Dr. Pascal Peter — WS 2023
Mathematical Image Analysis Group — Saarland University

Classroom Work Assignment C1

Problem C1.1 (Partial Derivatives)

(a) Calculate the gradient, the Hessian, and the Laplacian of the function

$$f(x,y) \;=\; \exp\left(xy - x^2 y\right).$$

(b) Calculate the divergence of

$$\boldsymbol{g}(x,y) \;:=\; \begin{pmatrix} \cos\left(x+y^2\right) \\ \ln\left(x^2+y\right) \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{h}(x,y) \;:=\; \begin{pmatrix} \ln\left(x^2+y\right) \\ \cos\left(x+y^2\right) \end{pmatrix}.$$

Problem C1.2 (Otsu's Axiomatic Derivation of Gaussian Scale-Space)

Your tutor will show that a transformation $\tilde{f}$ of a 2-D image $f : \mathbb{R}^2 \to \mathbb{R}$ which satisfies the four axioms

1. *Representation as a linear operator:*
   There exists a function $W : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that
   $$\tilde{f}(\boldsymbol{r}) \;=\; \int_{\mathbb{R}^2} W(\boldsymbol{r},\boldsymbol{r}')f(\boldsymbol{r}')\,d\boldsymbol{r}' \quad \text{for all } \boldsymbol{r} \in \mathbb{R}^2 \;.$$
2. *Translation invariance:*
   For all $\boldsymbol{r}, \boldsymbol{a} \in \mathbb{R}^2$ it is required that
   $$\tilde{f}(\boldsymbol{r}-\boldsymbol{a}) \;=\; \int_{\mathbb{R}^2} W(\boldsymbol{r},\boldsymbol{r}')\,f(\boldsymbol{r}'-\boldsymbol{a})\,d\boldsymbol{r}'.$$
3. *Rotation invariance (of the kernel):*
   For all rotation matrices $\boldsymbol{T}_\Theta$ and for all $\boldsymbol{r} \in \mathbb{R}^2$ it holds that $W(\boldsymbol{T}_\Theta \boldsymbol{r}) = W(\boldsymbol{r})$.
4. *Separability:*
   There exists a function $u : \mathbb{R} \to \mathbb{R}$ such that $W(\boldsymbol{r}) = u(x)\,u(y)$ for all $\boldsymbol{r} = (x,y)^\top \in \mathbb{R}^2$.

is given by a convolution of f with a Gaussian-type kernel

$$W(\boldsymbol{r}) \;=\; k\,\exp\left(c \cdot \left(x^2+y^2\right)\right).$$

Homework Assignment H1

Problem H1.1 (Derivatives) 4+2P

(a) Calculate the derivatives $f_1'(x), f_2'(x^2), f_3'(x)$ and $f_4'(x)$ for the following positive functions with $\lambda > 0$:

$$f_1(x) = 2\lambda^2 \sqrt{1 + \tfrac{x}{\lambda^2}}, \qquad f_2(x^2) = \lambda^2 \ln\Big(1 + \tfrac{x^2}{\lambda^2}\Big),$$

$$f_3(x) = x \cdot \frac{1}{\sqrt{1 + \frac{x^2}{\lambda^2}}}, \qquad f_4(x) = x \cdot f_2'\,(x^2)\,.$$

(b) What can you say about the monotonicity behaviour of $f_1(x)$ and $f_2(x^2)$ on $\mathbb{R}^+$?
For which values of $x > 0$ do $f_3'(x)$ and $f_4'(x)$ vanish?

Problem H1.2 (Convolution, Derivatives, and Diffusion Equation) 5+5P

(a) The one-dimensional Gaussian $K_\sigma(x)$ is defined as

$$K_\sigma(x) := \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Verify that $u(x,t) = K_{\sqrt{2t}}\,(x)$ is a solution of the homogeneous diffusion equation

$$\partial_t u = \partial_{xx} u \qquad (x \in \mathbb{R},\ t > 0).$$

(b) Prove that $u(x,t) := (K_{\sqrt{2t}} * f)(x)$ is a solution of the initial value problem

$$\begin{aligned} \partial_t u &= \partial_{xx} u, && (x \in \mathbb{R},\ t > 0) \\ u(x,0) &= f(x), && (x \in \mathbb{R}) \end{aligned}$$

for any given bounded continuous function f. You may use several properties of the convolution operator $*$ without proof. Namely, you can assume that $K_0 * f = f$ and that the following rules hold for the convolution operator $*$, functions f, g, h, and scalar values $\alpha \in \mathbb{R}$:

Commutativity : $f * g = g * f$

Associativity : $(f * g) * h = f * (g * h)$

Distributivity : $f * (g + h) = f * g + f * h$

Scalar associativity : $\alpha \cdot (f * g) = (\alpha \cdot f) * g = f * (\alpha \cdot g)$

Product rule for differentiation : $(f * g)' \;=\; (f' * g) = (f * g')$

Problem H1.3 (Homogeneous Diffusion and Gaussian Convolution)

8P

The program `hom_diff.c` implements an explicit finite difference scheme for homogeneous diffusion filtering. Under Linux, you can compile it (and similarly all other programs of this class) with

```
gcc -Wall -O2 -o hom_diff hom_diff.c -lm
```

and run it with `./hom_diff` . The program `gauss_conv.c` performs spatial convolution with a renormalised and sampled Gaussians that has been truncated at $\pm 5\sigma$. The image format we are using is the PGM (P5) format (consisting of a header and grey values between 0 and 255, represented by 1 byte per pixel). To visualise an image `fabric.pgm`, you can use e.g. `display fabric.pgm &`.

(a) Run the program `hom_diff.c` on the image `fabric.pgm`. It depicts a nonwoven fabric. One of its quality relevant parameters is the cloudiness. Visualise the cloudiness at different scales by choosing different iteration numbers. Is the evolution of the mean, the maximum, the minimum, and the variance in accordance with your expectation? What happens for time steps larger than 0.25?

(b) Cross-validate the program `hom_diff.c` with the program `gauss_conv.c`. For comparing two images, you can use the program `difference.c` which computes the absolute differences. Don't overdo it: Since PGM images encode a grey value by one byte, a maximum difference of 1 is normal. Note: The absolute difference image is rescaled to the greyscale range between 0 and 255.

Submission: Please create a directory Ex01_<your_name> with the following files (and nothing else):

- a pdf file – which can also be a scanned handwritten solution – that contains
  - the names of all people working together for this assignment
  - the solutions of the theoretical Problems 1–2 and answers to the questions in Problem 3
- for Problem 3(a): two representative diffusion-filtered images illustrating the cloudiness at different scales.
- for Problem 3(b): for both diffusion-filtered images, the corresponding Gaussian convolution counterparts and their difference images
- for Problem 3: state the parameters (iteration number, time step size or σ, respectively) for the submitted images.

Compress the directory to a zip file Ex01_<your_name>.zip.

Submit the file via CMS.
Deadline for submission is Friday, November 3, 14:00.