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Mathematical Image Analysis Group

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Saarland University

Classroom Work Assignment C4

Problem C4.1 (Diffusion Tensor of EED)

We know that all diffusion filters from the lecture can be cast under the form

$$\partial_t u \;=\; \mathbf{div}\,\left(\boldsymbol{D}\left(\boldsymbol{J}_\rho\left(\boldsymbol{\nabla} u_\sigma\right)\right)\boldsymbol{\nabla} u\right)$$

with some diffusion tensor $\boldsymbol{D}$. Show that for EED, this simplifies to

$$\partial_t u \;=\; \mathbf{div}\,\left(g\left(\boldsymbol{\nabla} u_\sigma \boldsymbol{\nabla} u_\sigma^\top\right)\,\boldsymbol{\nabla} u\right) \tag{1}$$

To this end, proceed in two steps.

1. Determine the eigenvectors and eigenvalues of the matrix $(\boldsymbol{\nabla} u_\sigma \boldsymbol{\nabla} u_\sigma^\top)^k$ for some $k \in \mathbb{N}$.
2. Assume that g can be represented by a power series with sufficiently large radius of convergence, i.e

   $$g(x) = \sum_{k=0}^{\infty} a_k x^k \qquad \text{with } x^0 := 1\,.$$

   Exploit this representation to determine the eigenvectors and eigenvalues of $g\left(\boldsymbol{\nabla} u_\sigma \boldsymbol{\nabla} u_\sigma^\top\right)$.

What is the effect of using anisotropic diffusion (1) without presmoothing, i.e. with $\sigma = 0$?
Is the resulting filter already known from the lecture?

Homework Assignment H4

Problem H4.1 (Anisotropic Diffusion Modelling) 1+2+3P

Edge- and coherence-enhancing diffusion (ECED) is an anisotropic diffusion model that unifies enhancement of edges and coherent structures. To model and investigate such a diffusion process, perform the following steps:

(a) Construct the diffusion tensor of ECED such that its eigenvectors are the eigenvectors of the structure tensor $\boldsymbol{J}_\rho(\boldsymbol{\nabla} u_\sigma)$. The eigenvalues are given by $g(\mu_1)$ and 1, where μ_1 is the larger eigenvalue of the structure tensor and g is a diffusivity as defined in Lecture 4. State the eigendecomposition of the resulting diffusion tensor.

(b) Consider the limiting case $\rho \to 0$. How do the eigenvalues of the diffusion tensor look like in that case? Compare the resulting diffusion process to the anisotropic diffusion models discussed in the lecture. What are your findings?

(c) Now assume that $\rho > 0$ is large and that the contrast parameter λ is small. In which direction does the process smooth preferably? Which effect do you expect when applying this diffusion process to an image of a pattern with bright and dark stripes?

Problem H4.2 (Directional Splitting of Anisotropic Diffusion) 6P

Consider the four directions

$$\boldsymbol{e}_0 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \boldsymbol{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \qquad \boldsymbol{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}, \qquad \boldsymbol{e}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}.$$

Show that the splitting

$$\mathbf{div}\left(\begin{pmatrix}a & b\\ b & c\end{pmatrix}\boldsymbol{\nabla} u\right) = \sum_{i=0}^{3} \partial_{\boldsymbol{e}_i}\left(w_i\, \partial_{\boldsymbol{e}_i} u\right)$$

into 1-D diffusions is satisfied if the directional diffusivities w_0, w_1, w_2, w_3 are given by

$$w_0 = a - \delta, \qquad w_1 = \delta + b, \qquad w_2 = c - \delta, \qquad w_3 = \delta - b$$

with some free parameter δ.

Hint: Remember that the directional derivative in direction $\boldsymbol{n}$ satisfies $\partial_{\boldsymbol{n}} u = \boldsymbol{n}^\top \boldsymbol{\nabla} u$.

Problem H4.3 (δ-Stencil for Isotropic Diffusions) 2+2P

(a) Write down the stencil resulting from the δ-stencil in the homogeneous diffusion case. Compare the stencil to the stencil from Problem C2.2 . What are your findings?

(b) State the δ-stencil for isotropic nonlinear diffusion. Compare your result to the discretisation of nonlinear isotropic diffusion from Lecture 5, slide 20. For which δ is the result equivalent?

Problem H4.4 (Anisotropic Diffusion) 2+3+3P

The program `eced_ex.c` is an almost complete C code for edge- and coherence-enhancing diffusion with a Weickert diffusivity and a delta stencil. It computes the structure tensor and the diffusion tensor at staggered grid locations $\left(i+\frac{1}{2},\, j+\frac{1}{2}\right)$.

(a) Complete the code in the subroutine `weights` such that the resulting stencil implements both

- the standard discretisation,
- and the WWW family with a free parameter $\alpha \in \left[0, \frac{1}{2}\right]$ and $\beta := (1-2\alpha)\,\mathrm{sgn}(b)$.

Note that `delta` also lives on the staggered grid, and its index `[i][j]` refers to $\left(i+\frac{1}{2},\, j+\frac{1}{2}\right)$.
Compile your program with

```
gcc -Wall -O2 -o eced_ex eced_ex.c -lm.
```

(b) Run your program on the image `pruebab.pgm` using the WWW stencil with $\alpha = 0.49$, contrast parameter $\lambda = 5.5$, noise scale $\sigma = 2$, integration scale $\rho = 0$, time step size $\tau = 0.24$, and 300 iterations. It gives you an EED result. How does this result change if you use $\sigma = 0$ and $\rho = 2$ instead? Thus, is denoising with CED-like filters a good idea?

(c) Compute CED-like evolutions of the van Gogh painting `cypress.pgm` with $\lambda = 1$, $\sigma = 1$, $\rho = 4$, $\tau = 0.2$, and 1000 iterations. Use both the standard discretisation and the WWW stencil with $\alpha = 0.49$. Which stencil do you prefer? Justify your reasoning.

Submission: Please create a directory `Ex04_<your_name>` with the following files (and nothing else):

- a pdf file – which can also be a scanned handwritten solution – that contains
  - the names of all people working together for this assignment
  - the solutions of the theoretical Problems 1–3 and answers to the questions in Problem 4
- for Problem 4: the completed file `eced_ex.c` and the four requested images

Compress the directory to a zip file `Ex04_<your_name>.zip`.

Submit the file via CMS.
Deadline for submission is Friday, November 24, 14:00.