Classroom Work Assignment C2

Problem C2.1 (Homogeneous Diffusion)

(a) Write down the 1-D homogeneous diffusion equation.

(b) Embed this equation into an initial boundary value problem that initialises a homogenous diffusion process with a bounded signal $f(x)$ and uses reflecting boundary conditions.

(c) Discretise the 1-D homogenous diffusion equation using finite differences with the time step size τ and the grid size h.

(d) Derive the explicit finite difference scheme for the 1-D case.

(e) State the corresponding 1-D stencil.

(f) For which time step size τ is this scheme stable?

Problem C2.2 (Rotationally Invariant Approximations)

While the continuous Laplacian Δu is invariant under rotations, this only holds approximatively for its discretisations. To improve the rotation invariance of the discrete Laplacian on a pixel grid with size h in x and y direction, one can use the stencil

$$\frac{1-\delta}{h^2}\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & -4 & 1 \\ \hline 0 & 1 & 0 \\ \hline\end{array} + \frac{\delta}{(\sqrt{2}\,h)^2}\begin{array}{|c|c|c|}\hline 1 & 0 & 1 \\ \hline 0 & -4 & 0 \\ \hline 1 & 0 & 1 \\ \hline\end{array}$$

with some weight parameter $\delta \in [0,1]$. This approximation is a weighted average of approximations along the axial and diagonal directions.
What is the time step size limit for an explicit finite difference scheme for 2-D homogeneous diffusion filtering if one uses this discretisation ?
Experimentally, one observes a good approximation of rotation invariance over all frequencies for $\delta = \sqrt{2} - 1$. What is the corresponding time step size limit?

Problem C2.3 (Lyapunov Functional in 1-D)

Let us consider the 1-D nonlinear diffusion filter

$$\begin{array}{rcll} \partial_t u & = & \partial_x \left(g(|u_x|^2) u_x \right) & \text{on} \quad [a,b] \times (0,\infty), \\ u(x,0) & = & f(x) & \text{on} \quad [a,b], \\ \partial_x u & = & 0 & \text{for} \quad x \in \{a,b\} \quad \text{and} \quad t \in (0,\infty). \end{array}$$

Show that for its solution $u(x,t)$ and a convex function $r \in \mathrm{C}^2(\mathbb{R})$, the expression

$$V(t) := \int_a^b r(u(x,t))\, dx$$

satisfies $V'(t) \le 0$ for all $t > 0$.

Hint: A convex function $r \in \mathrm{C}^2(\mathbb{R})$ satisfies $r''(x) \ge 0$.

Homework Assignment H2

Problem H2.1 (Finite Difference Approximation) 4+2+1+2P

For a smooth function $u(x,t)$, let u_i^k denote an approximation in $x_i = (i-\frac{1}{2})h$ at time $t_k = k\tau$, where h is the grid size, τ the time step size, and i and k are integer numbers.

(a) Discretise the PDE

$$\partial_t u = -\partial_{xxxx} u$$

with a forward difference for the time derivative and a difference approximation for the space derivative that involve the values u_{i-2}^k, u_{i-1}^k, u_i^k, u_{i+1}^k and u_{i+2}^k.
It suffices to state the system of equations whose solution gives the desired coefficients.

(b) Determine the order of consistency of

$$\partial_{xxxx} u = \frac{u_{i-2}^k - 4u_{i-1}^k + 6u_i^k - 4u_{i+1}^k + u_{i+2}^k}{h^4}.$$

(c) Embed the discretisation of part (b) in an explicit scheme.

(d) Does this scheme satisfy a discrete maximum–minimum principle? Justify your answer or give a counterexample.

Problem H2.2 (α-Scale-Spaces) 4P

Duits et al. (2004) have advocated *α-scale-spaces* as a family of linear scale-spaces with a free (not necessarily integer-valued) parameter $\alpha > 0$. Their scale-space representation $u(\boldsymbol{x}, t)$ of a greyscale image $f : \mathbb{R}^n \to \mathbb{R}$ in n dimensions solves

$$\begin{aligned} \partial_t u &= -(-\Delta)^\alpha u \qquad \text{for } t > 0, \\ u(\boldsymbol{x}, 0) &= f(\boldsymbol{x}). \end{aligned}$$

The pseudodifferential operator $(-\Delta)^\alpha$ is defined by its behaviour in the Fourier domain:

$$\mathcal{F}\left[(-\Delta)^\alpha u\right](\boldsymbol{\nu}) := (2\pi\,|\boldsymbol{\nu}|)^{2\alpha}\,\mathcal{F}[u](\boldsymbol{\nu}),$$

where $\mathcal{F}[u](\boldsymbol{\nu}) =: \hat{u}(\boldsymbol{\nu}, t)$ is the Fourier transform of $u(\boldsymbol{x}, t)$ with frequency $\boldsymbol{\nu} = (\nu_1, ..., \nu_n)$.
Transform the evolution to the Fourier domain using the linearity of the Fourier transform.
Show that the resulting problem is solved by

$$\hat{u}(\boldsymbol{\nu}, t) \;=\; \hat{f}(\boldsymbol{\nu}) \, \exp\left(-(2\pi\,|\boldsymbol{\nu}|)^{2\alpha}\, t\right).$$

(This problem is surprisingly easy to solve. It implies that α-scale-spaces perform a convolution with a kernel that has a simple and elegant representation in the Fourier domain. For $\alpha = \frac{1}{2}$, the result includes the Poisson scale-space, for $\alpha = 1$ the Gaussian scale-space, and for $\alpha = 2$ the "biharmonic" scale-space from Problem 1(a).)

Problem H2.3 (Diffusivities) — 4+3P

Let the following diffusivities be given:

1. Charbonnier diffusivity: $\quad g(s^2) := \dfrac{1}{\sqrt{1 + s^2/\lambda^2}}$
2. Rational Perona-Malik diffusivity: $\quad g(s^2) := \dfrac{1}{1 + s^2/\lambda^2}$
3. Unbounded total variation (TV) diffusivity: $\quad g(s^2) := \dfrac{1}{|s|}$
4. Balanced forward-backward (BFB) diffusivity: $\quad g(s^2) := \dfrac{1}{s^2}$

(a) Analyse these diffusivities with respect to the possibility of forward-backward diffusion. To this end, consider the sign of $\Phi'(s)$ for the flux function $\Phi(s) = s\,g(s^2)$. For which values of s is local contrast enhancement possible?

(b) Sketch the graphs of the diffusivities $g(s^2)$ and the corresponding flux functions $\Phi(s)$ for $\lambda = 5$ and $s \in [0, 20]$.

Problem H2.4 (Homogeneous Diffusion Inpainting) 2+1+1P

Homogeneous diffusion can also be used for inpainting (i.e. filling in) missing data in images. To this end, we need an inpainting mask $c(\boldsymbol{x})$, which is a binary image that tells us where are the pixels we can trust. These pixels are unchanged. Everywhere else we perform homogeneous diffusion. The steady state for $t \to \infty$ gives the desired reconstruction. In contrast to classical homogeneous diffusion filtering, the unchanged pixels in homogeneous diffusion inpainting allow a non-flat steady state.

(a) Implement homogeneous diffusion inpainting by supplementing the file `hd_inpainting.c` with the missing code. Compile your final program with the command

```
gcc -Wall -O2 -o hd_inpainting hd_inpainting.c -lm
```

(b) To test your program, you can use `klein-mask.pgm` as inpainting mask. It specifies the (carefully optimised) locations of only 5 % of all pixels. Study the influence of the initialisation at the non-mask pixels with two initial images: `klein-data.pgm` initialises with black pixels (grey value 0), while `klein-data2.pgm` uses uniformly distributed random values in the range $[0, 255]$. Which results do you obtain for 50000 iterations with time step size $\tau = 0.24$?

(c) Use `difference.c` to display the difference of your two reconstructions, as well as the difference between your first reconstruction and the original image `klein.pgm`. What are your observations?

(*This problem introduces you to inpainting-based compression, a research area pioneered by our group. Finding a good inpainting mask as well as good and fast inpainting algorithms are hot research topics.*)

Submission: Please create a directory `Ex02_<your_name>` with the following files (and nothing else):

- a pdf file – which can also be a scanned handwritten solution – that contains
  - the names of all people working together for this assignment
  - the solutions of the theoretical Problems 1–3 and answers to the questions in Problem 4
- for Problem 4(a): the completed inpainting code `hd_inpainting.c`
- for Problem 4(b): the two reconstructions `klein-data-inpainted.pgm` and `klein-data2-inpainted.pgm`
- for Problem 4(c): the difference image `inpainting-difference.pgm` between the two reconstrutions
- for Problem 4(c): the difference image `reconstruction-error.pgm` between `klein-data-inpainted.pgm` and `klein.pgm`

Compress the directory to a zip file `Ex02_<your_name>.zip`.

Submit the file via CMS.
Deadline for submission is Friday, November 10, 14:00.