Differential Equations in Image Processing and Computer Vision

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Classroom Work Assignment C5

Problem C5.1 (Convex Functionals and Forward Diffusion)

In Lecture 10 we have learned that minimising the 1-D energy functional

$$E(u) \;=\; \int_a^b \Big((u-f)^2 + \alpha\Psi(u_x^2)\Big)\,\mathrm{d}x$$

can be seen as an implicit time discretisation of the diffusion process $\partial_t u \;=\; \partial_x(\Psi'(u_x^2)\,u_x)$.
Assume that the regulariser Ψ is twice differentiable and convex in s, which implies that $\frac{d^2}{ds^2}\Psi(s^2) \geq 0$.
Show that the corresponding diffusion process always performs forward diffusion. Thus, it cannot act contrast-enhancing.

Homework Assignment H5

Problem H5.1 (Parameter Adaptation Under Rescaling) 3+1P

Assume you have a C^1 image on a quadratic domain Ω of size 128×128, and its grey values are in the interval $[0, 255]$. The spatial grid size is $h_1 = h_2 = 1$. Filtering it with an explicit scheme for EED with parameters $\lambda = 4, \sigma = 1$, $T = 248$ and $\tau = 0.2$ takes 12.4 seconds on your computer.

(a) How would you adapt the parameters σ, λ, and T, when your image is rescaled to 256×256 pixels, and its grey value range to $[0, 1]$? The spatial grid size remains 1.

(b) Estimate the required times for computing the result with the explicit scheme.

Problem H5.2 (Continuous Variational Regularisation) 5+1+3P

Consider the nonconvex 2-D energy functional

$$E(u) = \frac{1}{2} \int_{\Omega} \left((u - f)^2 - 2\alpha\lambda^2 \exp\left(-\frac{|\boldsymbol{\nabla} u|^2}{2\lambda^2} \right) \right) \mathrm{d}\boldsymbol{x}.$$

(a) State the Euler-Lagrange equation as well as the corresponding boundary condition of the above functional. Which diffusion filter is approximated by the energy functional?

(b) What can you say about the uniqueness of the minimiser?

(c) Determine the diffusivity and show that λ is the boundary between forward and backward diffusion.

Problem H5.3 (Energy Functional for Nonlinear Diffusion) 3P

Consider the 2-D nonlinear isotropic diffusion process with rational Perona-Malik diffusivity

$$\partial_t u = \mathbf{div} \left(\frac{1}{1 + |\boldsymbol{\nabla} u|^2 / \lambda^2} \boldsymbol{\nabla} u \right).$$

Compute the energy functional that corresponds to this diffusion process.

Problem H5.4 (Decorrelation Criterion) 1+2+5P

The file `charb_decorr.c` is similar to `iso_non_diff.c` from Assignment H3 and implements isotropic nonlinear diffusion filtering, but with the Charbonnier diffusivity (instead of a Perona-Malik diffusivity) and without Gaussian smoothing. However, this time we want to incorporate and test the decorrelation criterion that frees us from the need to specify the number of iterations.

(a) Find arguments that support the following claim: For small $\lambda > 0$, the Charbonnier diffusion

$$\partial_t u \;=\; \mathbf{div}\left(\frac{1}{\sqrt{1+|\boldsymbol{\nabla} u|^2/\lambda^2}}\,\boldsymbol{\nabla} u\right)$$

at time t approximates TV flow at time λt.

(b) Supplement the file `charb_decorr.c` with the missing code such that it incorporates the decorrelation criterion from Lecture 9. Compile it with `gcc -Wall -O2 -o charb_decorr charb_decorr.c -lm`

(c) The images `acros.pgm` and `peppers-noise.pgm` have been degraded by additive Gaussian noise with mean 0. Denoise them with `charb_decorr` with time step size $\tau = 0.2$. Use the correlation coefficient to estimate also the best contrast parameter λ. What are the corresponding λ values? Why is this heuristics justified?

Submission: Please create a directory `Ex05_<your_name>` with the following files (and nothing else):

- a pdf file – which can also be a scanned handwritten solution – that contains
  - the names of all people working together for this assignment
  - the solutions of the theoretical Problems 1–3 and answers to the questions in Problem 4
- for Problem 4: the completed file `charb_decorr.c` and the two resulting images with optimised λ

Compress the directory to a zip file `Ex05_<your_name>.zip`.

Submit the file via CMS.
Deadline for submission is Friday, December 1, 14:00.