Differential Equations in Image Processing and Computer Vision

Dr. Pascal Peter
Mathematical Image Analysis Group
WS 2023
Saarland University

Classroom Work Assignment C10

Problem C10.1 (Transition Matrix for 1-D Osmosis)

(a) Write down the 1-D drift-diffusion equation for osmosis with homogeneous Neumann boundary conditions.

(b) Derive the explicit finite difference scheme for the 1-D case.

(c) Write down the corresponding transition matrix $\boldsymbol{Q}$ for a one-dimensional discrete linear osmosis process with $\boldsymbol{f} = (f_1, f_2, f_3, f_4, f_5)^\top \in \mathbb{R}^5_+$ as initial signal.
Is $\boldsymbol{Q}$ symmetric? Does it have row sum one or column sum one?

Problem C10.2 (Energy Functional of Osmosis)

Let $\boldsymbol{d} = \frac{\boldsymbol{\nabla} v}{v}$ be the canonical drift vector field to some positive guidance image v.

Show that the steady state equation

$$\Delta u - \mathbf{div}\,(\boldsymbol{d} u) = 0$$

is equivalent to the Euler-Lagrange equation of the energy functional

$$E(u) = \int_\Omega v \left| \boldsymbol{\nabla} \left(\tfrac{u}{v}\right)\right|^2 d\boldsymbol{x}\,.$$

Hint: Consider $\mathbf{div}\left(v\,\boldsymbol{\nabla}\left(\frac{u}{v}\right)\right)$ and use the identity

$$\mathbf{div}\,(\boldsymbol{d} u) = (\mathbf{div}\,\boldsymbol{d}) u + \boldsymbol{d}^\top \boldsymbol{\nabla} u$$

for a scalar-valued function u and a vector field $\boldsymbol{d}$.

Homework Assignment H10

Problem H10.1 (Average Grey Value Invariance of Osmosis) 6P

Let $f : [a, b] \to \mathbb{R}^+$ be a 1-D signal, $d : [a, b] \to \mathbb{R}$ be a drift coefficient field with $d(a) = d(b) = 0$, and consider the continuous 1-D osmosis process

$$\partial_t u = \partial_{xx} u - \partial_x (du)$$

with f as initial state and homogeneous Neumann boundary conditions:

$$\partial_x u(a, t) = \partial_x u(b, t) = 0.$$

Show that a classical solution $u(x, t)$ preserves the average grey value of the initial signal.

Hint: Represent the average grey value of the evolving image as a continuous function in t and show that it is constant.

Problem 10.2 (Analysis of the Explicit Osmosis Scheme) 2+2+4P

Assume that the matrix $\boldsymbol{A} = (a_{i,j}) \in \mathbb{R}^{N \times N}$ satisfies the requirements (SLO1)–(SLO3) for a semidiscrete osmosis process.

(a) Show that this implies that $a_{i,i} \leq 0$ for all $i \in \{1, ..., N\}$.

(b) Prove that there exists at least one $i \in \{1, ..., N\}$ with $a_{i,i} < 0$.

(c) Use these results to establish that the explicit osmosis scheme $\boldsymbol{u}^{k+1} = (\boldsymbol{I} + \tau \boldsymbol{A})\boldsymbol{u}^k$ satisfies (DLO1)–(DLO4) if

$$\tau < \frac{1}{\max_i |a_{i,i}|}.$$

Problem H10.3 (Osmosis Filtering) 3+4+3P

The file `osmosis.c` is an almost complete implementation of linear osmosis filtering with the canonical drift vector field of some guidance image v.

(a) Supplement it with the missing code for the canonical drift vector field $(d_1, d_2)^\top$. Take into account that `d1[i][j]` approximates $d_{1,i+1/2,j}$, and `d2[i][j]` approximates $d_{2,i,j+1/2}$. Compile your program with `gcc -Wall -O2 -o osmosis osmosis.c -lm`.

(b) Use the program `osmosis` to compute the steady state when the initial images `const142.pgm`, `noise.pgm` and `const070.pgm` are used in conjunction with `peppers2.pgm` as guidance image. The parameter `offset` is used to guarantee the positivity of the images. It can be set to 1. Can you explain your observations?

(c) The image `shadow.pgm` simulates a shadow. Its boundary is located between the pixels with the x coordinates `i=128` and `i=129`. Modify the canonical drift vector field such that the corresponding osmosis process removes this shadow when you use `shadow.pgm` as initial and guidance image. Compute results for the stopping times $T = 10$ and $T = 60000$.

Submission: Please create a directory `Ex10_<your_name>` with the following files (and nothing else):

- a pdf file – which can also be a scanned handwritten solution – that contains
  - the names of all people working together for this assignment
  - the solutions of the theoretical Problems 1–2 and answers to the questions in Problem 3
- for Problem 3: the completed file `osmosis-complete.c` and the resulting images `peppers-const070.pgm`, `peppers-const142.pgm`, `peppers-noise.pgm`, `shadow10.pgm` and `shadow60000.pgm`

Compress the directory to a zip file `Ex10_<your_name>.zip`.

Submit the file via CMS.
Deadline for submission is Friday, January 12, 14:00.