Homework Assignment H2

7055053 Honglu Ma

7057573 Camilo Martínez Martínez

DIC H_2.

a) by useing taylor expansion we get

$$u_{i-2} = \frac{(-2h)^0}{0!}u_i + \frac{(-2h)^1}{1!}u'_i + \frac{(-2h)^2}{2!}u''_i + \frac{(-2h)^3}{3!}u'''_i + \frac{(-2h)^4}{4!}u''''_i$$

$$= u_i - 2h u'_i + 2h^2 u''_i - \frac{8}{6}h^3 u'''_i + \frac{16}{24}h^4 u''''_i$$

$$= u_i - 2h u'_i + 2h^2 u''_i - \frac{4h^3}{3}u'''_i + \frac{2h^4}{3}u''''_i + \cdots$$

$$u_{i-1} = u_i - h u'_i + \frac{h^2}{2}u''_i - \frac{h^3}{6}u'''_i + \frac{h^4}{24}u''''_i + \cdots$$

$$u_i = u_i$$

$$u_{i+1} = u_i + h u'_i + \frac{h^2}{2}u''_i + \frac{h^3}{6}u'''_i + \frac{h^4}{24}u''''_i + \cdots$$

$$u_{i+2} = u_i + 2h u'_i + 2h^2 u''_i + \frac{4}{3}h^3 u'''_i + \frac{2h^4}{3}u''''_i + \cdots$$

assume that $u''''_i \approx a u_{i-2} + b u_{i-1} + c u_i + d u_{i+1} + e u_{i+2}$

$$0\cdot u_i + 0\cdot u'_i + 0\cdot u''_i + 0\cdot u'''_i + 1\cdot u''''_i =$$

$$a\left(u_i - 2h u'_i + 2h^2 u''_i - \frac{4}{3}h^3 u'''_i + \frac{2h^4}{3}u''''_i\right)$$

$$+ b\left(u_i - h u'_i + \frac{h^2}{2}u''_i - \frac{h^3}{6}u'''_i + \frac{h^4}{24}u''''_i\right)$$

$$+ c\, u_i$$

$$+ e\left(u_i + 2h u'_i + 2h^2 u''_i + \frac{4h^3}{3}u'''_i + \frac{2h^4}{3}u''''_i\right)$$

$$+ d\left(u_i + h u'_i + \frac{h^2}{2}u''_i + \frac{h^3}{6}u'''_i + \frac{h^4}{24}u''''_i\right)$$

$$= (a+b+c+d+e)\, u_i$$

$$+ h(-2a - b + 2e + d)\, u'_i$$

$$h^2\left(2a + \frac{b}{2} + 2e + \frac{d}{2}\right) u''_i$$

$$h^3\left(-\frac{4}{3}a - \frac{1}{6}b + \frac{4}{3}e + \frac{1}{6}d\right) u'''_i$$

$$h^4\left(\frac{2}{3}a + \frac{1}{24}b + \frac{2}{3}e + \frac{1}{24}d\right) u''''_i$$

the linear system is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & 2 \\ -\frac{4}{3} & -\frac{1}{6} & 0 & \frac{1}{6} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{24} & 0 & \frac{1}{24} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{h^4} \end{pmatrix}$$

b).

to simplify notation u_i means u_i^k

$$u_{i-2}+u_{i+2} = 2\left(u_i + 2h^2 u_i'' + \frac{2}{3}h^4 u_i''''\right) + O(h^5)$$

$$-4(u_{i-1}+u_{i+1}) = 2\left(u_i + \frac{h^2}{2}u_i'' + \frac{h^4}{24}u_i''''\right)\cdot(-4) + O(h^5)$$

so. $\dfrac{(u_{i-2}+u_{i+2}) - 4(u_{i-1}+u_{i+1}) + 6u_i}{h^4}$

$$= \frac{1}{h^4}\Big((-6+6)\,u_i$$
$$+ (4-4)\,h^2 u_i''$$
$$+ \left(\frac{4}{3} - \frac{1}{3}\right) h^4 u_i'''' + O(h^5)\Big)$$

$$= \frac{1}{h^4}\left(h^4 u_i'''' + O(h^5)\right)$$

$$= u_i'''' + O(h)$$

consistency order of 1.

c) $\partial_t u = -\partial_{xxxx} u$

$$\frac{u_i^{k+1} - u_i^k}{T} = \frac{u_{i-2}^k - 4u_{i-1}^k + 6u_i^k - 4u_{i+1}^k + u_{i+2}^k}{h^4}$$

$$u_i^{k+1} = \frac{T}{h^4}\left(u_{i-2}^k - 4u_{i-1}^k + 6u_i^k - 4u_{i+1}^k + u_{i+2}^k\right) + u_i^k$$

$$= \frac{T}{h^4}u_{i-2}^k + \left(-\frac{4T}{h^4}\right)u_{i-1}^k + \left(\frac{6T}{h^4}+1\right)u_i^k + \left(\frac{-4T}{h^4}\right)u_{i+1}^k + \frac{T}{h^4}u_{i+2}^k$$

d) stencil:

$\frac{T}{h^4}$	$-\frac{4T}{h^4}$	$\frac{6T}{h^4}+1$	$\frac{-4T}{h^4}$	$\frac{T}{h^4}$

Pixel

0	255	0	255	0

will give a negative value.!

thus it ~~is no~~ does not satify max-min principal

Problem H2.2

With $\alpha > 0$, $u(x,t)$ being a greyscale image representation of $f: \mathbb{R}^n \to \mathbb{R}$ in n dimensions solves:

$$\partial_t u = -(-\Delta)^\alpha u, \ \forall t > 0$$

$$u(x,0) = f(x)$$

In the Fourier domain:

$$\mathcal{F}[\partial_t u](\nu) = \mathcal{F}[-(-\Delta)^\alpha u](\nu) \to \text{Linearity of Fourier domain} \to$$

$$\to \partial_t \mathcal{F}[u](\nu) = -\mathcal{F}[(-\Delta)^\alpha u](\nu) \overset{\text{def}}{=} -(2\pi|\nu|)^{2\alpha} \mathcal{F}[u](\nu)$$

Let $\mathcal{F}[u](\nu) := \hat{u}(\nu,t)$, then the resulting equation becomes:

$$\boxed{\partial_t \hat{u}(\nu,t) = -(2\pi|\nu|)^{2\alpha} \hat{u}(\nu,t)}$$ Evolution in the Fourier domain

If $g = \hat{u}(\nu,t)$, then this equation is of the form $\frac{\partial}{\partial t} g = -(2\pi|\nu|)^{2\alpha} g$
Which can be solved like:

$$\partial g = -(2\pi|\nu|)^{2\alpha} g \, \partial t \to \frac{1}{g}\partial g = -(2\pi|\nu|)^{2\alpha} \partial t \to \int \frac{1}{g}\partial g = \int -(2\pi|\nu|)^{2\alpha} \partial t$$

$$\to \ln(g) + c = -(2\pi|\nu|)^{2\alpha} t \to e^{\ln(g)+c} = e^{-(2\pi|\nu|)^{2\alpha} t} \to$$

$$\to e^c g = K g = \exp(-(2\pi|\nu|)^{2\alpha} t), \text{ where } c, K \in \mathbb{R} \text{ constants}$$

Then replacing back:

$$\boxed{K \hat{u}(\nu,t) = \exp(-(2\pi|\nu|)^{2\alpha} t)}$$

From the initial conditions: $u(x,0) = f(x) \Rightarrow \hat{u}(\nu,0) = \hat{f}(\nu)$

So

$$K \hat{u}(\nu,0) = \exp(-(2\pi|\nu|)^{2\alpha} \cdot 0) = e^0 = 1$$

$$\to \boxed{K = \frac{1}{\hat{u}(\nu,0)} = \frac{1}{\hat{f}(\nu)}}$$

So the solution becomes:

$$\frac{1}{\hat{f}(\nu)} \hat{u}(\nu,t) = \exp(-(2\pi|\nu|)^{2\alpha} t) \to$$

$$\to \boxed{\hat{u}(\nu,t) = \hat{f}(\nu) \exp(-(2\pi|\nu|)^{2\alpha} t)}$$

a) *Local contrast enhancement condition will be marked with ☆

3 1)

$$g(s^2) = \frac{1}{\sqrt{1+\frac{s^2}{\lambda^2}}} \qquad \Phi(s) = \frac{s}{\sqrt{1+\frac{s^2}{\lambda^2}}} = s\left(1+\frac{s^2}{\lambda^2}\right)^{-\frac{1}{2}}$$

$$\Phi'(s) = \left(1+\frac{s^2}{\lambda^2}\right)^{-\frac{1}{2}} - \frac{1}{\cancel{2}}\left(1+\frac{s^2}{\lambda^2}\right)^{-\frac{3}{2}} \cdot \left(\frac{\cancel{2}s}{\lambda^2}\right) \cdot s$$

$$= \left(1+\frac{s^2}{\lambda^2}\right)^{-\frac{1}{2}} - \frac{s^2}{\lambda^2}\left(1+\frac{s^2}{\lambda^2}\right)^{-\frac{3}{2}}$$

$$= \left(1+\frac{s^2}{\lambda^2}\right)^{-\frac{3}{2}}\left(1+\frac{s^2}{\lambda^2}-\frac{s^2}{\lambda^2}\right)$$

$$= \left(1+\frac{s^2}{\lambda^2}\right)^{-\frac{3}{2}}$$

As $\Phi'(s) > 0$, for $s > 0$

there is only forward diffusion,

no local contrast enhancement ☆

2) $$g(s^2) = \frac{1}{1+\frac{s^2}{\lambda^2}} \qquad \Phi(s) = s\left(1+\frac{s^2}{\lambda^2}\right)^{-1}$$

$$\Phi'(s) = \left(1+\frac{s^2}{\lambda^2}\right)^{-1} - \left(1+\frac{s^2}{\lambda^2}\right)^{-2} s \cdot \left(\frac{2s}{\lambda^2}\right)$$

$$= \left(1+\frac{s^2}{\lambda^2}\right)^{-2}\left(1+\frac{s^2}{\lambda^2}-\frac{2s^2}{\lambda^2}\right)$$

$$= \left(1+\frac{s^2}{\lambda^2}\right)^{-2}\left(1-\frac{s^2}{\lambda^2}\right)$$

$$= \frac{\left(1-\frac{s^2}{\lambda^2}\right)}{\left(1+\frac{s^2}{\lambda^2}\right)^2}$$

$$\Phi'(s) > 0 \Rightarrow 1-\frac{s^2}{\lambda^2} > 0, \quad s>0$$

$$\Rightarrow -\frac{s^2}{\lambda^2} > -1, \quad s>0$$

$$\Rightarrow s^2 < \lambda^2, \quad s>0$$

$$\Rightarrow s < |\lambda|, \quad s>0$$

$\Phi'(s) > 0$ when $s < |\lambda|$ (forward diffusion)

$\Phi'(s) < 0$ when $s > |\lambda|$ ☆ (backward diffusion)

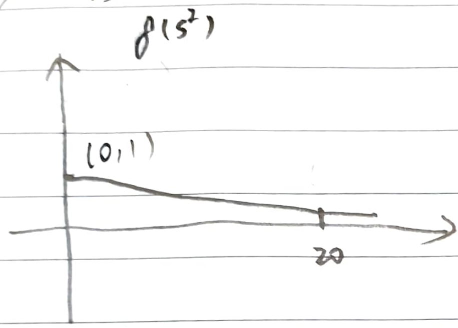
because $\frac{\partial u}{\partial t} = \Phi'(s) \cdot u_{xx}$ and $\Phi' = 0$
so the image remain unchanged.
no local contrast enhancement

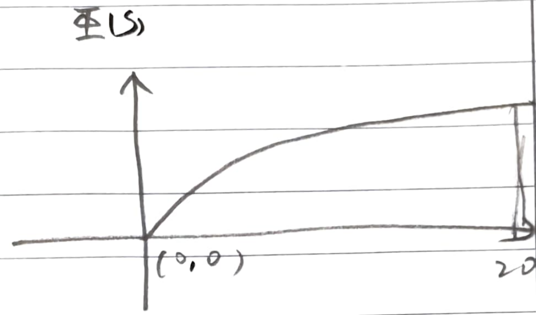
3) $g(s^2) = \frac{1}{|s|}$ $\quad \Phi(s) = 1 \quad \Phi'(s) = 0$

4) $g(s^2) = \frac{1}{s^2}$ $\quad \Phi(s) = \frac{s}{s^2} = \frac{1}{s} \quad \Phi'(s) = -s^{-2}$
$= -\frac{1}{s^2}$

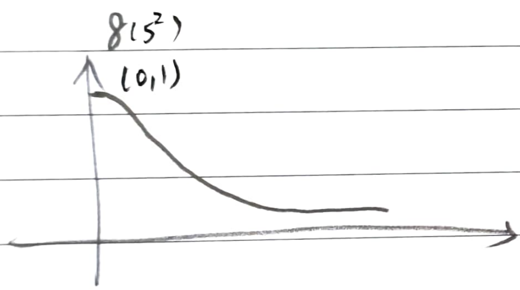
$\Phi'(s) < 0$ for $s > 0$ ☆ (backward diffusion)

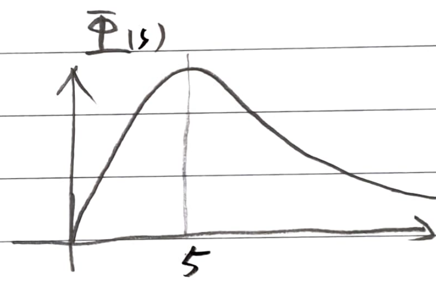
b) 1)



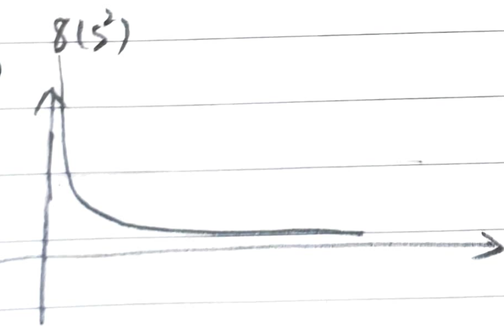


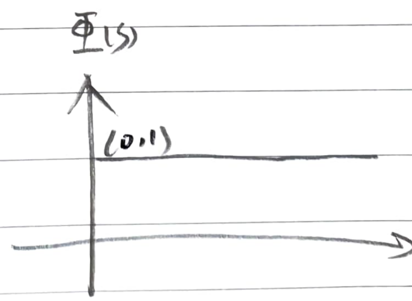
2)



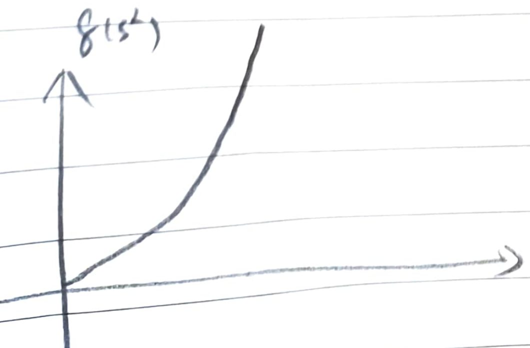


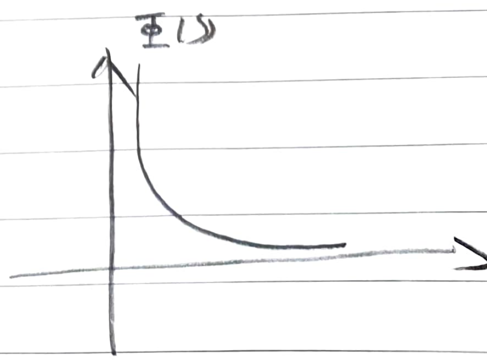
3)





4)





Problem H2.4:

(a) *Code in zip file.*

(b) With *klein-data* and *klein-mask*, we obtain the following results (iteration number: 16310):

initial image minimum: 0.00 maximum: 255.00 mean: 5.65 standard dev.: 27.32	Iteration number: 50000 minimum: 0.00 maximum: 255.00 mean: 117.22 (↑) standard dev.: 53.84 (↑)

With *klein-data2* and *klein-mask*, we obtain the following results (iteration number: 14100):

initial image minimum: 0.00 maximum: 255.00 mean: 126.19 standard dev.: 72.64	Iteration number: 50000 minimum: 0.00 maximum: 255.00 mean: 117.22 (↓) standard dev.: 53.83 (↓)

We can see there is no difference whatsoever between the 2 versions. This should be expected because we're using the mask to filter out the pixel positions that should not change, and the mask is the same in both examples. Those pixels that do have to change consider the neighboring pixels to obtain their value. As the diffusion process iterates, the less the values that the image started with take importance. But it's worth noting that the first one (black pixels initialization) converged to the final values at $\approx$16,000 iterations, while the second one (random pixel values initialization) did it at $\approx$14,000. This is probably due to the fact that the random values initialization has a probability that the random values chosen for a pixel is already close to its original value and thus the iteration process converges faster.

(c) We found no difference in the 2 reconstructions. That is, the maximum and the average difference were 0.0. On the other hand, the difference between the original image *klein.pgm* and the reconstruction was:

a. Average absolute difference: 2.6484
b. Maximum absolute difference: 17

The reconstruction looks like a smoothed-out version of the original image, which makes sense given the process of homogeneous diffusion applied to it. It looks quite like the original image to the naked eye, but it certainly has differences as noted by the *reconstruction_error.pgm*.