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H3.1

Stencils which violetes , which de-satifies

D2 : d) | convergence. to a steady-state

D3 : a), | Average grey level Invariance

D4 : g) | Max min Principal

D5 : f) |

D6 : c) | Convergence to a steady-state

to fullfill D2 : expended 1d stencil must be equal to its inverse

D3 : sum of stencil must be 1

D4 : the center of stencil must be positive

D5 ; rows of stencil cannot be completely zero.

H3.2 a) By induction on K

Base step : $K=0$, $u^0 = f$ where $\min_{j\in J} f_j \le f_i \le \max_{j\in J} f_j$ for all f_i in f ✓

Induction step : suppose $\min_{j\in J} f_j \le u_i^k \le \max_{j\in J} f_j$ $\forall\, i\in J,\ \exists\, k\ge 0$

$$u_i^{k+1} = (Q(u^k))_i^T\, u^k$$

$$u_i^{k+1} = \sum_{h=1}^{n} (q_{ih}\, u_h^k),$$

consider u_m^k as min of u^k then we have

$$u_i^k = u_m^k + p_i$$

where $p_i \ge 0$,

$$u_i^{k+1} - u_m^k = \sum_{h=1}^{n} \left(q_{ih}(u_m^k + p_h)\right) - u_m^k$$

$$= \sum_{h=1}^{n} q_{ih}\, u_m^k + \sum_{h=1}^{n} q_{ih} p_h - u_m^k$$

$$= u_m^k - u_m^k + \sum_{h=1}^{n} q_{ih} p_h \ge 0$$

thus $u_i^{k+1} \ge u_m^k$ ✓

similarily let u_x^k be the max of u^k then we have

$u_i^k = u_x^k + g_i$ where $g_i \le 0$

$$u_i^{k+1} - u_x^k = \underbrace{\cdots}_{\text{similar steps as the min case}} = u_x^k - u_x^k + \sum_{h=1}^{n} q_{ih} g_h \le 0$$ ✓

b) the max part is still valid when row sums ≤ 1

the min part is still valid when row sums ≥ 1

c) By induction on k

base step : $\frac{1}{N} \sum_{j=1}^{N} u_j^0 = \mu$ where μ is the mean of f ✓

induction step: assume that

$$\frac{1}{N} \sum_{j=1}^{N} u_j^k = \mu ,$$

we have $$\frac{1}{N} \sum_{i=1}^{N} u_i^{k+1} = \frac{1}{N} \sum_{i=1}^{N} \left(\sum_{j=1}^{N} (q_{ij} \; u_j^k) \right)$$

$$= \frac{1}{N} \sum_{j=1}^{N} \left(\left(\sum_{i=1}^{N} q_{ij} \right) u_j^k \right)$$ ← factor out all u_j^k

$$= \frac{1}{N} \sum_{j=1}^{N} 1 \cdot u_j^k$$

$$= \mu$$ ✓

H33

a) 1D ODE (semidiscretisation):

$$\frac{du_i}{dt} = \frac{1}{2h^2}\left((g_{i+1}+g_i)(u_{i+1}-u_i)-(g_i+g_{i-1})(u_i-u_{i-1})\right)$$

$$= \frac{1}{2h^2}\left((g_{i+1}+g_i)u_{i+1} - (g_{i+1}\ \cancel{+g_i}\ \cancel{+g_i}\ +g_i+g_i+ -g_{i-1})u_i + (g_i+g_{i-1})u_{i-1}\right)$$

$$\frac{1}{2h^2}\left((g_{i+1}+g_i)u_{i+1} - (g_{i+1}+g_{i-1})u_i + (g_i+g_{i-1})u_{i-1}\right)$$

$$= \frac{1}{2h^2}\left((g_{i+1}+g_i)u_{i+1} - ((g_{i+1}+g_i) + (g_i+g_{i-1}))u_i + (g_i+g_{i-1})u_{i-1}\right)$$

thus $N(k)$ denotes neighbor of k

$$b_{k,l} := \begin{cases} \frac{\tau(g_k+g_l)}{2h^2} & (l \in N(k)) \\ 1 - \frac{\tau}{2h^2}\sum_{l\in N(k)} (g_k+g_l) & (l=k) \\ 0 & \text{else} \end{cases}$$

b)
$$b_{kk} - \sum_{\substack{l=1\\ k\neq l}}^{N} b_{kl} = 1 - \frac{\tau}{2h^2}\left(\sum_{l\in N(k)}(g_k+g_l) - \sum_{l\in N(k)}(g_k+g_l)\right)$$

$$= 1 - \frac{\tau}{2h^2}\cdot 0$$

$$= 1 > 0 \quad \text{thus} \quad b_{kk} > \sum_{\substack{l=1\\ l\neq k}}^{N} b_{kl}.$$

c)
$$B(u^k)w = b_{kk} + \sum_{\substack{l=1\\ k\neq l}}^{N} b_{kl} = 1 \quad \text{thus row sum} = 1$$