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Mathematical Image Analysis Group

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Saarland University

Classroom Work Assignment C3

Problem C3.1 (Stencil Representation versus Matrix Representation)

Assume you want to approximate a homogeneous diffusion process with the stencil

0	r	0
r	$1-4r$	r
0	r	0

where $r := \frac{\tau}{h^2}$ with time step size τ and grid size h in x- and y-direction. Furthermore, you apply the stencil to a given 4×3 image where the pixels are numbered in the following way:

u_9	u_{10}	u_{11}	u_{12}
u_5	u_6	u_7	u_8
u_1	u_2	u_3	u_4

Derive its iteration formula in the matrix-vector notation $\boldsymbol{u}^{k+1} = \boldsymbol{Q}(\boldsymbol{u}^k)\, \boldsymbol{u}^k$ with $\boldsymbol{Q} \in \mathbb{R}^{12 \times 12}$.
Assume reflecting boundary conditions.

Hint: For simplicity, start with the inner pixels u_6 and u_7. Then consider boundary and corner pixels.

Homework Assignment H3

Problem H3.1 (Stencils and Discrete Diffusion Processes) 7P

Analyse which of the following stencils, applied to an inner pixel, satisfy the requirements (D1)–(D6) of a discrete diffusion process following Lecture 5. You may ignore the boundaries in your considerations.

If some of these requirements are violated, which scale-space properties are no longer satisfied when the corresponding filter is applied iteratively?

(a)

$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{5}$
0	$\frac{1}{5}$	0
$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{5}$

(b)

0	$\frac{2}{13}$	0
$\frac{4}{13}$	$\frac{1}{13}$	$\frac{4}{13}$
0	$\frac{2}{13}$	0

(c)

0	0	0
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	0	0

(d)

$\frac{1}{4}$	$\frac{1}{8}$	0
$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
0	0	$\frac{1}{4}$

(e)

$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
0	$\frac{1}{7}$	0
$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$

(f)

0	$\frac{1}{4}$	0
$\frac{1}{4}$	0	$\frac{1}{4}$
0	$\frac{1}{4}$	0

(g)

0	$-\frac{1}{4}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	$-\frac{1}{4}$	0

Hint: Which properties does a stencil need to fulfil for the corresponding matrix to satisfy (D1)–(D6)?

Problem H3.2 (Maximum–Minimum Principle and Average Grey Level Invariance) 2+1+2P

Let $\boldsymbol{f} \in \mathbb{R}^N$ and $\boldsymbol{Q}(\boldsymbol{u}^k) \in \mathbb{R}^{N\times N}$. Consider a discrete filter

$$\begin{aligned} \boldsymbol{u}^0 &= \boldsymbol{f}, \\ \boldsymbol{u}^{k+1} &= \boldsymbol{Q}(\boldsymbol{u}^k)\,\boldsymbol{u}^k \qquad (k \geq 0). \end{aligned}$$

(a) Prove that for $J = \{1, \dots, N\}$ the maximum–minimum principle

$$\min_{j\in J} f_j \leq u_i^k \leq \max_{j\in J} f_j \qquad \forall i \in J,\ \forall k \geq 0$$

holds, if all row sums of $\boldsymbol{Q}(\boldsymbol{u}^k)$ are 1, and $q_{i,j}(\boldsymbol{u}^k) \geq 0$ for all $i, j \in J$ and all $k \geq 0$.

(b) Which part of this principle remains valid if all row sums are smaller or equal than 1?
Which part of this principle remains valid if all row sums are greater or equal than 1?

(c) Prove that the average grey value

$$\frac{1}{N}\sum_{i=1}^{N} u_i^k$$

is preserved for all $k \geq 0$, if all column sums of $\boldsymbol{Q}(\boldsymbol{u}^k)$ are 1.

Problem H3.3 (Discrete Scale-Space Properties of the Semi-implicit Scheme)

1+2+1+2P

The semi-implicit scheme for nonlinear isotropic diffusion filtering is given by

$$\boldsymbol{u}^{k+1} = \boldsymbol{Q}(\boldsymbol{u}^k)\boldsymbol{u}^k \quad \text{with} \quad \boldsymbol{Q}(\boldsymbol{u}^k) := \boldsymbol{B}^{-1}(\boldsymbol{u}^k) \quad \text{and} \quad \boldsymbol{B}(\boldsymbol{u}^k) := \boldsymbol{I} - \tau \boldsymbol{A}(\boldsymbol{u}^k)\,.$$

The goal of this assignment is to show that $\boldsymbol{Q}(\boldsymbol{u}^k)$ satisfies (D1)–(D6) for arbitrary $\tau > 0$.

(a) Write down the matrix $\boldsymbol{B}(\boldsymbol{u}^k)$ for a 1-D signal with homogeneous Neumann boundary conditions.

(b) Show that this matrix is strictly diagonally dominant for $\tau \geq 0$.
Recall: An $N \times N$ matrix $\boldsymbol{A} = (a_{i,j})$ is strictly diagonally dominant, if
$|a_{i,i}| > \sum\limits_{\substack{j=1 \\ j \neq i}}^{N} |a_{i,j}|$ for $1 \leq i \leq N$.
Is such a matrix always invertible? Justify your answer.
You can use *Gershgorin's circle theorem*:

> Let $\boldsymbol{A} = (a_{i,j}) \in \mathbb{R}^{N \times N}$ be a matrix, then the union of all discs $K_i := \left\{ \mu \in \mathbb{C} \;\middle|\; |\mu - a_{i,i}| \leq \sum\limits_{\substack{j=1 \\ j \neq i}}^{N} |a_{i,j}| \right\}$ contains all eigenvalues of $\boldsymbol{A}$.

Use your findings to argue that (D1) and (D2) are satisfied for $\boldsymbol{Q}(\boldsymbol{u}^k)$.

(c) Consider $\boldsymbol{w} = (1, ..., 1)^\top \in \mathbb{R}^N$ and compute $\boldsymbol{B}(\boldsymbol{u}^k)\boldsymbol{w}$. Deduct (D3) from this computation.

(d) Show (D4)–(D6). You can use the following theorem without proof:

> Let $\boldsymbol{A} = (a_{i,j}) \in \mathbb{R}^{N\times N}$ be a strictly diagonally dominant, irreducible matrix with $a_{i,j} \leq 0$
> for all $i \neq j$, and $a_{i,i} > 0$ for all $i = 1, ..., N$. Then all entries of $\boldsymbol{A}^{-1}$ are positive.

Problem H3.4 (Nonlinear Isotropic Diffusion) 3+3P

(a) Implement the explicit finite difference scheme for the regularised isotropic nonlinear diffusion filter.

To this end, open the file `iso_non_diff.c` and supplement it with the missing code for creating a 2D array `g` with the diffusivities at each pixel. Use the exponential Perona-Malik diffusivity

$$g\left(|\boldsymbol{\nabla} u_\sigma|^2\right) \;:=\; \exp\left(-\frac{|\boldsymbol{\nabla} u_\sigma|^2}{2\lambda^2}\right)$$

and be sure that its argument is the Gaussian-smoothed image.

(b) Compile the program with

```
gcc -Wall -O2 -o iso_non_diff.c -o iso_non_diff -lm
```

and use it for denoising the image `noisy.pgm`.
Find values for t, σ, and λ that minimise the reconstruction error as measured by the mean squared error (MSE) between the diffused image $\boldsymbol{u} \in \mathbb{R}^N$ and the noise-free original $\boldsymbol{f} \in \mathbb{R}^N$:

$$\mathrm{MSE}(\boldsymbol{u}, \boldsymbol{f}) \;:=\; \frac{1}{N}\sum_{i=1}^{N}(u_i - f_i)^2\,.$$

Submission: Please create a directory `Ex03_<your_name>` with the following files (and nothing else):

- a pdf file – which can also be a scanned handwritten solution – that contains
  - the names of all people working together for this assignment
  - the solutions of the theoretical Problems 1–3 and your optimised parameters in Problem 4
- for Problem 4: the completed file `iso_non_diff.c` and the denoised image

Compress the directory to a zip file `Ex03_<your_name>.zip`.

Submit the file via CMS.
Deadline for submission is Friday, November 17, 14:00.