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H3.1

Stencils which violates , which de-satifies

D2 : d) | convergence. to a steady-state

D3 : a), d | Average grey level Invariance

D4 : g) | Max min Principal

D5 : f) |

D6 : c) | Convergence to a steady state

d fails symmetry so no columnsums

to fullfill D2 : ~~expended~~ 1d stencil must be equal to its inverse

D3 : sum of stencil must be 1

D4 : the center of stencil must be positive

D5 ; rows of stencil cannot be completely zero.

H3.2 a) By induction on K

Base step : $k=0$, $u^0 = f$ where $\min_{j\in J} f_j \le f_i \le \max_{j\in J} f_j$ for all f_i in f

Induction step : suppose $\min_{j\in J} f_j \le u_i^k \le \max_{j\in J} f_j$ $\forall i \in J$, $\exists k \ge 0$

$$u_i^{k+1} = \left(Q(u^k)\right)_i^T u^k$$

$$u_i^{k+1} = \sum_{h=1}^{n} (q_{ih} u_h^k) ,$$

consider u_m^k as min of u^k then we have

$$u_i^k = u_m^k + P_i$$

where $P_i \ge 0$,

$$u_i^{k+1} - u_m^k = \sum_{h=1}^{n} \left(q_{ih}(u_m^k + P_h)\right) - u_m^k$$

$$= \sum_{h=1}^{n} q_{ih} u_m^k + \sum_{h=1}^{n} q_{ih} P_h - u_m^k$$

$$= u_m^k - u_m^k + \sum_{h=1}^{n} q_{ih} P_h \ge 0$$

thus $u_i^{k+1} \ge u_m^k$

similarily let u_x^k be the max of u^k then we have

$$u_i^k = u_x^k + \delta_i \quad \text{where} \quad \delta_i \le 0$$

$$u_i^{k+1} - u_x^k = \underbrace{\cdots}_{\text{similar steps as the min case}} = u_x^k - u_x^k + \sum_{h=1}^{n} q_{ih}\delta_h \le 0$$

b) the max part is still valid when row sums ≤ 1

the min part is still valid when row sums ≥ 1

c) By induction on k

base step : $\frac{1}{N}\sum_{j=1}^{N} u_j^0 = \mu$ where μ is the mean of f

induction step: assume that

$$\frac{1}{N}\sum_{j=1}^{N} u_j^k = \mu,$$

we have

$$\frac{1}{N}\sum_{i=1}^{N} u_i^{k+1} = \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}(q_{ij}\ u_j^k)\right)$$

$$= \frac{1}{N}\sum_{j=1}^{N}\left(\left(\sum_{i=1}^{N} q_{ij}\right)u_j^k\right) \quad \leftarrow \text{factor out all } u_j^k$$

$$= \frac{1}{N}\sum_{j=1}^{N} 1\cdot u_j^k$$

$$= \mu$$

H33

a) 1D ODE (semidiscretisation):

$$\frac{du_i}{dt} = \frac{1}{2h^2}\left(\left(g_{i+1}+g_i\right)\left(u_{i+1}-u_i\right)-\left(g_i+g_{i-1}\right)\left(u_i-u_{i-1}\right)\right)$$

$$= \frac{1}{2h^2}\left(\left(g_{i+1}+g_i\right)u_{i+1} - \left(g_{i+1}\cancel{+g_i+g_i}-g_{i-1}\right)u_i + \left(g_i+g_{i-1}\right)u_{i-1}\right)$$

$$\frac{1}{2h^2}\left(\left(g_{i+1}+g_i\right)u_{i+1} - \left(g_{i+1}+g_{i-1}\right)u_i + \left(g_i+g_{i-1}\right)u_{i-1}\right)$$

$$= \frac{1}{2h^2}\left(\left(g_{i+1}+g_i\right)u_{i+1} - \left(\left(g_{i+1}+g_i\right) + \left(g_i+g_{i-1}\right)\right)u_i + \left(g_i+g_{i-1}\right)u_{i-1}\right)$$

thus $N(k)$ denotes neighbor of k

$$b_{k,l} := \begin{cases} -\tau\dfrac{\left(g_k+g_l\right)}{2h^2} & (l \in N(k)) \\ 1 + \dfrac{\tau}{2h^2}\displaystyle\sum_{l\in N(k)} \left(g_k+g_l\right) & (l=k) \\ 0 & \text{else} \end{cases}$$

5

b) $$b_{kk} - \sum_{\substack{l=1\\ k\neq l}}^{N} b_{kl} = 1 - \frac{\tau}{2h^2}\left(\sum_{l\in N(k)}\left(g_k+g_l\right) - \sum_{l\in N(k)}\left(g_k+g_l\right)\right)$$

$$= 1 - \frac{\tau}{2h^2}\cdot 0$$

$$= 1 > 0 \quad \text{thus} \quad b_{kk} > \sum_{\substack{l=1\\ l\neq k}}^{N} b_{kl}.$$

c) $$B(u^k)w = b_{kk} + \sum_{\substack{l=1\\ k\neq l}}^{N} b_{kl} = 1 \quad \text{thus row sum} = 1$$