DIC: Homework Assignment H4

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Problem H4.1 (Anisotropic Diffusion Modelling)

(a)

$$D(\nabla u) := (v_1 \mid v_2)\ diag(g(\mu_1), 1)(v_1 \mid v_2)^\top$$

where v_1 and v_2 are eigenvectors of $J_\rho(\nabla u_\sigma)$

(b)

(c)

Problem H4.2 (Directional Splitting of Anisotropic Diffusion)

First, we consider the right hand of the equation:

$$\sum_{i=0}^{3} \partial_{\boldsymbol{e_i}}(w_i \partial_{\boldsymbol{e_i}} u)$$

Where we know that $\partial_{\boldsymbol{n}} u = \boldsymbol{n}^\mathsf{T} \nabla u$, the directional diffusivities w_0, w_1, w_2, w_3 are given by

$$w_0 = a - \delta,\ w_1 = \delta + b,\ w_2 = c - \delta,\ w_3 = \delta - b$$

And the directions are given by

$$\boldsymbol{e_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \boldsymbol{e_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \boldsymbol{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \boldsymbol{e_3} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $i = 0$, we have:

$$\partial_{\boldsymbol{e_0}}(w_0 \partial_{\boldsymbol{e_0}} u) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^\mathsf{T} \nabla \left[(a - \delta) \begin{pmatrix} 1 \\ 0 \end{pmatrix}^\mathsf{T} \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} \right] = \partial_x(a \partial_x u) - \partial_x(\delta \partial_x u)$$

Similarly, for $i = 2$, we get:

$$\partial_{\boldsymbol{e_2}}(w_2 \partial_{\boldsymbol{e_2}} u) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\mathsf{T} \nabla \left[(c - \delta) \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\mathsf{T} \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} \right] = \partial_y(c \partial_y u) - \partial_y(\delta \partial_y u)$$

For $i = 1$:

$$\begin{aligned}
\partial_{\boldsymbol{e_1}}(w_1 \partial_{\boldsymbol{e_1}} u) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\mathsf{T} \nabla \left[\tfrac{1}{\sqrt{2}}(\delta + b) \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\mathsf{T} \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} \right] \\
&= \frac{1}{2}\partial_x(\delta \partial_x u) + \frac{1}{2}\partial_x(b \partial_x u) + \frac{1}{2}\partial_x(\delta \partial_y u) + \frac{1}{2}\partial_x(b \partial_y u) + \frac{1}{2}\partial_y(\delta \partial_x u) + \frac{1}{2}\partial_y(b \partial_x u) + \frac{1}{2}\partial_y(\delta \partial_y u) + \frac{1}{2}\partial_y(b \partial_y u)
\end{aligned}$$

Finally, for $i = 3$:

$$\partial_{\boldsymbol{e_3}}(w_3\partial_{\boldsymbol{e_3}}u) = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}^\intercal \nabla\left[\tfrac{1}{\sqrt{2}}(\delta-b)\begin{pmatrix}-1\\1\end{pmatrix}^\intercal\begin{pmatrix}\partial_x u\\\partial_y u\end{pmatrix}\right]$$
$$= \frac{1}{2}\partial_x(\delta\partial_x u) - \frac{1}{2}\partial_x(b\partial_x u) - \frac{1}{2}\partial_x(\delta\partial_y u) + \frac{1}{2}\partial_x(b\partial_y u) - \frac{1}{2}\partial_y(\delta\partial_x u) + \frac{1}{2}\partial_y(b\partial_x u) + \frac{1}{2}\partial_y(\delta\partial_y u) - \frac{1}{2}\partial_y(b\partial_y u)$$

Summing up the terms for $i = 1$ and $i = 3$, we get:

$$\partial_{\boldsymbol{e_2}}(w_2\partial_{\boldsymbol{e_2}}u) + \partial_{\boldsymbol{e_3}}(w_3\partial_{\boldsymbol{e_3}}u) = \partial_x(\delta\partial_x u) + \partial_x(b\partial_y u) + \partial_y(b\partial_x u) + \partial_y(\delta\partial_y u)$$

Then, summing up the resulting terms with the ones obtained for $i = 0$ and $i = 2$, we get:

$$\begin{aligned}\sum_{i=0}^{3}\partial_{\boldsymbol{e_i}}(w_i\partial_{\boldsymbol{e_i}}u) &= \partial_{\boldsymbol{e_0}}(w_0\partial_{\boldsymbol{e_0}}u) + \partial_{\boldsymbol{e_1}}(w_1\partial_{\boldsymbol{e_1}}u) + \partial_{\boldsymbol{e_2}}(w_2\partial_{\boldsymbol{e_2}}u) + \partial_{\boldsymbol{e_3}}(w_3\partial_{\boldsymbol{e_3}}u)\\ &= \partial_x(\delta\partial_x u) + \partial_x(b\partial_y u) + \partial_y(b\partial_x u) + \partial_y(\delta\partial_y u)\\ &+ \partial_x(a\partial_x u) - \partial_x(\delta\partial_x u) + \partial_y(c\partial_y u) - \partial_y(\delta\partial_y u)\\ &= \partial_x(a\partial_x u) + \partial_x(b\partial_y u) + \partial_y(b\partial_x u) + \partial_y(c\partial_y u)\end{aligned} \tag{1}$$

On the other hand, let us consider the following derivation which uses the mathematical definition of the divergence of a vector:

$$\begin{aligned}\textbf{div}\left(\begin{pmatrix}a & b\\ b & c\end{pmatrix}\nabla u\right) &= \textbf{div}\left(\begin{pmatrix}a & b\\ b & c\end{pmatrix}\begin{pmatrix}\partial_x u\\\partial_y u\end{pmatrix}\right)\\ &= \textbf{div}\begin{pmatrix}a\partial_x u + b\partial_y u\\ b\partial_x u + c\partial_y u\end{pmatrix}\\ &= \partial_x(a\partial_x u) + \partial_x(b\partial_y u) + \partial_y(b\partial_x u) + \partial_y(c\partial_y u)\end{aligned} \tag{2}$$

Comparing (1) and (2) term by term, we see that they are equal. Therefore,

$$\sum_{i=0}^{3}\partial_{\boldsymbol{e_i}}(w_i\partial_{\boldsymbol{e_i}}u) = \textbf{div}\left(\begin{pmatrix}a & b\\ b & c\end{pmatrix}\nabla u\right)$$

Problem H4.3 (δ-Stencil for Isotropic Diffusions)

Problem H4.4 (Anisotropic Diffusion)

Problem H4.5

Give an appropriate positive constant c such that $f(n) \le c \cdot g(n)$ for all $n > 1$.

1. $f(n) = n^2 + n + 1$, $g(n) = 2n^3$
2. $f(n) = n\sqrt{n} + n^2$, $g(n) = n^2$
3. $f(n) = n^2 - n + 1$, $g(n) = n^2/2$

Solution
We solve each solution algebraically to determine a possible constant c.

Part One

$$\begin{aligned} n^2 + n + 1 = & \\ & \leq n^2 + n^2 + n^2 \\ & = 3n^2 \\ & \leq c \cdot 2n^3 \end{aligned}$$

Thus a valid c could be when $c = 2$.

Part Two

$$\begin{aligned} n^2 + n\sqrt{n} = & \\ & = n^2 + n^{3/2} \\ & \leq n^2 + n^{4/2} \\ & = n^2 + n^2 \\ & = 2n^2 \\ & \leq c \cdot n^2 \end{aligned}$$

Thus a valid c is $c = 2$.

Part Three

$$\begin{aligned} n^2 - n + 1 = & \\ & \leq n^2 \\ & \leq c \cdot n^2/2 \end{aligned}$$

Thus a valid c is $c = 2$.

Problem H4.6

Let $\Sigma = \{0, 1\}$. Construct a DFA A that recognizes the language that consists of all binary numbers that can be divided by 5.

Let the state q_k indicate the remainder of k divided by 5. For example, the remainder of 2 would correlate to state q_2 because $7 \mod 5 = 2$.

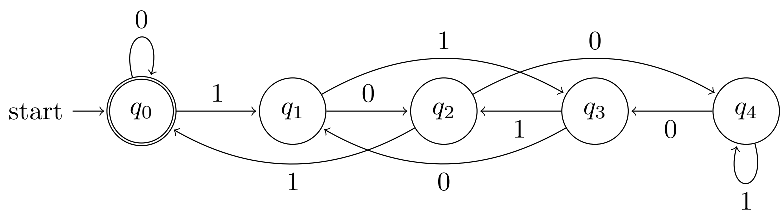

Figure 1: DFA, A, this is really beautiful, ya know?

Justification

Take a given binary number, x. Since there are only two inputs to our state machine, x can either become $x0$ or $x1$. When a 0 comes into the state machine, it is the same as taking the binary number and multiplying it by two. When a 1 comes into the machine, it is the same as multipying by two and adding one.

Using this knowledge, we can construct a transition table that tell us where to go:

	$x \mod 5 = 0$	$x \mod 5 = 1$	$x \mod 5 = 2$	$x \mod 5 = 3$	$x \mod 5 = 4$
$x0$	0	2	4	1	3
$x1$	1	3	0	2	4

Therefore on state q_0 or ($x \mod 5 = 0$), a transition line should go to state q_0 for the input 0 and a line should go to state q_1 for input 1. Continuing this gives us the Figure 1.

Problem H4.7

Write part of **Quick-Sort(**$list, start, end$**)**

function QUICK-SORT($list, start, end$)
  if $start \geq end$ **then**
    return
  end if
  $mid \leftarrow$ PARTITION($list, start, end$)
  QUICK-SORT($list, start, mid - 1$)
  QUICK-SORT($list, mid + 1, end$)
end function

Algorithm 1: Start of QuickSort

Problem H4.8

Suppose we would like to fit a straight line through the origin, i.e., $Y_i = \beta_1 x_i + e_i$ with $i = 1, \ldots, n$, $\mathrm{E}[e_i] = 0$, and $\mathrm{Var}[e_i] = \sigma_e^2$ and $\mathrm{Cov}[e_i, e_j] = 0, \forall i \neq j$.

Part A

Find the least squares esimator for $\hat{\beta}_1$ for the slope β_1.

Solution

To find the least squares estimator, we should minimize our Residual Sum of Squares, RSS:

$$\begin{aligned} RSS &= \sum_{i=1}^{n} \left(Y_i - \hat{Y}_i\right)^2 \\ &= \sum_{i=1}^{n} \left(Y_i - \hat{\beta}_1 x_i\right)^2 \end{aligned}$$

By taking the partial derivative in respect to $\hat{\beta}_1$, we get:

$$\frac{\partial}{\partial \hat{\beta}_1}(RSS) = -2 \sum_{i=1}^{n} x_i (Y_i - \hat{\beta}_1 x_i) = 0$$

This gives us:

$$\begin{aligned} \sum_{i=1}^{n} x_i (Y_i - \hat{\beta}_1 x_i) &= \sum_{i=1}^{n} x_i Y_i - \sum_{i=1}^{n} \hat{\beta}_1 x_i^2 \\ &= \sum_{i=1}^{n} x_i Y_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 \end{aligned}$$

Solving for $\hat{\beta}_1$ gives the final estimator for β_1:

$$\hat{\beta}_1 = \frac{\sum x_i Y_i}{\sum x_i^2}$$

Problem H4.8 continued on next page...

Part B

Calculate the bias and the variance for the estimated slope $\hat{\beta}_1$.

Solution

For the bias, we need to calculate the expected value $\mathrm{E}[\hat{\beta}_1]$:

$$\begin{aligned}
\mathrm{E}[\hat{\beta}_1] &= \mathrm{E}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\
&= \frac{\sum x_i \mathrm{E}[Y_i]}{\sum x_i^2} \\
&= \frac{\sum x_i (\beta_1 x_i)}{\sum x_i^2} \\
&= \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\
&= \beta_1 \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\
&= \beta_1
\end{aligned}$$

Thus since our estimator's expected value is β_1, we can conclude that the bias of our estimator is 0.

For the variance:

$$\begin{aligned}
\mathrm{Var}[\hat{\beta}_1] &= \mathrm{Var}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\
&= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{1}{\sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{1}{\sum x_i^2} \sigma^2 \\
&= \frac{\sigma^2}{\sum x_i^2}
\end{aligned}$$

Problem H4.9

Prove a polynomial of degree k, $a_kn^k + a_{k-1}n^{k-1} + \ldots + a_1n^1 + a_0n^0$ is a member of $\Theta(n^k)$ where $a_k \ldots a_0$ are nonnegative constants.

Proof. To prove that $a_kn^k + a_{k-1}n^{k-1} + \ldots + a_1n^1 + a_0n^0$, we must show the following:

$$\exists c_1 \exists c_2 \forall n \geq n_0,\ c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

For the first inequality, it is easy to see that it holds because no matter what the constants are, $n^k \leq a_kn^k + a_{k-1}n^{k-1} + \ldots + a_1n^1 + a_0n^0$ even if $c_1 = 1$ and $n_0 = 1$. This is because $n^k \leq c_1 \cdot a_kn^k$ for any nonnegative constant, c_1 and a_k.

Taking the second inequality, we prove it in the following way. By summation, $\sum_{i=0}^{k} a_i$ will give us a new constant, A. By taking this value of A, we can then do the following:

$$\begin{aligned}
a_kn^k + a_{k-1}n^{k-1} + \ldots + a_1n^1 + a_0n^0 = & \\
& \leq (a_k + a_{k-1} \ldots a_1 + a_0) \cdot n^k \\
& = A \cdot n^k \\
& \leq c_2 \cdot n^k
\end{aligned}$$

where $n_0 = 1$ and $c_2 = A$. c_2 is just a constant. Thus the proof is complete. $\square$

Problem H4.18

Evaluate $\sum_{k=1}^{5} k^2$ and $\sum_{k=1}^{5}(k-1)^2$.

Problem H4.19

Find the derivative of $f(x) = x^4 + 3x^2 - 2$

Problem H4.6

Evaluate the integrals $\int_0^1 (1-x^2)\mathrm{d}x$ and $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$.