Homework Assignment H1

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H 1.1

(a) • $f_1(x) = 2\lambda^2\sqrt{1+x/\lambda^2} \Rightarrow f_1'(x) = \frac{\cancel{2}\cancel{\lambda^2}(1/\cancel{\lambda^2})}{\cancel{2}(1+x/\lambda^2)^{1/2}} = \boxed{(1+x/\lambda^2)^{-1/2}}$

• $f_2(x) = \lambda^2 \ln\left(1+\frac{x^2}{\lambda^2}\right) \rightarrow f_2'(x^2) = \cancel{\lambda^2}\frac{1}{1+x^2/\lambda^2}\cdot\left(\frac{1}{\cancel{\lambda^2}}\right) = \boxed{\frac{1}{1+x^2/\lambda^2}}$

• $f_3(x) = x\cdot\frac{1}{\sqrt{1+\frac{x^2}{\lambda^2}}} \Rightarrow f_3'(x) = \frac{1}{\sqrt{1+x^2/\lambda^2}} - \frac{\cancel{2}x^2(1/\lambda^2)}{\cancel{2}\left(1+\frac{x^2}{\lambda^2}\right)^{3/2}} = \boxed{\frac{1}{(1+x^2/\lambda^2)^{3/2}}}$

• $f_4(x) = x\cdot f_2'(x^2) = \frac{x}{1+x^2/\lambda^2} \Rightarrow f_4'(x) = \frac{1}{1+x^2/\lambda^2} - \frac{2x^2(1/\lambda^2)}{(1+x^2/\lambda^2)^2}$

$\rightarrow f_4'(x) = \frac{1+x^2/\lambda^2 - 2x^2(1/\lambda^2)}{(1+x^2/\lambda^2)^2} = \boxed{\frac{1-x^2/\lambda^2}{(1+x^2/\lambda^2)^2}}$

(b) - Monotonicity of $f_1(x)$ on $\mathbb{R}^+$: $f_1(x)$ is always ~~decreasing~~ increasing.

$f_1'(x)$ does not change sign: $f_1'(x) = (1+x/\lambda^2)^{-1/2}$ for $x>0 \Rightarrow f_1'(x) \rightarrow 0$

but $f_1'(x) > 0$ always.

- Monotonicity of $f_2(x^2)$ on $\mathbb{R}^+$: $f_2(x^2)$ is always increasing.

$f_2'(x^2)$ does not change sign: $f_2'(x^2) = \frac{1}{1+x^2/\lambda^2}$ for $x>0 \Rightarrow f_2'(x^2) \rightarrow 0$

but $f_2'(x^2) > 0$ always.

- $f_3'(x)$ vanishes when $x \rightarrow +\infty$ for $x>0$. It's trivial that $\lim_{x\to\infty} f_3'(x) = 0$

- $f_4'(x)$ vanishes when $x \rightarrow +\infty$ for $x>0$.

$$\lim_{x\to\infty} f_4'(x) = \lim_{x\to\infty}\frac{1-\frac{x^2}{\lambda^2}}{\left(1+\frac{x^2}{\lambda^2}\right)^2} \overset{L'H}{=} \lim_{x\to\infty}\frac{-\cancel{2x}\left(\frac{1}{\lambda^2}\right)}{2\left(1+\frac{x^2}{\lambda^2}\right)\cancel{2x}} = \lim_{x\to\infty}\frac{-1}{2\lambda^2\left(1+\frac{x^2}{\lambda^2}\right)}$$

$= 0$ from bottom.

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1.2

(a) $$K_\delta(x) := \frac{1}{\delta\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\delta^2}\right)$$

$$K_{\sqrt{2t}}(x) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

If $u(x,t) = K_{\sqrt{2t}}(x) \rightarrow \partial_t u = -\frac{\pi}{4(\pi t)^{3/2}} \exp\left(-\frac{x^2}{4t^2}\right) + \frac{x^2}{8\sqrt{\pi t}\,t^2} \exp\left(-\frac{x^2}{4t}\right)$

and $\partial_x u = \left[\frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)\left(-\frac{\cancel{2}^{1} x}{\cancel{4}_{2} t}\right)\right] = -\frac{x}{4\sqrt{\pi t}\,t} \exp\left(-\frac{x^2}{4t}\right)$

and $\partial_{xx} u = -\frac{1}{4\sqrt{\pi t}\,t} \exp\left(-\frac{x^2}{4t}\right) + \left(-\frac{x}{4\sqrt{\pi t}\,t}\right)\left(-\frac{\cancel{2}^{1} x}{\cancel{4}_{2} t}\right) \exp\left(-\frac{x^2}{4t}\right)$

$-\frac{\pi}{4(\pi t)^{3/2}} =$ $= \left(-\frac{1}{4\sqrt{\pi t}\,t}\right) \exp\left(-\frac{x^2}{4t}\right) + \frac{x^2}{8\sqrt{\pi t}\,t^2} \exp\left(-\frac{x^2}{4t}\right)$

$\Rightarrow$ It's clear that $\partial_{xx} u = \partial_t u$.

(b) If $u(x,t) := \left(K_{\sqrt{2t}} * f\right)(x) \Rightarrow$

Then $\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[K_{\sqrt{2t}} * f\right] = \frac{\partial K_{\sqrt{2t}}}{\partial t} * f = \partial_t K_{\sqrt{2t}} * f$

Then $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[K_{\sqrt{2t}} * f\right]\right) = \frac{\partial}{\partial x}\left(\frac{\partial K_{\sqrt{2t}}}{\partial x} * f\right) = \partial_{xx} K_{\sqrt{2t}} * f$

But we already proved in (a) that if $u(x,t) = K_{\sqrt{2t}}(x) \Rightarrow \partial_t u = \partial_{xx} u$ $(x \in \mathbb{R}, t > 0)$

So, if $V(x,t) = K_{\sqrt{2t}}(x) \Rightarrow$
$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \underbrace{\partial_t V}_{= \text{ by (a)}} * f = \partial_t u \\ \frac{\partial^2 u}{\partial x^2} &= \overbrace{\partial_{xx} V} * f = \partial_{xx} u \end{aligned}\right\}$$

So, it must hold true that $\partial_t u = \partial_{xx} u$ $(x \in \mathbb{R}, t > 0)$

On the other hand, assuming $K_0 * f = f$

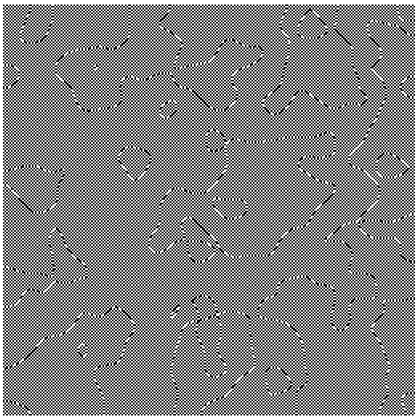
Then $u(x,0) = \left(K_{\sqrt{2 \cdot 0}} * f\right)(x) = \left(K_0 * f\right)(x) = f(x)$ $(x \in \mathbb{R})$

Problem H1.3:

(a) The evolution of the mean, the maximum, the minimum and the variance is exactly what one would expect with a process of homogeneous diffusion on an image. This process gradually reduces peaks in grey values of the image so that it becomes smoother. This produces the result of a "cloudy image". This smoothing process mathematically decreases the bigger grey values in the image, while increasing the smallest. That way, peaks get smaller. This in turn means that the maximum grey value in the original image will decrease while the minimum grey value will increase. Moreover, on average the variance will become smaller, since now bigger grey values are closer to the smallest grey values or, in other words, peaks in the image are no longer as pronounced. For example, with timestep $ts = 0.24$ and iterations $\text{iter} = 100$, this is how the values evolve:

initial image	iteration: 100
minimum: 51.00	minimum: 79.95 (↑)
maximum: 191.00	maximum: 165.81 (↓)
mean: 123.79	mean: 123.79 (=)
standard dev.: 20.38	standard dev.: 12.87 (↓)

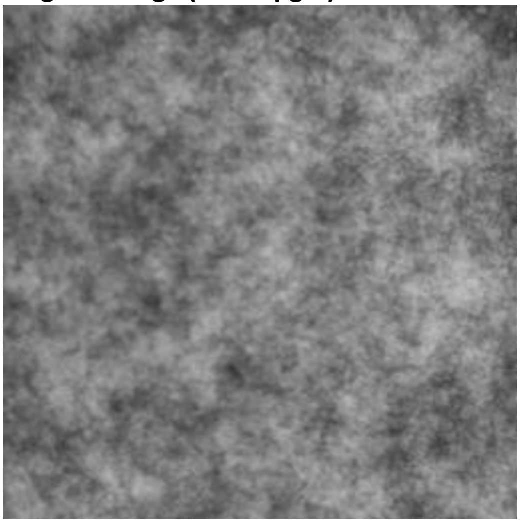
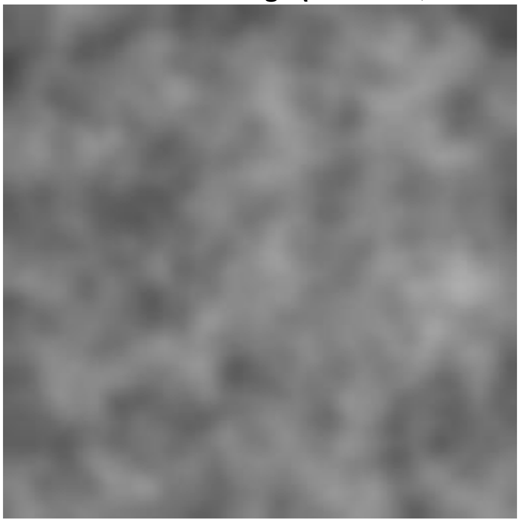
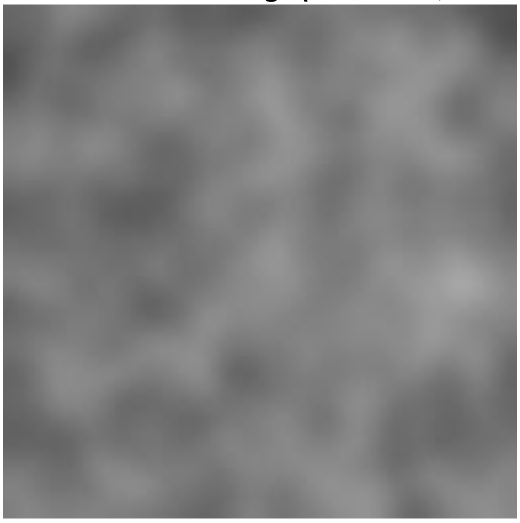
On the other hand, for values greater than $ts = 0.24$, the homogeneous diffusion explodes (does not converge) producing images like the following ($ts = 0.3; \text{iter} = 100$):



And an evolution of values like the following:

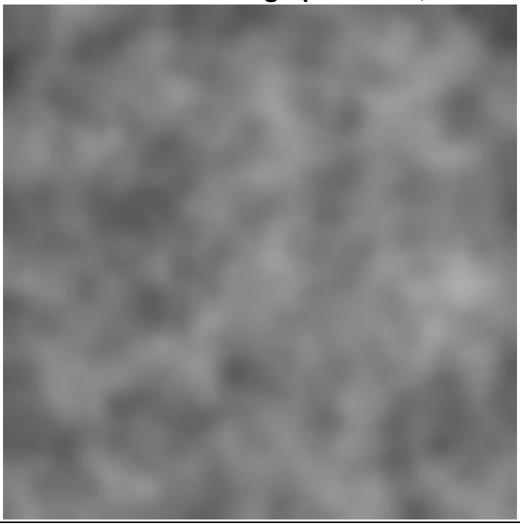
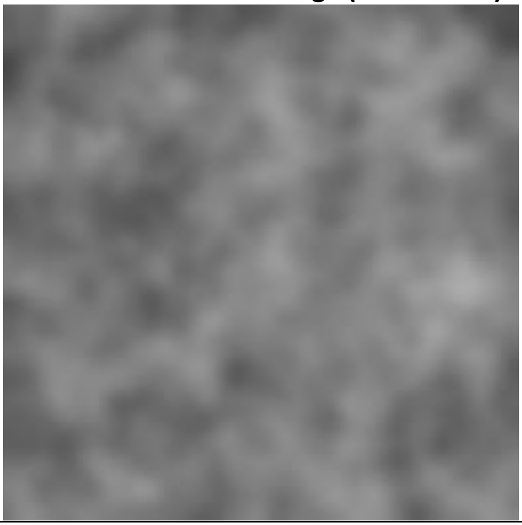
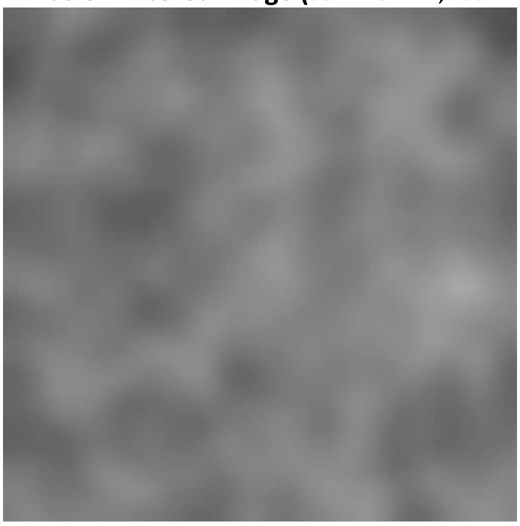
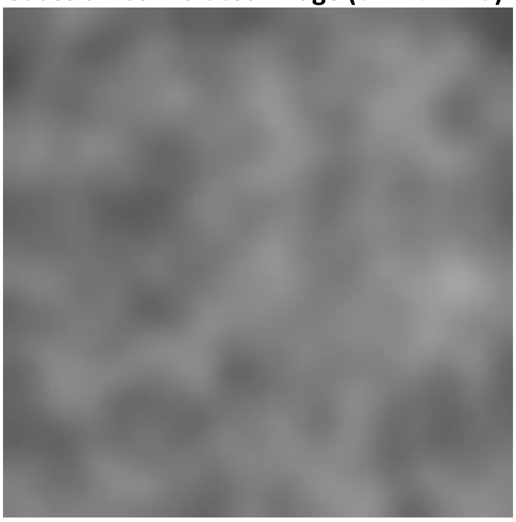
initial image	iteration: 100
minimum: 51.00	minimum: -122523368455836.78
maximum: 191.00	maximum: 121898099410075.66
mean: 123.79	mean: 123.79
standard dev.: 20.38	standard dev.: 20088437495767.09

Original image and the corresponding diffusion-filtered images illustrating the cloudiness at different scales:

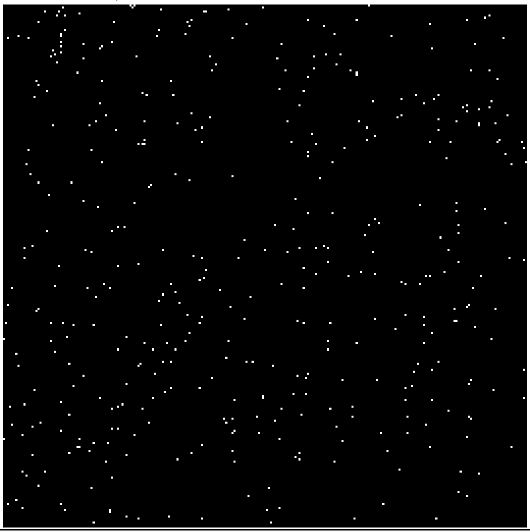
Original Image (fabric.pgm): 	**Diffusion-filtered image ($ts = 0.1$; $\mathbf{iter} = 100$):** 
	Diffusion-filtered image ($ts = 0.24$; $\mathbf{iter} = 100$): 

(b) The corresponding Gaussian convolution counterparts for the previous diffusion-filtered images. The standard deviation σ was calculated with the following formula:

$$t = \frac{1}{2}\sigma^2 \rightarrow ts * \text{iter} = \frac{1}{2}\sigma^2 \rightarrow \sigma = \sqrt{2ts\ * \text{iter}}$$

Diffusion-filtered image ($\boldsymbol{ts = 0.1; \text{iter} = 100}$): 	**Gaussian-convoluted image ($\boldsymbol{\sigma \approx 4.472}$):** 
Diffusion-filtered image ($\boldsymbol{ts = 0.24; \text{iter} = 100}$): 	**Gaussian-convoluted image ($\boldsymbol{\sigma \approx 6.928}$):** 

The difference images between the diffusion-filtered image and the corresponding gaussian-convoluted image:

$\boldsymbol{ts = 0.1; \text{iter} = 100\ vs\ \sigma \approx 4.472}$: 	$\boldsymbol{ts = 0.24; \text{iter} = 100\ vs\ \sigma \approx 6.928}$: