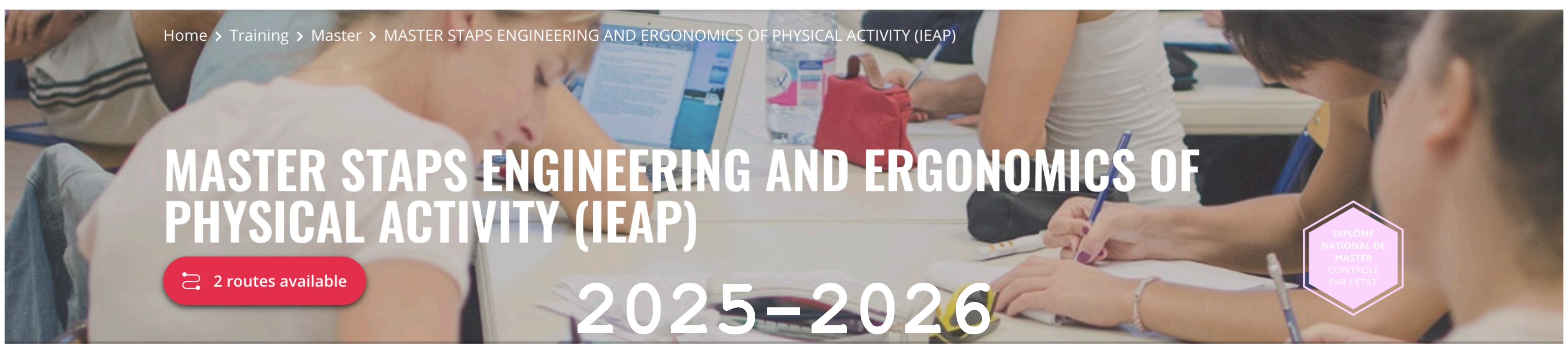
Modeling using Dynamical Systems

SAMID1MU

Mathieu Desroches



https://www-sop.inria.fr/members/Mathieu.Desroches/

https://team.inria.fr/mathneuro/

The program (see Moodle)

Master IEAP, SAMID1MU: **Modeling Using Dynamical Systems**
Mathieu Desroches, Inria Branch at the University of Montpellier[1]

Program of the course

Dynamical systems: The mathematics of change / Discrete systems [week 1]

- *Input/Output* relationships, *dynamical system*
- *State*, *State variable*, *State space*
- How to deal with time? *Discrete* vs. *continuous*
- Systems: *Linear* vs. *nonlinear*
- *Cobwebs* (discrete time)
- **Examples:** basic maps (traffic light, walking/running, logistic, ...)
- **Homework:** TBA

Derivatives & 1D differential equations [week 2]

- What is a *change equation* ? a matter of *stocks* and *flows*
- *instantaneous rate of change* vs. *average rate of change* → *derivatives*
- from an *old state* to a *new state* via *Euler scheme*
- *Euler scheme* as an approximate (discretized) version of *(ordinary) differential equations*
- **Examples:** bathtub, logistic ODE, heart frequency, standing/lying down ...
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2D differential equations [weeks 3]

- 1D systems cannot oscillate! Intermediate: *phase oscillator* (example: overdamped pendulum, Kuramoto model, quadratic-integrate-and-fire model)
- Notion of *nullclines* and *phase-plane analysis*
- Equilibria with complex eigenvalues: *damped* oscillations; linear systems classification with *tr.* and *det.*
- *Robust* oscillations without friction: harmonic oscillator, pendulum
- *Linearization* of a 2D linear system: *Jacobian* matrix, stability (*eigenvalues*, *eigenvectors*)
- Robust oscillations: *limit cycles*
- **Examples:** van der Pol / FitzHugh-Nagumo (from electrical circuit to neural membrane), population models (e.g., Lotka-Volterra), ...
- **Homework:** TBA

Mechanisms of oscillations [weeks 4]

- *Feedback* loops
- Restorative vs regenerative forces
- Destabilization of equilibria or *bifurcations*
- Oscillations in *nature*
- **Examples:** genes/hormone regulation, neuronal spikes, muscle tremor, HKB model of human coordination
- **Homework:** TBA

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The program `[week 2]`

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Change equations

A matter of stocks and flows

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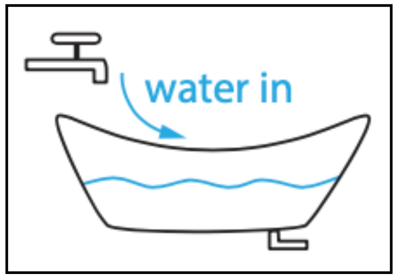
example: 

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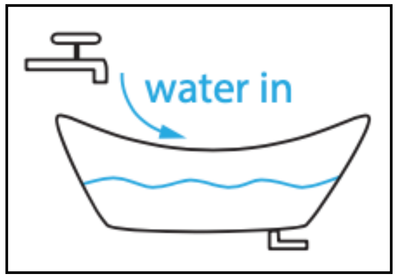
<u>example</u>: 

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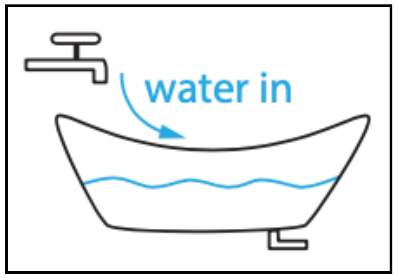
example: 

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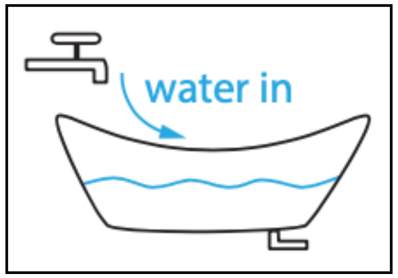
example: 

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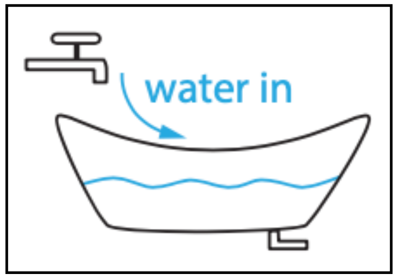
example: 

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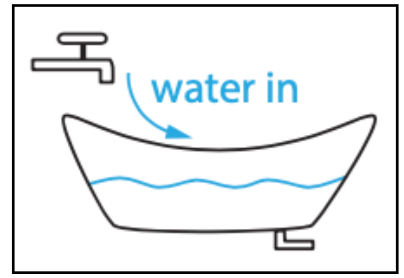
<u>example</u>: 

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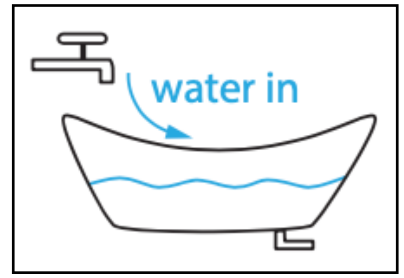
<u>example</u>: 

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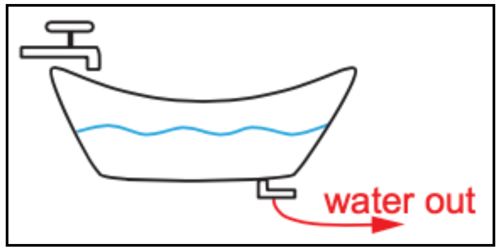
<u>example</u> (…): 

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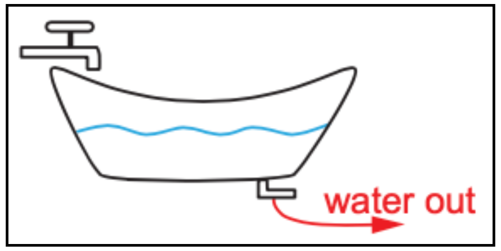
example (…): 

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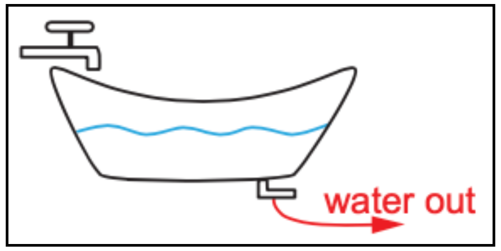
example (…): 

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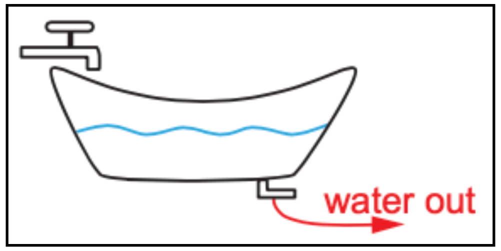
<u>example</u> (…): 

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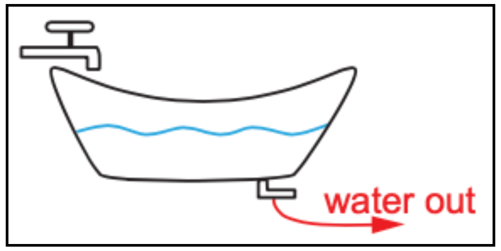
<u>example</u> (…): 

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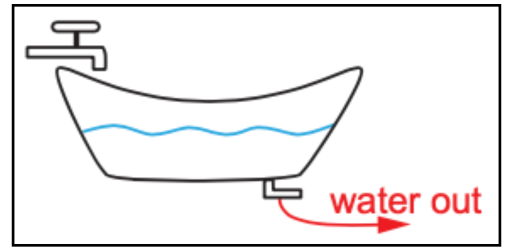
example (…): 

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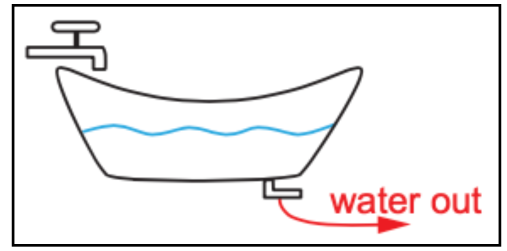
example (…): 

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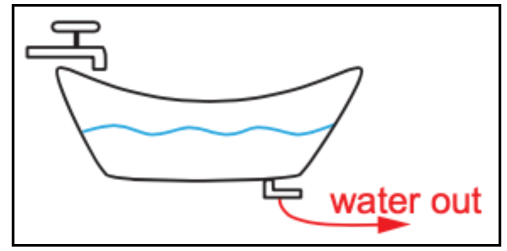
example (…): 

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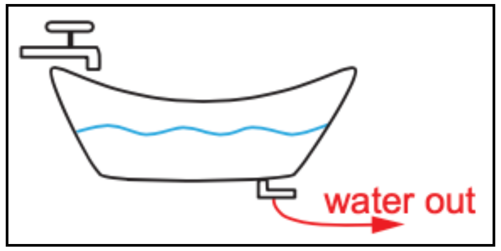

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!!UNITS!! X in litres | X' in litres per time unit | hence k in 1/time unit

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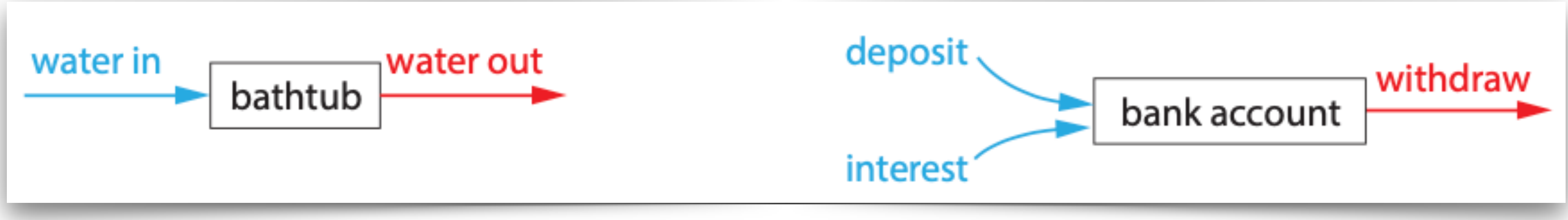

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what we really would like to do:

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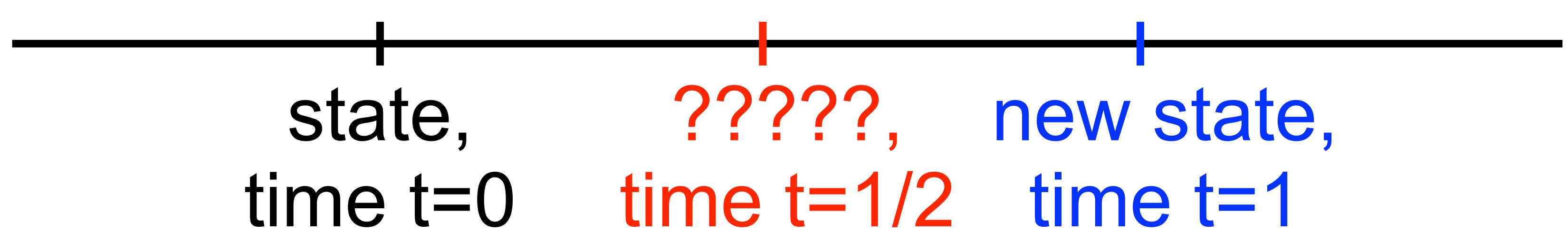

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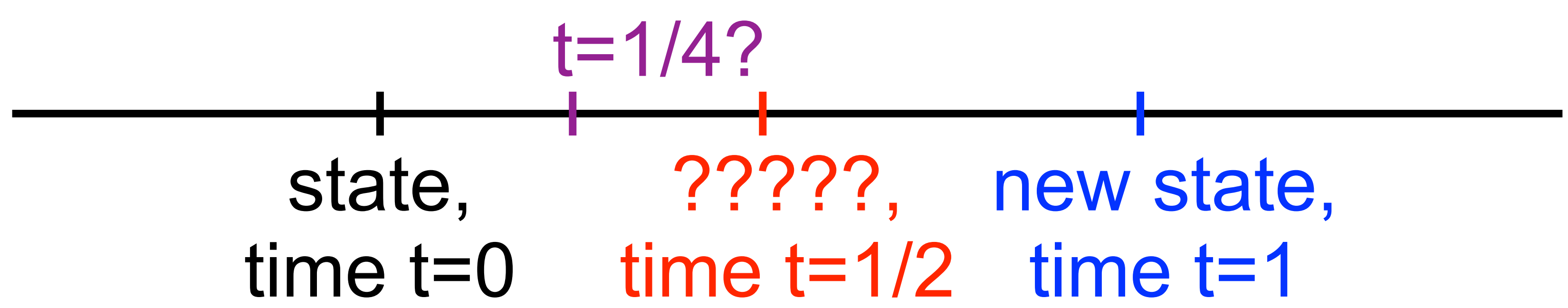

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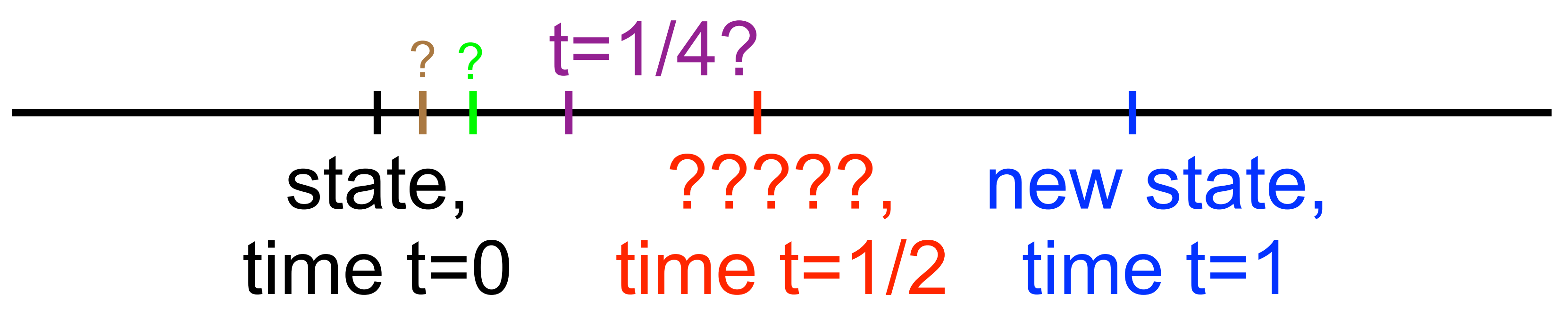

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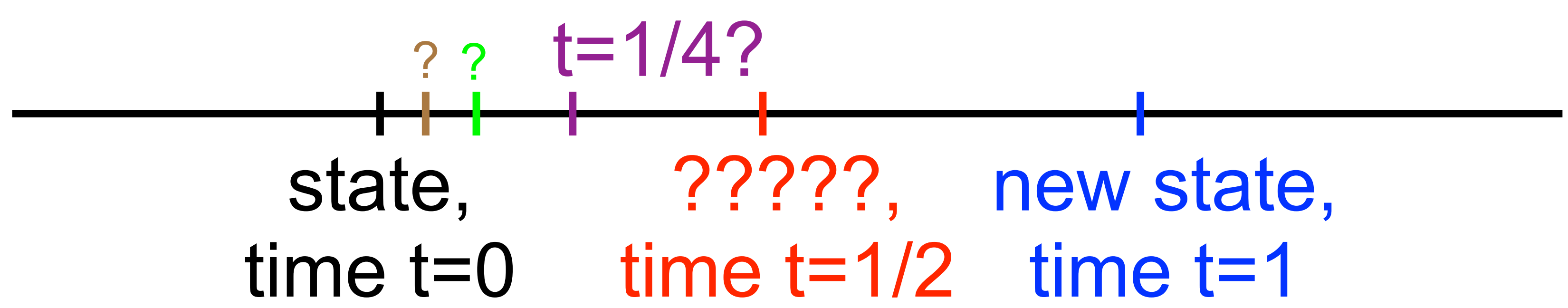

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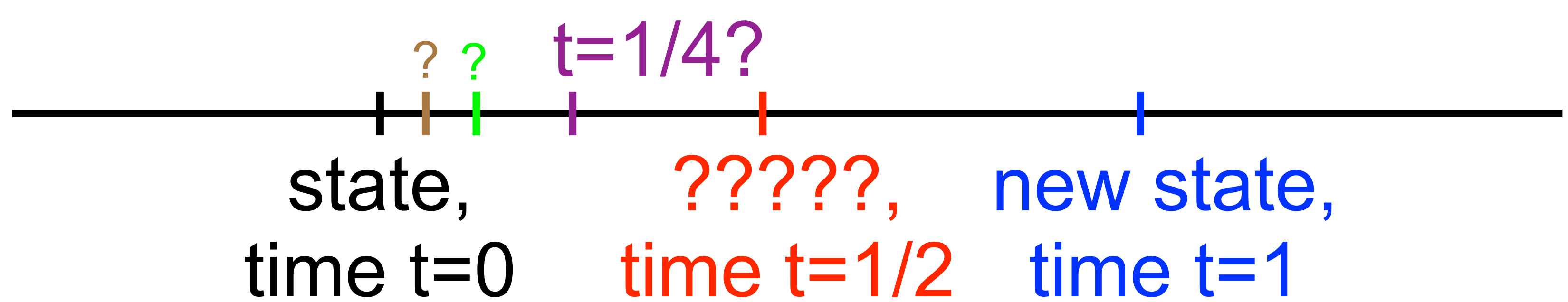

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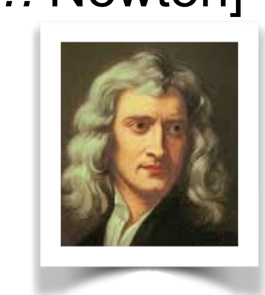

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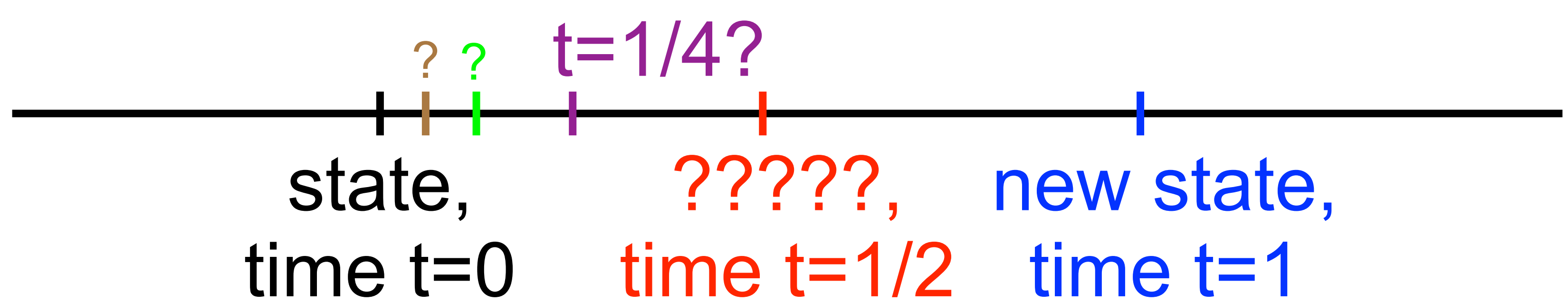

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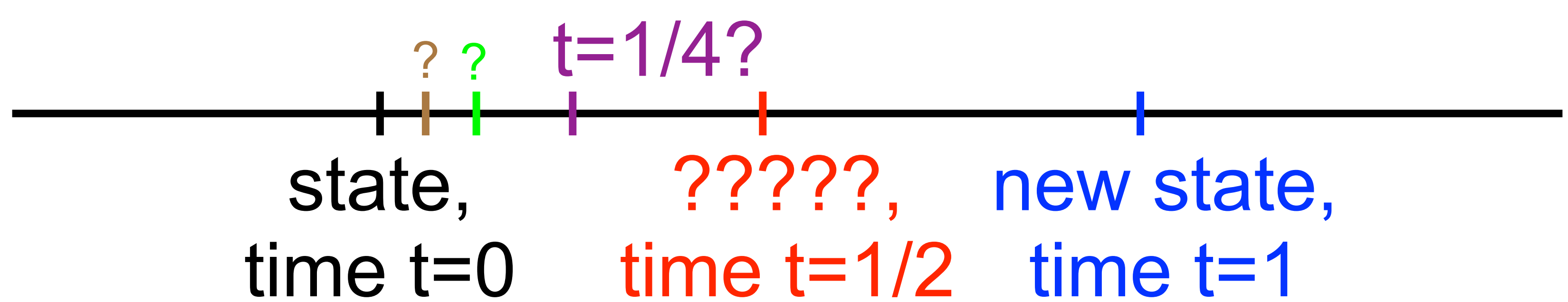

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new *X* = old *X* + Δt · *X*'

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d e r i v a t i v e

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So if we want to understand X' then we rearrange as:

$$X'(t) \approx \frac{X(t + \Delta t) - X(t)}{\Delta t}$$

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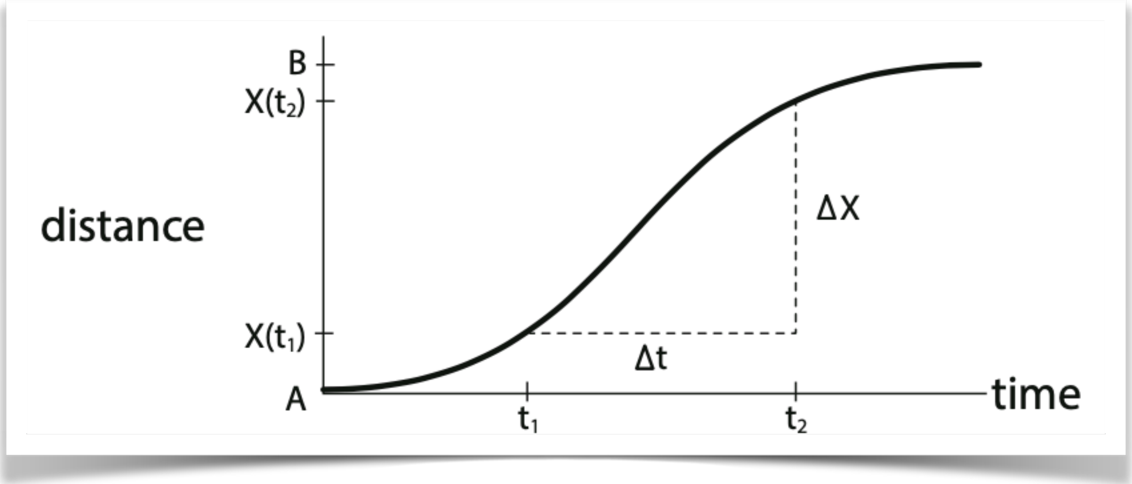
Derivatives: Rates of change

$$\text{rate of change} = \frac{\text{change in quantity}}{\text{change in time}}$$



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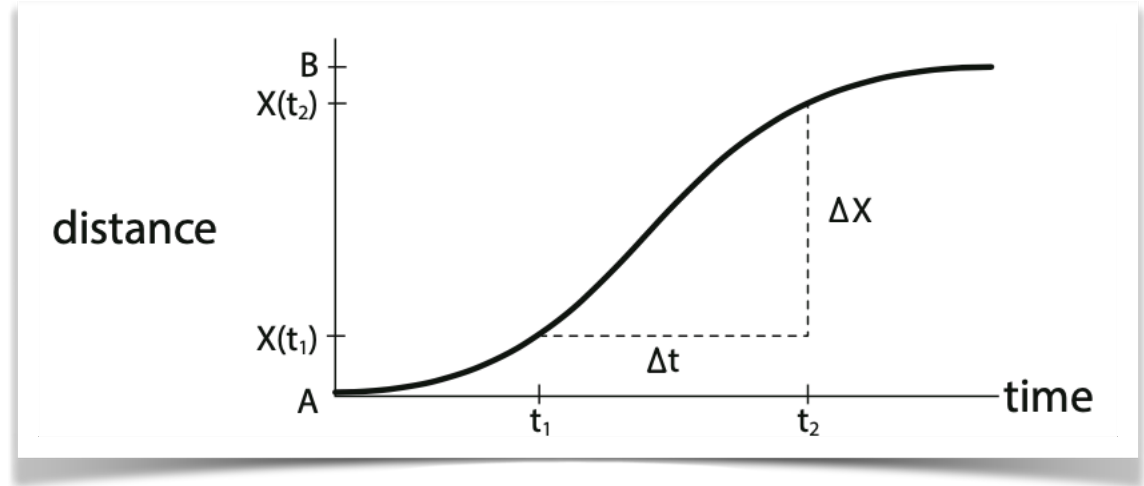

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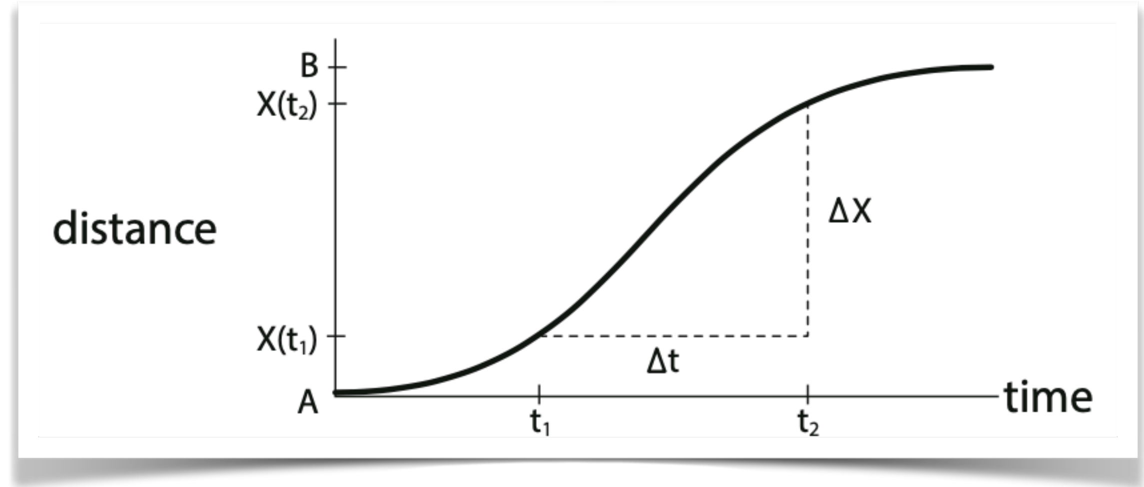

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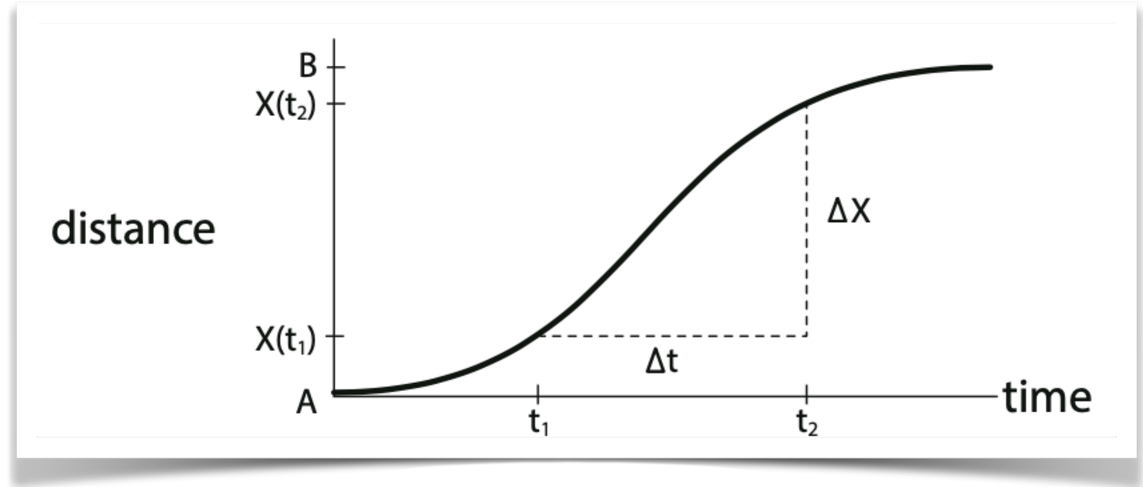

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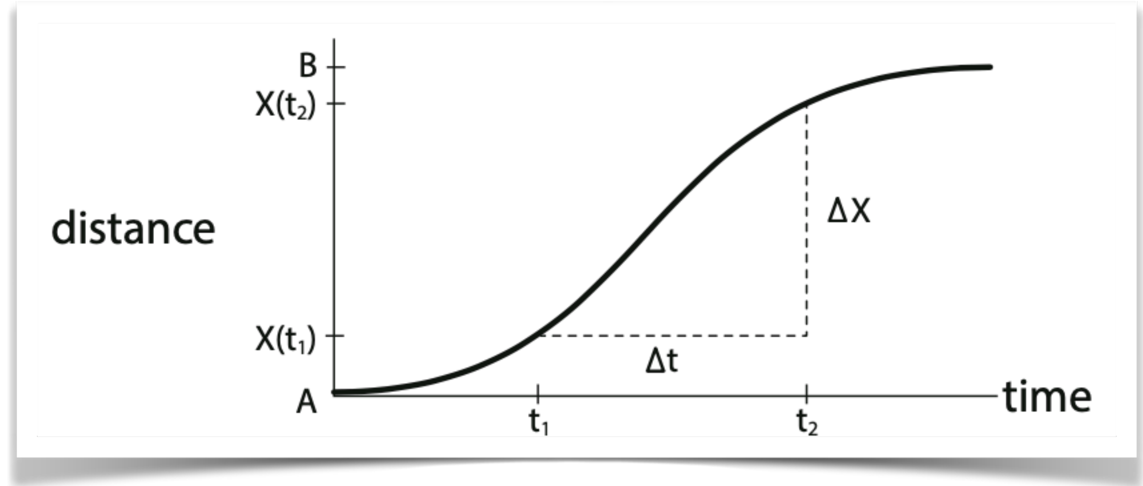

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What happens when Δt gets closer & closer to 0?

Derivatives: instantaneous rates of change

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Derivative of a function

Given a function $Y = f(X)$, we now understand the concept of the derivative of f

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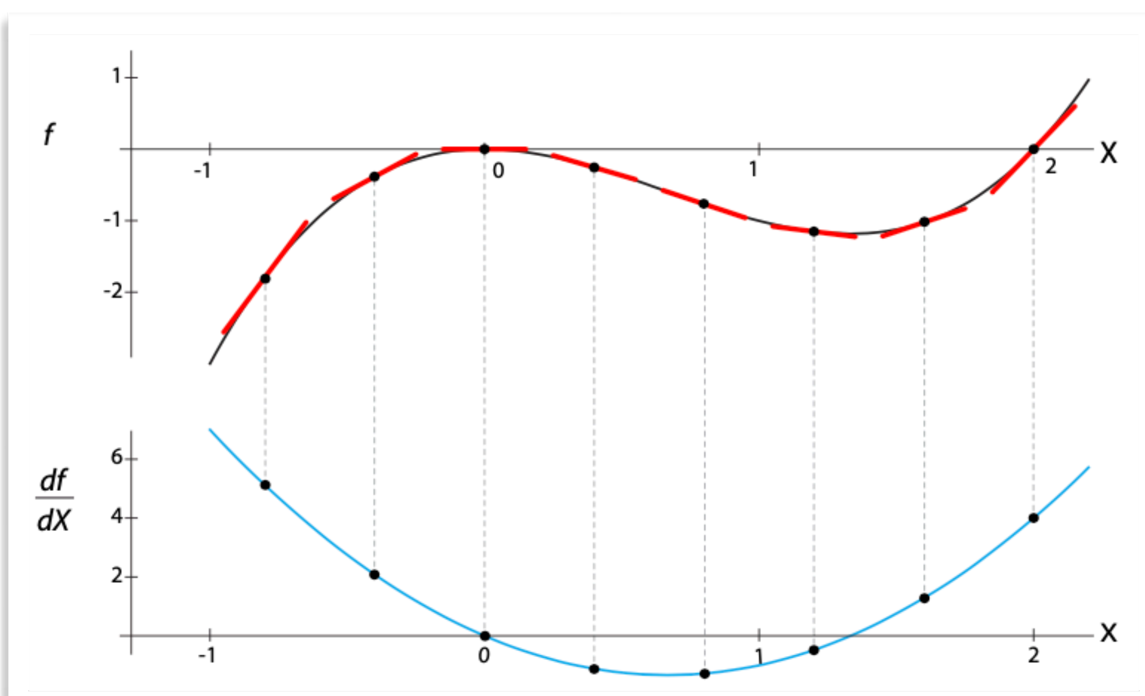
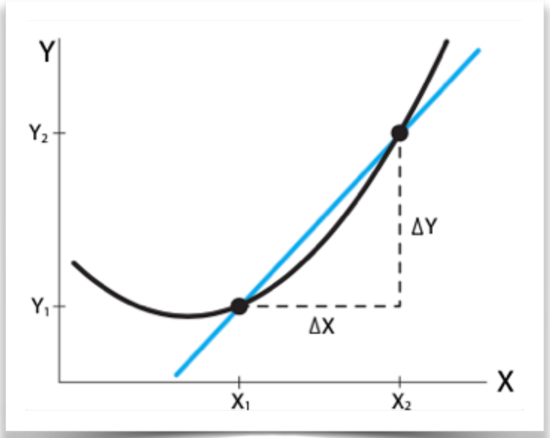


Derivatives: Geometric interpretation

average rate of change: $\dfrac{Y_2 - Y_1}{X_2 - X_1} = \dfrac{\Delta Y}{\Delta X}$



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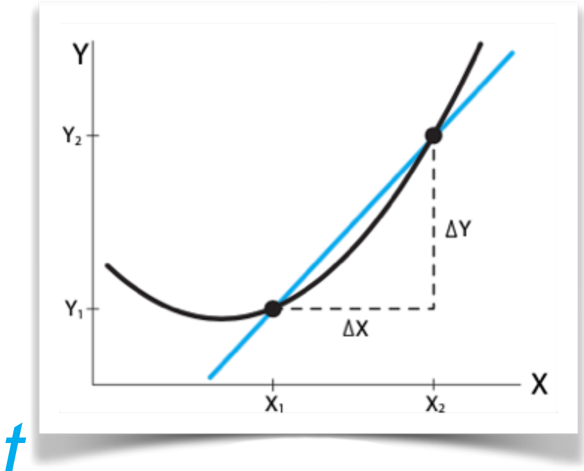
Slope of the *secant*



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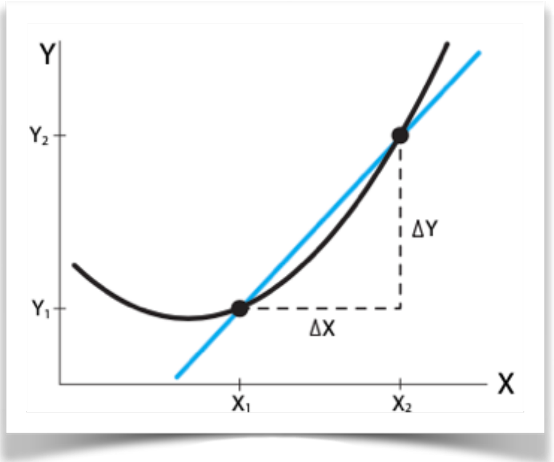

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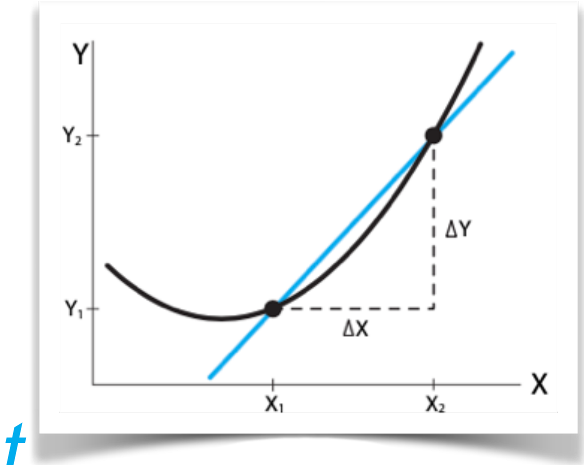

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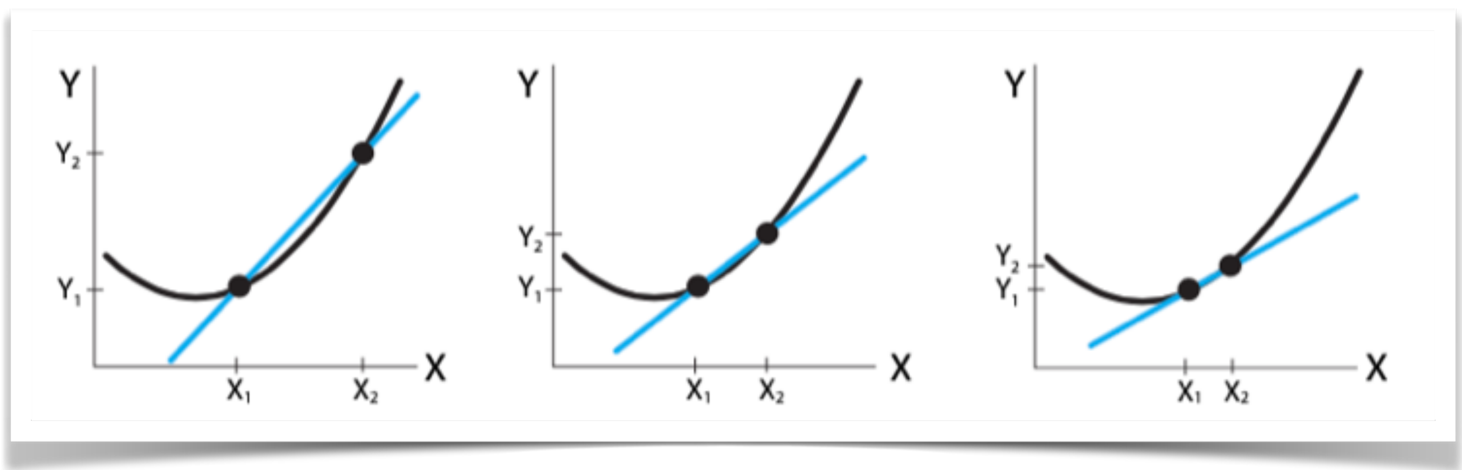

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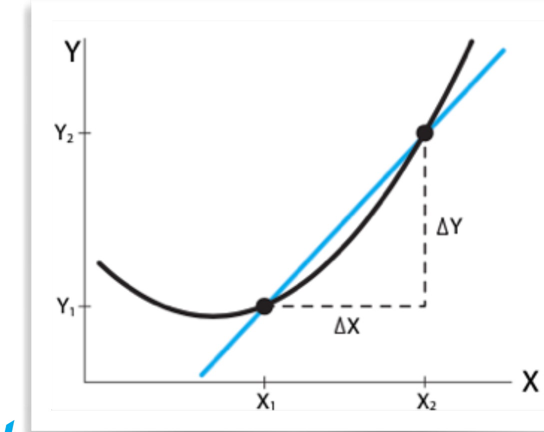

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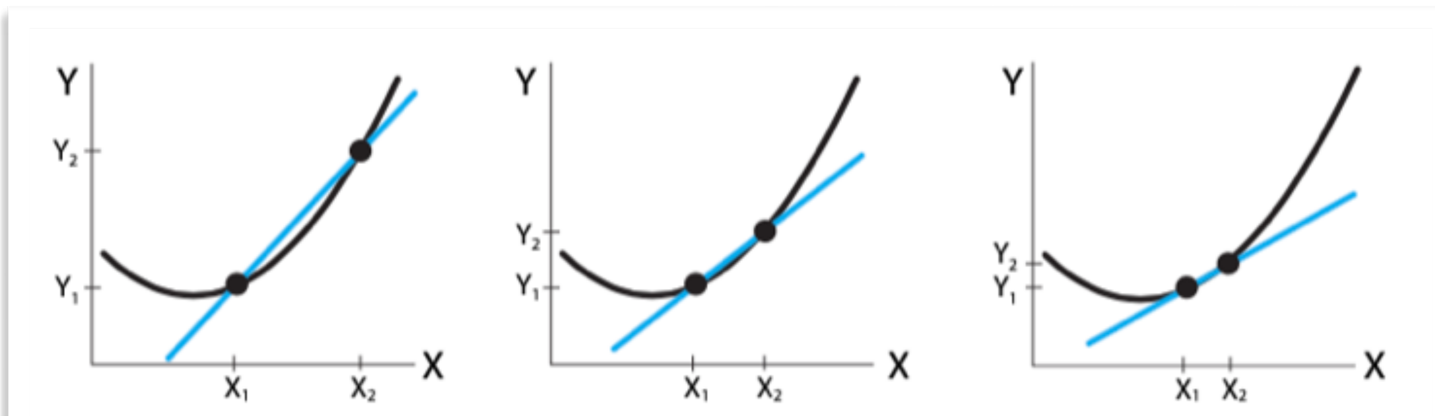
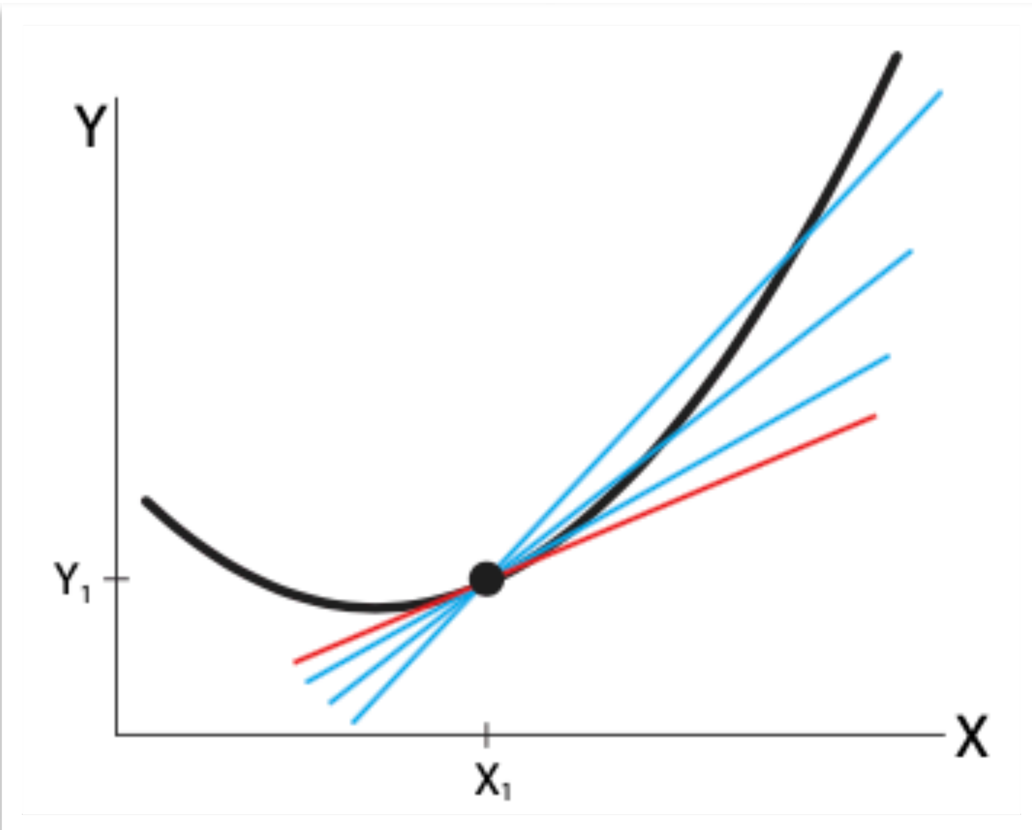


→ as ΔX gets smaller and smaller, the blue secant lines cut through smaller and smaller portions of the curve near X_1

→ it approaches a line that "just touches" the curve at the point (X_1, Y_1): *tangent line* to the curve $Y = f(X)$ at the point (X_1, Y_1)

Derivatives: Geometric interpretation (…)



In summary:

as Δx	$\longrightarrow$	0
secant lines	$\longrightarrow$	tangent line
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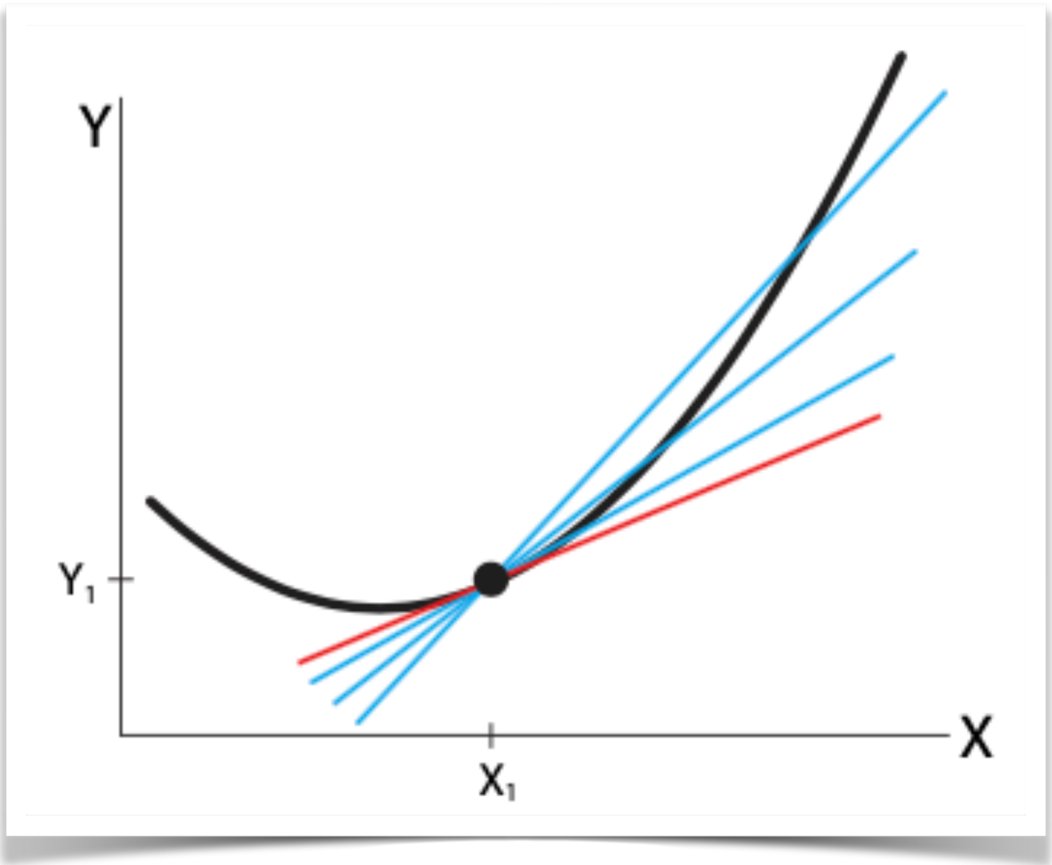

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derivative at X_1

$$X' = f(X)$$

d i f f e r e n t i a l

e q u a t i o n s

Linear ODEs (‘O’ for ‘Ordinary’)

these are the simplest ones:

$X' = kX$, where: k is a real number

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<u>central element</u>: the **exponential** function

Backbone state: Equilibria

no change / no “speed”:

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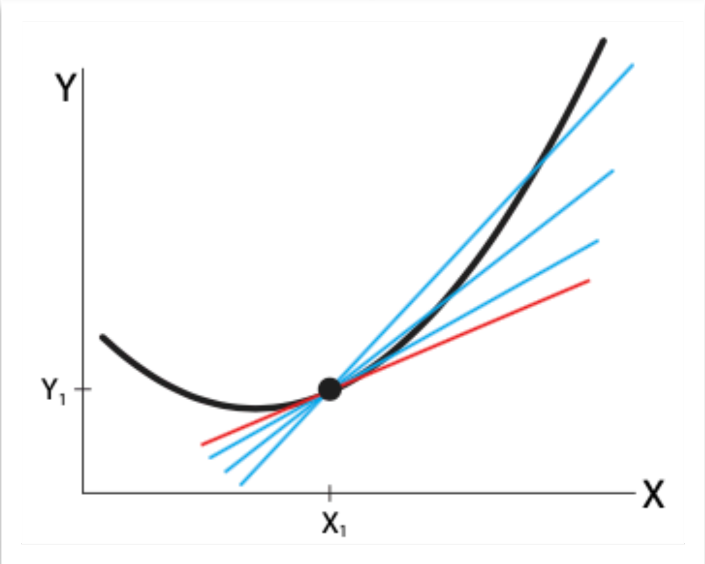
… **`python NOW!`**

Graphical Recap



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Homework for **next** week

bistable potential

On to [week 3]

physics-motivated
nonlinear ODE

two stable equilibria

standing → lying down