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Classroom Work Assignment C11

Problem C11.1 (Discretisation of the 1-D Erosion PDE)

Discretise the 1-D erosion equation $\partial_t u = -|\partial_x u|$ in three different ways:

- Use forward differences in time and central differences in space.
- Use forward differences in time and backward differences in space.
- Use forward differences in time and forward differences in space.

(a) Compute the result after 1 iteration step with $h = 1$ and $\tau = 1$ for the increasing discrete initial signal

$$f_i = \begin{cases} 0 & (i \leq 0), \\ 1 & (i \geq 1). \end{cases}$$

Sketch the results.

(b) Which discretisation performs best?
Interpret your answer in terms of the transport direction of the process.

(c) What happens when you use the best-performing discretisation and perform two iterations with $\tau = 0.5$? Consider the pixels $i = 1$ and $i = 2$. In particular, interpret the results in terms of dissipative effects (blur artefacts).

Problem C11.2 (Slope Transform)

Verify the following properties for the slope transform of a function $f : \mathbb{R} \to \mathbb{R}$ and a parameter $a \in \mathbb{R}$:

(a) $\mathcal{S}[af(x)](\nu) = a\mathcal{S}[f](\frac{\nu}{a})$

(b) $\mathcal{S}[f(ax)](\nu) = \mathcal{S}[f](\frac{\nu}{a})$

(c) $\mathcal{S}[a + f(x)](\nu) = a + \mathcal{S}[f](\nu)$

(d) $\mathcal{S}[f(x + a)](\nu) = a\nu + \mathcal{S}[f](\nu)$

Homework Assignment H11

Problem H11.1 (Ellipses as Structuring Elements) 1+1+2+3+1P

Let $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ be a symmetric matrix with normalised eigenvectors $\boldsymbol{v}_1 := (v_{1,1}, v_{1,2})^\top$ and $\boldsymbol{v}_2 := (-v_{1,2}, v_{1,1})^\top$ in direction of the half-axes a and b of an ellipse, and let its eigenvalues be given by a^2 and b^2.

(a) Determine the entries of $\boldsymbol{A}$.

(b) The ellipse can now be defined by the equation $\boldsymbol{x}^\top \boldsymbol{A}^{-1} \boldsymbol{x} = 1$. Determine the inverse of $\boldsymbol{A}$.

(c) Show that the parametric form

$$\boldsymbol{x}(t) = (x_1(t), x_2(t))^\top = \boldsymbol{R}\,(a\cos(t), b\sin(t))^\top$$

with the rotation matrix $\boldsymbol{R} := (\boldsymbol{v}_1|\boldsymbol{v}_2)$ satisfies the equation $\boldsymbol{x}^\top \boldsymbol{A}^{-1} \boldsymbol{x} = 1$.

(d) Let $\partial_t \boldsymbol{c} = \beta \cdot \boldsymbol{n}$ denote a curve evolution. The velocity β in each point of the curve is given by the maximal projection of the boundary of the structuring element B onto the outer normal vector $\boldsymbol{n}$ of the curve $\boldsymbol{c}$, i.e.

$$\beta = \sup_{\boldsymbol{x}\in B} \left(\boldsymbol{x}^\top \boldsymbol{n}\right).$$

Show that for a continuous-scale dilation evolution with an ellipse $\boldsymbol{x}^\top \boldsymbol{A}^{-1} \boldsymbol{x} = 1$ as structuring element one obtains $\beta = \sqrt{\boldsymbol{n}^\top \boldsymbol{A} \boldsymbol{n}}$.

(e) How does the corresponding image evolution look like, if one embeds $\boldsymbol{c}$ as a level line using the distance transformation? Make sure that you express $\boldsymbol{n}$ in terms of the image u.

Problem 11.2 (Upwind Scheme for the 1-D Erosion PDE) 2+2+1P

Rouy and Tourin have suggested the following scheme for the 1-D erosion equation $\partial_t u = -|\partial_x u|$:

$$\frac{u_i^{k+1} - u_i^k}{\tau} = -\max\left\{0,\ \frac{u_i^k - u_{i-1}^k}{h},\ \frac{u_i^k - u_{i+1}^k}{h}\right\}.$$

(a) Show that this scheme has upwind behaviour w.r.t. the local transport direction. To this end, analyse its behaviour at a locally increasing ($u_{i-1}^k \le u_i^k \le u_{i+1}^k$) and a locally decreasing ($u_{i-1}^k \ge u_i^k \ge u_{i+1}^k$) part of the signal.

(b) What happens in a local minimum and in a local maximum?

(c) Prove that the scheme is stable in terms of a discrete maximum-minimum principle for $\frac{\tau}{h} \le 1$.

Problem H11.3 (Slope Transform and Paraboloids) 3+1P

Consider a paraboloid of the form $p(\boldsymbol{x}) = -\frac{\boldsymbol{x}^\top \boldsymbol{x}}{4t}$ with $t > 0$.

(a) Verify that the slope transform of p is again a paraboloid.

(b) Gaussians are separable in the standard algebra. Verify the corresponding property for a 2-D paraboloid in the $(\max, +)$-algebra.

Problem H11.4 (Morphology with a Disk) 2+2P

(a) The program `dilation.c` performs continuous-scale dilation with a disc-shaped structuring element. Have a look at it and supplement the corresponding program `erosion.c` with the missing code for a Rouy-Tourin scheme. Compile it with `gcc -Wall -O2 -o erosion erosion.c -lm`.

(b) Apply `dilation` and `erosion` to the image `dolphin.pgm` with 1 iteration with $\tau = 0.5$. Take the modulus of the difference between both results using the program `difference`. The result is an edge detector *("morphological gradient")* which can be scaled by varying the evolution time of both underlying processes. Thus, try it also using 10 iterations with $\tau = 0.5$.

Problem H11.5 (Euclidean Distance Transformation) 2+1P

(a) The program `distance.c` is an almost complete program for computing the Euclidean distance transformation via erosion with a quadratic structuring function. Complete it and compile it with
`gcc -Wall -O2 -o distance distance.c -lm`.

(b) Apply it to the map of EV chargers in Germany `ev-chargers.pgm`. To visualise the result, use a wave length of 16.

Submission: Please create a directory Ex11_<your_name> with the following files (and nothing else):

- a pdf file – which can also be a scanned handwritten solution – that contains
  - the names of all people working together for this assignment
  - the solutions of the theoretical Problems 1–3
- for Problem 4(a): the completed file `erosion-complete.c`,
- for Problem 4(b): the resulting images `dolphin-dilation-1.pgm`, `dolphin-erosion-1.pgm`, `dolphin-difference-1.pgm` obtained with 1 iteration, as well as `dolphin-dilation-10.pgm`, `dolphin-erosion-10.pgm`, `dolphin-difference-10.pgm` for 10 iterations,
- for Problem 5(a): the completed file `distance-complete.c`
- for Problem 5(b): the resulting image `ev-chargers-distance.pgm`.

Compress the directory to a zip file Ex11_<your_name>.zip.

Submit the file via CMS.
Deadline for submission is Friday, January 19, 14:00.