Classroom Work Assignment C7

Problem C7.1 (Vector-valued Edge-Enhancing Anisotropic Nonlinear Diffusion)

Consider the vector-valued EED filter with two channels

$$\partial_t u_i \;=\; \mathbf{div}\left(g\Big(\sum_{k=1}^{2} \boldsymbol{\nabla} u_{k,\sigma}\, \boldsymbol{\nabla} u_{k,\sigma}^\top\Big)\, \boldsymbol{\nabla} u_i\right) \qquad \forall\, i \in \{1,2\}\,. \tag{1}$$

Name a sufficient condition such that (1) becomes isotropic for $\sigma = 0$. State one concrete example for $\boldsymbol{\nabla} u_1$, $\boldsymbol{\nabla} u_2$ such that the behaviour in a certain point is anisotropic for $\sigma = 0$.

Homework Assignment H7

Problem H7.1 (Multiple Choice) 6P

Mark which of the following statements are true and which ones are false. Justify your answers briefly. Correct answers without justification will not receive any points.

(a) The larger the diffusivity, the smaller is the time step size limit of an explicit scheme.

(b) A continuous Perona-Malik filter with a smooth, positive diffusivity always performs forward diffusion in extrema.

(c) The stencils for homogeneous and nonlinear isotropic diffusion from the lecture are stable in the Euclidean norm.

(d) Doubling the resolution of an image requires to quadruple the evolution time for the Poisson scale-space in order to obtain comparable results.

(e) The larger the regularisation parameter, the more will the result of a variational method differ from its corresponding diffusion filter.

(f) The gradient descent scheme for Whittaker–Tikhonov regularisation converges to a flat steady state given by the average grey value of the initialisation.

Problem H7.2 (Half-Quadratic Regularisation) 6P

Derive the half-quadratic regularisation formulation corresponding to the (nonconvex) energy function

$$E_f(u) := \int_\Omega \left((u-f)^2 + \alpha \cdot \lambda^2 \ln\left(1 + \frac{|\boldsymbol{\nabla} u|^2}{\lambda^2}\right)\right) d\boldsymbol{x} \qquad \text{with } \alpha, \lambda > 0 \,.$$

Problem H7.3 (Matrix-Valued Diffusion Filtering) 3+3P

(a) Consider the matrix-valued diffusion filter without channel coupling

$$\partial_t \boldsymbol{u}_{i,j} = \mathbf{div}\, \left(g\left(|\boldsymbol{\nabla} \boldsymbol{u}_{i,j}|^2\right) \boldsymbol{\nabla} \boldsymbol{u}_{i,j}\right) \qquad \forall i, j \; .$$

Do typical discretisations lead to positive semidefinite matrices $\boldsymbol{u}_{i,j}$, if the process is initialised with a positive semidefinite matrix? Justify your answer.

(b) Consider the matrix-valued regularisation method

$$E_{\boldsymbol{f}}(\boldsymbol{u}) = \int_{\Omega} \left(\|\boldsymbol{u} - \boldsymbol{f}\|_2^2 + \alpha \sum_{k=1}^{m} \sum_{\ell=1}^{m} \Psi\left(|\boldsymbol{\nabla} \boldsymbol{u}_{k,\ell}|^2\right) \right) d\boldsymbol{x} \; .$$

What is the corresponding time-discrete diffusion filter?

Problem H7.4 (Ambrosio–Tortorelli Approximation) 2+2+2P

(a) Discretise the gradient descent equations of the Ambrosio–Tortorelli functional with a simple modified explicit scheme in the sense of Lecture 11. For the second equation, use an explicit approach for $|\boldsymbol{\nabla} u|^2$. State the resulting equations in the form $u_{i,j}^{k+1} = \dots$ and $v_{i,j}^{k+1} = \dots$.

(b) Supplement the missing code in `ambrosio.c` such that you obtain an implementation of this algorithm and compile it with

```
gcc -Wall -O2 -o ambrosio ambrosio.c -lm
```

(c) Use this program to process the image `house.pgm` with two parameter settings

- $\beta = 0.007$, $\alpha = 1$, and $c = 0.05$.
- $\beta = 0.001$, $\alpha = 1$, and $c = 0.01$.

In both cases, perform 50000 iterations with $\tau = 0.2$.
What are the resulting images for u and v ?

Submission: Please create a directory Ex07_<your_name> with the following files (and nothing else):

- a pdf file – which can also be a scanned handwritten solution – that contains
  - the names of all people working together for this assignment
  - the solutions of the theoretical Problems 1–3 and answers to the questions in Problem 4
- for Problem 4: the four images for the specified parameter settings

Compress the directory to a zip file Ex07_<your_name>.zip.

Submit the file via CMS.
Deadline for submission is Friday, December 15, 14:00.