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Mathematical Image Analysis Group — Saarland University

Classroom Work Assignment C6

Problem C6.1 (Gradient Domain Methods)

Some applications such as seamless image cloning benefit from manipulating the gradient of an image rather than its grey values: Discontinuities in the gradient data appear as continuous transitions in the image data. Since the manipulated vector field $\boldsymbol{p}$ is not necessarily a gradient vector field anymore, it cannot be integrated exactly.
As a remedy, one searches for a function u whose gradient approximates $\boldsymbol{p}$. It is found by minimising the energy

$$E(u) = \int_\Omega |\boldsymbol{\nabla} u - \boldsymbol{p}|^2 \, d\boldsymbol{x}.$$

(a) Give an example of a non-integrable vector field i.e. a vector field that is not the gradient of some function.

(b) Write down the Euler-Lagrange equation of the above energy functional. Also state the corresponding boundary condition.

Problem C6.2 (Forward-Backward Splitting)

In the forward backward splitting (FBS) method, the primal variable $\boldsymbol{u}$ and the dual variable $\boldsymbol{b}$ are obtained by the following iteration:

$$\begin{aligned} \boldsymbol{b}^{k+1} &= \underset{\boldsymbol{b}\in\mathbb{R}^{2N}}{\operatorname{argmax}} \left\{ -\iota_{C_\alpha}(\boldsymbol{b}) + \left\langle \boldsymbol{b}, \boldsymbol{\mathcal{D}}\boldsymbol{u}^k \right\rangle - \frac{1}{2\tau} \left\| \boldsymbol{b} - \boldsymbol{b}^k \right\|_2^2 \right\}, \\ \boldsymbol{u}^{k+1} &= \underset{\boldsymbol{u}\in\mathbb{R}^{N}}{\operatorname{argmin}} \left\{ \frac{1}{2} \left\| \boldsymbol{u} - \boldsymbol{f} \right\|_2^2 + \left\langle \boldsymbol{\mathcal{D}}^\top \boldsymbol{b}^{k+1}, \boldsymbol{u} \right\rangle \right\}. \end{aligned} \tag{1}$$

Show that FBS can be written as

$$\boldsymbol{b}^{k+1} = \underset{\boldsymbol{b}\in\mathbb{R}^{2N}}{\operatorname{argmin}} \left\{ \iota_{C_\alpha}(\boldsymbol{b}) + \frac{1}{2} \left\| \boldsymbol{b} - (\boldsymbol{b}^k - \tau(\boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{D}}^\top \boldsymbol{b}^k - \boldsymbol{f}))) \right\|_2^2 \right\}.$$

You can use the fact that $\boldsymbol{u}^{k+1} = \boldsymbol{f} - \boldsymbol{\mathcal{D}}^\top \boldsymbol{b}^{k+1}$ minimises Equation (1) without proof.

Homework Assignment H6

Problem H6.1 (Stability of Diffusion-Reaction Discretisations)

2+2+1+3P

Consider the diffusion-reaction equation

$$\frac{\partial u}{\partial t} = \mathbf{div}\left(g(|\boldsymbol{\nabla} u|^2)\boldsymbol{\nabla} u\right) - \frac{u-f}{\alpha} \qquad \text{with } \alpha > 0\,.$$

In this assignment, you can examine stability criteria for a discretisation with the *modified explicit scheme*

$$\frac{\boldsymbol{u}^{k+1}-\boldsymbol{u}^{k}}{\tau} = \boldsymbol{A}^{k}(\boldsymbol{u}^{k})\,\boldsymbol{u}^{k} - \frac{1}{\alpha}(\boldsymbol{u}^{k+1}-\boldsymbol{f}) \tag{2}$$

and the *fully explicit scheme*

$$\frac{\boldsymbol{u}^{k+1}-\boldsymbol{u}^{k}}{\tau} = \boldsymbol{A}^{k}(\boldsymbol{u}^{k})\,\boldsymbol{u}^{k} - \frac{1}{\alpha}(\boldsymbol{u}^{k}-\boldsymbol{f})\,. \tag{3}$$

(a) Show that the solution $\boldsymbol{u}^{k+1}$ of the modified explicit scheme (2) can be computed as

$$\boldsymbol{u}^{k+1} = \frac{\alpha\boldsymbol{v}^{k+1}+\tau\boldsymbol{f}}{\alpha+\tau}$$

where $\boldsymbol{v}^{k+1}$ denotes the solution of the explicit diffusion scheme without reaction term:

$$\frac{\boldsymbol{v}^{k+1}-\boldsymbol{u}^{k}}{\tau} = \boldsymbol{A}^{k}(\boldsymbol{u}^{k})\,\boldsymbol{u}^{k}\,.$$

(b) Show that the result from (a) implies stability of (2) in terms of the discrete maximum-minimum principle

$$\min_j f_j \;\le\; u_i^{k+1} \;\le\; \max_j f_j$$

for all i and all $k \ge 0$ if we use the initialisation $\boldsymbol{u}^0 := \boldsymbol{f}$ and if the explicit scheme without reaction term satisfies

$$\min_j u_j^k \;\le\; v_i^{k+1} \;\le\; \max_j u_j^k$$

for all i and all $k \ge 0$.

(c) Determine a stability criterion for τ for the modified explicit scheme (2), if $h_1 = h_2 = 1$, $0 < g(s^2) \le 5$, and $\alpha = 10$.

(d) For the same parameters as stated in Part (c), determine a stability criterion for τ for the fully explicit scheme (3).
Hint: Use Parts (a)–(c) as a guideline.

Problem H6.2 (Primal-Dual Hybrid Gradient Algorithm) 6P

In the primal-dual hybrid gradient (PDHG) algorithm, the primal variable $\boldsymbol{u}$ and the dual variable $\boldsymbol{b}$ are obtained by the following iteration:

$$\begin{aligned}
\boldsymbol{b}^{k+1} &= \underset{\boldsymbol{b}\in\mathbb{R}^{2N}}{\operatorname{argmax}} \left\{ -\iota_{C_\alpha}(\boldsymbol{b}) + \left\langle \boldsymbol{b}, \boldsymbol{\mathcal{D}}\boldsymbol{u}^k \right\rangle - \frac{1}{2\tau} \left\| \boldsymbol{b} - \boldsymbol{b}^k \right\|_2^2 \right\}, \\
\boldsymbol{u}^{k+1} &= \underset{\boldsymbol{u}\in\mathbb{R}^{N}}{\operatorname{argmin}} \left\{ \frac{1}{2} \left\| \boldsymbol{u} - \boldsymbol{f} \right\|_2^2 + \left\langle \boldsymbol{\mathcal{D}}^\top \boldsymbol{b}^{k+1}, \boldsymbol{u} \right\rangle + \frac{1}{2\sigma} \left\| \boldsymbol{u} - \boldsymbol{u}^k \right\|_2^2 \right\}.
\end{aligned} \tag{4}$$

Show that PDHG can be written as

$$\begin{aligned}
\boldsymbol{b}^{k+1} &= P_{C_\alpha}\left(\boldsymbol{b}^k + \tau \boldsymbol{\mathcal{D}}\boldsymbol{u}^k \right), \\
\boldsymbol{u}^{k+1} &= \frac{1}{1+\frac{1}{\sigma}} \left(\boldsymbol{f} - \boldsymbol{\mathcal{D}}^\top \boldsymbol{b}^{k+1} + \frac{1}{\sigma}\boldsymbol{u}^k \right).
\end{aligned}$$

In particular, show that $\boldsymbol{u}^{k+1}$ is a unique minimiser of Equation (4).

Problem H6.3 (Primal-Dual Methods for TV Regularisation)

1+1+5+3P

(a) The program `tv-kacanov.c` implements the differentiable ε-approximation of TV regularisation by means of the Kačanov method (cf. Lecture 11). The Gauss-Seidel algorithm serves as simple (and fairly slow) iterative solver for the linear systems of equations.
Compile it with `gcc -Wall -O2 -o tv-kacanov tv-kacanov.c -lm` and use it to denoise the image `pruebab1.pgm` with regularisation parameter $\alpha = 140$ and approximation parameter $\varepsilon = 0.01$. Use 50 outer fixed point iterations and 30000 inner Gauss-Seidel iterations to obtain the filtered image `pruebab1-kacanov.pgm`.

(b) The program `tv-fbs.c` is a forward-backward splitting algorithm for TV regularisation that does not require any ε-approximation. Compile it with `gcc -Wall -O2 -o tv-fbs tv-fbs.c -lm` and use it for denoising `pruebab1.pgm` with $\alpha = 140$, $\tau = 0.2$, and 100000 iterations. This creates your reference solution `pruebab1-ref.pgm`.
(However, please keep in mind that this cannot be a perfect reference solution, since our discrete models for the primal–dual methods have a directional bias.)

(c) Supplement the missing code in `tv-fista.c` such that you obtain an implementation of the FISTA algorithm for TV regularisation. Compile it with `gcc -Wall -O2 -o tv-fista tv-fista.c -lm`

(d) Run both `tv-fbs` and `tv-fista` on `pruebab1.pgm` with $\alpha = 140$ and $\tau = 0.2$ (for FBS), but use only 100 iterations. This gives the images `pruebab1-fbs100.pgm` and `pruebab1-fista100.pgm`. Check their accuracy and the accuracy of `pruebab1-kacanov.pgm` w.r.t. the reference solution `pruebab1-ref.pgm` by means of the program `difference.c`. This creates the error images `error-fbs100.pgm`, `error-fista100.pgm`, and `error-kacanov.pgm`. What are your conclusions?

Submission: Please create a directory `Ex06_<your_name>` with the following files (and nothing else):

- a pdf file – which can also be a scanned handwritten solution – that contains
  - the names of all people working together for this assignment
  - the solutions of the theoretical Problems 1–2 and answers to the questions in Problem 3
- for Problem 3: the completed file `tv-fista.c` and the seven images `pruebab1-kacanov.pgm`, `pruebab1-ref.pgm`, `pruebab1-fbs100.pgm`, `pruebab1-fista100.pgm`, `error-fbs100.pgm`, `error-fista100.pgm`, and `error-kacanov.pgm`

Compress the directory to a zip file `Ex06_<your_name>.zip`.

Submit the file via CMS.
Deadline for submission is Friday, December 8, 14:00.