Dr. Pascal Peter
Mathematical Image Analysis Group

WS 2023
Saarland University

Classroom Work Assignment C8

Problem C8.1 (Eigenvalues and Eigenvectors of Nagel's Method)

(a) Let $\boldsymbol{v} \in \mathbb{R}^n$ be a vector with $\boldsymbol{v} \neq \boldsymbol{0}$.
Determine the eigenvectors and eigenvalues as well as the rank of the quadratic matrix $\boldsymbol{v}\boldsymbol{v}^\top$.

(b) Nagel's optic flow method uses the regulariser

$$S_{AI}(\boldsymbol{\nabla} f, \boldsymbol{\nabla u}) \;:=\; \sum_{i=1}^{2} \boldsymbol{\nabla} u_i^\top \, \boldsymbol{D}(\boldsymbol{\nabla} f)\, \boldsymbol{\nabla} u_i$$

where $\boldsymbol{D}(\boldsymbol{\nabla} f)$ is a regularised projection matrix on $\boldsymbol{\nabla} f^\perp := (f_{x_2}, -f_{x_1})^\top$:

$$\boldsymbol{D}(\boldsymbol{\nabla} f) \;:=\; \frac{1}{|\boldsymbol{\nabla} f|^2 + 2\lambda^2} \left(\boldsymbol{\nabla} f^\perp \boldsymbol{\nabla} f^{\perp\top} + \lambda^2 \boldsymbol{I} \right).$$

Show that $\boldsymbol{D}$ has the eigenvectors $\boldsymbol{v}_1 = \boldsymbol{\nabla} f$ and $\boldsymbol{v}_2 = \boldsymbol{\nabla} f^\perp$ with corresponding eigenvalues

$$\lambda_1(|\boldsymbol{\nabla} f|) \;=\; \frac{\lambda^2}{|\boldsymbol{\nabla} f|^2 + 2\lambda^2}\,, \qquad \lambda_2(|\boldsymbol{\nabla} f|) \;=\; \frac{|\boldsymbol{\nabla} f|^2 + \lambda^2}{|\boldsymbol{\nabla} f|^2 + 2\lambda^2}\,.$$

Problem C8.2 (Photometric Invariants)

Consider the HSV colour space. Show that

- the hue channel H is invariant under local additive and multiplicative illumination changes.
- the saturation channel S is invariant under local multiplicative illumination changes.
- the value channel V is not invariant under local additive or multiplicative illumination changes.

Homework Assignment H8

Problem H8.1 (Anisotropic Image-Driven Regularisation with Robustified Data Term) 5P

Show that the steepest descent evolution for the energy functional

$$E(\boldsymbol{u}) \;=\; \frac{1}{2}\int_\Omega \Big(\Psi\big(\boldsymbol{u}^\top \boldsymbol{J}\boldsymbol{u}\big) + \alpha \sum_{i=1}^{2} \boldsymbol{\nabla} u_i^\top \boldsymbol{D}\,\boldsymbol{\nabla} u_i\Big)\, d\boldsymbol{x}$$

with $\boldsymbol{J} = (J_{i,k}) \in \mathbb{R}^{3\times 3}$, $\boldsymbol{D} := \boldsymbol{D}(\boldsymbol{\nabla} f)$ and speed factor $\gamma = \frac{1}{\alpha}$ is given by the diffusion–reaction equations

$$\begin{aligned}
\partial_t u_1 \;&=\; \textbf{div}\,(\boldsymbol{D}\,\boldsymbol{\nabla} u_1) \;-\; \frac{1}{\alpha}\,\Psi'\big(\boldsymbol{u}^\top \boldsymbol{J}\boldsymbol{u}\big)\,\big(J_{1,1}\,u_1 + J_{1,2}\,u_2 + J_{1,3}\big),\\
\partial_t u_2 \;&=\; \textbf{div}\,(\boldsymbol{D}\,\boldsymbol{\nabla} u_2) \;-\; \frac{1}{\alpha}\,\Psi'\big(\boldsymbol{u}^\top \boldsymbol{J}\boldsymbol{u}\big)\,\big(J_{1,2}\,u_1 + J_{2,2}\,u_2 + J_{2,3}\big).
\end{aligned}$$

Problem H8.2 (Flow-Driven Anisotropic Regularisation) 7P

Show that the steepest descent diffusion-reaction system for the energy functional

$$E(\boldsymbol{u}) = \int_\Omega \Big(\big(f_{x_1}u_1 + f_{x_2}u_2 + f_{x_3}\big)^2 + \alpha\;\mathrm{tr}\;\Psi\Big(\sum_{i=1}^{2} \boldsymbol{\nabla} u_i \boldsymbol{\nabla} u_i^\top\Big)\Big)\, d\boldsymbol{x}$$

is given by

$$\begin{aligned}
\partial_t u_1 \;&=\; \textbf{div}\,(\boldsymbol{D}\,\boldsymbol{\nabla} u_1) - \frac{1}{\alpha}\, f_{x_1}\big(f_{x_1}u_1 + f_{x_2}u_2 + f_{x_3}\big),\\
\partial_t u_2 \;&=\; \textbf{div}\,(\boldsymbol{D}\,\boldsymbol{\nabla} u_2) - \frac{1}{\alpha}\, f_{x_2}\big(f_{x_1}u_1 + f_{x_2}u_2 + f_{x_3}\big),
\end{aligned}$$

with $\boldsymbol{D} := \boldsymbol{D}(\boldsymbol{\nabla}\boldsymbol{u}) = \Psi'\left(\boldsymbol{\nabla} u_1 \boldsymbol{\nabla} u_1^\top + \boldsymbol{\nabla} u_2 \boldsymbol{\nabla} u_2^\top\right)$.

Hint: You may use the following identities without proof:

$$\begin{aligned}
\partial_{a_i}\big(\boldsymbol{a}\boldsymbol{a}^\top\big) \;&=\; \boldsymbol{e}_i\,\boldsymbol{a}^\top + \boldsymbol{a}\,\boldsymbol{e}_i^\top, && (1)\\
\mathrm{tr}\big(\boldsymbol{a}\boldsymbol{b}^\top\big) \;&=\; \boldsymbol{a}^\top\boldsymbol{b}, && (2)\\
\mathrm{div}(\boldsymbol{a}) \;&=\; \sum_i \partial_{x_i}\big(\boldsymbol{e}_i^\top \boldsymbol{a}\big), && (3)
\end{aligned}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors, and $\boldsymbol{e}_i$ is the unit vector in direction x_i.

Moreover, remember that

$$\Psi'\Big(\sum_i \boldsymbol{\nabla} u_i \boldsymbol{\nabla} u_i^\top\Big) \;=\; \sum_i \Psi'(\mu_i)\, \boldsymbol{v}_i \boldsymbol{v}_i^\top, \tag{4}$$

where $\boldsymbol{v}_i$ denotes the i-th unit eigenvector of $\sum_i \boldsymbol{\nabla} u_i \boldsymbol{\nabla} u_i^\top$, and μ_i its corresponding eigenvalue.

Problem H8.3 (Design of Global Optic Flow Methods) 6P

Design an energy functional based on spatio-temporal regularisation that allows an accurate computation of the optic flow for rotational motion under local additive illumination changes in a colour image sequence. Use a smoothness term that is based on isotropic flow-driven regularisation. Justify all your modelling choices and pay attention to specify correctly the domain of integration as well as differential operators and integration variables. How do the energy functional and the Euler–Lagrange equations look like?

Problem H8.4 (Motion Tensors) 6P

Derive the motion tensors for the following constancy assumptions:

(a) constancy of the squared gradient magnitude for a RGB colour image

(b) constancy of the Hessian for a greyscale image

(c) constancy of the hue channel of an image in the HSV colour space

Is there a specific angle where you would expect problems in (c)?
Make sure to circumvent these problems.

State the entries of the motions tensors explicitly.

Submission: Please submit solutions to the Problems 1–4 as a pdf file `Ex08_<your_name(s)>.pdf` which can also be a scanned handwritten solution.

Submit the file via CMS.
Deadline for submission is Friday, December 22, 14:00.