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SCALING LIMITS OF LOOP-ERASED RANDOM WALKS AND UNIFORM SPANNING TREES

BY

ODED SCHRAMM*

Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel
e-mail: schramm@wisdom.weizmann.ac.il
http://www.wisdom.weizmann.ac.il/~schramm/

ABSTRACT

The uniform spanning tree (UST) and the loop-erased random walk (LERW) are strongly related probabilistic processes. We consider the limits of these models on a fine grid in the plane, as the mesh goes to zero. Although the existence of scaling limits is still unproven, subsequential scaling limits can be defined in various ways, and do exist. We establish some basic a.s. properties of these subsequential scaling limits in the plane. It is proved that any LERW subsequential scaling limit is a simple path, and that the trunk of any UST subsequential scaling limit is a topological tree, which is dense in the plane.

The scaling limits of these processes are conjectured to be conformally invariant in dimension 2. We make a precise statement of the conformal invariance conjecture for the LERW, and show that this conjecture implies an explicit construction of the scaling limit, as follows. Consider the Löwner differential equation

$$\frac{\partial f}{\partial t} = z\frac{\zeta(t)+z}{\zeta(t)-z}\frac{\partial f}{\partial z},$$

with boundary values $f(z,0)=z$, in the range $z \in \mathbb{U} = \{w \in \mathbb{C}\colon |w|<1\}$, $t \leqslant 0$. We choose $\zeta(t) := \mathsf{B}(-2t)$, where $\mathsf{B}(t)$ is Brownian motion on $\partial\mathbb{U}$ starting at a random-uniform point in $\partial\mathbb{U}$. Assuming the conformal invariance of the LERW scaling limit in the plane, we prove that the scaling limit of LERW from 0 to $\partial\mathbb{U}$ has the same law as that of the path $f(\zeta(t),t)$ (where $f(z,t)$ is extended continuously to $\partial\mathbb{U}\times(-\infty,0]$). We believe that a variation of this process gives the scaling limit of the boundary of macroscopic critical percolation clusters.

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