use a cylindrical coordinate system (because of the circular motion). For this part of the problem, it would be more efficient to analyze the situation by considering the two coins as a single combined object, which enables us to ignore internal forces (e.g. the normal and friction forces between the two coins). However, it will be helpful for the second part of this problem to treat the two coins separately. Thus, we will start by drawing separate free body diagrams for each coin (shown below). We have the weight of each coin $m\vec{g}$, the forces of static friction between the coins (both action $\vec{f}_{bt}$ and reaction $\vec{f}_{tb} = -\vec{f}_{bt}$), the force of static friction from the turntable on the bottom coin $\vec{f}_{gb}$, the normal forces between the coins (both action $\vec{N}_{bt}$ and reaction $\vec{N}_{tb} = -\vec{N}_{bt}$), and the normal force of the turntable on the bottom coin $\vec{N}_{gb}$. Here the top coin is denoted by the t subscript, the bottom coin by b, and the turntable by g.



To determine the magnitude of the radial force exerted by the turntable on the bottom coin $\vec{f}_{gb}$, we will apply Newton's second law to each coin. A key point is that the static friction between the coins form an action-reaction pair, so the static friction that makes the top coin accelerate inward also acts to push the bottom coin radially outward.

The radial component of Newton's second law for the bottom coin is

$$f_{tb} - f_{gb} = -mR\omega^2, \tag{1}$$

where we've used that the centripetal acceleration is $\vec{a} = -R\omega^2\hat{\rho}$. The radial component of Newton's second law on the top coin is given by

$$-f_{bt} = -mR\omega^2 \quad \Rightarrow \quad f_{bt} = mR\omega^2. \tag{2}$$

Since the static friction force of the bottom coin on the top coin f_{bt} and the static friction force of the top coin on the bottom coin f_{tb} are an action-reaction pair, Newton's third law requires that their magnitudes respect $f_{bt} = f_{tb}$. Substituting this into equation (2) gives

$$f_{tb} = mR\omega^2. \tag{3}$$

Then substituting equation (3) into equation (1) yields

$$mR\omega^2 - f_{gb} = -mR\omega^2. \tag{4}$$

Hence the turntable exerts a inward radial force on the bottom coin with a magnitude of

$$f_{gb} = 2mR\omega^2. \tag{5}$$

Comparing with equation (2), we see that the static friction force on the bottom coin from the turntable is twice as large as the static friction force on the top coin.

2. When two surfaces slip, it is because the static friction force required to hold them in place has exceeded its maximum possible strength of $f^{max} = \mu N$. To calculate this, we must first find the magnitude of the normal force N between the relevant surfaces. Applying Newton's second law to the top coin in the vertical $\hat{z}$ direction yields

$$N_{bt} - mg = 0, \tag{6}$$

where we note that none of the objects in this problem are accelerating vertically so $a_z = 0$. Thus, we can calculate the normal force of the bottom coin on the top coin to be

$$N_{bt} = mg. \tag{7}$$

Since this force forms an action-reaction pair with the normal force of the top coin on the bottom coin, we know that their magnitudes follow $N_{tb} = N_{bt} = mg$.

Substituting equation (7) into the form of the static friction force, we see that the top coin will slip when the static friction between the two coins reaches its maximum value of

$$f_{bt}^{max} = \mu_2 N_{bt} = \mu_2 mg. \tag{8}$$

We then substitute this result into equation (2) to find

$$\mu_2 mg = mR\left(\omega_t^{max}\right)^2, \tag{9}$$

where ω_t^{max} is the maximum angular speed for which the top coin does not slip. Rearranging this, we find

$$\omega_t^{max} = \sqrt{\frac{\mu_2 g}{R}}. \tag{10}$$

Newton's second law for the bottom coin in the $\hat{z}$ direction is

$$N_{gb} - N_{tb} - mg = 0. \tag{11}$$

Noting again from Newton's third law that $N_{tb} = N_{bt} = mg$, we can rearrange this equation to show that the normal force between the turntable and the bottom coin is

$$N_{gb} = 2mg. \tag{12}$$

Using this and the form of the static friction force, we see that the bottom coin will slip when the static friction between it and the turntable exceeds

$$f_{gb}^{max} = \mu N_{gb} = 2\mu_1 mg. \tag{13}$$

From equation (5), we can determine that the maximum angular speed after which the bottom coin slips will satisfy

$$2\mu_1 mg = 2mR\left(\omega_b^{max}\right)^2. \tag{14}$$

Rearranging we find that

$$\omega_b^{max} = \sqrt{\frac{\mu_1 g}{R}}. \tag{15}$$

Comparing equations (10) and (15) and remembering that $\mu_2 < \mu_1$, we see that

$$\omega_t^{max} < \omega_b^{max}. \tag{16}$$

Thus, as we increase the angular velocity of the turntable, the top coin will slip first.