Dr. Pascal Peter — Mathematical Image Analysis Group
WS 2023 — Saarland University

Classroom Work Assignment C13

Problem C13.1 (Mean and Median Filtering)

Let $\boldsymbol{f} = (f_i)_{i=1}^{N}$ be a discrete image.

(a) Show that the arithmetic mean

$$u = \frac{1}{N} \sum_{i=1}^{N} f_i$$

minimises the distance in the 2-norm:

$$E(u) = \left(\sum_{i=1}^{N} |u - f_i|^2 \right)^{\frac{1}{2}}$$

(b) Show that the median minimises the distance in the 1-norm:

$$E(u) = \sum_{i=1}^{N} |u - f_i|$$

You can assume that the image $\boldsymbol{f}$ has odd length N and the grey values of its entries are in increasing order.

Homework Assignment H13

Problem H13.1 (Self-snakes) 5+4+1P

Consider a self-snake evolution with $\sigma = 0$ and a decreasing positive edge-stopping function g:

$$\partial_t u \;=\; |\boldsymbol{\nabla} u| \, \textbf{div} \left(g\big(|\boldsymbol{\nabla} u|^2\big) \, \frac{\boldsymbol{\nabla} u}{|\boldsymbol{\nabla} u|} \right).$$

(a) Show that it can be written as

$$\partial_t u \;=\; g\big(|\boldsymbol{\nabla} u|^2\big)\, u_{\xi\xi} \;+\; 2g'\big(|\boldsymbol{\nabla} u|^2\big)\, |\boldsymbol{\nabla} u|^2 \, u_{\eta\eta}$$

with unit vectors $\boldsymbol{\xi} \perp \boldsymbol{\nabla} u$ and $\boldsymbol{\eta} \parallel \boldsymbol{\nabla} u$.

(b) What is the sign of the factors in front of $u_{\xi\xi}$ and $u_{\eta\eta}$?
Is this PDE of forward or backward parabolic type?
Does the basic property of edge enhancement depend on a contrast parameter in the function g?
State one reason why this PDE can be expected to satisfy an extremum principle.

(c) Rewrite the last term of the evolution in (a) such that it can be interpreted as a shock filter.

Problem H13.2 (Minmod Scheme for SILD) 1+1+3+3P

The goal of this assignment is to gain some basic insights into the minmod scheme for stabilised inverse linear diffusion (SILD) in the 1-D case.

(a) Write down the minmod scheme for SILD in 1D.

(b) What is its behaviour in signal extrema?

(c) Now assume that a signal $\boldsymbol{u}^k = (u_i^k)$ is monotonically increasing and concave.
Simplify the minmod scheme in this case.
Prove that for a suitable time step size restriction, this scheme is monotonicity preserving, i.e.

$$u_{i+1}^k - u_i^k \ge 0 \quad \text{for all } i \quad \Longrightarrow \quad u_{i+1}^{k+1} - u_i^{k+1} \ge 0 \quad \text{for all } i.$$

You can disregard boundary conditions by assuming that the signal is infinitely extended.
Moreover, you can use the fact that a concave signal $\boldsymbol{u}^k$ satisfies $u_{i+2}^k - u_{i+1}^k \leq u_{i+1}^k - u_i^k$.

(d) Show similar results to (c) also for signals that are
- monotonically decreasing and concave,
- monotonically increasing and convex,
- monotonically decreasing and convex.

(These properties are essential for proving the stability of the minmod scheme in the maximum norm, and for showing that it preserves the discrete total variation $\mathrm{TV}(\boldsymbol{u}^k) = \frac{1}{h} \sum_{i=1}^{N-1} |u_{i+1}^k - u_i^k|$ *.)*

Problem H13.3 (Mode Filter Evolution) 3+2+1P

The program `m_smoother.c` is an almost complete implementation of the family of PDEs that arise as scaling limits of M-smoothers with order-p means.

(a) Supplement the missing code and compile it.

(b) Use it to compute the mode filter evolution of the image `penguin.pgm` for the times 100, 500, 2500, 5000, and 6500. Use the diagonal stencil weight $\delta = \sqrt{2} - 1$ and the time step size $\tau = 0.1$, and be prepared that this will take some time.

(c) Which old philosophical problem is solved by your experiments :-) ?

Submission: Please create a directory Ex13_<your_name> with the following files (and nothing else):

- a pdf file – which can also be a scanned handwritten solution – that contains
  - the names of all people working together for this assignment
  - the solutions of the theoretical Problems 1–2 and answers to the questions in Problem 3
- for Problem 3(a): the completed file mode_filter_complete.c,
- for Problem 3(b): the resulting images penguin100.pgm, penguin500.pgm, penguin2500.pgm, penguin5000.pgm, and penguin6500.pgm

Compress the directory to a zip file Ex13_<your_name>.zip.

Submit the file via CMS.
Deadline for submission is Friday, February 2, 14:00.